A MODEL FOR THE DIFFUSION OF POPULATIONS IN ANNULAR PATCHY ENVIRONMENTS

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Abstract. A system of reaction-diffusion differential equations is utilized to model the diffusion of a population through annular patches with different carrying capacities. In the case of continuous solutions with continuous flux, it is shown that a unique, positive steady-state solution exists. In the case of radially symmetric initial conditions, it is shown that all solutions of the Cauchy problem approach this steady-state solution. Models involving nonsymmetric initial conditions are also considered.

1. Introduction. Recently, several papers have appeared in the literature concerning the effect of environmental heterogeneity on population growth. In order to study such effects [PR] and [SR] devised an efficient method of dividing the habitat into homogeneous patches, where the growth and diffusion coefficients for the given populations are constant in each patch, but may be different in different patches.

In studying the behaviour of a population in a patchy environment, [FST] obtained asymptotic stability of a unique positive steady-state in a two-patch environment with population diffusion along a linear transect. [FW] analyzed the steady-state in a linear three-patch environment. In [FK] a complete analysis of the population diffusion in an n -linear patch environment was obtained. However in natural settings, models that assume population diffusion along a line that is the transect of patches have their limitations. Hence [AC1] considered a model of two populations with competition and diffusion in a two-planar-patch environment, but with the same diffusion coefficients (and different growth rates) in a Lotka-Volterra setting. They derived sufficient conditions for persistence and extinction of the populations, as well as some convincing numerical interpretations for the biological phenomena.

Received January 9, 1997.

1991 *Mathematics Subject Classification.* Primary 92D40, 35K50.

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The second author's research was partially supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. NSERC A4823.

In this paper we are interested in the case where the environment changes radially from some point. Such change may e.g. be due to a pollution source (such as a smoke stack) or an enrichment source (such as bird droppings in a concentrated area of a lake), etc. To this end we model our system by means of annular patches with constant diffusion and growth laws in each patch (but of course they may be different from patch to patch).

We will obtain criteria for global stability of a unique radially symmetric positive steady-state solution. We divide the environment into several patches from a central area to a margin patch. A polluted water field around an oil well in a sea and the unpolluted water area would form an annular two-patch environment.

The model we utilize is a parabolic system of the form

$$\frac{\partial}{\partial t} u_i(x, y, t) = d_i \Delta u_i(x, y, t) + u_i(x, y, t) g_i(u_i(x, y, t)), \quad (1.1)$$

$$t > 0, \quad i = 1, 2, \dots, n, \quad (x, y) \in \Omega_i,$$

with boundary condition

$$u_n|_{\partial\Omega^{(n)}} = K_n, \quad (1.2)$$

and interface conditions

$$u_i|_{\partial\Omega^{(i)}} = u_{i+1}|_{\partial\Omega^{(i)}}, \quad (1.3)$$

$$d_i \left. \frac{\partial u_i}{\partial \nu} \right|_{\partial\Omega^{(i)}} = d_{i+1} \left. \frac{\partial u_{i+1}}{\partial \nu} \right|_{\partial\Omega^{(i)}}, \quad (1.4)$$

$$i = 1, 2, \dots, n-1,$$

and initial conditions

$$u_i(x, y, 0) = \eta_i(x, y), \quad (x, y) \in \Omega_i, \quad (1.5)$$

$$i = 1, 2, \dots, n,$$

where ν is the unit outward vector to Ω_i . Here Ω_i is the i th patch with positive radial annulus,

$$\Omega_i = \{(x, y) \mid r_{i-1}^2 \leq x^2 + y^2 \leq r_i^2\},$$

and we define

$$\Omega^{(i)} = \{(x, y) \mid 0 \leq x^2 + y^2 \leq r_i^2\},$$

and $0 = r_0 < r_1 < r_2 < \dots < r_n$ are constants. Δ is the Laplacian, $d_i > 0$, $i = 1, 2, \dots, n$ is the diffusion constant in the i th patch Ω_i . The function $g_i \in C'(R_+, R)$, $i = 1, 2, \dots, n$ is the specific growth rate in the i th patch and satisfies the standard assumptions for a Gause population model (see [F]), that is $g_i(0) > 0$, $g'_i(u) < 0$ and $g_i(K_i) = 0$ for some $K_i > 0$, the capacity of the i th patch. $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ with $\eta_i \in H^1(\Omega_i)$ is positive and radially symmetric and satisfies (1.2) and (1.3). Here $H^1(\Omega_i)$ is a Sobolev space.

The general r th-order Sobolev space on a region Ω (r a positive integer) is

$$H^m(\Omega) = \{\varphi \in \mathcal{L}^2(\Omega) \mid \varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(m)} \in \mathcal{L}^2(\Omega)\},$$

with norm

$$|\varphi|_{H^m} = \left(\sum_{i=0}^m \int_{\Omega} |\varphi^{(i)}|^2 d\Omega \right)^{1/2},$$

where $\varphi^{(i)}$, $i = 1, 2, \dots, m$ is a distributional derivative. We introduce a Sobolev space in order to discuss our model by employing well-developed skills in partial differential equations [Fr], [H], [J].

For the regularity of the solutions of our model we also require Hölder spaces, $C^m(\Omega)$, defined by

$$C^m(\Omega) = \{\varphi \mid \varphi \text{ is } m\text{-times continuously differentiable}\},$$

with norm

$$|\varphi|_{C^m(\Omega)} = \sum_{i=0}^m \sup_{s \in \Omega} |d^i \varphi(s)|, \quad (1.6a)$$

where m is a nonnegative integer. A fractional Hölder space, $C^\rho(\Omega)$, $0 < \rho < 1$, is defined as

$$C^\rho(\Omega) = \{\varphi \mid |\varphi|_{C^\rho} < +\infty\},$$

where

$$|\varphi|_{C^\rho(\Omega)} = |\varphi|_{C^0} + \sup_{s \neq t} \frac{|\varphi(s) - \varphi(t)|}{|s - t|^\rho}. \quad (1.6b)$$

We require in what follows the standard result that the inclusion $H^1(\Omega) \rightarrow C^{1/2}(\Omega)$ is continuous [Fr].

For the sake of conciseness, we shall write u instead of $(u_1, u_2, \dots, u_n)$ for the solution of (1.1)–(1.5), and treat u_i as a restriction of u to $\Omega_i \times R^+$, which is feasible if (1.3) is satisfied. A solution of (1.1)–(1.5) in the classical sense is a function $u \in C(\Omega^{(n)} \times R^+, R^+)$ such that

- (i) the partial derivatives $\frac{\partial}{\partial t} u_i$, $\frac{\partial^2}{\partial x^2} u_i$, $\frac{\partial^2}{\partial y^2} u_i$ of the restriction u_i of u are continuous on $\Omega_i \times R^+$ for $i = 1, 2, \dots, n$,
- (ii) the restriction u_i , $i = 1, 2, \dots, n$ satisfies (1.1)–(1.5).

For a complete analysis of model (1.1)–(1.5), we will consider the steady-state solution first, which is a solution of the elliptic system

$$\begin{aligned} d_i \Delta v_i + v_i g_i(v_i) &= 0 \quad \text{in } \Omega_i, \quad i = 1, 2, \dots, n, \\ v_n|_{\partial\Omega^{(n)}} &= K_n, \\ v_i|_{\partial\Omega^{(i)}} &= v_{i+1}|_{\partial\Omega^{(i)}}, \\ d_i \frac{\partial v_i}{\partial \nu} \Big|_{\partial\Omega^{(i)}} &= d_{i+1} \frac{\partial v_{i+1}}{\partial \nu} \Big|_{\partial\Omega^{(i)}}, \\ i &= 1, 2, \dots, n-1. \end{aligned} \quad (1.7)$$

Because of the symmetry of the patches, it is reasonable to consider a radially symmetric steady-state that can be solved from the radially symmetric polar form of (1.7):

$$d_i v_i''(r) + \frac{d_i}{r} v_i'(r) + v_i(r) g_i(v_i(r)) = 0, \quad r \in (r_{i-1}, r_i), \quad i = 1, 2, \dots, n, \quad (1.8)$$

$$\begin{aligned}v_n(r_n) &= K_n, \\v_i(r_i) &= v_{i+1}(r_i), \\d_i v'_i(r_i) &= d_{i+1} v'_{i+1}(v_i), \quad i = 1, 2, \dots, n-1.\end{aligned}$$

The next step is to treat (1.1)–(1.5) as a dynamical system in a suitable space and analyze the Cauchy problem. The third step is to apply Liapunov stability theory to obtain the convergence of all solutions with radially symmetric initial data to the radially symmetric positive steady-state solution. We basically adopt the method developed in [FK], [H], and [P]. Then we discuss the asymptotic behaviour of the solution with arbitrary initial data.

2. Existence and uniqueness of the positive radially symmetric steady-state solution. Before stating and proving our main results we require two lemmas, which deal with general properties of the radially symmetric steady-state solution of a parabolic equation in the two typical regions, namely Ω_i and $\Omega^{(1)}$. The equation under consideration will have the form

$$u_t = d\Delta u + h(u), \quad (2.1)$$

where $d > 0$ is a constant,

$$\begin{aligned}h(u) &= \begin{cases} ug(u), & u > 0, \\ 0, & \text{otherwise,} \end{cases} \\g'(u) &< 0 \quad \text{for } u > 0 \quad \text{and} \\g(K) &= 0 \quad \text{for some } K > 0.\end{aligned}$$

We introduce $h(u)$ in order to include the negative solution cases.

LEMMA 2.1. In Ω_i , $i > 1$, there exists a radially symmetric steady-state solution $v(\Omega)$ of (2.1) for the boundary value problem

$$v|_{\partial\Omega^{(i-1)}} = \tilde{K}, \quad \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega^{(i-1)}} = \alpha,$$

where $\tilde{K} > 0$ is a positive constant. Further, if $\tilde{K} > K$ and $\alpha \geq 0$, then v will be strictly increasing in r ; if $\tilde{K} < K$ and $\alpha \leq 0$, then v will be strictly decreasing in r .

Proof. The existence of a radially symmetric steady-state $v(r)$ of (2.1) is implied by the radially symmetric polar form of (2.1) which is a second-order ODE

$$dv''(r) + \frac{d}{r}v'(r) + h(v(r)) = 0, \quad r \in (r_{i-1}, r_i). \quad (2.2)$$

The monotonicity of $v(r)$ can easily be derived from the integral form of (2.2), namely

$$\begin{aligned}v'(r) &= \frac{r_{i-1}}{r}v'(r_{i-1}) - \frac{1}{rd} \int_{r_{i-1}}^r sh(v(s)) ds \\&= \frac{r_{i-1}\alpha}{r} - \frac{1}{rd} \int_{r_{i-1}}^r sh(v(s)) ds,\end{aligned} \quad (2.3)$$

and the properties of g , i.e., $h(v) > 0$ for $v < K$ and $h(v) < 0$ for $v > K$. □

LEMMA 2.2. In Ω_1 there exists a radially symmetric steady-state solution v of (2.1) satisfying the initial condition $v(0) = \tilde{K}$, where $\tilde{K}$ is any positive number. Further, v is strictly increasing if $\tilde{K} > K$, and v is strictly decreasing if $\tilde{K} < K$.

Proof. Clearly the precondition $v'(0) = 0$ must hold for the existence of a radially symmetric steady-state solution v , since from the radially symmetric polar form of (2.1)

$$dv''(r) + \frac{d}{r}v'(r) + h(v(r)) = 0, \quad 0 \leq r < r_1. \quad (2.4)$$

The monotonicity of v is obvious by the integral form of (2.4) in Ω_{r_1} , namely

$$v'(r) = -\frac{1}{rd} \int_0^r sh(v(s)) ds, \quad (2.5)$$

and the properties of h .

To prove existence, we will prove that the radially symmetric steady-state solution is a limit of solutions $\{v^{(m)}\}$ by the Ascoli-Arzelà lemma, where $v^{(m)}(r)$ is the radially symmetric steady-state of

$$\begin{aligned} d\Delta v^{(m)} + h(v^{(m)}) &= 0 \quad \text{in } \Omega_{r_{\epsilon_m} r_1}, \\ v|_{\partial\Omega_{r_{\epsilon_m}}} &= \tilde{K}, \\ \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega_{r_{\epsilon_m}}} &= 0. \end{aligned} \quad (2.6)$$

Here

$$\begin{aligned} \Omega_{r_{\epsilon_m} r_1} &= \{(x, y) \mid r_{\epsilon_m}^2 < x^2 + y^2 < r_1^2\}, \\ \Omega_{r_{\epsilon_m}} &= \{(x, y) \mid x^2 + y^2 < r_{\epsilon_m}^2\}, \\ r_1 &> r_{\epsilon_1} > r_{\epsilon_2} > r_{\epsilon_3} > \cdots > r_{\epsilon_m} \rightarrow 0. \end{aligned}$$

The radially symmetric polar integral form of (2.6) is

$$v^{(m)'}(r) = -\frac{1}{rd} \int_{r_{\epsilon_m}}^r sh(v^{(m)}(s)) ds, \quad r_{\epsilon_m} < r < r_1. \quad (2.7)$$

In order to use the Ascoli-Arzelà Lemma, we only need to check the uniform boundedness of $\{v^{(m)}\}$, since then its equi-continuity is implied by the uniform boundedness that results from (2.7).

When $\tilde{K} < K$, $v^{(m)}(r) < K$ by Lemma 2.1. By (2.7),

$$|v^{(m)'}(r)| \leq \frac{1}{rd} \frac{r^2 - r_{\epsilon_m}^2}{2} \cdot M_1 < \frac{M_1 r_1}{2d}, \quad r_{\epsilon_m} < r < r_1,$$

where $M_1 = \sup_{v \leq K} |h(v)|$, which exists by the properties of h . Hence

$$\begin{aligned} |v^{(m)}(r)| &\leq |v^{(m)}(r_{\epsilon_m})| + |v^{(m)}(r) - v^{(m)}(r_{\epsilon_m})| \\ &\leq \tilde{K} + \frac{M_1 r_1}{2d} r_1, \quad r_{\epsilon_m} < r < r_1. \end{aligned}$$

When $\tilde{K} > K$, $v^{(m)}(r)$ is strictly increasing by Lemma 2.1. Since $h'(v) < 0$ and $g(v) < 0$ for $v > K$, we have from (2.7),

$$\begin{aligned} v^{(m)'}(r) &< -\frac{1}{rd} \int_{r_{\epsilon_m}}^r s v^{(m)}(s) g(v^{(m)}(s)) ds \\ &< -\frac{r_1}{d} v^{(m)}(r) g(v^{(m)}(r)), \quad r_{\epsilon_m} < r < r_1. \end{aligned} \quad (2.8)$$

Let $\tilde{v}(r)$ be the solution of

$$\begin{aligned} v'(r) &= -\frac{r_1}{d} v(r) g(v(r)), \\ v(0) &= \tilde{K}, \quad 0 \leq r \leq r_1. \end{aligned} \quad (2.9)$$

Comparing (2.8) to (2.9), we get that

$$v^{(m)}(r) < \tilde{v}(r), \quad r_{\epsilon_m} \leq r \leq r_1.$$

Therefore in both cases $\{v^{(m)}(r)\}$ is uniformly bounded.

By the Ascoli-Arzelà Lemma, there exists a uniformly convergent subsequence of $\{v^{(m)}(r)\}$ in $[r_{\epsilon_1}, r_1]$, denoted by $\{v^{(1,m)}(r)\}_{m=1}^\infty$, and similarly there exists a uniformly convergent subsequence of $\{v^{(1,m)}(r)\}_{m=1}^\infty$ in $[r_{\epsilon_2}, r_1]$, denoted by $\{v^{(2,m)}(r)\}_{m=1}^\infty$, etc. By this procedure we obtain a uniformly convergent subsequence of $\{v^{(l-1,m)}(r)\}_{m=1}^\infty$ in $[r_{\epsilon_l}, r_1]$, which we denote by $\{v^{(l,m)}(r)\}_{m=1}^\infty$, $l = 1, 2, \dots$. It is obvious that $\{v^{(m,m)}(r)\}_{m=1}^\infty$ uniformly converges in any interval $[\alpha, \beta] \subset (0, r_1]$. Without loss of generality, we assume that $\{v^{(m)}(r)\}$ is this subsequence and that $v^*(r)$ is its limit in $(0, r_1]$. It is easy to see that $v^*(r)$ is continuous and bounded in $(0, r_1]$, and we can extend the definition of $v^*(r)$ to $r = 0$. Actually from the construction of $\{v^{(m)}(r)\}$, we have that $v^*(0) = \tilde{K}$.

A direct consequence of (2.7) is that $\{v^{(m)'}(r)\}$ is uniformly convergent in $[\alpha, \beta] \subset (0, r_1]$. Actually (2.7) can be written in two parts:

$$\begin{aligned} v^{(m)'}(r) &= -\frac{1}{rd} \int_{r_{\epsilon_m}}^{\max\{r_{\epsilon_m}, r^*\}} s h(v^{(m)}(s)) ds \\ &\quad - \frac{1}{rd} \int_{\max\{r_{\epsilon_m}, r^*\}}^r s h(v^{(m)}(s)) ds, \end{aligned} \quad (2.10)$$

where $r^* < \alpha$ does not depend on m and is small enough to make the first part as uniformly small as we want by the uniform boundedness of $\{v^{(m)}(r)\}$. The second part is uniformly convergent to

$$-\frac{1}{rd} \int_{r^*}^r s h(v^*(s)) ds,$$

in $[\alpha, \beta]$ as $m \rightarrow \infty$ by the uniform convergence of $\{v^{(m)}\}$ in $[r^*, r_1]$. Furthermore by this argument, we obtain that the right side of (2.7) is uniformly convergent to

$$\frac{1}{rd} \int_0^r s h(v^*(s)) ds$$

in $[\alpha, \beta]$ as $m \rightarrow \infty$. Therefore $\{v^{(m)'}(r)\}$ is uniformly convergent in $[\alpha, \beta]$, and hence $v^*(r)$ is differentiable and $v^{(m)'}(r) \rightarrow v^{*'}(r)$ uniformly in $[\alpha, \beta]$, and (2.7) becomes

$$v^{*'}(r) = -\frac{1}{rd} \int_0^r sh(v^*(s)) ds,$$

i.e., $v^*(r)$ satisfies (2.5) and thus is a radially symmetric steady-state solution of (2.1). This finishes the proof of Lemma 2.2. $\square$

The next two theorems are concerned with the existence and uniqueness of a positive radially symmetric steady-state solution of (1.8).

THEOREM 2.1 (Existence). There exists a positive radially symmetric steady-state solution of (1.8) that is located in $[\underline{K}, \overline{K}]$, where

$$\overline{K} = \max_{1 \leq i \leq n} \{K_i\} \quad \text{and} \quad \underline{K} = \min_{1 \leq i \leq n} \{K_i\}.$$

Proof. We assume that the growth function is $h_i(u_i)$ instead of $u_i g_i(u_i)$ by defining

$$h_i(u_i) = \begin{cases} u_i g_i(u_i), & u_i \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.11)$$

This is in case there exist negative solutions. Now consider two special radially symmetric steady-state solutions. The first will be $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ with initial condition $\bar{v}_1(0) = \overline{K}$. The solution component $\bar{v}_1(r)$ is determined by Lemma 2.1 and components $\bar{v}_i(r)$ are determined consecutively by $\bar{v}_{i-1}(r)$, $i = 2, 3, \dots, n$ by Lemma 2.2. The other solution is $\underline{v} = (\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n)$ with initial condition $\underline{v}_1(0) = \underline{K}$, and the $\underline{v}_i(r)$ are determined similarly. It is easy to see that $\underline{v}_n(r_n) \leq K_n \leq \bar{v}_n(r_n)$ is the monotonicity in Lemmas 2.1 and 2.2. By the continuous dependence of the solution on initial data we will get a radially symmetric steady-state solution $(v_1, v_2, \dots, v_n)$ with $v_n(r_n) = K_n$ and $\underline{K} \leq v_1(0) \leq \overline{K}$. This solution must be located in $[\underline{K}, \overline{K}]$, since if any component of $v(r)$ leaves the domain $[\underline{K}, \overline{K}]$ it will never return by the monotonicity of Lemmas 2.1 and 2.2. $\square$

THEOREM 2.2 (Uniqueness). The positive radially symmetric steady-state solution of (1.8) is unique.

Proof. Suppose $(\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n)$ and $(\tilde{\tilde{v}}_1, \tilde{\tilde{v}}_2, \dots, \tilde{\tilde{v}}_n)$ are two positive radially symmetric steady-state solutions. Define the function $w = (w_1, w_2, \dots, w_n)$ on $[0, r_n]$ by

$$w_i(r) = \frac{d_i \tilde{\tilde{v}}_i'(r)}{\tilde{\tilde{v}}_i(r)} - \frac{d_i \tilde{v}_i'(r)}{\tilde{v}_i(r)}, \quad r \in [r_{i-1}, r_i], \quad i = 1, 2, \dots, n.$$

If $w(r_n) = 0$, then $\tilde{\tilde{v}}_n'(r_n) = \tilde{v}_n'(r_n)$ since $\tilde{\tilde{v}}_n(r_n) = \tilde{v}_n(r_n) = K_n$, and hence $\tilde{\tilde{v}} \equiv \tilde{v}$ in $[0, r_n]$ by considering the uniqueness of the solution of (1.8) in $[\epsilon, r_n]$ for arbitrary $\epsilon > 0$. If $w(r_n) \neq 0$, assume without loss of generality that $w(r_n) < 0$. Let

$$r^* = \sup_{0 \leq r \leq r_n} \{w(r) = 0\},$$

which exists since $w(0) = 0$. It is easy to see that $\tilde{v}(r) > \tilde{v}(r)$ for $r \in (r^*, r_n)$. We calculate by (1.8),

$$w'_i(r) = g_i(\tilde{v}_i) - g_i(\tilde{v}_i) - d_i \left(\frac{1}{r} + \frac{\tilde{v}'_i}{\tilde{v}_i} + \frac{\tilde{v}'_i}{\tilde{v}_i} \right) w_i(r).$$

Since $w(r^*) = 0$ and $\tilde{v} < \tilde{v}$ in (r^*, r_n) and $g'_i < 0$, we have

$$w(r) > 0 \quad \text{in} \quad (r^*, r_n),$$

which contradicts $w(r_n) < 0$. This completes the proof of Theorem 2.2. $\square$

In our next result, we show that when the carrying capacities are monotone, the radially symmetric steady-state solutions are also monotone.

THEOREM 2.3. When the carrying capacities are decreasing, the positive radially symmetric steady-state solution is also decreasing, and vice versa.

Proof. Suppose $K_1 > K_2 > \cdots > K_n$, and let $v = (v_1, v_2, \dots, v_n)$ be the positive radially symmetric steady-state solution. Since $v'(0) = 0$, we can pick

$$r^* = \max\{r \leq r_n \mid v' \leq 0 \quad \text{in} \quad [0, r]\}.$$

If $r^* < r_n$, say $r^* \in [r_{i-1}, r_i]$, then by the monotonicity in Lemmas 2.1 and 2.2, we have that the inequality $v(r^*) > K_i$ will result in $v(r) > K_i$ for $r \geq r^*$, which contradicts $v(r_n) = K_n < K_i$. On the other hand, inequality $v(r^*) \leq K_i$ and $v'(r^*) = 0$ will contradict the definition of r^* . Therefore $r^* = r_n$, i.e., $v(r)$ must be decreasing. Similarly the case $K_1 < K_2 < \cdots < K_n$ can be treated. $\square$

3. The Cauchy problem with radially symmetric initial data. The existence of a unique, globally defined solution to (1.1)–(1.5) with any radially symmetric set of initial data is proved in this section. The regularity and boundedness of the solution are also given in order to carry out the stability analysis of the next section.

Let $v^*(r) = (v_1^*(r), v_2^*(r), \dots, v_n^*(r))$ be a radially symmetric steady-state solution of (1.8) and define $w = (w_1, w_2, \dots, w_n)$ by

$$\begin{aligned} w(x, y, t) &= u(x, y, t) - v^*(r), \\ t \geq 0, \quad 0 \leq r \leq r_n, \quad x^2 + y^2 &= r^2, \end{aligned}$$

for the solution $u(x, y, t)$ of (1.1)–(1.5). $w(x, y, t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} w_i(x, y, t) &= d_i \Delta w_i(x, y, t) + h_i(w_i(x, y, t) + v_i^*(r)) - h_i(v_i^*(r)), \\ t > 0, \quad (x, y) &\in \Omega_i, \quad x^2 + y^2 = r^2, \end{aligned} \tag{3.1}$$

the homogeneous boundary condition

$$w_n|_{\partial\Omega(n)} = 0, \tag{3.2}$$

the interface conditions

$$w_i|_{\partial\Omega^{(i)}} = w_{i+1}|_{\partial\Omega^{(i)}}, \quad (3.3)$$

$$d_i \frac{\partial w_i}{\partial \nu} \Big|_{\partial\Omega^{(i)}} = d_{i+1} \frac{\partial w_{i+1}}{\partial \nu} \Big|_{\partial\Omega^{(i)}}, \quad (3.4)$$

and radially symmetric initial data

$$w(x, y, 0) = \gamma(x, y) \in H_0^1(\Omega^{(n)}), \quad (3.5)$$

where h_i has the same meaning as that in (2.11), $i = 1, 2, \dots, n$, and

$$H_0^1(\Omega_{r_n}) = \{\varphi \in H^1(\Omega^{(n)}) \mid \varphi|_{\partial\Omega^{(n)}} = 0\}.$$

Now we analyze (3.1)–(3.5) in a Hilbert space in the distributional sense, and employ the abstract semigroup theory of operators to obtain the existence of a distributional solution. We then utilize the regularity of the solution with an embedding theorem in order to prove that this distributional solution is also a classical solution. Here a classical solution of (3.1)–(3.5) means that $w(x, y, t) + v(r)$ is a solution of (1.1)–(1.5) in the classical sense.

$X_0 = \mathcal{L}^2(\Omega_1) \times \mathcal{L}^2(\Omega_2) \times \dots \times \mathcal{L}^2(\Omega_n)$ is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle = \sum_{i=1}^n \langle \varphi_i, \psi_i \rangle_{\mathcal{L}^2(\Omega_i)},$$

where

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n), \quad \psi = (\psi_1, \psi_2, \dots, \psi_n).$$

All of the almost everywhere radially symmetric elements compose a Hilbert subspace X with inner product $\langle \cdot, \cdot \rangle_X$.

$$Y_0 = \{\varphi \in H^1(\Omega_1) \times H^1(\Omega_2) \times \dots \times H^1(\Omega_n) \mid \varphi_i|_{\partial\Omega^{(i)}} = \varphi_{i+1}|_{\partial\Omega^{(i)}}, i = 1, 2, \dots, n-1, \varphi_n|_{\partial\Omega^{(n)}} = 0\}$$

is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle = \sum_{i=1}^n \langle \nabla \varphi_i, \nabla \psi_i \rangle_{\mathcal{L}^2(\Omega_i)}.$$

Actually Y_0 is a subspace of $H_0^1(\Omega^{(n)})$ and the gradient norm on $H_0^1(\Omega^{(n)})$ is equivalent to the $\mathcal{L}^2$ -norm (see [J], 4.(5.19)). Therefore Y_0 is complete.

We denote by Y the set of almost everywhere radially symmetric elements of Y_0 . Then Y is a Hilbert space whose inner product is $\langle \cdot, \cdot \rangle_Y$.

We define a linear operator $A: X \rightarrow X$ by

$$A(\varphi_1, \varphi_2, \dots, \varphi_n) = (d_1 \Delta \varphi_1, d_2 \Delta \varphi_2, \dots, d_n \Delta \varphi_n),$$

where $\Delta \varphi_i$ denotes the second distributional Laplacian derivative of φ_i , and with domain

$$D(A) = \{\varphi \in H^2(\Omega_1) \times \dots \times H^2(\Omega_n) \mid \varphi \in Y, \quad d_i \frac{\partial \varphi_i}{\partial \nu} \Big|_{\partial\Omega^{(i)}} = d_{i+1} \frac{\partial \varphi_{i+1}}{\partial \nu} \Big|_{\partial\Omega^{(i)}}, \quad i = 1, 2, \dots, n-1\}.$$

LEMMA 3.1. A will generate a compact, analytic, uniformly bounded semigroup $T(t)$, such that $|T(t)| \leq M_0$ and $0 \in \rho(A)$.

Proof. $\prod_{i=1}^n C^\infty(\Omega^i) \cap D(A)$ is dense in X and therefore $D(A)$ is dense in X .

We prove $R(A) = X$ and $0 \in \rho(A)$ by considering $A\varphi = \psi$ for any $\psi \in X$. Its polar form is

$$d_i \varphi_i'' + \frac{d_i}{r} \varphi_i' = \psi_i, \quad r_{i-1} \leq r \leq r_i, \quad i = 1, 2, \dots, n-1, \quad \varphi \in D(A). \quad (3.6)$$

Integrating (3.6) we get

$$\begin{aligned} \varphi_1'(r) &= \frac{1}{r} \int_0^r s d_1^{-1} \psi_1(s) ds, \\ \varphi_1(r) &= \varphi_1(0) + \int_0^r \frac{1}{\tau} \int_0^\tau \frac{s}{d_1} \psi_1(s) ds d\tau, \quad 0 < r \leq r_1, \\ \varphi_i'(r) &= \frac{r_{i-1}}{r} \varphi_i'(r_{i-1}) + \frac{1}{r} \int_0^r \frac{s}{d_i} \psi_i(s) ds, \quad r_{i-1} < r < r_{i+1}, \\ \varphi_i(r) &= \varphi_i(r_{i-1}) + r_{i-1} \varphi_i'(r_{i-1}) \ln \frac{r}{r_{i-1}} + \int_{r_{i-1}}^r \frac{1}{\tau} \int_0^\tau \frac{s}{d_i} \psi_i(s) ds, \\ i &= 2, 3, \dots, n. \end{aligned}$$

By the interface condition and boundary condition, $\varphi_n(r_n) = 0$. Hence $\varphi_1(0)$ can be uniquely determined, and the integral representation of φ will be smooth enough; therefore $A\varphi = \psi$ has a unique solution $\varphi \in D(A)$ and $0 \in \rho(A)$, and A^{-1} is compact.

For $\varphi, \psi \in D(A)$, $\langle \varphi, A\psi \rangle_X = \langle A\varphi, \psi \rangle_X$ by integrating by parts, and thus A is symmetric. $R(A) = X$ and the symmetry of A imply that A is selfadjoint, and hence A is closed. A is also negative definite, because for $\varphi \in D(A)$,

$$\begin{aligned} \langle A\varphi, \varphi \rangle_X &= \sum_{i=1}^n \int_{\Omega_i} d_i \Delta \varphi_i \varphi_i d\Omega \\ &= - \sum_{i=1}^n d_i \int_{\Omega_i} |\nabla \varphi_i|^2 d\Omega \\ &\leq -C|\varphi|_X^2 \end{aligned} \quad (3.7)$$

for some constant $C > 0$ as a result of the equivalence of the gradient norm to the $\mathcal{L}^2$ -norm in Y .

Now A is selfadjoint, closed, densely defined and bounded above. Therefore A will generate an analytic semigroup $T(t)$; $t \geq 0$. (3.7) implies that $|T(t)| \leq M_0$ for some $M_0 > 0$. The compactness of $T(t)$ comes from the analyticity of $T(t)$ and the compactness of A^{-1} (see [P], Corollary 2.3.4). This finishes the proof of Lemma 3.1. $\square$

We now consider the fractional power $(-A)^{1/2}$ (see [Fr], [H], [P]). $X^{1/2}$ is the domain of $(-A)^{1/2}$, which is a Banach space with norm $|\varphi|_{X^{1/2}} = |(-A)^{1/2} \varphi|_X$. Since $-A$ is positive definite, $\langle -A\varphi, \varphi \rangle_X = |(-A)^{1/2} \varphi|_X^2$ for $\varphi \in D(A)$ and by (3.7), $|\cdot|_Y$ and $|\cdot|_{X^{1/2}}$ are equivalent norms on $D(A)$. The denseness of $D(A)$ in both Y and $X^{1/2}$ implies that $Y = X^{1/2}$.

Let the operator $f: X \rightarrow X$ be induced by the mapping

$$\begin{aligned} & h_i(\varphi_i(x; y) - v_i^*(r)) - h_i(v_i^*(r)), \\ & (x, y) \in \Omega_i, \quad x^2 + y^2 = r^2, \quad i = 1, 2, \dots, n. \end{aligned}$$

f is locally Lipschitzian by the definition of h_i . Therefore (3.1)–(3.5) can be written in the abstract form

$$\begin{aligned} w'(t) &= Aw(t) + f(w(t)), \quad t > 0, \\ w(0) &= w_0 \end{aligned} \tag{3.8}$$

where $w_0 \in Y$. A function $w: [0, T] \rightarrow X$ is a solution of (3.8) on $[0, T]$ if $w \in C([0, T], X) \cap C'((0, T), X)$ and (3.8) holds for $t \in [0, T]$. We write the local existence, uniqueness, continuity and compactness results in the next theorem, the proof of which is routine (see [P], Theorem 6.3.1, [H], Theorems 3.3.4, 3.3.6, 3.4.1).

THEOREM 3.1. For any $w_0 \in Y$, the Cauchy problem of (3.8) has a unique solution on $[0, T]$ for some $T > 0$ such that

- (i) $w \in C([0, T], Y)$,
- (ii) if $T < \infty$, then there exists a sequence $\{t_n\}$ such that $t_n \rightarrow T - 0$ and $|w(t_n)|_Y \rightarrow \infty$ as $n \rightarrow \infty$,
- (iii) if $T = \infty$ and $\{w(t) \mid t \geq 0\}$ is bounded in Y , then $\{w(t) \mid t \geq 0\}$ is precompact in Y ,
- (iv) the mapping $w_0 \mapsto w(t)$ is continuous from Y to Y uniformly on a compact subinterval of $[0, T]$.

By a standard technique in ([H], Secs. 3.5 and 3.6), we can prove that $w(t)$ is actually a classical solution. We write this result in the next lemma for a later purpose.

LEMMA 3.2. Let w be the unique solution of (3.8) on $[0, T]$. Then $\frac{\partial}{\partial t} w_i$, $\frac{\partial}{\partial x} w_i$, $\frac{\partial}{\partial y} w_i$, $\frac{\partial^2}{\partial x^2} w_i$, $\frac{\partial^2}{\partial y^2} w_i$, $\frac{\partial^2}{\partial x \partial t} w_i$, $\frac{\partial^2}{\partial y \partial t} w_i$, $\frac{\partial^2}{\partial t \partial x} w_i$, $\frac{\partial^2}{\partial t \partial y} w_i$, $i = 1, 2, \dots, n$, $0 < t < T$, all exist and are continuous in the corresponding patch.

Actually, elements of X are equivalence classes of functions. A more exact statement of Lemma 3.2 is that there is a function with the stated smoothness properties in the equivalence class of the solution. Therefore, we have the existence and uniqueness of the solution $u(x, y, t)$ of (1.1)–(1.5) in some interval $t \in [0, T]$. According to the maximum principle for parabolic equations, from the boundary condition (1.2)–(1.4) and properties of g_i , it follows that

$$\begin{aligned} \min_{1 \leq i \leq n} \left\{ K_i, \min_{(x, y) \in \Omega_i} \eta_i(x, y) \right\} &\leq u(x, y, t) \\ &\leq \max_{1 \leq i \leq n} \left\{ K_i, \max_{(x, y) \in \Omega_i} \eta_i(x, y) \right\} \end{aligned} \tag{3.9}$$

for all $(x, y) \in \Omega^{(n)}$ and $t \in [0, T]$. Therefore $|f(w(t))|_X$ is bounded on $[0, T]$.

By the variation of parameter formula,

$$w(t) = T(t)w_0 + \int_0^t T(t-s)f(w(s))ds,$$

and by some results on fractional powers ([P], Theorem 2.6.13)

$$(-A)^{1/2}w_0 \in X, \quad |(-A)^{1/2}T(t)| \leq M_{1/2}t^{-1/2}, t > 0 \text{ and } |T(t)| \leq M_0.$$

We have

$$\begin{aligned} |w(t)|_Y &= |w(t)|_{X^{1/2}} \leq |T(t)w_0|_{X^{1/2}} + \int_0^t |T(t-s)f(w(s))|_{X^{1/2}} ds \\ &\leq |(-A)^{1/2}T(t)w_0|_X + \int_0^t |(-A)^{1/2}T(t-s)f(w(s))|_X ds \\ &\leq M_0|(-A)^{1/2}w_0|_X + M_{1/2} \cdot \sup_{t \in [0, T]} |f(w(s))|_X \cdot \int_0^t (t-s)^{-1/2} ds \\ &\leq M < \infty \end{aligned}$$

for all $t \in [0, T)$. Hence $T = \infty$ by Theorem 3.1(ii), and consequently we get:

THEOREM 3.2. (1.1)–(1.5) has a unique solution defined on $\Omega^{(n)} \times R^+$ satisfying (3.9).

4. Global stability of the positive radially symmetric steady-state solution.

In this section, we treat (1.1)–(1.5) as a dynamical system $S(t)$ in a complete metric space

$$Z = \{\varphi \mid \varphi - v^* \in Y, \varphi \geq 0\}$$

with metric $d_Z(\varphi, \psi) = |\varphi - \psi|_Y$, and then prove the convergence of all solutions to the radially symmetric steady-state solution by the Lyapunov invariance principle. $S(t)$ is defined by $S(t)\varphi = w(\varphi - v^*)(t) + v^*$, where v^* is the positive radially symmetric steady-state solution of (1.8) and $w(\varphi - v^*)(t)$ is a solution of (3.8) with initial condition $w_0 = \varphi - v^*$. Theorems 3.1 and 3.2 guarantee that $S(t)$ is a dynamical system on Z ([H], Chapter 4).

Define the functional $V: Z \rightarrow R$ by

$$V(\varphi) = \sum_{i=1}^n \left(\int_{\Omega_i} \frac{d_i}{2} |\nabla \varphi_i|^2 d\Omega - \int_{\Omega_i} \int_0^{\varphi_i} h_i(s) ds d\Omega \right).$$

V is continuous by the continuity of the inclusion $Y \rightarrow C(\Omega^{(n)})$. We will show that V is a Lyapunov functional, i.e., its derivative is nonpositive,

$$\dot{V}(\varphi) = \limsup_{t \rightarrow 0^+} \frac{1}{t} [V(S(t), \varphi) - V(\varphi)] \leq 0, \quad \varphi \in Z. \quad (4.1)$$

For $\varphi \in Z$, $u(x, y, t) = (S(t)\varphi)(x, y)$ satisfies (1.1)–(1.5) with $\eta = \varphi$ and $u(x, y, t) = w(x, y, t) + v^*(x, y)$, where $w(x, y, t)$ is the solution of (3.8) with $w_0 = \varphi - v^*$. By Lemma 3.2, the mixed partial derivatives of u with respect to t and x or y are continuous, and so the order for the derivatives can be exchanged. Therefore, by the interface and boundary

conditions,

$$\begin{aligned}
 \frac{d}{dt}V(u(\cdot, \cdot, t)) &= \sum_{i=1}^n \left(\int_{\Omega_i} d_i \nabla u_i \nabla u_{it} d\Omega - \int_{\Omega_i} h_i(u_i) u_{it} d\Omega \right) \\
 &= \sum_{i=1}^n \left(\int_{\partial\Omega_i} d_i u_{it} \frac{\partial u_i}{\partial \nu} dS - \int_{\Omega_i} d_i u_{it} \Delta u_i d\Omega - \int_{\Omega_i} h_i(u_i) u_{it} d\Omega \right) \\
 &= - \sum_{i=1}^n \int_{\Omega_i} u_{it} (d_i \Delta u_i + h_i(u_i)) d\Omega \\
 &= - \sum_{i=1}^n \int_{\Omega_i} (d_i \Delta u_i + h_i(u_i))^2 d\Omega.
 \end{aligned} \tag{4.2}$$

Hence (4.1) holds, because

$$\frac{1}{t} [V(S(t)\varphi) - V(\varphi)] = \frac{d}{dt}V(u(\cdot, \cdot, t^*))$$

for some $t^* \in (0, t)$ by the Mean Value Theorem.

We define the w -limit set $w(\psi)$ for $\psi \in Z$ by

$$w(\psi) = \{\varphi \in Z \mid \exists t_n \rightarrow \infty \ni S(t_n)\psi \rightarrow \varphi\}.$$

By Theorem 3.1, $\{S(t)\psi \mid t \geq 0\}$ is precompact in Z . Therefore $w(\psi)$ is nonempty, compact, invariant, connected and $\text{dist}(S(t)\psi, w(\psi)) \rightarrow 0$ as $t \rightarrow \infty$ ([H], Theorem 4.3.3). Moreover, $w(\psi) \subset E = \{\varphi \in Z \mid \dot{V}(\varphi) = 0\}$ ([H], Theorem 4.3.4).

We are going to prove that E consists of the radially symmetric steady-state solution. By (4.2), $\dot{V}(\varphi) = 0$ implies that there exists a sequence $t_n \rightarrow 0+$ such that $\sum_{i=1}^n \int_{\Omega_i} (d_i \Delta u_i + h_i(u_i))^2 d\Omega \rightarrow 0$ where $u = S(t)\varphi$. This implies that

$$Aw(t_n) + f(w(t_n)) \rightarrow 0 \quad \text{in } X \quad \text{as } n \rightarrow \infty;$$

here $w(\cdot, \cdot, t) = u(\cdot, \cdot, t) - v^*(\cdot, \cdot)$. By Theorem 3.1, $w(t_n) \rightarrow w(0)$ in Y as $n \rightarrow \infty$, and so $w(t_n) \rightarrow w(0)$ also in X , and furthermore $f(w(t_n)) \rightarrow f(w(0))$ in X by the continuity of $f: Y \rightarrow X$. Therefore $Aw(t_n) \rightarrow -f(w(0))$. Because A is closed, $w(0) \in D(A)$ and $Aw(0) = -f(w(0))$, i.e., $Aw(0) + f(w(0)) = 0$. This means that $w(0)$ is a steady-state solution of (3.8). Hence $w(0) + v^*$ is a steady-state solution of (1.7). Because $w(0) \in D(A) \subset Y$, $w(0)$ is radially symmetric, and thus $\varphi = w(0) + v^*$ is a radially symmetric steady-state solution of (1.8) and is positive by (3.9). By the uniqueness of the positive radially symmetric steady-state solution, we have $w(0) = 0$, or $\varphi = v^*$, $E = \{v^*\}$. By the above, we have shown:

THEOREM 4.1. $\|u(\cdot, \cdot, t) - v^*\|_Y \rightarrow 0$ as $t \rightarrow \infty$ for all solutions u of (1.1)–(1.5) with positive radially symmetric initial data.

Because the gradient norm in Y is stronger than the $\mathcal{L}^2$ -norm, the conclusion of Theorem 4.1 can be made stronger, namely

$$\sum_{i=1}^n \int_{\Omega_i} (|u_i - v_i^*|^2 + |\nabla u_i - \nabla v_i^*|^2) d\Omega \rightarrow 0$$

as $t \rightarrow \infty$. By (1.6), the global convergence can be described in Hölder space, i.e.,

$$\sum_{i=1}^n |u_i - v_i^*|_{C^{1/2}(\Omega_i)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

5. The asymptotic behaviour of the solution with arbitrary initial data. In the previous sections, we have given the entire picture of the behaviour of the solution with radially symmetric initial data. We get the global attractivity of the positive radially symmetric steady-state solution with respect to all the solutions with radially symmetric initial data. In this section, we will prove the attractivity of this radially symmetric steady-state solution with respect to all solutions with arbitrary initial data under certain conditions.

Before proving the stability, we generalize the analysis of the Cauchy problem in Sec. 3 to those problems with arbitrary initial data. We will use spaces X_0 and Y_0 instead of X and Y . The domain and range of operator A need to be extended correspondingly. We do so in the following lemma.

LEMMA 5.1. Operator A will generate a compact, analytic, uniformly bounded semi-group $T(t)$ with $|T(t)| \leq M_0$ and $0 \in \rho(A)$.

Proof. Comparing to the proof of Lemma 3.1, we only need to show that $R(A) = X_0$, $0 \in \rho(A)$ and A^{-1} is compact. The rest of the analysis can be easily carried over similarly.

We prove that $R(A) = X$ and $0 \in \rho(A)$ by considering the weak form of $A\varphi = \psi$ for any $\psi \in X_0$, i.e., $\forall \tilde{\varphi} \in C_0^\infty(\Omega)$

$$\int_{\Omega} A\varphi \tilde{\varphi} \, d\Omega = \int_{\Omega} \psi \tilde{\varphi} \, d\Omega.$$

By the matching conditions in $D(A)$ and Green's formula, this is equivalent to

$$\sum_{i=1}^n \int_{\Omega_i} d_i \nabla \varphi_i \cdot \nabla \tilde{\varphi}_i \, d\Omega = \sum_{i=1}^n \int_{\Omega_i} \psi_i \tilde{\varphi}_i \, d\Omega.$$

By the theory of weak solutions of elliptic equations [A], $A\varphi = \psi$ has a unique solution, $A^{-1}\psi$, in $D(A)$, and A^{-1} is compact. Thus $0 \in \rho(A)$. This finishes the proof of Lemma 5.1. $\square$

All the other lemmas and theorems in Sec. 3 can be developed parallelly without changing the statements in their proofs except for substituting X and Y by X_0 and Y_0 respectively. Hence we have:

THEOREM 5.1. System (1.1)–(1.5) has a unique solution defined on $\Omega^{(n)} \times R^+$ satisfying (3.9) for any set of initial data.

Let $Z_0 = \{\varphi \mid \varphi - v^* \in Y_0, \varphi \geq 0\}$. Similarly all the analyses on Z in Sec. 4 can be carried over to Z_0 . Therefore all solutions will approach E in Z_0 , where E is the set of all positive steady-state solutions. We give a condition for the uniqueness of our steady-state solution.

THEOREM 5.2 (Uniqueness). If $K^* \leq \underline{K}$, then the positive steady-state solution of (1.1) is unique, where $K^* = \max\{K_i^*\}$, $K_i^* = \sup\{v_i \mid [v_i g_i(v_i)]' > 0\}$, $i = 1, 2, \dots, n$, and $\underline{K}$ is defined in Theorem 2.1.

Proof. Suppose $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ and $\bar{\bar{v}} = (\bar{\bar{v}}_1, \bar{\bar{v}}_2, \dots, \bar{\bar{v}}_n)$ are two positive steady-state solutions, i.e.,

$$\begin{aligned} d_i \Delta \bar{v}_i + \bar{v}_i g_i(\bar{v}_i) &= 0, \\ d_i \Delta \bar{\bar{v}}_i + \bar{\bar{v}}_i g_i(\bar{\bar{v}}_i) &= 0, \quad i = 1, 2, \dots, n. \end{aligned} \quad (5.1)$$

We claim that $\bar{v} \geq \underline{K}$, for otherwise let

$$\begin{aligned} \bar{m} &= \min_{\Omega^{(n)}} \bar{v}(x, y) = \bar{v}_i(x_0, y_0), \quad (x_0, y_0) \in \bar{\Omega}_i, \\ \bar{H}(x, y) &= -y_i(\bar{v}_i(x, y)), \\ \bar{V}_i &= \bar{m} - \bar{v}_i(x, y), \quad (x, y) \in \Omega_i. \end{aligned}$$

Then

$$d_i \Delta \bar{V}_i + \bar{H}(x, y) \bar{V}_i = \bar{m} g_i(\bar{v}_i). \quad (5.2)$$

Since $\bar{m} < \underline{K}$, there exists a neighborhood $U(x_0, y_0)$ of (x_0, y_0) in Ω_i , in which

$$\bar{H}(x, y) \leq 0, \quad \bar{m} g_i(\bar{v}_i) > 0.$$

Therefore (5.2) satisfies the conditions of a maximum principle [PW]. If $(x_0, y_0) \in \text{int } \Omega_i$, then by the maximum principle, $\bar{v} \equiv \bar{m}$ in $U(x_0, y_0)$. If $(x_0, y_0) \in \partial\Omega^{(i)}$, by boundary condition (1.2), we know that $i < n$, and furthermore by the interface condition (1.4) and the minimum at (x_0, y_0) , we have

$$\left. \frac{\partial \bar{v}_i}{\partial \nu} \right|_{\partial\Omega^{(i)}} = 0.$$

Again by the maximum principle of Hopf [PW], $\bar{v} \equiv \bar{m}$ in $U(x_0, y_0)$. We can extend this conclusion to the entire $\Omega^{(n)}$. This contradicts the boundary condition (1.2) and hence $\bar{v} \geq \underline{K}$. Similarly $\bar{\bar{v}} \geq \underline{K}$.

From (5.1), we have

$$d_i \Delta (\bar{\bar{v}}_i - \bar{v}_i) + \int_0^1 [v_i g_i(v_i)]' \Big|_{v_i = \bar{v}_i + \xi(\bar{\bar{v}}_i - \bar{v}_i)} d\xi \cdot (\bar{\bar{v}}_i - \bar{v}_i) = 0. \quad (5.3)$$

Since $\bar{v}, \bar{\bar{v}} \geq \underline{K}$, then $\bar{v}, \bar{\bar{v}} \geq K^*$, giving

$$\int_0^1 [v_i g_i(v_i)]' \Big|_{v_i = \bar{v}_i + \xi(\bar{\bar{v}}_i - \bar{v}_i)} d\xi \leq 0.$$

By applying the boundary condition and the maximum principle to (5.3) we get that $\bar{v} \equiv \bar{\bar{v}}$. $\square$

From the above statement and Theorem 5.2, we have:

THEOREM 5.3. If $K^* \leq \underline{K}$, then the positive radially symmetric steady-state is globally asymptotically stable, and

$$\|u(\cdot, \cdot, *) - v^*\|_{Y_0} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for any solution u of (1.1)–(1.5) with any positive set of initial data.

COROLLARY 5.1 (Equi-capacity). If $K_1 = K_2 = \cdots = K_n$, then $v \equiv K_1$ is globally asymptotically stable, i.e.,

$$\|u(\cdot, \cdot, t) - K_1\|_{Y_0} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Acknowledgments. The first author would like to thank Moxun Tang for a helpful discussion. The authors also wish to thank an anonymous referee for some helpful suggestions.

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