HELE-SHAW FLOWS
WITH TIME-DEPENDENT FREE BOUNDARIES
INVOLVING AN INFINITE STRIP OF FLUID

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Abstract. We present a new family of analytic solutions for Hele-Shaw flows with time-dependent free boundaries, a constant-pressure condition being supposed relevant on each free boundary. The basic starting configuration has the plan-view of the region occupied by fluid within the Hele-Shaw cell as an infinite strip bounded by parallel lines, these lines being the free boundaries. The subsequent motion is produced by the injection (or suction) of fluid at points within the strip and, perhaps, the imposition of a pressure difference between the two boundaries.

1. Introduction. In an earlier paper [1], the author considered a class of Hele-Shaw flows with free boundaries in which the plan-view of the region occupied by fluid was confined to an infinite strip by barriers within the Hele-Shaw cell in the form of two infinite parallel lines. With the fluid initially occupying a bounded, simply-connected region that touched both barriers along a single line segment, the motion was produced by imposing a pressure difference between the two free boundaries. The analysis supposed the pressure along each free boundary to be constant.

A different class of such flows was considered in [2]. In that paper, there were no barriers within the cell and the motions discussed could all be regarded as beginning with the plan-view of the fluid region in the form of a concentric circular annulus, with the fluid surrounding what we will refer to as an air hole. The subsequent evolution was then driven by the injection of further fluid into the annulus (or suction therefrom) and, perhaps, the addition or removal of air from the air hole. Again, the pressure was supposed to be constant along each free boundary.

The solutions obtained in [1] and [2] displayed a number of interesting phenomena. For example, in both settings we were able to present instances of a soliton-like interaction. With the scenario discussed in [1], a blob of fluid with a rectangular outline

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evidently moves undeformed along the infinite strip; if it encounters stationary blobs of fluid placed within the strip then, modulo multiply-connected complications, these are first absorbed into the advancing front of the rectangular blob and then disgorged from its retreating rear, leaving behind stationary blobs of exactly the same shape in exactly the same place as those originally present, but consisting of different fluid particles. With the scenario discussed in [2], adding air to the air hole within the basic concentric circular annulus simply results in an expansion that maintains the plan-view as that of a concentric circular annulus; if this expanding annulus encounters stationary blobs of fluid then, modulo multiply-connected complications, they are absorbed and disgorged in an interaction that is the same in all essential details as that occurring with the rectangular blob moving along the infinite strip.

It was also noted in [2] that the families of flows considered in [1] and [2] could be exploited to yield interesting examples of cusp formation in the free boundaries. Typically, the mathematical solutions obtained for Hele-Shaw flows when a constant-pressure condition is assumed to hold at a free boundary show the development of cusps in the boundary whose appearance signals a breakdown of the solution. However, Howison [3] noted that a cusp may develop in a free boundary but *not* lead to a breakdown of the solution; the motion passes *through* the cusped state, the free boundary being smooth both before and after the instant when the cusp is present. With the nomenclature of [3], the usual form of cusp leading to breakdown is of $3/2$ -power type, while a $5/2$ -power cusp does not lead to breakdown; more generally, with n a positive integer, we have breakdown if a $(4n - 1)/2$ -power cusp evolves, but the motion continues beyond the cusped state if it is a $(4n + 1)/2$ -power cusp. Hohlov and Howison [4] also discuss motions that pass through cusped states.

The solutions obtained in [1] were expressed in terms of the standard functions associated with the theory of Jacobian elliptic functions, while those in [2] were expressed in terms of special functions defined with the particular needs of the analysis that arose there in mind. In the present paper, we consider a class of Hele-Shaw flows with free boundaries where the solutions can be expressed in terms of trigonometric functions. Within this class, we can still demonstrate soliton-like behaviour and manufacture flows that involve cusps of various kinds, much as is possible within the contexts of [1] and [2], but the simpler mathematical setting here makes a detailed analysis of these phenomena somewhat more straightforward. In fact, the solutions to be considered here may be regarded as limiting cases of those in [1] or of those in [2]; we may either let the width of the infinite strip confining the motion in [1] tend to infinity, or let the radius of the basic concentric circular annulus exploited in [2] tend to infinity. However, we do not adopt this approach here but derive the results in a more direct manner.

Suppose we begin with a Hele-Shaw cell containing fluid that occupies a region whose plan-view is an infinite strip bounded by parallel lines, these lines being free boundaries. We then contemplate injection (or suction) of fluid at a finite number of points within the strip followed, perhaps, by the imposition of a pressure difference between the two free boundaries. This is the sequence of events that should be kept in mind as we develop the solutions, but we will find that the results we obtain are applicable in other circumstances.

2. Basic formulae. With a Cartesian (x, y) -plane parallel to the plane of the Hele-Shaw cell, the geometry may be described via a projection in terms of x and y . We suppose the infinite strip of fluid with which we begin to have its boundaries parallel to the y -axis, and choose the length scale so that this initial strip has width π .

Suppose now that we inject an area πr_j^2 into the strip at the point $z = a_j$ for $j = 1, 2, \dots, N$; some of the areas πr_j^2 may actually be *negative* if suction rather than injection has been applied at $z = a_j$. We argue as in [1]: regard the unbounded region occupied by fluid as the limit of a bounded region, and consider the limit of the Cauchy transform of that bounded region. This transform tends to a function $h_l(z)$ that is an analytic function of the complex variable $z = x + iy$ to the left of the deformed strip occupied by fluid, and a function $h_r(z)$ that is an analytic function of z to the right of this strip; we are here using “left” and “right” as adjectives on the assumption that the (x, y) -plane is oriented in the usual manner. In fact, we have

$$\left. \begin{aligned} h_l(z) &= -\pi + \sum_{j=1}^N \frac{r_j^2}{z - a_j}, \\ h_r(z) &= +\pi + \sum_{j=1}^N \frac{r_j^2}{z - a_j}. \end{aligned} \right\} \quad (2.1)$$

The function given by $h_l(z)$ to the left of the strip and by $h_r(z)$ to the right of the strip, together with a function of the form $\bar{z} - h_i(z)$ within the strip, where $h_i(z)$ is analytic interior to the strip and an overbar denotes a complex conjugate, is now to yield a continuous function in the entire z -plane; this is a direct consequence of the fact that the Cauchy transform is a continuous function of position.

The original parallel-sided strip can, of course, be made to move sideways by imposing a pressure difference across it. Following injection at the N points $z = a_j$, the deformed strip still approaches a parallel-sided strip of width π as $y \rightarrow \pm\infty$, and these distant portions continue to move as before under the influence of a pressure difference. To determine the geometry completely, we evidently need to specify the position of the strip for large $|y|$, a specification that is not inherent in (2.1). In fact, we shall see that Eqs. (2.1), together with the continuity property inherited from the Cauchy transform and a knowledge of the position of the strip at large distances, do enable us to determine the outline of the blob.

Let $z = f(\zeta)$ map the infinite strip $-\pi/4 < \xi < +\pi/4$ in the plane of the complex variable $\zeta = \xi + i\eta$ conformally onto the region occupied by fluid in the z -plane. We suppose that $y \rightarrow \pm\infty$ corresponds to $\eta \rightarrow \pm\infty$; this does not quite determine the map uniquely, for a translation of the strip in the ζ -plane in the η -direction remains arbitrary. We exploit this arbitrariness later; when we know more about the structure of the function $f(\zeta)$, we will see that there is a natural way to enforce uniqueness.

Let $\zeta = \gamma_j$ map to the injection point $z = a_j$ so that

$$f(\gamma_j) = a_j \quad \text{for } j = 1, 2, \dots, N. \quad (2.2)$$

The continuity of the Cauchy transform implies that $h_l(z) + h_i(z) = \bar{z}$ on the left-hand free boundary. Transferring to the ζ -plane and noting that on the corresponding

boundary there we have $\zeta + \bar{\zeta} = -\pi/2$, we obtain

$$-\pi + \sum_{j=1}^N \frac{r_j^2}{f(\zeta) - a_j} + h_i(f(\zeta)) = \overline{f(-\bar{\zeta} - \pi/2)}. \quad (2.3)$$

This implies that $f(\zeta)$ can be meromorphically continued into the strip $-3\pi/4 < \xi < -\pi/4$.

Arguing similarly with conditions on the right-hand free boundary, we obtain

$$+\pi + \sum_{j=1}^N \frac{r_j^2}{f(\zeta) - a_j} + h_i(f(\zeta)) = \overline{f(-\bar{\zeta} + \pi/2)}, \quad (2.4)$$

which implies that $f(\zeta)$ can be meromorphically continued to the right of the line $\xi = \pi/4$. In fact, subtracting (2.3) from (2.4), conjugating, and replacing ζ by $-\bar{\zeta} - \pi/2$, we have

$$f(\zeta + \pi) - f(\zeta) = 2\pi, \quad (2.5)$$

which tells us that $f(\zeta)$ can be meromorphically continued into the entire ζ -plane. Differentiating (2.5) we obtain

$$f'(\zeta + \pi) = f'(\zeta), \quad (2.6)$$

so that the derivative of the mapping function is a meromorphic function of period π .

It follows from (2.4) that the only singularities of $f(\zeta)$ in the strip $-\pi/4 < \xi < 3\pi/4$ are simple poles at $\zeta = -\bar{\gamma}_j + \pi/2$ for $j = 1, 2, \dots, N$. Suppose that the residue of $f(\zeta)$ at $\zeta = -\bar{\gamma}_j + \pi/2$ is $-\bar{\beta}_j$ so that

$$f(\zeta) = -\frac{\bar{\beta}_j}{\zeta - (-\bar{\gamma}_j + \pi/2)} + O(1) \quad \text{as } \zeta \rightarrow -\bar{\gamma}_j + \pi/2. \quad (2.7)$$

Then

$$\overline{f(-\bar{\zeta} + \pi/2)} = \frac{\beta_j}{\zeta - \gamma_j} + O(1) \quad \text{as } \zeta \rightarrow \gamma_j,$$

and a comparison of residues on both sides of (2.4) tells us that we must have

$$\beta_j f'(\gamma_j) = r_j^2 \quad \text{for } j = 1, 2, \dots, N. \quad (2.8)$$

Now (2.7) implies that

$$f'(\zeta) = \frac{\bar{\beta}_j}{\{\zeta - (-\bar{\gamma}_j + \pi/2)\}^2} + O(1) \quad \text{as } \zeta \rightarrow -\bar{\gamma}_j + \pi/2,$$

so that the sum $\sum_{j=1}^N \bar{\beta}_j \sec^2(\zeta + \bar{\gamma}_j)$ has precisely the same singularities as $f'(\zeta)$. Since $f'(\zeta)$ tends to a constant as $\eta \rightarrow \pm\infty$, we see that we must have

$$f'(\zeta) = \sum_{j=1}^N \bar{\beta}_j \sec^2(\zeta + \bar{\gamma}_j) + \delta \quad (2.9)$$

for some constant δ , so that

$$f(\zeta) = \sum_{j=1}^N \bar{\beta}_j \tan(\zeta + \bar{\gamma}_j) + \delta\zeta + \varepsilon \quad (2.10)$$

for some constant ε . We now see the natural way to secure the uniqueness of the mapping: we suppose ε to be real.

With $f(\zeta)$ given by (2.10), we have (2.5) satisfied if and only if $\delta = 2$, so that

$$f(\zeta) = \sum_{j=1}^N \bar{\beta}_j \tan(\zeta + \bar{\gamma}_j) + 2\zeta + \varepsilon. \quad (2.11)$$

This implies that

$$f(\zeta) \sim 2\zeta + \varepsilon \pm i \sum_{j=1}^N \bar{\beta}_j \quad \text{as } \eta \rightarrow \pm\infty. \quad (2.12)$$

But if we substitute (2.9)—with the dummy index j replaced by k , say, and $\delta = 2$ —into (2.8) and sum over j we find that $\sum_{j=1}^N \beta_j$ is real, so that (2.12) yields

$$\operatorname{Re}\{f(\zeta)\} \sim 2\xi + \varepsilon \quad \text{as } \eta \rightarrow \pm\infty.$$

Thus the region occupied by fluid in the z -plane approaches a strip of width π as $y \rightarrow \pm\infty$ that lies in $\varepsilon - \pi/2 < x < \varepsilon + \pi/2$; the number ε in the map (2.11) gives the x -coordinate of the mid-point of the strip at large distances. If we begin with the basic parallel-sided strip and inject at some points with equal pressures on both free boundaries, then ε is constant throughout. If a pressure difference is subsequently applied between the two boundaries, ε changes in a determinate manner. When solving for a particular configuration, we may thus suppose ε to be known; it enjoys the same status as the data a_j and r_j^2 , which specify the injection points and amounts injected, respectively.

We now have all the equations we need. If our original strip of width π has been subjected to injection of an area πr_j^2 at $z = a_j$ for $j = 1, 2, \dots, N$, and has possibly been moved by a pressure difference so that its position at large distances is specified by ε as noted above, then it occupies a region that is the image of the strip $-\pi/4 < \xi < +\pi/4$ under a map given by the function $f(\zeta)$ in (2.11), where the $2N$ parameters β_j and γ_j are determined as solutions of the $2N$ equations (2.2) and (2.8). In general, the parameters and equations are complex and the equations are independent, but if any symmetry is present in the problem this may be exploited to advantage.

3. Examples. The simplest examples evidently arise with $N = 1$. We may then suppose a_1 to be real so that we have symmetry about the x -axis. Both β_1 and γ_1 are then real, while (2.2) and (2.8) yield two real equations. Taking $a_1 = \varepsilon = 0$, we have injection (or suction) at the origin into an initial strip lying in $-\pi/2 < x < +\pi/2$, the pressures being maintained equal on the free boundaries. This is of interest because the equations solve explicitly; we have $\gamma_1 = 0$ (also evident because we have symmetry about the y -axis) and the mapping is given by

$$f(\zeta) = (\{1 + r_1^2\}^{1/2} - 1) \tan \zeta + 2\zeta.$$

If suction is applied ($\pi r_1^2 < 0$), we find that $3/2$ -power cusps appear when an area π has been removed, and there is no continuation of the solution beyond this instant under the conditions we have assumed. The outline of the blob during this motion is sketched in Fig. 1 (see p. 206).

FIG. 1. Injection and suction at the origin $(0,0)$ with fluid initially occupying the strip $-\pi/2 < x < +\pi/2$; the pressures on both free boundaries are equal. With an area πr_1^2 injected, the outlines are drawn for increments of 1 in r_1 . Also drawn is the outline after an area π has been removed from $(0,0)$, when cusps appear in the free boundary at the points $\pm(\pi/2 - 1, 0)$. The figure shows the region $-10 < x < +10$, $-10 < y < +10$, and the origin is marked with a dot.

If the injection point is not at the origin, say $a_1 > 0$, but we still have the initial strip as in Fig. 1, with the pressures equal on both free boundaries so that $\varepsilon = 0$ throughout, the two real equations provided by (2.2) and (2.8) may be solved numerically. Figure 2 shows the growth of the blob outline when $a_1 = \pi/2$ and we are injecting on the edge of the initial strip lying in $-\pi/2 < x < +\pi/2$.

We can, in fact, consider $a_1 > \pi/2$ in this context, corresponding to injection *outside* the initial strip; a circular blob disjoint from the strip then grows about the point $(a_1, 0)$ to begin with, until it touches the strip when $r_1 = a_1 - \pi/2$. The present solution is then applicable for $r_1 > a_1 - \pi/2$. As noted for similar circumstances in [1], the appropriate form of the functions $h_l(z)$ and $h_r(z)$ in (2.1) is actually relevant here from the moment injection begins, with a suitable interpretation; it is the connectivity properties of the fluid region that decide how these functions must be exploited to determine the geometry.

It is helpful to bear this shift of viewpoint concerning $h_l(z)$ and $h_r(z)$ in mind when we contemplate a modification of the motion in Fig. 1: what happens if we inject a certain area πr_1^2 of fluid as there, stop this injection, and then try to move the deformed strip sideways by imposing a pressure difference between the free boundaries? In fact, the strip leaves behind a circular disc of fluid of radius r_1 centered on the injection point, and then continues on its way as an infinite parallel-sided strip of width π just like the initial strip

FIG. 2. The same initial strip as in Fig. 1 with equal pressures on both free boundaries, but injection is now at the point $(\pi/2, 0)$ on the edge of this strip. With an area πr_1^2 injected, the outlines are drawn for increments of 1 in r_1 . Note that when $r_1 = 1$ the distance moved by the left-hand free boundary is so small that it is not discernible as a separate curve in this figure; the first curve on the left that can be distinguished from the corresponding boundary of the original strip corresponds to $r_1 = 2$. The figure shows the same region $-10 < x < +10$, $-10 < y < +10$ as does Fig. 1.

with which we began in Fig. 1. This is so because the configuration with the strip plus the disjoint circular disc corresponds to precisely the same functions $h_l(z)$ and $h_r(z)$ as does the outline with this value of r_1 in Fig. 1. Invoking the time-reversible nature of the motions, it is more natural to view this scenario in a different light: the basic strip driven by a pressure difference encounters a stationary disc of fluid, first absorbs it and then disgorges it, and then continues on its way. So far as our equations with $N = 1$ are concerned, while the motions in Figs. 1 and 2 correspond to fixed a_1 and ε but variable r_1 , we are now contemplating fixed a_1 and r_1 but variable ε .

A particular example of this behaviour is shown in Fig. 3 (see p. 208). Here $a_1 = 0$ and $r_1 = 2$, while ε increases from the value $-(2 + \pi/2)$ when the strip is to the left of the circular disc of radius 2 and just touching it, to the value $+(2 + \pi/2)$ when it is to the right of the disc and just touching it. Thinking of the motion being from left to right, as we always will henceforth, the advancing right-hand free boundary is drawn solid in Fig. 3, while the retreating left-hand free boundary is drawn dashed.

Suppose now that, instead of encountering just *one* circular disc of fluid as it does in Fig. 3, the strip encounters *two* disjoint such discs. To be specific, take the discs to be of approximately the same radius and let their centres have approximately the

FIG. 3. The strip of width π absorbs and disgorges a circular disc of fluid of radius 2 as it is driven, say left to right, by a pressure difference between the free boundaries. The outline is drawn for 10 equal increments of the parameter ε , corresponding to 10 equal increments of time if the pressure difference remains constant, between the instants of touching and separation. The advancing boundary is drawn solid while the retreating boundary is drawn dashed. The centre of the circular disc is marked with a dot and the sketch shows the region $-4 < y < +4$.

same x -coordinate. If these discs are far apart, they are each absorbed and disgorged by the strip, the presence of one disc having very little effect on the motion in the vicinity of the other. On the other hand, if they are very close together, the attempt by the strip to absorb them traps an air pocket: even when they are placed so as to be almost touching, the motion in the immediate neighbourhood of each circular disc just after it has been hit by the strip is largely unaffected by the proximity of the other, until topological changes intervene as two different portions of the same free boundary come into contact. We have an interesting situation when the separation between the discs is carefully chosen between these two extremes, on the borderline separating these qualitatively different behaviours: the discs are absorbed without an air pocket being trapped, but the advancing free boundary passes through a cusped configuration.

To determine the precise value of the separation needed to yield this cusped arrangement, we can employ arguments analogous to those used in [2] for a similar purpose. If the cusp in the advancing free boundary corresponds to $\zeta = \zeta_0 = \pi/4 + i\eta_0$, say, then the extra conditions to be satisfied for this cusped configuration are

$$f'(\zeta_0) = 0 \quad (3.1)$$

and

$$\operatorname{Re} \left\{ \frac{f'''(\zeta_0)}{f''(\zeta_0)} \right\} = 0. \quad (3.2)$$

Equation (3.1) is the usual condition for a cusp: since $f''(\zeta_0) \neq 0$ because the map must be univalent, it merely forces the angle between curves meeting at ζ_0 to be doubled during the transformation to the z -plane. Equation (3.2) forces the cusp to be of higher power than $3/2$ and is the relevant form of Eq. (8.6) in [2] for the present circumstances; it is somewhat simpler than the latter equation because the free boundary here is the image of a straight line—a curve of zero curvature—while in [2] it was the image of a circle.

Our scenario now requires $N = 2$ in our formulation. If we take the size and position of our discs to be given (that is, a_1 , r_1 , a_2 , and r_2 are specified), then we solve for increasing ε as the strip moves left to right, determining β_1 , γ_1 , β_2 , and γ_2 from the four equations given by (2.2) and (2.8). If we wish to choose the data so that we pass through a cusped configuration, we can let the imaginary part of a_2 become an unknown so that the separation may vary. We also have ε as an extra unknown because we do not know beforehand where the strip will be when the cusp is present, and η_0 needs to be determined. We thus have 3 extra real parameters to be found, and adding the three real equations furnished by (3.1) and (3.2) to our set, we can solve for all the unknowns; Eq. (3.1) is a complex equation with two real equations as its real and imaginary parts. Once we know a value of a_2 that yields a cusp, we can fix a_2 at this value and solve the four equations (2.2) and (2.8) for β_1 , γ_1 , β_2 , and γ_2 as we increase ε to determine the complete motion.

Since the cusps are the features of primary interest, the figures to be presented here focus on their immediate neighbourhood. Bearing the evolution in Fig. 3 in mind, one may readily deduce the principal features of the motion outside the field of view shown.

In the following, numerical values given to four decimal places are correctly rounded to that accuracy; those given with fewer decimal places are exact.

In Fig. 4 (see p. 210), we have a first (lower) circular disc of radius 2 centered on the origin (so $r_1 = 2$ and $a_1 = 0$) and a second (upper) circular disc of radius 1.8 (so $r_2 = 1.8$) with its centre on the line $x = 0.2$. For the motion to pass through a cusped state as the discs are absorbed, we have $a_2 = 0.2 + 3.9814i$ and the cusp is present when $\varepsilon = -2.7179$. With this configuration, the moving strip first touches the disc of radius 2 when $\varepsilon = -(2 + \pi/2) = -3.5708$ and the disc of radius 1.8 when $\varepsilon = -3.1744$; with the disc of radius 2 absent, the disc of radius 1.8 would be touched by the strip when $\varepsilon = -(1.6 + \pi/2) = -3.1708$.

The upper sketch of Fig. 4 shows the advancing free boundary passing through the state with the $5/2$ -power cusp when $\varepsilon = -2.7179$; the curves are drawn for ε equal to this value, and for two steps of 0.03 on either side. The cusp is at the point $(-0.1617, 2.1507)$.

If we had chosen both centres in Fig. 4 to lie on $x = 0$, we would have needed $a_2 = 3.9780i$ for a cusp to appear in the advancing free boundary and the cusp would have been present, at the point $(-0.2626, 2.1124)$, when $\varepsilon = -2.8256$. In this symmetric situation, the retreating free boundary also passes through a cusped state as the strip prepares to extricate itself from the discs; the picture of the retreating free boundary passing through this state is just the reflection in the y -axis of the picture showing the advancing boundary passing through *its* cusped state.

FIG. 4. The strip of width π encounters a disc of radius 2 centered on $(0,0)$ and a disc of fluid of radius 1.8 centered on $(0.2, 3.9814)$. The upper sketch covers the region $-0.9 < x < 0.7$, $1.8 < y < 2.4$ while the lower covers $-0.5 < x < 1.1$, $1.8 < y < 2.4$; the vertical lines mark the y -axis in each case and the portions of the boundaries of the original discs visible through the appropriate window are also drawn. The upper sketch shows the advancing boundary for ε increasing from -2.7779 to -2.6579 in increments of 0.03 ; the curve for $\varepsilon = -2.7179$ has a $5/2$ -power cusp. The lower sketch shows the retreating boundary for ε increasing from 2.84 to 2.99 in increments of 0.03 .

For Fig. 4, we have intentionally chosen data so that this symmetry is not present, and the lower sketch shows the behaviour of the retreating free boundary. It remains smooth throughout, though the curvature becomes very large. The increments in ε for the curves in the lower sketch are 0.03 , as for the upper, with ε increasing from 2.84 to 2.99 .

Note that the two sketches in Fig. 4 show the same range of y -values, but different ranges of x -values to enable the interesting features to be displayed more clearly. The vertical lines mark the y -axis in each sketch, so that the lateral displacement involved can be judged readily.

In Fig. 4, the disc of radius 1.8 has its centre on the line $x = 0.2$. The sole change made in moving to Fig. 5 is to put this centre on the line $x = -0.2$ instead. For the motion to pass through the cusped state as the discs are absorbed, we now have $a_2 = -0.2 + 3.9702i$ and the cusp is present when $\varepsilon = -2.9286$. With this configuration, the moving strip touches both discs simultaneously when $\varepsilon = -(2 + \pi/2) = -3.5708$.

The upper sketch of Fig. 5 shows the advancing free boundary passing through the state with the $5/2$ -power cusp when $\varepsilon = -2.9286$; the curves are drawn for ε equal to this value, and for two steps of 0.03 on either side. The cusp is at the point $(-0.3652, 2.0724)$.

If we now consider the motion of the retreating free boundary as the strip tries to extricate itself from the discs, we find that it fails in its attempt. A $3/2$ -power cusp appears when $\varepsilon = 2.7261$ at the point $(0.0453, 2.1333)$ and the solution breaks down.

FIG. 5. The strip of width π encounters a disc of fluid of radius 2 centered on $(0,0)$ and a disc of fluid of radius 1.8 centered on $(-0.2, 3.9702)$. The upper sketch covers the region $-1.1 < x < 0.5$, $1.8 < y < 2.4$ while the lower covers $-1 < x < 0.6$, $1.8 < y < 2.4$; the vertical lines mark the y -axis in each case and the portions of the boundaries of the original discs visible through the appropriate window are also drawn. The upper sketch shows the advancing boundary for ε increasing from -2.9886 to -2.8686 in increments of 0.03 ; the curve for $\varepsilon = -2.9286$ has a $5/2$ -power cusp. The lower sketch shows the retreating boundary for ε increasing from 2.6661 to 2.7261 in increments of 0.03 ; the curve for $\varepsilon = 2.7261$ has a $3/2$ -power cusp which signifies a breakdown of the solution.

This is shown in the lower sketch of Fig. 5 which displays the curve with the cusp, plus two earlier curves with ε two steps of 0.03 smaller.

4. Concluding remarks. It has been the purpose of this paper to show how one may obtain analytic solutions to a particular class of problems involving Hele-Shaw flows with free boundaries, and to present a few examples to demonstrate the efficacy of the solution scheme and the varied phenomena that can occur in such flows. It is evident that one could easily produce more complicated examples; for instance, the blobs of fluid that are absorbed in Figs. 3, 4, and 5 can be made other than circular, either convex or non-convex, by invoking more injection points. Some of the complications that can arise when non-convex blobs are present are illustrated in [1].

Even in the relatively simple situation where two circular discs are encountered by the advancing strip and we have $N = 2$, as in Figs. 4 and 5, no attempt has been made to catalogue all possible types of behaviour, let alone locate the boundaries in the space of the variables a_1 , r_1 , a_2 , and r_2 across which this behaviour undergoes a qualitative change. To give just one hint of the complexities that can arise, consider again our preamble to Figs. 4 and 5. We took discs of approximately the same radius whose centres had approximately the same x -coordinate, and were able to adjust their separation to produce $5/2$ -power cusps in the advancing free boundary as the discs were absorbed. If we

reposition the discs so that they have approximately the same y -coordinate instead (that is, they are almost one-behind-the-other so far as the approaching strip is concerned) there is obviously no possibility of any cusps appearing; see Figs. 9 and 10 in [1] for pertinent sketches in a similar environment. What happens on the borderline between these two extremes?

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