EIGENOSCILLATIONS OF MECHANICAL SYSTEMS WITH BOUNDARY CONDITIONS CONTAINING THE FREQUENCY

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Abstract. The problem of eigenoscillations of beam-mass systems is investigated and four examples are developed. For such systems the corresponding Sturm-Liouville problems contain the eigenvalue parameter in the boundary conditions. It is shown that the eigenfunctions for the systems considered form a basis of the appropriate Hilbert space. Rayleigh-Ritz formulas are also developed. Some lower bound estimations for eigenfrequencies are also found.

1. Introduction. As is well known, the set of eigenfunctions of a positive selfadjoint second-order differential operator on $L^2[0, d]$ forms a basis of the space involved [1, p. 167]. There exists diverse literature [1] devoted to the problem of determining when a system of eigenfunctions forms a basis for the appropriate Hilbert space. The goal of this work is to contribute to this eigenfunction problem for certain mechanical systems that are defined by a Sturm-Liouville problem with an eigenvalue appearing in the boundary conditions. The intention of this paper is to show how known methods and results from the theory of operators on a Hilbert space can be applied to a class of eigenoscillation problems in mechanics. Appropriate subspaces of the Sobolev spaces $H^k[0, d]$, $k = 1, 2$ are used with alternative inner products that are equivalent to the Sobolev inner product and yet that allow the operator to be selfadjoint on the space. The examples presented are taken from the recent engineering and physics literature (see, for example, [2]–[5]) and, it is thought, progressively show the applicability of these operator-theoretic methods to such eigenoscillation problems.

The basic mechanical system considered is that of free oscillations of a string with fixed end points. The vertical displacement $u(z)$, $z \in [0, d]$, is the solution of the Sturm-Liouville problem

$$Tu'' + \omega^2 \mu u = 0, \quad z \in (0, d), \quad u(0) = u(d) = 0. \quad (1.1)$$

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Here T and μ are the parameters of a string of length $d > 0$, and the parameter ω is the frequency of oscillations. If the characteristics T and μ of the model are constant, the Sturm-Liouville problem (1.1) is quite straightforward and the eigenfunctions $u_n(z) = \sin(n\pi z/d)$, $n = 1, 2, \dots$, form an orthogonal basis of $L^2[0, d]$.

In order to consider more general oscillation problems two definitions of relevant Sobolev spaces [1] are recalled.

DEFINITION 1.1. The Sobolev space $H^k[0, d]$ is the Hilbert space consisting of all functions $u \in L^2[0, d]$ that have generalized derivatives $u^{(j)} \in L^2[0, d]$ for each $j = 1, \dots, k$, with the inner product

$$\langle u_1, u_2 \rangle_{H^k} = \int_0^d \sum_{j=0}^k u_1^{(j)} \bar{u}_2^{(j)} dz \quad (1.2)$$

and corresponding norm $\|u\|_{H^k} = \langle u, u \rangle_{H^k}^{1/2}$.

The complex conjugation of u is denoted by $\bar{u}$. The notation $\langle u_1, u_2 \rangle$ is used for the standard inner product in $L^2[0, d]$, and $\|u\|$ is used for the corresponding norm. In various sections of this work, additional inner products will be introduced to fit the situation being discussed in that section. Each of these inner products will be denoted by $\langle u, v \rangle_1$.

DEFINITION 1.2. The Sobolev space $H^{0,k}[0, d]$ is a subspace of $H^k[0, d]$ obtained by the closure, in the norm generated by the inner product (1.2), of the set of functions in $H^k[0, d]$ that are equal to zero at both end points.

REMARK 1.1. According to the embedding theorems for the Sobolev spaces [1], [6], [7], the functions in $H^2[0, d]$ are continuously differentiable on $[0, d]$, though the functions in $H^1[0, d]$ can only be assumed to be continuous.

A method is now developed to show that the eigenfunctions for the Sturm-Liouville problem

$$u'' + q(z)u = \lambda u, \quad z \in (0, d), \quad u(0) = u(d) = 0 \quad (1.3)$$

form a basis of $L^2[0, d]$ (see also [1, p. 169]). The approach used in Section 2 will follow this method. It is assumed that the continuous function $q(z)$ keeps its sign on the interval $[0, d]$. In fact, after the shift of the spectral parameter

$$\lambda \rightarrow \lambda + Q, \quad Q = \max q(z) \quad \text{for } z \in [0, d],$$

the function $q(z)$ can be assumed to be negative everywhere on $[0, d]$.

The concept of a generalized solution [1], [6], [7] is fundamental to this work. To derive this concept for system (1.3) multiply the differential equation by the conjugate of an arbitrary $\eta \in H^{0,1}[0, d]$ and integrate by parts over the interval $[0, d]$. Applying the boundary conditions produces the following.

DEFINITION 1.3. The element $u \in H^{0,1}[0, d]$ is called a generalized solution of the Sturm-Liouville problem (1.3) if for any $\eta \in H^{0,1}[0, d]$

$$-\int_0^d u' \bar{\eta}' dz + \int_0^d q(z) u \bar{\eta} dz = \lambda \int_0^d u \bar{\eta} dz. \quad (1.4)$$

Thus, a solution of the Sturm-Liouville problem (1.3) satisfies (1.4) for all $\eta \in H^{0,1}[0, d]$. To consider the converse, suppose a generalized solution u belongs to $C^2[0, d]$.

Then integrate the first term in the integral identity (1.4) by parts. Since η is arbitrary, it follows from the Variational Lemma [8, p. 18] that u will solve the Sturm-Liouville problem (1.3).

Now introduce a new inner product in $H^{0,1}[0, d]$

$$\langle u, \eta \rangle_1 = \int_0^d (u' \bar{\eta}' - q(z) u \bar{\eta}) dz. \quad (1.5)$$

This inner product generates a norm that is equivalent to the standard norm produced by (1.2) for $k = 1$. The Riesz representation theorem [1] guarantees that

$$\int_0^d u \bar{\eta} dz = \langle Au, \eta \rangle_1 \quad (1.6)$$

for an appropriate linear operator $A: H^{0,1}[0, d] \rightarrow H^{0,1}[0, d]$. The operator A is positive because of the integral definition (1.6). The compactness of A can be shown using the embedding theorems for Sobolev spaces (Sobolev inequalities) [1], [6], [7]. As an example, by using the explicit representation for A in terms of the Green's function for the operator $L = d^2/dz^2 + q(z)$ it can be shown that all properties of A mentioned hold.

Equations (1.4)–(1.6) imply

$$-\langle u, \eta \rangle_1 = \lambda \langle Au, \eta \rangle_1$$

for all $\eta \in H^{0,1}[0, d]$. Thus, the above discussion yields the following result for the operator A on the Hilbert space $H^{0,1}[0, d]$, [1, pp. 115, 118].

THEOREM 1.1. The generalized solution u of the Sturm-Liouville problem (1.3) satisfies the equation

$$-u = \lambda Au \quad (1.7)$$

in $H^{0,1}[0, d]$, where A is the positive compact operator on $H^{0,1}[0, d]$ defined by (1.6). This equation has a real, discrete spectrum with the only point of accumulation at infinity. The set of eigenfunctions forms an orthogonal basis of $H^{0,1}[0, d]$.

REMARK 1.2. The density of $H^{0,1}[0, d]$ in $L^2[0, d]$ implies that the set of eigenfunctions of (1.7) also forms a basis of $L^2[0, d]$. Equations (1.6)–(1.7) imply that this basis is orthogonal in $L^2[0, d]$.

The mechanical systems considered in Section 2 have the characteristic property that their eigenoscillations are formulated as a Sturm-Liouville problem with an eigenvalue parameter contained in the boundary conditions. Therefore, the previous classical techniques do not directly apply. In this case the spectral problem (1.7) must be replaced by a problem for an operator polynomial according to the terminology introduced by M. V. Keldysh [9], [10].

In the paper [9], Keldysh considers operator polynomials of an arbitrary order m . These operator polynomials will be important in the examples of Section 2. The main idea of [9] is the conversion to a new Hilbert space, denoted by Ξ^m , which is the orthogonal sum of m copies of the original space Ξ . As a result, an equation is obtained in which the spectral parameter appears linearly. In case $m = 1$, the simplest operator polynomial

arises as

$$L(\lambda)u = 0, \quad L(\lambda) = I - T - \lambda V. \quad (1.8)$$

Here I is the identity; T and V are operators acting on Ξ . This operator polynomial will be central to the results of Section 2.

Some terminology is needed to state Keldysh's result. Namely, a selfadjoint operator V in Ξ is complete if the equation $V\psi = 0$ has only a trivial solution. The operator V has finite order if for its eigenvalues λ_n , $n = 1, 2, \dots$, there exists some finite p such that the following series converges:

$$\sum_{n \geq 1} \lambda_n^{-p} < \infty.$$

The vectors u_n , $n = 1, 2, \dots, k$, in Ξ are said to be associated with the eigenvector u defined by $L(\lambda)u = 0$ provided that

$$L(\lambda)u_r = Vu_{r-1}, \quad r = 1, \dots, k, \quad u_0 = u.$$

THEOREM 1.2 (Keldysh). If V is a selfadjoint complete operator of finite order and T is a compact operator, then the system of eigenvectors and associated vectors of the spectral problem (1.8) is complete in Ξ .

The literature for Sturm-Liouville problems where the spectral parameter appears in the boundary conditions is diverse. The case where the spectral parameter appears linearly in the boundary conditions is discussed first.

Fulton in [11], [12] employed the residue calculus in a manner similar to Titchmarsh [13] to give a direct proof of the convergence properties of the eigenfunction expansion. In [12] the well-known problem of cooling a thin solid bar one end of which is placed in contact with a finite amount of liquid is considered. It is assumed that heat flows from the bar only into the liquid and is convected from the liquid to a surrounding medium; Newton's law of cooling is assumed at the liquid-solid interface. One special case of this problem was considered by Langer [14] (see also Friedman [15]). The general problem was the motivation for [11]–[12], [14]–[16], as well as other authors. Walter [16] obtained the expansion theorem using the selfadjointness of the operator associated with the Sturm-Liouville problem. Note, it is important to emphasize that the Hilbert space $\Xi = L^2[0, d] \oplus C$ is used systematically in [11], [12], [15], [16] to get a selfadjoint operator. Bobrovnikskii [17] considered oscillations of some mechanical systems and established the orthogonality relations for eigenfunctions of the corresponding Sturm-Liouville problems with the spectral parameter in the boundary conditions. Those relations contain the terms outside the integral. Note, the structure of functional space mentioned above evidently suggests the presence of these terms. The problem of completeness of eigenfunctions is not discussed in [17]. Hinton [18], [19] considered the same family of Sturm-Liouville problems as Fulton [11], [12], but he included the case of a singular end point. He obtained the convergence theory, uniform and absolute on compact intervals, for the eigenfunction expansions associated with the operator of the Sturm-Liouville problem. For the case where all eigenvalues of this operator are nonnegative, he proved that the class of functions where convergence takes place is the domain of

the square root of the operator (energy space). In [20], [21] Hinton and Shaw consider the Sturm-Liouville problem for a vector with the spectral parameter in the boundary conditions. They prove the existence of a Titchmarsh-Weyl coefficient $M(\lambda)$ and use it to construct the Green's function and resolvent operator. The spectrum of the operator is found to be connected with the singular structure of $M(\lambda)$. Hinton and Shaw also give a list of some engineering models where such Sturm-Liouville problems appear. Some of those problems will be mentioned below. Steinrück [22] considered the singular perturbed Sturm-Liouville problem for a couple of functions. He gave an asymptotic analysis of the problem but did not consider completeness of eigenfunctions. Binding and Ye [23] considered the operator polynomial $L(\lambda) = A - \lambda B$, which is similar to (1.8). Here A and B are selfadjoint. The pair (A, B) is simultaneously diagonalizable, in a certain sense, and a variational approach is developed.

The case where the spectral parameter appears nonlinearly in the boundary conditions is more complex. The general results about operator polynomials including the multidimensional case can be found in monographs by Gohberg and Krein [10] and Rodman [24] (see also Radzievskii [25]). Greenberg and Babuska [26], [27] considered Sturm-Liouville problems for second-order and fourth-order differential equations. Here the eigenvalue may occur nonlinearly in the differential operator and in the boundary conditions. Using variational properties of eigenvalues and the Sturm comparison theorems, the authors found bounds for eigenvalues. Among many other papers, we would like to mention only Kostyuchenko and Shkalikov [28], [29] who considered an operator-differential equation in connection with oscillations of an elastic semi-infinite cylinder. They reduced the problem to an operator polynomial equation of the second order

$$L(\lambda)u = 0 \quad \text{with} \quad L(\lambda) = A + \lambda B + \lambda^2 C. \quad (1.9)$$

Here A is bounded and selfadjoint, and B and C are compact and selfadjoint. The authors assume the existence of a constant c such that the operator $L(c)$ is positive and invertible. They proved the completeness of eigenfunctions by studying the resolvent (see [13], [11], [12]). Note, this assumption will be proved for the operator polynomials considered in Section 2. Additional related references from the Russian literature can be found in [10, 25, 28, 29].

Chen et al [30] analyzed the spectrum of two identical coupled beams with dissipative joints. In particular, the asymptotic behavior of the eigenvalues is found. Chen did not consider the problem of the completeness of eigenfunctions. However, the case of dissipation of energy is not considered in this paper. The technique developed here does not apply (at least not directly) to problems where the corresponding operators are not selfadjoint.

The above description shows that diverse experience has been accumulated in operator theory for the proof that a set of eigenfunctions of the Sturm-Liouville problem with the spectral parameter in the boundary conditions forms a basis. Note especially the methods of [10].

The last result to be mentioned deals with Riesz bases [10].

THEOREM 1.3. Every bounded invertible operator transforms any orthonormal basis of Ξ into another basis of Ξ , a so-called Riesz basis.

The main objective of this paper is to prove that the set of eigenfunctions for certain Sturm-Liouville problems arising from mechanics forms a basis for the corresponding space. Note that the Sturm-Liouville problems considered in this work cannot be reduced to the problems considered in [11], [12], [15], [16], though the equations in these references also contain the spectral parameter in boundary conditions. In Section 2, a cable-mass and several beam-mass systems are considered. The main result from abstract operator theory used in that section is Theorem 1.3. The development is such that the associated vectors of Keldysh are not needed in order to obtain a basis of the space Ξ . The examples are arranged in order of increasing complexity.

The following result is a direct consequence of the polar identity [31, p. 4]. This inequality is used below in various places to show that an operator is positive.

LEMMA 1.1. $\pm 2 \operatorname{Re}(\int u \bar{\eta} dz) \geq -\|u\|^2 - \|\eta\|^2$.

2. The eigenfunctions of cable-mass and beam-mass systems. Various cable-mass and beam-mass systems are considered in [2]–[5]. However, these authors deal only with eigenfrequencies and their calculations are formal. No existence theorems for eigenfrequencies are proved and no properties of eigenfunctions are discussed except for some numerical results.

In this work, the corresponding Sturm-Liouville problem for each of the cable-mass and beam-mass systems of [2]–[5] is used to define the generalized solution utilizing integral identities. The identity is then reduced to an operator equation in the appropriate Hilbert space. The operator techniques mentioned above are used to prove that the set of eigenfunctions of each system forms a basis of the corresponding Hilbert space. Some estimations for eigenfrequencies from below are found.

2.1. *A beam of non-uniform cross section with one end elastically restrained and the other end carrying a guided mass.* Bambill and Laura [2] conduct numerical experiments for determining the fundamental frequency of vibration for this system using the Rayleigh-Ritz method.

Such a system is described by the Sturm-Liouville problem

$$(\beta(z)u'')' - \omega^2 q(z)u = 0, \quad z \in (0, d), \quad (2.1a)$$

$$u'(0) = u(d) = 0, \quad EI_0 u'''(0) = \omega^2 M u(0), \quad -\Phi EI_1 u''(d) = u'(d). \quad (2.1b)$$

The mechanical parameters E , I_0 , I_1 , Φ , M are assumed to be positive constants that do not contain the spectral parameter ω . The function $\beta(z)$ is assumed to be bounded, measurable, Lebesgue integrable, positive definite on the interval $[0, d]$ and reduces to a (positive) constant near the point $z = 0$. Note that β was taken to be piecewise constant by Bambill and Laura [2]. The function $q(z)$ is assumed to be bounded, measurable, Lebesgue integrable, and positive on the interval $[0, d]$. The Hilbert space Ξ is taken to be the subspace of functions in $H^2[0, d]$ satisfying $u'(0) = u(d) = 0$. (See Remark 1.1.)

The generalized eigenfunctions of the Sturm-Liouville problem (2.1) in the space Ξ are now defined as in Definition 1.3. (See Remark 1.1.)

DEFINITION 2.1. The element $u \in \Xi$ is called a generalized solution of the Sturm-Liouville problem (2.1) if for any $\eta \in \Xi$ the following identity holds:

$$\begin{aligned} \int_0^d \beta(z) u'' \bar{\eta}'' dz + \frac{\beta(d) u'(d) \bar{\eta}'(d)}{\Phi E I_1} \\ = \omega^2 \left(\int_0^d q(z) u \bar{\eta} dz + \frac{M \beta(0) u(0) \bar{\eta}(0)}{E I_0} \right). \end{aligned} \quad (2.2)$$

Now introduce a new inner product on Ξ as follows:

$$\langle u, \eta \rangle_1 = \int_0^d \beta(z) u'' \bar{\eta}'' dz + \frac{\beta(d) u'(d) \bar{\eta}'(d)}{\Phi E I_1}. \quad (2.3)$$

This inner product generates a norm that is equivalent to the original norm of $u \in \Xi$; see (1.2) for $k = 2$.

LEMMA 2.1. For the bilinear functional

$$\Lambda(u, \eta) = \int_0^d q(z) u \bar{\eta} dz + \frac{M \beta(0) u(0) \bar{\eta}(0)}{E I_0} \quad (2.4)$$

the following representation holds:

$$\Lambda(u, \eta) = \langle Au, \eta \rangle_1 \quad (2.5)$$

with a positive compact operator A .

The proof is given in Section 3.

The representation (2.5) and the identity (2.2) result in the generalized eigenfunctions of (2.1) satisfying the following equation in Ξ :

$$u = \omega^2 Au. \quad (2.6)$$

THEOREM 2.1. The eigenfrequencies of the Sturm-Liouville problem (2.1) form a real discrete set with the only point of accumulation at infinity. The system of generalized eigenfunctions forms an orthogonal basis of Ξ , and the corresponding eigenfunction expansion converges in the Ξ -norm.

Though the spectral parameter ω^2 is contained in the boundary conditions (2.1b), the final equation (2.6) has the simple structure of (1.7). Theorem 2.1 then follows immediately as in Theorem 1.1.

Let $\eta = u$ in (2.2) and assume the eigenfunctions are real valued. Then Eq. (2.2) implies that

$$\begin{aligned} \int_0^d \beta(z) (u'')^2 dz + \frac{\beta(d) u'(d)^2}{\Phi E I_1} \\ = \omega^2 \left(\int_0^d q(z) u^2 dz + \frac{M \beta(0) u(0)^2}{E I_0} \right). \end{aligned}$$

This formula coincides with the formula used by Bambill and Laura [2, Eq. (6)] for numerical calculations of the eigenfrequencies. Using this equation, a Rayleigh-Ritz

procedure (e.g., see Stakgold [32]) can be developed as

$$\omega^2 = \frac{\int_0^d \beta(z)(u'')^2 dz + \frac{\beta(d)u'(d)^2}{\Phi EI_1}}{\int_0^d q(z)u^2 dz + \frac{M\beta(0)u(0)^2}{EI_0}}. \quad (2.7)$$

2.2. *A rotating beam with a tip mass.* Transverse vibrations of a rotating beam with a tip mass, such as a helicopter blade or bobbed pendulum, have been considered by many authors (see [33], [34], [35]). By using the notation from [20] the corresponding Sturm-Liouville problem has the form

$$(z''' - q(x)z')' = \lambda z, \quad x \in (0, 1), \quad (2.8a)$$

$$z''(0) = 0, \quad z(1) = z'(1) = 0, \quad z'''(0) - c^2\gamma^2 z'(0) = \lambda\gamma^2 z(0). \quad (2.8b)$$

Here c and γ are constants such that $c^2\gamma^2 = q(0)$. The method developed by Hinton and Shaw [20], [21] allows them to include this problem in the general scheme and to get important results about the spectrum. Here the basis property of the eigenfunctions is discussed briefly because the construction is similar to those used in Section 2.1. The generalized solution of the problem (2.8a), (2.8b) is defined in the space Ξ , which is the subspace of functions in $H^2[0, 1]$ satisfying $z(1) = z'(1) = 0$. The identity similar to (2.2) has the form

$$\int_0^1 (z''\bar{\eta}'' + q(x)z'\bar{\eta}') dx = \lambda \left(\int_0^1 z\bar{\eta} dx + \gamma^2 z(0)\bar{\eta}(0) \right). \quad (2.9)$$

The inner product $\langle z, \eta \rangle_1$ is introduced as the left-hand side of the identity (2.9). The bilinear functional $\Lambda(z, \eta)$ in the right-hand side can be represented as follows:

$$\Lambda(z, \eta) = \langle Az, \eta \rangle_1 \quad (2.10)$$

with an appropriate positive linear compact operator A . As a result the following equation for the generalized eigenfunctions in Ξ appears:

$$z = \lambda Az \quad (2.11)$$

and the theorem similar to 2.1 holds. The formula for the eigenvalues similar to (2.7) has the form

$$\lambda = \frac{\int_0^1 ((z'')^2 + q(x)(z')^2) dx}{\int_0^1 z^2 dx + \gamma^2 z(0)^2}.$$

2.3. *A sagged cable supporting a discrete mass.* Cheng and Perkins [3] derive the system of equations for an asymptotic model for such a system with small attached mass. A dispersion equation is considered, and frequencies and the shape of the eigenfunctions (modes) are numerically calculated. Additional references related to the problem of behavior of an oscillating cable with tip mass can be found in [20]. In particular, this problem is associated with the feedback control of large flexible space structures [36].

Consider the Sturm-Liouville problem

$$u'' + \omega^2 u = s^2 \left(\int_0^d u \, dz + mu(z_0) \right), \quad z \in (0, z_0) \cup (z_0, d), \quad (2.12a)$$

$$u(0) = u(d) = 0, \quad u(z_0^+) = u(z_0^-), \quad (2.12b)$$

$$u'(z_0^+) - u'(z_0^-) + m\omega^2 u(z_0) = ms^2 \left(\int_0^d u \, dz + mu(z_0) \right). \quad (2.12c)$$

The mechanical constants m and s^2 are positive and do not contain ω .

DEFINITION 2.2. The Hilbert space Ξ consists of all functions u such that $u \in H^1[0, z_0]$, $u \in H^1[z_0, d]$, $u(0) = u(d) = 0$, and $u(z_0^+) = u(z_0^-)$. (See Remark 1.1.)

DEFINITION 2.3. The element $u \in \Xi$ is the generalized solution of the Sturm-Liouville problem (2.12) if for any $\eta \in \Xi$ the following identity holds:

$$\begin{aligned} \int_0^d u' \bar{\eta}' \, dz + m^2 s^2 u(z_0) \bar{\eta}(z_0) + s^2 \int_0^d u \, dz \int_0^d \bar{\eta} \, dz \\ + ms^2 \left(u(z_0) \int_0^d \bar{\eta} \, dz + \bar{\eta}(z_0) \int_0^d u \, dz \right) = \omega^2 \left(\int_0^d u \bar{\eta} \, dz + mu(z_0) \bar{\eta}(z_0) \right). \end{aligned} \quad (2.13)$$

Let us introduce the new inner product in Ξ :

$$\langle u, \eta \rangle_1 = \int_0^d u' \bar{\eta}' \, dz + m^2 s^2 u(z_0) \bar{\eta}(z_0) + s^2 \int_0^d u \, dz \int_0^d \bar{\eta} \, dz \quad (2.14)$$

and the corresponding norm $\|u\|_1 = \langle u, u \rangle_1^{1/2}$, which is equivalent to the original norm (see (1.2) for $k = 1$).

LEMMA 2.2. For the bilinear functionals

$$\Lambda_1(u, \eta) = \int_0^d u \bar{\eta} \, dz + mu(z_0) \bar{\eta}(z_0), \quad (2.15a)$$

$$\Lambda_2(u, \eta) = ms^2 \left(u(z_0) \int_0^d \bar{\eta} \, dz + \bar{\eta}(z_0) \int_0^d u \, dz \right), \quad (2.15b)$$

the following representations hold:

$$\Lambda_k(u, \eta) = \langle A_k u, \eta \rangle_1, \quad k = 1, 2, \quad (2.16)$$

with a positive operator A_1 . Both A_1 and A_2 are compact and selfadjoint.

Proof. The proof is discussed in Section 3.

The representations (2.16) and the identity (2.13) result in the identity

$$\langle u, \eta \rangle_1 + \langle A_2 u, \eta \rangle_1 = \omega^2 \langle A_1 u, \eta \rangle_1. \quad (2.17)$$

To summarize, if u is a generalized solution of the Sturm-Liouville problem (2.12), then (2.17) holds for all $\eta \in \Xi$. Therefore, an operator polynomial equation arises instead of the simplest equation (2.6), and the following result is valid.

LEMMA 2.3. The generalized eigenfunctions of the Sturm-Liouville problem (2.12) satisfy the operator equation in Ξ

$$L(\omega^2)u = 0, \quad L(\omega^2) = I + A_2 - \omega^2 A_1. \quad (2.18)$$

But, recall that this operator polynomial will not be analyzed using the Keldysh result in Theorem 1.2. Instead, the following result is needed in order to use Theorem 1.3.

LEMMA 2.4. The operator polynomial $L(-\lambda_0)$ is positive definite for sufficiently large positive values of λ_0 .

Proof. The equations (2.14), (2.15), (2.16), and (2.18) give the formula

$$\begin{aligned} \langle L(-\lambda_0)u, u \rangle_1 &= \|u'\|^2 + (\lambda_0 m + m^2 s^2) |u(z_0)|^2 + s^2 \left| \int_0^d u \, dz \right|^2 \\ &\quad + 2ms^2 \operatorname{Re} \left(u(z_0) \int_0^d \bar{u} \, dz \right) + \lambda_0 \|u\|^2. \end{aligned}$$

Recall that $\|\psi\|$ is the L^2 -norm according to notation in Section 1. The next estimate follows from Lemma 1.1 by noting that $u(z_0)$ is a constant function,

$$2 \operatorname{Re} \left(u(z_0) \int_0^d \bar{u} \, dz \right) \geq -\|u\|^2 - d |u(z_0)|^2.$$

Thus, utilizing (2.14), for sufficiently large positive λ_0 ,

$$\begin{aligned} \langle L(-\lambda_0)u, u \rangle_1 &\geq \|u'\|^2 + (\lambda_0 m + m^2 s^2 - ms^2 d) |u(z_0)|^2 \\ &\quad + (\lambda_0 - ms^2) \|u\|^2 + s^2 \left| \int_0^d u \, dz \right|^2 \geq \|u\|_1^2. \quad \square \end{aligned}$$

After the shift of the spectral parameter $\mu = \omega^2 + \lambda_0$ the equation (2.18) has the form

$$L(-\lambda_0)u = \mu A_1 u$$

with a new spectral parameter μ . Now the transformation $v = L(-\lambda_0)^{1/2} u$ is introduced in order to apply Theorems 1.1 and 1.3. Here $L(-\lambda_0)^{1/2}$ is the positive square root of the positive selfadjoint operator $L(-\lambda_0)$ [37; 38, p. 60].

LEMMA 2.5. The transformed functions $v = L(-\lambda_0)^{1/2} u$ satisfy the operator equation

$$v = \mu W v, \quad W = L(-\lambda_0)^{-1/2} A_1 L(-\lambda_0)^{-1/2},$$

which for sufficiently large, fixed λ_0 has a positive compact selfadjoint operator W . The system of eigenfunctions of this equation forms an orthogonal basis of Ξ .

THEOREM 2.2. The eigenfrequencies of the Sturm-Liouville problem (2.7) form a real discrete set with the only point of accumulation at infinity. The system of generalized eigenfunctions forms a Riesz basis of Ξ , and the corresponding eigenfunction expansion converges in the Ξ -norm.

Proof. The proof follows from Lemma 2.5 by applying Theorem 1.3 to the bounded invertible operator $L(-\lambda_0)^{1/2}$. The operator $L(-\lambda_0)^{1/2}$ is invertible since it is selfadjoint and the operator $L(-\lambda_0)$ is positive definite and bounded [38, p. 63]. $\square$

A formula for eigenfrequencies similar to (2.7) can be obtained from Eq. (2.13) as follows:

$$\omega^2 = \frac{\int_0^d (u')^2 dz + s^2 \left(mu(z_0) + \int_0^d u dz \right)^2}{\int_0^d u^2 dz + mu(z_0)^2}. \quad (2.19)$$

2.4. *A Timoshenko-Mindlin beam carrying a concentrated mass and a moment of inertia on the end.* Maurizi and Bellés ([4], see also [39]) derive the frequency equation for such a system and numerically obtain the natural frequencies of the first three eigenfunctions (modes).

Consider the Sturm-Liouville problem

$$D\psi'' - \alpha^2 Gh(\psi + \xi') + \frac{\rho h^3}{12} \omega^2 \psi = 0, \quad z \in (0, d), \quad (2.20a)$$

$$\alpha^2 Gh(\xi'' + \psi') + \rho h \omega^2 \xi = 0, \quad \psi(0) = \xi(0) = 0, \quad (2.20b)$$

$$D\psi'(d) - J\omega^2 \psi(d) = 0, \quad \alpha^2 Gh(\psi(d) + \xi'(d)) - M\omega^2 \xi(d) = 0. \quad (2.20c)$$

The mechanical constants D , $\alpha^2 Gh$, ρ , J , and M are positive and do not contain ω .

DEFINITION 2.4. The Hilbert space Ξ is the subspace of $H^1[0, d] \oplus H^1[0, d]$ consisting of all two-component functions $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with $\psi_1(0) = \psi_2(0) = 0$ and inner product

$$\langle \Psi^{(1)}, \Psi^{(2)} \rangle_{H^1} = \langle \psi_1^{(1)}, \psi_1^{(2)} \rangle_{H^1} + \langle \psi_2^{(1)}, \psi_2^{(2)} \rangle_{H^1}, \quad (2.21)$$

where $\langle u, v \rangle_{H^1}$ denotes the standard scalar product (1.2) of $H^1[0, d]$.

DEFINITION 2.5. The element $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is the generalized solution of the Sturm-Liouville problem (2.20) if for any $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \Xi$ the following identities hold:

$$D \int_0^d \psi_1' \bar{\eta}_1' dz + \alpha^2 Gh \int_0^d (\psi_1 + \psi_2') \bar{\eta}_1 dz - w^2 \left(\frac{\rho h^3}{12} \int_0^d \psi_1 \bar{\eta}_1 dz + J \psi_1(d) \bar{\eta}_1(d) \right) = 0, \quad (2.22a)$$

$$\alpha^2 Gh \left(\int_0^d \psi_2' \bar{\eta}_2' dz + \int_0^d \psi_1 \bar{\eta}_2' dz \right) - w^2 \left(\rho h \int_0^d \psi_2 \bar{\eta}_2 dz + M \psi_2(d) \bar{\eta}_2(d) \right) = 0. \quad (2.22b)$$

Introduce the new inner product in both copies of the space $H^1[0, d]$ and in Ξ ,

$$\langle \psi_k, \eta_k \rangle_1 = \int_0^d \psi_k' \bar{\eta}_k' dz, \quad k = 1, 2, \quad (2.23a)$$

$$\langle \Psi, \eta \rangle_1 = \langle \psi_1, \eta_1 \rangle_1 + \langle \psi_2, \eta_2 \rangle_1. \quad (2.23b)$$

Here $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. The corresponding norm $\|\Psi\|_1 = \langle \Psi, \Psi \rangle_1^{1/2}$ is equivalent to the original norm (see (2.21)).

LEMMA 2.6. The following representations are valid:

$$\int_0^d \psi_k \bar{\eta}_k dz = \langle A\psi_k, \eta_k \rangle_1, \quad k = 1, 2, \quad (2.24a)$$

$$\int_0^d \psi'_2 \bar{\eta}_1 dz = \langle B\psi_2, \eta_1 \rangle_1, \quad (2.24b)$$

$$\psi_k(d) \bar{\eta}_k(d) = \langle C\psi_k, \eta_k \rangle_1, \quad k = 1, 2, \quad (2.24c)$$

$$\int_0^d \psi_1 \bar{\eta}'_2 dz = \langle B^*\psi_1, \eta_2 \rangle_1 \quad (2.24d)$$

with the positive operators A and C . The operators A , B , and C are compact. B^* here and below denotes an operator conjugate to B .

Proof. The proof can be done in the same way as for Lemmas 2.1, 2.2. A direct proof is also possible. For example, an explicit representation for B is

$$B\psi_2 = \psi_2(d)z - \int_0^z \psi_2(t) dt.$$

Now the compactness of B is clear. □

The representations (2.24) and identities (2.22) result in the following operator polynomial.

LEMMA 2.7. The generalized eigenfunctions of the Sturm-Liouville problem (2.20) satisfy the operator equation in Ξ ,

$$L(\omega^2)\Psi = 0, \quad L(\omega^2) = I + U - \omega^2 V, \quad (2.25a)$$

$$U = \begin{pmatrix} a_1 A & a_1 B \\ B^* & 0 \end{pmatrix}, \quad V = \begin{pmatrix} a_2 A + a_3 C & 0 \\ 0 & a_4 A + a_5 C \end{pmatrix} \quad (2.25b)$$

where $a_1 = \frac{\alpha^2 Gh}{D}$, $a_2 = \frac{\rho h^3}{12D}$, $a_3 = \frac{J}{D}$, $a_4 = \frac{\rho h}{\alpha^2 Gh}$, $a_5 = \frac{M}{\alpha^2 Gh}$.

First, the notation is slightly changed. Set $\psi_2 \rightarrow \kappa\psi_2$, $\kappa = (a_1)^{-1/2}$, and divide the second equation by κ . As a result the operator polynomial equation becomes

$$L_1(\omega^2)\Psi = 0, \quad L_1(\omega^2) = I + U_1 - \omega^2 V, \quad U_1 = \begin{pmatrix} a_1 A & \kappa^{-1} B \\ \kappa^{-1} B^* & 0 \end{pmatrix}. \quad (2.26)$$

The following result is needed in order to use Theorem 1.3.

LEMMA 2.8. The operator polynomial $L_1(-\lambda_0)$ is a positive-definite operator for all positive values of λ_0 .

Proof. The definitions and representations (2.23)–(2.26) result in the formula

$$\begin{aligned} \langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 &= \|\psi'_1\|^2 + \|\psi'_2\|^2 + (\lambda_0 a_2 + a_1)\|\psi_1\|^2 + 2\kappa^{-1} \operatorname{Re} \langle \psi'_2, \psi_1 \rangle \\ &\quad + \lambda_0 a_3 |\psi_1(d)|^2 + \lambda_0 a_4 \|\psi_2\|^2 + \lambda_0 a_5 |\psi_2(d)|^2. \end{aligned}$$

The next estimate follows from Lemma 1.1:

$$\begin{aligned} 2 \operatorname{Re} \langle \psi'_2, \psi_1 \rangle &= 2 \operatorname{Re} \langle \varepsilon^{1/2} \psi'_2, \varepsilon^{-1/2} \psi_1 \rangle \\ &\geq -\varepsilon \|\psi'_2\|^2 - \varepsilon^{-1} \|\psi_1\|^2. \end{aligned}$$

Here ε is an arbitrary positive number. Thus

$$\begin{aligned} \langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 &\geq \|\psi'_1\|^2 + (1 - \kappa^{-1}\varepsilon)\|\psi'_2\|^2 + (a_1 + \lambda_0 a_2 - \kappa^{-1}\varepsilon^{-1})\|\psi_1\|^2 \\ &\quad + \lambda_0 a_3 |\psi_1(d)|^2 + \lambda_0 a_4 \|\psi_2\|^2 + \lambda_0 a_5 |\psi_2(d)|^2. \end{aligned}$$

Taking ε in the interval

$$\kappa > \varepsilon > \kappa^{-1}(a_1 + \lambda_0 a_2)^{-1} \quad (2.27)$$

yields

$$\langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 > \|\psi'_1\|^2 + (1 - \kappa^{-1}\varepsilon)\|\psi'_2\|^2 > (1 - \kappa^{-1}\varepsilon)\|\Psi\|_1^2.$$

The inequality (2.27) is true for any $\lambda_0 > 0$ since $a_2 > 0$ implies that $\kappa^{-1}(a_1 + \lambda_0 a_2)^{-1} < \kappa^{-1}a_1^{-1} = \kappa$. Thus, for any $\lambda_0 > 0$ the operator $L_1(-\lambda_0)$ is positive definite. $\square$

The proof that the eigenfunctions form a basis of the space can be concluded now utilizing a shift in the spectral parameter and the transformation $\Upsilon = L(-\lambda_0)^{1/2}\Psi$, as in Section 2.2.

THEOREM 2.3. The eigenfrequencies of the Sturm-Liouville problem (2.20) form a real discrete set with the only point of accumulation at infinity. The system of generalized eigenfunctions forms a Riesz basis of Ξ , and the corresponding eigenfunction expansion converges in the Ξ -norm.

Assume the eigenfunctions are real valued. Then (2.24), (2.25b), and (2.26) yield

$$\begin{aligned} \omega^2 &= \frac{\langle (I + U_1)\Psi, \Psi \rangle_1}{\langle V\Psi, \Psi \rangle_1} \\ &= \frac{\int_0^d (\psi'_1)^2 dz + \int_0^d (\psi'_2 + \kappa^{-1}\psi_1)^2 dz}{a_2 \int_0^d (\psi_1)^2 dz + a_4 \int_0^d (\psi_2)^2 dz + a_3 (\psi_1(d))^2 + a_5 (\psi_2(d))^2}. \end{aligned} \quad (2.28)$$

2.5. A Timoshenko-Mindlin beam with translational and rotational springs and with a tip mass. Abramovich and Hamburger [5] numerically derive the natural frequencies of such a cantilever Timoshenko beam.

A Sturm-Liouville problem similar to (2.20) is considered. The differential equations are identical but are satisfied for $z \in (0, z_0) \cup (z_0, d)$. The boundary conditions are

$$\psi(0) = \xi(0) = 0, \quad (2.29a)$$

$$D\psi'(d) - J_1\omega^2\psi(d) + J_2\omega^2\xi(d) = 0, \quad (2.29b)$$

$$\alpha^2 Gh(\psi(d) + \xi'(d)) - M\omega^2\xi(d) + J_2\omega^2\psi(d) = 0. \quad (2.29c)$$

In addition, the conditions at the interior point z_0 are

$$\psi(z_0^+) = \psi(z_0^-), \quad \xi(z_0^+) = \xi(z_0^-),$$

$$D(\psi'(z_0^+) - \psi'(z_0^-)) = k_r\psi(z_0),$$

$$\alpha^2 Gh(\xi'(z_0^+) - \xi'(z_0^-)) = k_t\xi(z_0).$$

The constants D , $\alpha^2 Gh$, M , J_1 , J_2 , k_r , k_t are positive, do not contain ω , and satisfy the inequality

$$J_1 M > J_2^2, \quad (2.30)$$

which can be checked easily with the help of formulas from [5]. Other notation for functional spaces, their elements, and operators remains the same as that used in Section 2.4. Except for Lemma 2.11 (below), only a brief description of results and proofs is given since they are similar to those in Section 2.4.

DEFINITION 2.6. The Hilbert space Ξ consists of all two-component functions $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ such that for each $k = 1, 2$, $\psi_k \in H^1[0, z_0]$, $\psi_k \in H^1[z_0, d]$ with $\psi_1(0) = \psi_2(0) = 0$, and ψ_k is continuous at z_0 . The inner product is introduced according to (2.23).

DEFINITION 2.7. The element $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a generalized solution of the Sturm-Liouville problem under consideration if for any $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \Xi$ the following identities hold:

$$\begin{aligned} D \int_0^d \psi_1' \bar{\eta}_1' dz + \alpha^2 Gh \int_0^d \psi_1 \bar{\eta}_1 dz + \alpha^2 Gh \int_0^d \psi_2' \bar{\eta}_2' dz + k_r \psi_1(z_0) \bar{\eta}_1(z_0) \\ = w^2 \left(\frac{\rho h^3}{12} \int_0^d \psi_1 \bar{\eta}_1 dz + J_1 \psi_1(d) \bar{\eta}_1(d) - J_2 \psi_2(d) \bar{\eta}_1(d) \right), \end{aligned} \quad (2.31a)$$

$$\begin{aligned} \alpha^2 Gh \int_0^d \psi_2' \bar{\eta}_2' dz + \alpha^2 Gh \int_0^d \psi_1 \bar{\eta}_2' dz + k_t \psi_2(z_0) \bar{\eta}_2(z_0) \\ = w^2 \left(\rho h \int_0^d \psi_2 \bar{\eta}_2 dz + M \psi_2(d) \bar{\eta}_2(d) - J_2 \psi_1(d) \bar{\eta}_2(d) \right). \end{aligned} \quad (2.31b)$$

LEMMA 2.9. The following representations are valid:

$$\psi_k(z_0) \bar{\eta}_k(z_0) = \langle A_1 \psi_k, \bar{\eta}_k \rangle_1, \quad k = 1, 2,$$

where the operator A_1 is positive and compact.

LEMMA 2.10. The generalized eigenfunctions of the Sturm-Liouville problem under consideration satisfy the operator equation in Ξ :

$$\begin{aligned} L(\omega^2) \Psi = 0, \quad L(\omega^2) = I + U - \omega^2 V, \\ U = \begin{pmatrix} a_1 A + a_6 A_1 & a_1 B \\ B^* & a_7 A_1 \end{pmatrix}, \quad V = \begin{pmatrix} a_2 A + a_8 C & -a_9 C \\ -a_{10} C & a_4 A + a_5 C \end{pmatrix}, \end{aligned}$$

where $a_6 = \frac{k_r}{D}$, $a_7 = \frac{k_t}{\alpha^2 Gh}$, $a_8 = \frac{J_1}{D}$, $a_9 = \frac{J_2}{D}$, $a_{10} = \frac{J_2}{\alpha^2 Gh}$, and the operators and other constants are given in Lemmas 2.6 and 2.7.

Substituting $\psi_2 \rightarrow \kappa \psi_2$, $\kappa = (a_1)^{-1/2}$ and dividing the second equation by κ , yields the following equation:

$$L_1(\omega^2) \Psi = 0, \quad L_1(\omega^2) = I + U_1 - \omega^2 V_1. \quad (2.32)$$

Here, instead of U and V as in (2.25), the selfadjoint operators are

$$U_1 = \begin{pmatrix} a_1 A + a_6 A_1 & \kappa^{-1} B \\ \kappa^{-1} B^* & a_7 A_1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} a_2 A + a_8 C & -\kappa_1 C \\ -\kappa_1 C & a_4 A + a_5 C \end{pmatrix} \quad (2.33)$$

with $\kappa_1 = a_9\kappa = a_{10}\kappa^{-1}$, since $a_1 = a_9/a_{10}$.

LEMMA 2.11. The operator polynomial $L_1(-\lambda_0)$ is a positive-definite operator for all positive values of λ_0 .

Proof. The beginning of the proof coincides with the proof of Lemma 2.8 (recall that $\langle u, v \rangle$ is in the inner product of $L^2[0, d]$ with norm $\|u\|$):

$$\begin{aligned} \langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 &= \|\psi'_1\|^2 + \|\psi'_2\|^2 + (a_1 + \lambda_0 a_2)\|\psi_1\|^2 + \lambda_0 a_4\|\psi_2\|^2 \\ &\quad + a_6|\psi_1(z_0)|^2 + a_7|\psi_2(z_0)|^2 + \lambda_0 a_8|\psi_1(d)|^2 + \lambda_0 a_5|\psi_2(d)|^2 \\ &\quad - 2\kappa_1\lambda_0 \operatorname{Re}(\psi_2(d)\bar{\psi}_1(d)) + 2\kappa^{-1} \operatorname{Re}\langle \psi'_2, \psi_1 \rangle. \end{aligned}$$

The next estimates follow from Lemma 1.1:

$$\begin{aligned} 2 \operatorname{Re}\langle \psi'_2, \psi_1 \rangle &= 2 \operatorname{Re}\langle \varepsilon^{1/2}\psi'_2, \varepsilon^{-1/2}\psi_1 \rangle \\ &\geq -\varepsilon\|\psi'_2\|^2 - \varepsilon^{-1}\|\psi_1\|^2, \end{aligned} \tag{2.34a}$$

$$\begin{aligned} -2 \operatorname{Re}(\psi_2(d)\bar{\psi}_1(d)) &= -2 \operatorname{Re}(\varepsilon_1^{1/2}\psi_2(d)\varepsilon_1^{-1/2}\psi_1(d)) \\ &\geq -\varepsilon_1|\psi_1(d)|^2 - \varepsilon_1^{-1}|\psi_2(d)|^2. \end{aligned} \tag{2.34b}$$

Here ε and ε_1 are two arbitrary positive numbers. Using (2.34), yields

$$\begin{aligned} \langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 &\geq \|\psi'_1\|^2 + (1 - \kappa^{-1}\varepsilon)\|\psi'_2\|^2 + (a_1 + \lambda_0 a_2 - \kappa^{-1}\varepsilon^{-1})\|\psi_1\|^2 \\ &\quad + \lambda_0 a_4\|\psi_2\|^2 + a_6|\psi_1(z_0)|^2 + a_7|\psi_2(z_0)|^2 \\ &\quad + \lambda_0(a_8 - \kappa_1\varepsilon_1)|\psi_1(d)|^2 + \lambda_0(a_5 - \kappa_1\varepsilon_1^{-1})|\psi_2(d)|^2. \end{aligned}$$

Let us take ε in the interval (2.27) and ε_1 in the interval

$$a_8\kappa_1^{-1} > \varepsilon_1 > \kappa_1 a_5^{-1}. \tag{2.35}$$

As was mentioned in the proof of Lemma 2.8, the inequalities (2.27) do not impose any restrictions on the positive parameters a_1, a_2 . The choice in (2.35) of the parameter ε_1 is possible if $a_5 a_8 > \kappa_1^2$. The representations of all these parameters (see Lemmas 2.7 and 2.10) and the inequality (2.30) show that this inequality is true. This finally yields

$$\langle L_1(-\lambda_0)\Psi, \Psi \rangle_1 > (1 - \kappa^{-1}\varepsilon)\|\Psi\|_1^2. \quad \square$$

THEOREM 2.4. The eigenfrequencies of the Sturm-Liouville problem (2.20) under boundary conditions (2.29) form a real discrete set with the only point of accumulation at infinity. The system of generalized eigenfunctions forms a Riesz basis of Ξ , and the corresponding eigenfunction expansion converges in the Ξ -norm.

The following formula for eigenfrequencies, which is similar to (2.28), can be obtained from Eqs. (2.24), (2.32), (2.33), and Lemma 2.9:

$$\begin{aligned}\omega^2 &= \frac{\langle (I + U_1)\Psi, \Psi \rangle_1}{\langle V_1\Psi, \Psi \rangle_1} \\ &= \frac{\int_0^d ((\psi'_1)^2 + (\psi'_2)^2) dz + a_1 \int_0^d (\psi_1)^2 dz + a_6(\psi_1(z_0))^2 + a_7(\psi_2(z_0))^2 + \frac{2}{\kappa} \int_0^d \psi'_2 \psi_1 dz}{a_2 \int_0^d (\psi_1)^2 dz + a_8(\psi_1(d))^2 + a_4 \int_0^d (\psi_2)^2 dz + a_5(\psi_2(d))^2 - 2\kappa_1 \psi_1(d)\psi_2(d)} \\ &= \frac{\int_0^d (\psi'_1)^2 dz + a_6(\psi_1(z_0))^2 + a_7(\psi_2(z_0))^2 + \int_0^d (\psi'_2 + \kappa^{-1}\psi_1)^2 dz}{\int_0^d (a_2(\psi_1)^2 + a_4(\psi_2)^2) dz + a_8(\psi_1(d))^2 - 2\kappa_1 \psi_1(d)\psi_2(d) + a_5(\psi_2(d))^2}.\end{aligned}\quad (2.36)$$

Recall that $a_1 = \kappa^{-2}$ and that $a_8 a_5 > (\kappa_1)^2$ from (2.30). Thus, the denominator is positive.

2.6. *Estimations for eigenfrequencies from below.* The formulas (2.7), (2.19), (2.28), and (2.36) allow the estimation of lower bounds for the eigenfrequencies

$$\omega \geq \omega_*.\quad (2.37)$$

This estimation for problems in Sections 2.1 and 2.3 is now derived explicitly in terms of mechanical parameters of the corresponding systems. Such estimations exist for the problems in Sections 2.4 and 2.5 but are sufficiently more complex to derive. The corresponding estimation for the problem in Section 2.2 is similar to those in 2.1.

For the beam (Section 2.1) the derivation is based on the representation (2.7) and the inequalities

$$\int_0^d q(z)u^2 dz \leq C_1 \langle u, u \rangle_1, \quad (2.38)$$

$$u(0)^2 \leq C_2 \langle u, u \rangle_1 \quad (2.39)$$

for appropriate constants C_1 and C_2 , which are independent of the function $u(z)$, $u \in \Xi$. Recall that the inner product is given by (2.3). Combining (2.7), (2.38), and (2.39) yields

$$\omega^2 \geq \left(C_1 + \frac{M\beta(0)C_2}{EI_0} \right)^{-1}. \quad (2.40)$$

The inequalities (2.38), (2.39) are just particular cases of the embedding theorems for the Sobolev space $H^2[0, d]$ (e.g., see [1], [6], [7]). The functions from this space are continuously differentiable on $[0, d]$. The goal is to find the explicit representation for C_1 and C_2 . Using the second boundary condition (2.1b) one finds

$$u(z) = \int_d^z u'(t) dt. \quad (2.41)$$

The Schwarz Inequality yields

$$u(z)^2 = \left(\int_d^z u'(t) dt \right)^2 \leq (d - z) \int_z^d (u'(t))^2 dt \quad (2.42)$$

for any $z \in [0, d]$. Further, with help of the first of the boundary conditions (2.1b) it follows that

$$u'(z) = \int_d^z u''(t) dt. \quad (2.43)$$

Arguments similar to those mentioned above yield

$$\begin{aligned} (u'(z))^2 &= \left(\int_0^z \frac{1}{\sqrt{\beta(t)}} \left(\sqrt{\beta(t)} u''(t) \right) dt \right)^2 \\ &\leq \int_0^z \frac{1}{\beta(t)} dt \int_0^z \beta(t) (u''(t))^2 dt \\ &\leq \int_0^z \frac{1}{\beta(t)} dt \langle u, u \rangle_1. \end{aligned} \quad (2.44)$$

Therefore,

$$\int_z^d (u'(t))^2 dt \leq \int_z^d \int_0^t \frac{1}{\beta(\tau)} d\tau dt \langle u, u \rangle_1. \quad (2.45)$$

Combining inequalities (2.42) and (2.44) yields

$$\begin{aligned} u(z)^2 &\leq (d-z) \int_z^d \int_0^t \frac{1}{\beta(\tau)} d\tau dt \langle u, u \rangle_1 \\ &= (d-z) \int_0^d y(\tau, z) \frac{1}{\beta(\tau)} d\tau \langle u, u \rangle_1 \end{aligned} \quad (2.46)$$

where the function $y(\tau, z) = d - \max(z, \tau)$ is introduced. In particular, inequality (2.39) holds with

$$C_2 = d \int_0^d \frac{d-\tau}{\beta(\tau)} d\tau. \quad (2.47)$$

Further, inequality (2.46) shows that

$$\int_0^d q(z) u^2 dz \leq \int_0^d q(z) (d-z) \int_0^d \frac{y(\tau, z)}{\beta(\tau)} d\tau dz \langle u, u \rangle_1 \quad (2.48)$$

and, therefore, the inequality (2.38) holds with

$$C_1 = \int_0^d q(z) (d-z) \int_0^d \frac{y(\tau, z)}{\beta(\tau)} d\tau dz. \quad (2.49)$$

Combining (2.37), (2.40), (2.47), and (2.49) yields

$$\omega_* = \left\{ \int_0^d q(z) (d-z) \int_0^d \frac{y(\tau, z)}{\beta(\tau)} d\tau dz + \frac{M\beta(0)d}{EI_0} \int_0^d \frac{d-\tau}{\beta(\tau)} d\tau \right\}^{-1/2}. \quad (2.50)$$

Note that this estimation does not depend on the parameter ΦEI . The following inequality is a simpler, but more rough, estimate:

$$\frac{1}{\beta(\tau)} \leq \left(\min_{[0, d]} \beta(z) \right)^{-1}. \quad (2.51)$$

Finally,

$$\omega \geq \omega_* = \sqrt{2 \min_{[0,d]} \beta(z)} \left\{ \int_0^d q(z)(d-z)(d^2-z^2) dz + \frac{M\beta(0)d^3}{EI_0} \right\}^{-1/2}. \quad (2.52)$$

For the cable (Section 2.3) similar arguments to those used above produce the inequalities

$$\int_0^d u^2 dz \leq C_3 \langle u, u \rangle_1, \quad C_3 = \frac{d^2}{2}, \quad (2.53)$$

$$u(z_0)^2 \leq C_4 \langle u, u \rangle_1, \quad C_4 = \min(z_0, d - z_0) \quad (2.54)$$

where the inner product is given by (2.14). The inequality (2.53) follows from (2.42) by integration, and the inequality (2.54) follows from (2.42). Finally,

$$\omega \geq \omega_* = \left\{ \frac{d^2}{2} + m \min(z_0, d - z_0) \right\}^{-1/2}. \quad (2.55)$$

Note, this estimation does not depend on the cable parameter s .

REMARK 2.1. The results of this paper can be further developed in three directions. First, the formulas (2.7), (2.19), (2.28), and (2.36) allow the development of variational techniques for the numerical analysis of eigenfunctions and eigenfrequencies [32]. This approach is especially important when the parameters of the system are not constant. Second, the spectral analysis of the above mechanical systems is the first step in the development of a theory of control for those models. In particular, the asymptotic representation of eigenfrequencies for large numbers is important in such an analysis. Third, the corresponding eigenfunction expansions allow the consideration of the oscillations of systems under the action of random forces [40]. Current research on these three topics by the authors uses the techniques and results developed in this paper.

REMARK 2.2. After this work was complete the authors became aware of the work of Laura, Reyes, and Rossi [41] on cable-payload systems. The corresponding Sturm-Liouville problem contains a spectral parameter in the boundary conditions and presents a sixth example to which the technique developed in this paper can be applied. The methods of Fulton [11, 12] and Hinton [18, 19] can also be applied to this system.

3. Proofs of compactness.

1. The proof of Lemma 2.1. Recall, the Hilbert space Ξ is taken to be the subspace of functions in $H^2[0, d]$ satisfying $u'(0) = u(d) = 0$ (see Remark 1.1) and the inner product is given by (2.3). The functional $\Lambda(u, \eta)$ allows the following obvious estimate:

$$|\Lambda(u, \eta)| \leq c_1 \|u\| \|\eta\| + c_2 |u(0)| |\eta(0)|. \quad (3.1)$$

Here and below, c_k ($k = 1, 2, \dots$) denote different positive constants whose exact values are not important for the proof.

The inequalities (2.38) and (2.39) imply

$$\|u\| \leq c_3 \|u\|_1, \quad (3.2)$$

$$|u(0)| \leq c_4 \|u\|_1. \quad (3.3)$$

Hence, the functional $\Lambda(u, \eta)$ allows the following estimate:

$$|\Lambda(u, \eta)| \leq c_5 \|u\|_1 \|\eta\|_1. \quad (3.4)$$

The estimate (3.4) shows that $\Lambda(u, \eta)$ is a linear functional in $\eta \in \Xi$ for any given $u \in \Xi$. The Riesz representation theorem leads to the representation (2.5) with a bounded linear operator $A: \Xi \rightarrow \Xi$. The selfadjointness of A and its positivity are obvious because the function $q(z)$ is positive and bounded and the parameter $c_2 = M\beta(0)/(EI_0)$ is positive (see (2.4)). Now the compactness of the operator A will be proven [6]. Let the sequence $\{u_k\}$ converge weakly in Ξ to an element $u \in \Xi$. It will be shown that the sequence $\{Au_k\}$ converges strongly in Ξ . The boundedness of A implies the weak convergence of $\{Au_k\}$ to Au in Ξ . Moreover, the compactness of the embedding operator of Ξ into $L^2[0, d]$ [6] implies the strong convergence of the sequences $\{u_k\}$ and $\{Au_k\}$ in $L^2[0, d]$ to u and Au , respectively. Further, the compactness of the embedding operator of Ξ into $C[0, d]$ [6] implies the strong convergence of the sequences $\{u_k(0)\}$ and $\{(Au_k)(0)\}$ in $\mathbb{C}$ to $u(0)$ and $(Au)(0)$, respectively. The representation (2.5) and the inequality (3.1) imply

$$\begin{aligned} \|A(u_k - u_m)\|_1^2 &= \Lambda(u_k - u_m, A(u_k - u_m)) \\ &\leq c_1 \|u_k - u_m\| \|A(u_k - u_m)\| + c_2 |(u_k - u_m)(0)| |(A(u_k - u_m))(0)| \\ &\rightarrow 0 \quad \text{as } k, m \rightarrow \infty. \end{aligned} \quad (3.5)$$

Hence, the sequence $\{Au_k\}$ converges strongly in Ξ . The compactness of the operator A is proven. $\square$

2. The proof of Lemma 2.2. Recall, the Hilbert space Ξ consists of all functions u such that $u \in H^1[0, z_0]$, $u \in H^1[z_0, d]$, $u(0) = u(d) = 0$, and $u(z_0^+) = u(z_0^-)$ (see Remark 1.1) and the inner product is given by (2.14).

The functional $\Lambda_2(u, \eta)$ allows the following bound:

$$|\Lambda_2(u, \eta)| \leq c_6 \left(|u(z_0)| \int_0^d |\eta| dz + |\eta(z_0)| \int_0^d |u| dz \right). \quad (3.6)$$

Hölder's inequality implies

$$|\Lambda_2(u, \eta)| \leq c_7 (|u(z_0)| \|\eta\| + |\eta(z_0)| \|u\|). \quad (3.7)$$

The inequality (3.7) and the inequalities (2.53), (2.54) imply

$$|\Lambda_2(u, \eta)| \leq c_8 \|u\|_1 \|\eta\|_1.$$

The further proof is essentially the same as for Lemma 2.1. As a result, the representation (2.16) is justified for $k = 2$. The proof for the case $k = 1$ is easier and does not differ essentially from the proof for Lemma 2.1.

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