CONSERVATION LAWS AND SYMMETRIZATION OF THE EQUATIONS OF INCOMPRESSIBLE INVISCID FLUIDS

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Abstract. Many physical processes are described by conservative quasilinear systems of equations that admit a supplementary independent conservation law. A typical example is the equations of incompressible fluid dynamics. If for basic equations the conservation laws of the momentum and the volume are chosen then the conservation law of the energy is an additional conservation law. In the present paper all conservation laws for incompressible fluid dynamics are derived and the symmetric forms of its equations are obtained. Godunov's method of symmetrization is generalized for the case of an arbitrary quasilinear system of equations.

1. Introduction. Because of the importance of the incompressible fluid flow equation, the investigations concerned with the existence, uniqueness, and stability of solutions of the initial-boundary value problems and their numerical implementations are numerous [1]–[4]. As a rule, results in those directions are obtained by means of the well-known conservation laws: of energy, momentum, mass, [1]–[4].

Traditional methods for obtaining conservation laws are based on Noether's theorem coupled with group analysis, differential forms, etc., [5]–[8]. Here we use a method proposed by Roždestvenskii [9]. It relies on a direct analysis for compatibility of an overdetermined system of first-order partial differential equations for density and fluxes of an arbitrary conservation law. Let us mention in passing that the methods of the type we have mentioned are those employed when all conservation laws for other important systems of continuum mechanics [9]–[15] are derived.

The remainder of the paper consists of two parts. In the first part (Sec. 2) the full set of conservation laws for the Euler equations of an incompressible fluid is obtained. Part of these conservation laws are derived in [7] using Noether's theorem; other preliminary

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papers study the group properties of this system. In [10] it is shown that all first-order integral invariants are linear combinations of the momentum, energy, and vorticity.

In the second part (Secs. 3, 4) of the paper the system of equations for an incompressible fluid is written in a symmetric form. Such forms are known for a number of hyperbolic systems of mathematical physics (for example, gas dynamics [16], [17], and magnetohydrodynamics [18]). An important example of a system that is evolutionary but not hyperbolic is provided by the Euler equations for incompressible fluids. In Sec. 4, Godunov's symmetrization method is generalized for an arbitrary conservative quasi-linear system of equations with a complementary conservation law.

2. Conservation laws in incompressible fluid dynamics. The equations of an incompressible, inviscid fluid in the absence of body forces read

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \quad i = 1, \dots, n, \quad (2.1)$$

$$\sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0, \quad (2.2)$$

where u_i , $i = 1, \dots, n$, are the components of the velocity and p is the pressure.

Here and henceforth we assume that the functions considered possess all necessary derivatives. Also we shall not explicitly specify the domain of the variables.

The classical conservation laws are obtained for (2.1), (2.2) as follows. We multiply (2.2) by u_i ($i = 1, \dots, n$) and add the equations that express the conservation law of the momentum

$$\frac{\partial u_i}{\partial t} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\partial u_i u_j}{\partial x_j} + \frac{\partial (\frac{1}{2} u_i^2 + p)}{\partial x_i} = 0, \quad i = 1, \dots, n. \quad (2.3)$$

We multiply the i th ($i = 1, \dots, n$) equation of (2.1) by u_i , the equation (2.2) by $(p + 0.5(u_1^2 + \dots + u_n^2))$ and take the sum of the resulting equations and obtain the conservation law of the energy

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \sum_{i=1}^n u_i^2 \right) + \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(u_i \left(p + \frac{1}{2} \sum_{j=1}^n u_j^2 \right) \right) = 0. \quad (2.4)$$

Could any other conservation laws be derived from the equations (2.1), (2.2)? Before answering this question we shall specify the notion of conservation law [9].

DEFINITION 2.1. An equation of the type

$$\frac{\partial T}{\partial t} + \sum_{i=1}^n \frac{\partial X^i}{\partial x_i} = 0 \quad (2.5)$$

that is a consequence of the system (2.1), (2.2) is called a first-order conservation law (or simply a conservation law) of the system (2.1), (2.2), where the functions T , X^i are

called the density and fluxes of the conservation law (2.5). The conservation laws of the system (2.1), (2.2) are independent when the corresponding vector functions

$$\vec{Z}_j(t, x_1, \dots, x_n, u_1, \dots, u_n, p) = (T, X_1^j, \dots, X_n^j)$$

are linearly independent.

THEOREM 1. The full system of conservation laws for incompressible fluid dynamics is the following:

$$T = \frac{\alpha}{2} \sum_{j=1}^n u_j^2 + \sum_{j=1}^n \left(\sum_{\substack{l=1 \\ l \neq j}}^n \beta_j^l(t) x_l + \beta_j^j(t) \right) u_j + T_0(s), \quad s \equiv (t, x_1, \dots, x_n), \quad (2.6)$$

$$\begin{aligned} X^i = & \alpha u_i \left(p + \frac{1}{2} \sum_{j=1}^n u_j^2 \right) + \left(\sum_{\substack{l=1 \\ l \neq i}}^n \beta_i^l(t) x_l + \beta_i^i(t) \right) \left(\frac{1}{2} u_i^2 + p \right) \\ & + u_i \sum_{j=1}^n \left(\sum_{\substack{l=1 \\ l \neq j}}^n \beta_j^l(t) x_l + \beta_j^j(t) \right) u_j \\ & - \left(\sum_{\substack{k,l=1 \\ k \neq l}}^n \beta_k^l(t) x_l x_k + \sum_{l=1}^n \beta_l^l(t) x_l + \gamma_0(t) \right) u_i + \delta_i(s), \end{aligned} \quad (2.7)$$

where α is an arbitrary constant, the functions β_i^j satisfy the conditions $\beta_i^j = -\beta_j^i$, $i \neq j$, and $\beta_i^j(t)$, $\gamma_0(t)$ are arbitrary functions. $T_0(t)$ and $\delta_i(x)$ satisfy the equation

$$\frac{\partial T_0}{\partial t} + \sum_{i=1}^n \frac{\partial \delta_i}{\partial x_i} = 0.$$

COROLLARY. The only linearly independent conservation laws that do not depend explicitly on time and space are the classical conservation laws: the conservation laws of the volume (2.1), the momentum (2.3), and the energy (2.4).

Proof. In (2.5), $T, X^1, \dots, X^n$ are the desired functions of the variables $t, x_1, \dots, x_n, u_1, \dots, u_n, p$, and each of the variables $u_1, \dots, u_n, p$ depends on $t, x_1, \dots, x_n$. The subscripts further show partial derivatives in the system of the independent variables $t, x_1, \dots, x_n, u_1, \dots, u_n, p$. If in (2.5) we differentiate $T, X^1, \dots, X^n$ as compound functions and eliminate the derivatives with respect to t and x with the help of (2.1), (2.2), we obtain a linear relation with respect to space partial derivatives, which must be zero for arbitrary values of the solution derivatives. As a result there arises a system of $n(n+1)+1$ first-order partial differential equations for the functions $T, X^1, \dots, X^n$. The first group of equations is

$$-u_j T u_1 + X_{u_1}^j = 0, \quad j = 2, \dots, n; \quad (1.1.j)$$

$$-u_i T u_j + X_{u_j}^i = 0, \quad i = 1, \dots, n, \quad i \neq j, \quad (1.i.j)$$

$$u_1 T_{u_1} - u_j T_{u_j} - X_{u_1}^1 + X_{u_j}^j = 0, \quad j = 2, \dots, n; \quad (1.j.j)$$

the second one is

$$T_p = 0, \quad (2.i.0)$$

$$-T_{u_i} + X_p^i = 0, \quad i = 1, \dots, n; \quad (2.i.i)$$

and the third one is

$$T_t + \sum_{i=1}^n X_{x_i} = 0. \quad (2.8)$$

It follows from (2.i.0), (2.i.i) that $x_{pp}^i = 0$, $i = 1, \dots, n$, and from (1.i.j) that $X_{u_j p} = 0$, $i, j = 1, \dots, n$, $i \neq j$. Therefore,

$$X^i = X_1^i(u_i, s)p + X_2^i(u_1, \dots, u_n, s), \quad i = 1, \dots, n. \quad (2.9)$$

Then, from (2.i.i) we get n representations for T :

$$T = \int X_1^i(u_i, s) du_i + T^i(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, s), \quad i = 1, \dots, n.$$

For $i \neq j$ we differentiate with respect to u_i the equality

$$\begin{aligned} & \int X_1^i(u_i, s) du_i + T^i(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, s) \\ &= \int X_1^j(u_j, s) du_j + T^j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n, s) \end{aligned}$$

to establish that

$$T^j = \int X_1^i(u_i, s) du_i + T_i^j(u_1, \dots, u_n),$$

where T_i^j does not depend on u_i, u_j . Since i, j were arbitrary, we conclude that

$$T = \sum_{k=1}^n \int X_1^k(u_k, s) du_k + T_0(s). \quad (2.10)$$

Combining (1.1.j), (2.9), (2.10), we get

$$X_2^i = u_i \int X_2^j(s, u_j) du_j + X_2^{ij}(s, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n), \quad i, j = 1, \dots, n. \quad (2.11)$$

We will show that

$$X_2^i = u_i \sum_{j=1}^n \int X_1^j(s, u_j) du_j + \bar{X}_2^i(s, u_i), \quad i = 1, \dots, n, \quad (2.12)$$

where $\bar{X}^i$ is an arbitrary function. Indeed, let us fix i and let $k \neq j$, $j \neq i$, $1 \leq k, j \leq n$. We introduce (2.11) into (1.i.j) and obtain

$$u_i X_1^k(s, u_k) = \frac{\partial X_2^{i,j}}{\partial u_k},$$

or, what is the same,

$$X_2^{i,j} = u_i \int X_1^k(s, u_k) du_k + \bar{X}_2^{ij},$$

where $\overline{X}^{ij}$ is an arbitrary function that does not depend on u_k, u . We substitute (2.12) into (2.9) and conclude that, at this stage of the proof, for the function X^i we have the representations

$$X^i = X_1^i(s, u_i) + u_i \sum_{j=1}^n X_1^j(s, u_j) + \overline{X}_2(s, u_i), \quad i = 1, \dots, n. \quad (2.13)$$

We substitute (2.10) and (2.13) into Eqs. (i.j.j) and, after a suitable rearranging, we get

$$p \left(\frac{\partial X_1^j(s, u_1)}{\partial u_j} - \frac{\partial X_1^1(s, u_j)}{\partial u_1} \right) = F_j(s, u_1, \dots, u_n),$$

where

$$\begin{aligned} F_j &= u_1 X_1^1(s, u_1) - u_j X_1^j(s, u_j) - \int X_1^j(s, u_j) du_j \\ &\quad + \int X_1^1(s, u_1) du_1 + \frac{\partial \overline{X}_2^1}{\partial u_1} + \frac{\partial X_2^j}{\partial u_j}, \quad j = 2, \dots, n. \end{aligned}$$

Since the variable p can take arbitrary values, we have

$$\frac{\partial X_i^j(s, u_j)}{\partial u_i} = \frac{\partial X_1^1(s, u_1)}{\partial u_1} = \alpha(s), \quad j = 2, \dots, n, \quad (2.14)$$

$$F_j = 0, \quad j = 2, \dots, n, \quad (2.15)$$

where $\alpha(s)$ is an arbitrary function. From (2.14) we get

$$X_1^j = \alpha(s)u_j + \beta_j(s), \quad j = 2, \dots, n, \quad (2.16)$$

where $\alpha(s), \beta(s)$ are arbitrary functions. We substitute (2.16) into (2.14) and, after suitable rearranging, we get

$$2\beta_1(s)u_1 + \frac{3}{2}\alpha(s)u_1^2 - \frac{\partial \overline{X}_2}{\partial u_1} = 2\beta_j(s)u_j^2 + \frac{3}{2}\alpha(s)u_j^2 - \frac{\partial \overline{X}_2(s, u_j)}{\partial u_j}, \quad j = 1, \dots, n.$$

These equalities are true if and only if

$$2\beta_j(s)u_j + \frac{3}{2}\alpha(s)u_j^2 - \frac{\partial \overline{X}_2(s, u_j)}{\partial u_j} = \gamma(s), \quad j = 1, \dots, n,$$

where $\gamma(s)$ is an arbitrary function. Hence

$$\overline{X}_2^j = \gamma(s)u_j + \alpha(s)\frac{u_j^3}{2} + \beta_j(s)u_j^2 + \delta_j(s), \quad j = 1, \dots, n,$$

where $\delta(s)$ is an arbitrary function. At this stage of the proof, the functions $T, X^1, \dots, X^n$ have the representations

$$\begin{aligned} T &= \frac{\alpha(s)}{2} \sum_{j=1}^n u_j^2 + \sum_{j=1}^n \beta_j(s) u_j + T_0(s), \\ X^i &= \alpha(s) u_i \left(p + \frac{1}{2} \sum_{j=1}^n u_j^2 \right) + \beta_i(s) \left(\frac{1}{2} u_i^2 + p \right) \\ &\quad + u_j \sum_{j=1}^n \beta_j(s) u_j + \gamma(s) u_i + \delta_i(s). \end{aligned} \quad (2.17)$$

Further, we substitute T and X^i , $i = 1, \dots, n$, from (2.17) into (2.1) and equate the coefficients of the like monomials of $u_1, \dots, u_n, p$. Thus we get the following system of partial differential equations ($i, j = 1, \dots, n$):

$$\begin{aligned} (1) \quad & \beta_{it} + \gamma_{x_i} = 0, \\ (2) \quad & \beta_{ix_i} = 0, \\ (3) \quad & \frac{1}{2} \alpha_t + \beta_{jx_j} = 0, \\ (4) \quad & \beta_{jx_i} + \beta_{ix_j} = 0, \\ (5) \quad & \alpha_{x_i} = 0, \end{aligned} \quad (2.18)$$

and the equation in the theorem for $T_0, \delta_1, \dots, \delta_n$.

From (2.18.2), (2.18.5) it follows that $\alpha = \text{const.}$ The formulas for β_i and γ are obtained from (2.18.2)–(2.18.4) and (2.18.1) respectively. The theorem is proved.

3. Symmetric forms for the incompressible fluid flow equations. The conservation laws of the volume (2.2) and the momentum (2.3) generate a conservative system of $(n+1)$ equations with $(n+1)$ unknowns:

$$-\frac{\partial P_t^0}{\partial t} + \sum_{j=1}^n \frac{\partial P_i^j}{\partial x_j} = 0, \quad i = 0, 1, \dots, n, \quad (3.1)$$

where

$$\begin{aligned} P_0^0 &= 0, & P_i^0 &= u_i, & i &= 1, \dots, n, \\ P_i^j &= u_i u_j, & i &\neq j, & P_i^i &= \frac{1}{2} u_i^2 + p, & i, j &= 1, \dots, n. \end{aligned}$$

The conservation law of the energy

$$-\frac{\partial Q^0}{\partial t} + \sum_{j=1}^n \frac{\partial Q^j}{\partial x_j} = 0, \quad (3.2)$$

where

$$Q^0 = \frac{1}{2} \sum_{j=1}^n u_j^2, \quad Q^j = u_j \left(p + \frac{1}{2} u_j^2 \right), \quad j = 1, \dots, n,$$

is a complementary conservation law that is linearly independent with the conservation laws (3.1). It turns out that the presence of complementary conservation laws implies a number of special properties of the system (3.1).

THEOREM 2. The system (3.1) for smooth solutions admits the following forms:

$$\frac{\partial}{\partial t} \left(\frac{\partial R^0}{\partial q_0} \right) + \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial R^i}{\partial q_k} \right) = 0, \quad k = 1, \dots, n, \quad (3.3)$$

where

$$\begin{aligned} q_0 &= p - \frac{1}{2} \sum_{j=1}^n u_j^2, & q_j &= u_j, \quad j = 1, \dots, n, \\ R^0 &= \frac{1}{2} \sum_{s=1}^n q_s^2, & R^i &= q_i \left(q_0 + \frac{q_i^3}{3} + \frac{1}{2} \sum_{\substack{s=1 \\ s \neq i}}^n q_s^2 \right), \quad i = 1, \dots, n; \\ A^0 \frac{\partial q}{\partial t} &+ \sum_{i=1}^n A^i \frac{\partial q}{\partial x_i} &= 0, \end{aligned} \quad (3.4)$$

where $q = (q_0, q_1, \dots, q_n)^{\text{Tr}}$, and $A^i = \|a_{lm}^i\|$ are symmetric matrices with components

$$\begin{aligned} a_{lm}^i &= \frac{\partial^2 R^i}{\partial q_l \partial q_m}, \quad i, l, m = 0, 1, \dots, n; \\ B^0 \frac{\partial q}{\partial t} &+ \frac{\partial(B^0 q)}{\partial t} + \sum_{i=1}^n \left(B^0 \frac{\partial q}{\partial t} + \frac{\partial(B^i q)}{\partial x_i} \right) &= 0, \end{aligned} \quad (3.5)$$

where $B^i = \|b_{lm}^i\|$ are symmetric matrices with components

$$\begin{aligned} b_{00}^0 &= \frac{\partial^2 g^0}{\partial q_0^2} + L^0 \frac{1}{q_0^2}, & b_{lm}^0 &= \frac{\partial^2 g^0}{\partial q_l \partial q_m}, \quad l \neq m, \quad m = 0, 1, \dots, n, \\ b_{ll}^0 &= \frac{1}{2} + \frac{\partial^2 g^0}{\partial q_l^2} + \frac{L^l}{q_l^2}, & l &= 1, \dots, n, \\ b_{ii}^i &= \frac{3}{2} q_i + \frac{\partial^2 g_i}{\partial q_i^2} + \frac{L^l}{q_l^2}, & i &= 1, \dots, n, \\ b_{0m}^i &= \frac{\partial^2 g_i}{\partial q_0 \partial q_m}, & m &= 1, \dots, n, \\ b_{im}^i &= \frac{1}{3} q_s + \frac{\partial^2 g_i}{\partial q_i \partial q_m}, & m &= 1, \dots, n, \quad i = 1, \dots, n, \\ b_{0i} &= \frac{1}{2} + \frac{\partial^2 g_i}{\partial q_0 \partial q_i}, & i &= 1, \dots, n, \\ b_{ll}^i &= \frac{1}{3} q_i + \frac{\partial^2 g_i}{\partial q_l^2} + \frac{L^i}{q_l^2}, & l \neq i, \quad l &= 1, \dots, n, \quad i = 1, \dots, n, \end{aligned}$$

where

$$q_i = G_i \left(\frac{q_i}{q_0}, \dots, \frac{q_n}{q_0} \right),$$

G_i are arbitrary functions, and L^i are arbitrary constants.

REMARK. If $n = 2$, then

$$q_0 = p - \frac{1}{2}(u_1^2 + u_2^2), \quad q_1 = u_1, \quad q_2 = u_2,$$

$$R^0 = \frac{1}{2}(q_1^2 + q_2^2), \quad R^1 = q_1 \left(q_0 + \frac{q_1}{3} + \frac{q_2}{3} \right), \quad R^2 = q_2 \left(q_0 + \frac{q_1}{2} + \frac{q_2^3}{3} \right),$$

$$A^0 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad A^1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 4q_1^2 & q_2 \\ 0 & q_2 & q_1 \end{vmatrix}, \quad A^2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & q_2 & q_1 \\ 1 & q_1 & 4q_2^2 \end{vmatrix}.$$

Now (also in the case $n = 3$) it is reasonable to use the canonical quasilinear form (3.4) for construction of difference schemes (or finite element approximations) that satisfy discrete analogs of (3.3). Difference schemes with such conservative properties for the compressible Euler equations are served in [19, 20] and, for the equations of nonlinear elasticity, in [21].

Proof. Following the idea of Godunov [16], to find the functions $q_i(u_1, \dots, u_n, p)$, we multiply the corresponding equations (3.1) by q_i , add them and compare with (3.2):

$$q_0 \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} + \sum_{i=1}^n q_i \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^n \frac{\partial (u_i u_j)}{\partial x_j} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_i^2 + p \right) \right) \\ = \frac{\partial}{\partial t} \left(\frac{1}{2} \sum_{j=1}^n u_j^2 \right) + \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(u_j \left(p + \frac{1}{2} \sum_{j=1}^n u_j^2 \right) \right).$$

The equality will be satisfied if the following equalities for differentials hold:

$$\sum_{i=1}^n q_i du_i = \frac{1}{2} d \sum_{j=1}^n u_j^2, \quad (3.6)$$

$$q_0 du + q_i d \left(\frac{1}{2} u_i^2 + p \right) + \sum_{j=1}^n q_j du_i u_j = d \left(u_i \left(p + \frac{1}{2} \sum_{j=1}^n u_j^2 \right) \right), \quad i = 1, \dots, n. \quad (3.7)$$

From (3.7) at $i = 1$, after equating the expressions in front of the corresponding differentials, we get the formulas for q_i from (3.3). It is easy to check that these q_i satisfy the other equalities in (3.6), (3.7). Now, for P_i^j we get

$$P_0^0 = 0, \quad P_i^0 = q_i, \quad i = 1, \dots, n,$$

$$P_j^i = q_i q_j, \quad i \neq j, \quad P_i^i = q_0 + q_i^2 + \frac{1}{2} \sum_{s=1}^n q_s^2, \quad i, j = 1, \dots, n.$$

As one can easily verify,

$$\frac{\partial P_j^s}{\partial q_i} = \frac{\partial P_i^s}{\partial q_j}, \quad s, i, j = 1, \dots, n.$$

Hence, the expressions $\sum_{j=1}^n P_j^s dq_j$, $s = 1, \dots, n$, are total differentials of some function, namely, R^s , $s = 1, \dots, n$.

To obtain (3.5) from (3.3) we have to solve the system of equations

$$b_{rs}^i + \sum_{k=0}^n \frac{\partial(q_k b_{rk}^i)}{\partial q_s} = \frac{\partial^2 R^i}{\partial q_i \partial q_s}, \quad i, r, s = 0, \dots, n. \quad (3.8)$$

It is easy to show that

$$\frac{\partial b_{rs}^i}{\partial q_m} = \frac{\partial b_{rm}^i}{\partial q_s}.$$

Hence, the expressions $\sum_{s=0}^n b_{rs}^i dq_s$ are total differentials from some function ψ_r^i . Then, the system (3.8) can be recast as follows:

$$\frac{\partial \psi^i}{\partial q_s} + \sum_{k=0}^n \frac{\partial}{\partial q_s} \left(q_k \frac{\partial \psi^i}{\partial q_k} \right) = \frac{\partial^2 R^i}{\partial q_r \partial q_s}$$

or, after an integration,

$$\varphi_r^i + \sum_{k=0}^n q_k \frac{\partial \varphi_r^i}{\partial q_k} = \frac{\partial^2 R^i}{\partial q_k^2} + K_r^i \quad (K_r^i = \text{constant}). \quad (3.9)$$

Since the matrices B^i are symmetric,

$$\frac{\partial \varphi_r^i}{\partial q_s} = b_{rs}^i = b_{sr}^i = \frac{\partial \varphi_s^i}{\partial q_r}.$$

Therefore, the expressions $\sum_{r=0}^n \varphi_r^i dq_r$ are total differentials of some functions ψ^i . For these functions we obtain from (3.9) the independent equations

$$\sum_{k=0}^n q_k \frac{\partial \psi^i}{\partial q_k} = R^i + \sum_{k=0}^n K^i q_k + L^i \quad (L^i = \text{constant}), \quad i = 1, \dots, n.$$

The solution of these equations can be written in the form

$$\psi^i = \int_0^1 \frac{1}{\tau} (q_0 \tau, \dots, q_n \tau) d\tau + \sum_{k=1}^n K_k^i q_k + L^i \ln(q_0, \dots, q_n) + g^i \left(\frac{q_1}{q_0}, \dots, \frac{q_n}{q_0} \right), \quad (3.10)$$

where g^i are arbitrary functions of their argument. Now, for b_{ks}^i we find that

$$\begin{aligned} b_{ks}^i &= \frac{\partial \varphi^i}{\partial q_s} = \frac{\partial}{\partial q_s} \left(\frac{\partial \psi^i}{\partial q_k} \right) \\ &= \int_0^1 \frac{1}{\tau} \frac{\partial^2 R^i(q_0 \tau, \dots, q_n \tau)}{\partial q_k \partial q_s} d\tau + \frac{\partial^2 g^i}{\partial q_k \partial q_s} + L^i \frac{\partial^2 \delta_{ks}}{\partial q_k \partial q_s} \end{aligned} \quad (3.11)$$

(δ_{ks} is Kronecker's symbol). Finally, substituting the expressions for R^i into (3.1), we obtain the formulas for b_{em}^i in the theorem. The theorem is proved.

4. A generalization of the Godunov symmetrization method for an arbitrary conservative system. Consider the quasilinear system

$$\sum_{i=1}^m \sum_{l=1}^n A_{kl}^i(u) \frac{\partial u_l}{\partial x_i} = 0, \quad (4.1)$$

where $u = (u_1, \dots, u_n)$ is an n -dimensional vector and $A^i(u) = \|A_{kl}^i(u)\|$ are $m \times n$ matrices.

DEFINITION 4.1. If the equation

$$\sum_{i=1}^n \frac{\partial Q(u)}{\partial x_i} = 0 \quad (4.2)$$

is a consequence of the system (4.1), then (4.2) is called a conservation law for the system (4.1), explicitly independent of the independent variables x_i , $i = 1, \dots, n$. The system (4.1) is called conservative, if it possesses n independent conservation laws $Q_1, \dots, Q_n$, i.e., when there exists k , $1 < k < m$, such that

$$\frac{\partial[Q_1^k, \dots, Q_n^k]}{\partial[u_1, \dots, u_n]} \neq 0. \quad (4.3)$$

Further, we consider only conservative systems of equations

$$\sum_{i=1}^m \frac{\partial P_j^i(u)}{\partial x_i} = 0, \quad j = 1, \dots, n, \quad (4.4)$$

which admit complementary conservation laws of the form (4.2), linearly independent from the conservation laws (4.4).

If the system (4.4) is x_k -hyperbolic in the sense of Friedrichs [9] (very often x_k is time) and if Q^k is a convex function, then Q^k is called a generalized entropy [16], [17]. Then (4.4) can be written in terms of new dependent variables $q_i = q_i(u)$, $i = 1, \dots, n$, as a symmetric hyperbolic system; and, conversely, if the hyperbolic system (4.4) is symmetrizable, then it possesses a generalized entropy [16], [17], [18].

THEOREM 3. Let (4.4) be a conservative quasilinear system of equations possessing a complementary conservation law (4.2), linearly independent of the conservation laws (4.4). In addition, suppose that the algebraic system of equations for the unknowns X_j

$$\sum_{j=1}^n X_j \frac{\partial P_j^s}{\partial u_p} = \frac{\partial \Phi^s}{\partial u_p}, \quad p = 1, \dots, n, \quad s = 1, \dots, m, \quad (4.5)$$

has a solution. Then, there exists an image onto $q_i = q_i(u)$, $i = 1, \dots, n$, such that (4.4) admits the following form:

$$\sum_{i=1}^m \frac{\partial}{\partial x_i} \left(\frac{\partial R^i}{\partial q_k} \right) = 0, \quad k = 1, \dots, n, \quad (4.6)$$

where

$$R^i = \sum_{j=1}^n q_j P_j^i + Q^i, \quad i = 1, \dots, n;$$

$$\sum_{i=1}^n A^i(q) \frac{\partial q}{\partial x_i} = 0, \quad (4.7)$$

where $A^i(q)$ are symmetric matrices, and

$$\sum_{i=1}^n \left(B^i \frac{\partial q}{\partial x_i} + \frac{\partial(B^i q)}{\partial x_i} \right) = 0, \quad (4.8)$$

where $B^i = B^i(q)$ are also symmetric matrices.

Proof. One accomplishes this proof by the same scheme as the proof of Theorem 2. Now, for the unknown functions one obtain the equalities

$$\sum_{j=1}^n q_i dP_j^i = dQ^i, \quad i = 1, \dots, n. \quad (4.9)$$

Deriving them in detail, we get the over-determined system (4.5) and, from the condition of independency, it follows that there exists k , $1 \leq k \leq m$, such that

$$\det \left\| \frac{\partial P_j^k}{\partial u_s} \right\| \neq 0.$$

Then, from (4.5), when $s = k$ one obtains a system from which, in a unique way, one gets the unknown functions. Hence, this is the unique solution of (4.5). From the equality

$$\frac{D(p_1^k, \dots, p_n^k)}{D(q_1, \dots, q_n)} \cdot \frac{D(q_1, \dots, q_n)}{D(u_1, \dots, u_n)} = \frac{D(p_1^k, \dots, p_n^k)}{D(u_1, \dots, u_n)} \neq 0,$$

it follows that the image $q_i = q_i(u)$ is a one-to-one transformation (locally).

We will prove that, in the new variables q_i , the functions $P_i = P_i(q)$ satisfy the conditions

$$\frac{\partial P_i^s}{\partial q_j} = \frac{\partial P_j^s}{\partial q_i}, \quad i, j = 1, \dots, n, \quad s = 1, \dots, m. \quad (4.10)$$

For this purpose we will first prove the equalities

$$\sum_{k=1}^m q_k \frac{\partial P_k^s}{\partial q_e} = \frac{\partial \Phi^s}{\partial q_e}, \quad e, m = 1, \dots, n, \quad s = 1, \dots, m. \quad (4.11)$$

Indeed, in view of the relations (4.5) we obtain

$$\sum_{i=1}^n q_j \frac{\partial P_i^s}{\partial q_j} = \sum_{i=1}^n q_i \left(\sum_{l=1}^m \frac{\partial P_i^s}{\partial u_l} \cdot \frac{\partial u_l}{\partial q_j} \right) = \sum_{l=1}^m \frac{\partial u_l}{\partial q_j} \left(\sum_{i=1}^n q_i \frac{\partial P_i^s}{\partial u_l} \right) = \frac{\partial Q^s}{\partial q_j}.$$

Hence, after equalizing the second mixed derivatives of (4.11) we get the conditions (4.10). It follows from (4.10) that $\sum_{i=1}^n P_i^s dq_i$ is a total differential of some functions R^s , i.e., $P_i^s = \partial R^s / \partial q_i$, $i = 1, \dots, m$. Putting these expressions for P_i^s in (4.4), we get (4.6). From (4.9) the representations of R^s in the theorem follow. Conversely, if we multiply each k th equation of (4.6) by q_k and add the results, we will get (4.4).

One obtains (4.7) from (4.6) after a straightforward differentiation, and a_{ks}^i can be obtained in the same manner as in Theorem 2. Thus the theorem is proved.

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¹In these papers the Russian spelling Volkov was used for the author's name.