DELTA-SHOCK WAVES AS SELF-SIMILAR VISCOSITY LIMITS

By

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1. Introduction. We consider the Riemann problem for a system of conservation laws:

$$U_t + F(U)_x = 0; \quad (x, t) \in \mathbf{R}_+^2, \quad (1.1)$$

$$U(x, 0) = \begin{cases} U_l; & x < 0, \\ U_r; & x > 0. \end{cases} \quad (1.2)$$

It is natural to search for weak solutions for (1.1)–(1.2) that are *self-similar*, which in this context means that they are invariant under dilations of the independent variables of the form $(x, t) \mapsto (ax, at)$, $a > 0$, or in other words, that they are constant along the rays $x = at$, $t > 0$.

One of the standard methods for the study of the Riemann problem (1.1)–(1.2) is the vanishing viscosity method, which consists of introducing a term of the form εU_{xx} on the right-hand side of (1.1), which results in a parabolic system whose solutions depend on the parameter ε . One then proceeds to determine the behavior of the smooth solutions of this parabolic system when $\varepsilon \rightarrow 0^+$, with initial data (1.2) or some suitable regularization of it. One hopes to obtain existence and additional properties of the weak solutions of (1.1)–(1.2) this way. The vanishing viscosity regularization is often physically appropriate and, to an extent, preserves the Galilean invariance associated to the system (1.1) by approximating shock waves by travelling wave solutions of the regularized system. However, the self-similarity of the approximating solutions is lost.

If one wishes to preserve the self-similarity at the level of approximations, it is possible to employ a variant of the vanishing viscosity method that consists of introducing a term of the form $\varepsilon t U_{xx}$, on the right-hand side of (1.1). This gives rise to the regularized system:

$$U_t + F(U)_x = \varepsilon t U_{xx}; \quad (x, t) \in \mathbf{R}_+^2, \quad \varepsilon > 0. \quad (1.3)$$

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This system often admits solutions that are smooth and self-similar, that satisfy (1.2) and which are called *viscosity wavefans*. These solutions can be written in the form $U_\varepsilon = U_\varepsilon(\xi)$, with $\xi = x/t$ and U_ε satisfying the boundary value problem:

$$\varepsilon \ddot{U} = \dot{F}(u) - \xi \dot{U}; \quad \xi \in (-\infty, +\infty), \quad (1.4)$$

$$U(-\infty) = U_l, \quad U(+\infty) = U_r. \quad (1.5)$$

(The dot denotes $\frac{d}{d\xi}$.)

This approach was developed by C. Dafermos in [1]. Dafermos proved the existence of the viscosity wavefans $U_\varepsilon(\xi)$ using the Leray-Schauder fixed point theorem. Furthermore, he used Helly's lemma to prove the weak convergence of the viscosity wavefans under the hypothesis of uniform boundedness in L^∞ and in total variation of the family $\{U_\varepsilon\}$. The existence result remains the best available but it requires an a priori estimate in L^∞ which can be problematic to obtain. The application to 2×2 systems was developed by C. Dafermos and R. DiPerna in [2].

The theory of viscosity wavefans, as developed by Dafermos, has been applied to several examples of 2×2 systems, as in [7, 8, 11, 14], including systems that change type, [3, 4, 10]. Recently (see [15]), A. Tzavaras has employed the method of Dafermos, plus a careful local study of the behavior of the total variation of the solutions of (1.4)–(1.5) to obtain convergence of the viscosity wavefans for Riemann problems with small amplitude.

In the present work we introduce a general framework for obtaining weak convergence of the viscosity wavefans which relies basically on L^1 uniform bounds on the viscosity wavefans, plus some additional pointwise information. We apply this framework to the following class of strictly hyperbolic Riemann problems:

$$\begin{cases} u_t + f(u)_x = 0, \\ v_t + (vg(u))_x = 0, \end{cases} \quad (x, t) \in \mathbf{R}_+^2, \quad (1.6)$$

$$(u(x, 0), v(x, 0)) = \begin{cases} (u_l, v_l); & x < 0, \\ (u_r, v_r); & x > 0, \end{cases} \quad (1.7)$$

where f and g are smooth functions depending only on the variable u and satisfying

$$g' > 0, \quad f'' > 0, \quad f' < g. \quad (1.8)$$

The weak solutions of (1.6) may involve superpositions of a shock wave with a Dirac mass, occurring at the same place. This kind of wave, called a *delta-shock wave*, was introduced by D. J. Korchinski in [9] and further studied by D. Tan, T. Zhang, and Y. Zheng in [13] for the Riemann problem with $f(u) = u^2$ and $g(u) = u$. This nonstrictly hyperbolic problem has no standard weak solutions (composed by standard shock waves, rarefaction waves, and contact discontinuities) for certain values of initial data (u_l, v_l) and (u_r, v_r) , but the Riemann problem can be solved uniquely if one adds delta-shock waves with a suitable entropy condition to the list of allowed simple waves.

Also in [13], D. Tan, T. Zhang, and Y. Zheng stated that the Riemann problem (1.6)–(1.7) under the conditions (1.8) has no standard weak solutions when the initial data evolves to an overcompressive discontinuity. Furthermore, they claimed that the

appropriate weak solution for this case is formed by one delta-shock wave. This fact is proved here, as an application of our L^1 framework for viscosity wavefans.

With respect to the specific strictly hyperbolic system (1.6) with the conditions (1.8) we note that a general result of well-posedness for the Cauchy problem was recently proved by F. Huang in [6]. He built approximate solutions for the first equation in (1.6) of both sides of the possible points of discontinuities by mollification of the Oleinik solution for this equation. Then, by making a careful study of the characteristic curves of a smoothing out of the second equation, he found the solution of the Cauchy problem as a weak limit. This solution is obtained in a generalized sense that employs the Lebesgue-Stieltjes integral specific for the system. Huang's entropy criterion agrees with the entropy criteria found here and in the literature for the specific case of Riemann data.

Huang's work gives a very satisfactory account for the theory of weak solutions to system (1.6) subject to (1.8). In this sense, the current work may, at best, be regarded as a structure theorem, studying in detail certain waves that appear in Huang's weak solutions. However, the L^1 framework for the convergence of viscosity wavefans developed here, although initially motivated by the Riemann problem for system (1.6), is of independent interest.

Our analysis begins by proving (just assuming $g' > 0$) existence and uniqueness of the viscosity wavefan $U_\varepsilon(x/t) = (u_\varepsilon(x/t), v_\varepsilon(x/t))$, which is the solution of

$$\begin{cases} u_t + f(u)_x = \varepsilon t u_{xx}, \\ v_t + (vg(u))_x = \varepsilon t v_{xx} \end{cases} \quad (1.9)$$

satisfying (1.7). The uniqueness of the viscosity wavefans is a new observation, even for a single equation.

Our main result concerning systems of the form (1.9) is that, under the conditions (1.8) on f and g , the viscosity wavefans $U_\varepsilon = (u_\varepsilon, v_\varepsilon)$ converge weakly to a Riemann solution of (1.6) composed of conventional simple waves and delta-shocks as described above. We call attention to the fact that such Riemann solutions are excluded from the original Dafermos framework, since no bounds in L^∞ or in total variation can be expected to hold uniformly in the sequence. The best that one may expect is a uniform L^1_{loc} bound.

The existence and convergence results for the viscosity wavefans were proved in [12] by D. Tan for the particular system of the form (1.9) studied in [13]. His proof of existence of the viscosity wavefans does not generalize to the class of systems under consideration, because he used the fact that each solution of the first equation in (1.9) is also a solution of the second (when $f(u) = u^2$ and $g(u) = u$). In addition, in view of the uniqueness that we have proved, the viscosity wavefans obtained in [12] coincide with the ones obtained here.

In [5], J. Hu extended the existence and convergence results of D. Tan. His approach departs sharply from what was done in [12], by adapting the general framework of DiPerna and Dafermos to this problem instead of relying on the rather special algebraic properties of the solution of (1.9). In this sense, [5] is very close in spirit to the present work, and a substantial part of our work may be regarded as an independently obtained extension of the work in [5]. Aside from the fact that [5] and the present work

deal with different systems, there are two aspects in which this work goes beyond [5]. One is the general framework developed in Sec. 2 for an L^1 theory of convergence of viscosity wavefans, of which the work in [5] includes a very particular instance. The second aspect is that in the proof of Theorem 4.3 (i) and (ii) in [5], J. Hu used the Rankine-Hugoniot condition to obtain the convergence of the viscosity wavefans in the case where the Riemann solution is composed of a classical shock and a contact discontinuity. There is no reason for a weak limit that is not a weak solution to satisfy the Rankine-Hugoniot condition. The technique for bypassing the difficulty here is based on an observation by B. Keyfitz and H. Kranzer [8] (Theorem 2.3 in this paper), and which is part of the general L^1 framework that is the main purpose of this work.

The paper is organized as follows:

In Sec. 2 we present the general results about the viscosity wavefans U_ε associated to $n \times n$ Riemann problems (1.1)–(1.2) that constitute the L^1 framework mentioned above. The main results are the weak convergence of U_ε and $F(U_\varepsilon)$, as $\varepsilon \rightarrow 0^+$, for distributions involving Dirac delta functions, assuming that the viscosity wavefans are uniformly bounded in L^1_{loc} and satisfy certain pointwise estimates away from the shocks.

In Sec. 3 we present the global weak solution of the Riemann problem (1.6)–(1.7) under the conditions (1.8). This follows the closely related work [13].

In Sec. 4 we prove existence, uniqueness and *a priori* L^∞ and L^1_{loc} -estimates of the viscosity wavefans associated with (1.6)–(1.7). For this purpose we make a study of critical points. All results of this section can be proved for a system similar to (1.6) with the second equation replaced by a more general equation that is nonlinear in v (see Remark 4.1).

Finally, in Sec. 5 we prove the weak convergence of the viscosity wavefans associated with (1.6)–(1.7). We consider only the case $u_l > u_r$ in which the solution of (1.6)–(1.7) is composed of one shock wave or one delta-shock wave followed by one contact discontinuity. Since our principal interest is the weak convergence for delta-shock waves, we give the proof in detail for this case and indicate the crucial points in the proof of the weak convergence for the classical shock waves. We observe that the L^∞ -estimate obtained in Sec. 4 is $O(1/\varepsilon)$ which is sufficient to prove existence but not to prove convergence by arguments based on Helly's lemma.

In the case $u_l \leq u_r$ the weak solution is composed of one rarefaction wave followed by one contact discontinuity and the weak convergence can be obtained via the Dafermos framework.

2. Viscosity wavefans. In this section we consider the general viscosity wavefans of order n , i.e., the smooth solutions U_ε of the boundary value problem (1.4)–(1.5) with a smooth function $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$.

We present some results that will be used in this work and prove that, under some conditions, U_ε converges weakly to a distribution D involving Dirac delta functions. This distribution is a weak solution of (1.4)–(1.5) providing that $F(D)$ is defined in an appropriate manner. Moreover, we prove that, under an additional condition, $F(U_\varepsilon)$ converges to $F(D)$.

We begin by stating two estimates on the first derivative of the viscosity wavefans which can be found in [15].

PROPOSITION 2.1. Let $U_\varepsilon = U_\varepsilon(\xi)$ be a smooth solution of (1.4)–(1.5). Then

$$\|\dot{U}_\varepsilon(\xi)\| \leq \|\dot{U}_\varepsilon(0)\| \exp\left(\frac{2\alpha_\varepsilon|\xi| - \xi^2}{2\varepsilon}\right); \quad \xi \in \mathbf{R}, \quad (2.1)$$

and

$$\|\dot{U}_\varepsilon(\xi)\| \leq \frac{C}{\varepsilon}[\beta_\varepsilon + (1 + |\xi|)\|U_\varepsilon\|_\infty]; \quad \xi \in \mathbf{R}, \quad (2.2)$$

where $\alpha_\varepsilon = \sup\{\|dF(U_\varepsilon(\xi))\| : \xi \in \mathbf{R}\}$, $\beta_\varepsilon = \sup\{\|F(U_\varepsilon(\xi))\| : \xi \in \mathbf{R}\}$ and C is a constant that does not depend on ε .

REMARK 2.1. The estimate (2.1) shows that the viscosity wavefans decay rapidly to zero when $|\xi| \rightarrow +\infty$, for each fixed $\varepsilon > 0$. On the other hand, the estimate (2.2) can be used on a closed interval I to obtain estimates of the form $\|\dot{U}_\varepsilon\|_{L^\infty(I)} = O(1/\varepsilon^m)$. For example, if $I = [a, b]$ then from (2.2) it follows that

$$\|\dot{U}_\varepsilon(\xi)\| \leq \frac{K_I}{\varepsilon}(\beta_\varepsilon + \|U_\varepsilon\|_\infty); \quad \xi \in I,$$

where $K_I = C(1 + \max\{|b|, |a|\})$ is a constant that depends only on I . Thus, if $\beta_\varepsilon = O(1/\varepsilon^p)$ and $\|U_\varepsilon\|_\infty = O(1/\varepsilon^p)$, then $\|\dot{U}_\varepsilon\|_{L^\infty(I)} = O(1/\varepsilon^{p+1})$ on I .

Next we state, without proof, a Theorem due to Dafermos [1] which is the only general result known about the existence of a smooth solution for (1.4)–(1.5). Its proof is based on the Leray-Schauder fixed-point theorem.

THEOREM 2.2. Consider the following boundary value problem with parameters $\mu \in [0, 1]$ and $L \geq 1$:

$$\begin{cases} \varepsilon \ddot{U} = \mu \dot{F}(U) - \xi \dot{U}; & \xi \in (-L, +L), \\ U(-L) = \mu U_l, & U(L) = \mu U_r. \end{cases} \quad (2.3)$$

Suppose there exists a constant M , depending at most on ε, U_l , and U_r (and thus independent of μ and L) such that

$$\sup_{\xi \in (-L, L)} \|U(\xi)\| \leq M, \quad (2.4)$$

for all smooth solutions $U(\xi)$ of (2.3).

Then there exists a smooth solution (not necessarily unique) of (1.4)–(1.5) satisfying (2.4) with $L = \infty$.

The following result, due to Keyfitz and Kranzer [8], provides an interesting relation, which is independent of ε , between the viscosity wavefans and the discontinuous functions of the form

$$W_s(\xi) = \begin{cases} U_l; & \xi < s, \\ U_r; & \xi > s. \end{cases} \quad (2.5)$$

THEOREM 2.3. Let $U_\varepsilon = U_\varepsilon(\xi)$ be a smooth solution of (1.4)–(1.5). Then, for each $s \in \mathbf{R}$, we have $\int_{-\infty}^{+\infty} [U_\varepsilon(\xi) - W_s(\xi)] d\xi = s[U_r - U_l] - [F(U_r) - F(U_l)]$, where W_s is defined by (2.5).

Now we prove the main results of this section which show that, under certain conditions, $U_\varepsilon \rightharpoonup D$ and $F(U_\varepsilon) \rightharpoonup F(D)$ when $\varepsilon \rightarrow 0^+$. Here D and $F(D)$ are the distributions defined respectively by

$$D = V + \sum_{j=1}^k C_j \delta_{\sigma_j} \quad (2.6)$$

and

$$F(D) = F(V) + \sum_{j=1}^k \sigma_j C_j \delta_{\sigma_j}, \quad (2.7)$$

where $\sigma_0 = -\infty < \sigma_1 < \dots < \sigma_k < \sigma_{k+1} = \infty$, $V = V(\xi)$ is a piecewise continuous function that is a classical solution of the system in (1.4) on each interval $I_j = (\sigma_j, \sigma_{j+1})$, δ_{σ_j} is the Dirac δ -function concentrated at σ_j and

$$C_j = \sigma_j [V(\sigma_j^+) - V(\sigma_j^-)] - [F(V(\sigma_j^+)) - F(V(\sigma_j^-))]. \quad (2.8)$$

THEOREM 2.4. For each $\varepsilon > 0$, let $U_\varepsilon = U_\varepsilon(\xi)$ be a smooth function of (1.4)–(1.5). Suppose that

- (i) U_ε converges uniformly to a function V as above on closed intervals contained in $I_j = (\sigma_j, \sigma_{j+1})$ for all $j \in \{0, 1, 2, \dots, k\}$;
- (ii) the family $\{U_\varepsilon\}$ is uniformly bounded in $L^1_{\text{loc}}(\mathbf{R})$, i.e., for each closed interval $I \subset \mathbf{R}$, there exists a constant M (which can depend on I , but not on ε) such that $\int_I \|U_\varepsilon\| d\xi \leq M$ for all $\varepsilon > 0$.

Then $U_\varepsilon \rightharpoonup D$, where D is the distribution defined by (2.6).

Proof. We must show that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} [U_\varepsilon - V] \phi d\xi = \sum_{j=1}^k C_j \phi(\sigma_j); \quad \phi \in C_0^\infty(\mathbf{R}). \quad (2.9)$$

Initially we prove (2.9) for $\phi \in C_0^\infty(\mathbf{R})$ such that $\text{supp } \phi \subset [\sigma_j - \alpha, \sigma_j + \alpha]$ for fixed $j \in \{1, 2, \dots, k\}$ with $\sigma_i \notin [\sigma_j - \alpha, \sigma_j + \alpha]$ if $i \neq j$ for some $\alpha > 0$. Thus, the equality in (2.9) reduces to

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} (U_\varepsilon - V) \phi d\xi = C_j \phi(\sigma_j). \quad (2.10)$$

Following [13], we fix ϕ as above and take $\psi \in C_0^\infty(\mathbf{R})$ such that

$$\psi(\xi) \equiv \phi(\sigma_j); \quad \xi \in [\sigma_j - \beta, \sigma_j + \beta] \subset (\sigma_j - \alpha, \sigma_j + \alpha), \quad (2.11)$$

$\text{supp } \psi \subset [\sigma_j - \alpha, \sigma_j + \alpha]$ and $\|\psi - \phi\|_\infty \leq \tau$, where τ is a small positive constant. The function ψ is a *sloping test function* near to ϕ .

Multiplying (1.4) by ψ and integrating by parts on $(-\infty, +\infty)$ we obtain

$$\int_{-\infty}^{+\infty} U_\varepsilon \psi d\xi = I_1^\varepsilon - I_2^\varepsilon, \quad (2.12)$$

where $I_1^\varepsilon = \int_{-\infty}^{+\infty} [F(U_\varepsilon) - \xi U_\varepsilon] \dot{\psi} d\xi$ and $I_2^\varepsilon = \varepsilon \int_{-\infty}^{+\infty} \dot{U}_\varepsilon \dot{\psi} d\xi = -\varepsilon \int_{-\infty}^{+\infty} U_\varepsilon \ddot{\psi} d\xi$.

We have $\lim_{\varepsilon \rightarrow 0^+} I_2^\varepsilon = 0$ since $\|I_2^\varepsilon\| \leq \varepsilon \|\ddot{\psi}\|_\infty \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} \|U_\varepsilon\| d\xi$ and U_ε is uniformly bounded in $L^1_{\text{loc}}(\mathbf{R})$.

From hypothesis (i) we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} I_1^\varepsilon &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\sigma_j - \alpha}^{\sigma_j - \beta} + \int_{\sigma_j + \beta}^{\sigma_j + \alpha} \right) (F(U_\varepsilon) - \xi U_\varepsilon) \dot{\psi} d\xi \\ &= \left(\int_{\sigma_j - \alpha}^{\sigma_j - \beta} + \int_{\sigma_j + \beta}^{\sigma_j + \alpha} \right) (F(V) - \xi V) \dot{\psi} d\xi = \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} (F(V) - \xi V) \dot{\psi} d\xi. \end{aligned}$$

Using integration by parts on $[\sigma_j - \alpha, \sigma_j] \cup (\sigma_j, \sigma_j + \alpha]$ and noting that $\psi(\sigma_j - \alpha) = \psi(\sigma_j + \alpha) = 0$ and $\psi(\sigma_j) = \phi(\sigma_j)$ we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} I_1^\varepsilon &= \phi(\sigma_j) [F(V(\xi)) - \xi V(\xi)]_{\xi=\sigma_j^+}^{\xi=\sigma_j^-} + \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} V \dot{\psi} d\xi \\ &= C_j \phi(\sigma_j) + \int_{-\infty}^{+\infty} V \dot{\psi} d\xi. \end{aligned}$$

Therefore, in view of (2.12) we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} (U_\varepsilon - V) \dot{\psi} d\xi = C_j \phi(\sigma_j). \quad (2.13)$$

Now, from hypothesis (ii) it follows that

$$\int_{\sigma_j - \alpha}^{\sigma_j + \alpha} \|U_\varepsilon - V\| d\xi \leq L, \quad (2.14)$$

for some positive constant L that is independent of $\varepsilon > 0$. So,

$$\begin{aligned} &\left\| \int_{-\infty}^{+\infty} (U_\varepsilon - V) \dot{\psi} d\xi - C_j \phi(\sigma_j) \right\| \\ &\leq \|\phi - \psi\|_\infty \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} \|U_\varepsilon - V\| d\xi + \left\| \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} (U_\varepsilon - V) \dot{\psi} d\xi - C_j \phi(\sigma_j) \right\| \\ &\leq \tau L + \left\| \int_{\sigma_j - \alpha}^{\sigma_j + \alpha} (U_\varepsilon - V) \dot{\psi} d\xi - C_j \phi(\sigma_j) \right\|. \end{aligned}$$

In view of (2.13) we have $\lim_{\varepsilon \rightarrow 0^+} \left\| \int_{-\infty}^{+\infty} (U_\varepsilon - V) \dot{\psi} d\xi - C_j \phi(\sigma_j) \right\| \leq \tau L$ which leads to (2.10) when $\tau \rightarrow 0^+$.

If $\phi \in C_0^\infty(\mathbf{R})$ is arbitrary, then we take $\phi_1, \phi_2, \dots, \phi_k \in C_0^\infty(\mathbf{R})$ such that $\sigma_i \notin \text{supp } \phi_j$ if $i \neq j$ and

$$\sum_{i=1}^k \phi_i = \phi. \quad (2.15)$$

Thus, we obtain (2.9) from (2.10) applied to each ϕ_i . $\square$

Under an additional hypothesis to the last theorem we obtain the weak convergence of $F(U_\varepsilon)$.

THEOREM 2.5. For each $\varepsilon > 0$ let U_ε be a solution of (1.4)–(1.5). Suppose (i)–(ii) of Theorem 2.4 and the additional hypothesis:

(iii) for each $\xi \in \mathbf{R} - \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ the sequence $\{\dot{U}_\varepsilon(\xi)\}$ is bounded.

Then, $F(U_\varepsilon) \rightharpoonup F(D)$, where $F(D)$ is the distribution defined by (2.7).

Proof. Let $j \in \{1, 2, \dots, k\}$ be fixed and let $\varphi \in C_0^\infty(\mathbf{R})$ be a test function with $\text{supp } \varphi \subset I = [a, b]$, $\sigma_j \in I$ and $\sigma_i \notin I$ if $i \neq j$.

Integrating (1.4) by parts we obtain

$$F(U_\varepsilon(\xi)) = [F(U_\varepsilon) - \varepsilon \dot{U}_\varepsilon - a U_\varepsilon]_{\xi=a} + \varepsilon \dot{U}_\varepsilon(\xi) + \xi U_\varepsilon(\xi) - \int_a^\xi U_\varepsilon(\eta) d\eta.$$

Therefore,

$$\int_{-\infty}^{+\infty} F(U_\varepsilon) \varphi d\xi = I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 - I_\varepsilon^4, \quad (2.16)$$

where $I_\varepsilon^1 = [F(U_\varepsilon) - \varepsilon \dot{U}_\varepsilon - \xi U_\varepsilon]_{\xi=a} \int_a^b \varphi d\xi$, $I_\varepsilon^2 = \varepsilon \int_a^b \dot{U}_\varepsilon \varphi d\xi = -\varepsilon \int_a^b U_\varepsilon \dot{\varphi} d\xi$, $I_\varepsilon^3 = \int_a^b \xi U_\varepsilon \varphi d\xi$, and $I_\varepsilon^4 = \int_a^b \varphi \int_a^\xi U_\varepsilon d\eta d\xi$.

From hypothesis (ii) we have immediately $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^2 = 0$ and from hypotheses (i) and (iii) it follows that $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 = [F(V) - \xi V]_{\xi=a} \int_a^b \varphi d\xi$.

Since $U_\varepsilon \rightarrow D$ (Theorem 2.4) we have $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^3 = \int_a^b \xi V \varphi d\xi + C_j \sigma_j \varphi(\sigma_j)$.

Now, from hypothesis (ii), we obtain

$$\left\| \varphi(\xi) \int_a^\xi U_\varepsilon(\eta) d\eta \right\| \leq \|\varphi\|_\infty \int_a^b \|U_\varepsilon(\eta)\| d\eta \leq L; \quad \xi \in I,$$

for some $L > 0$ that is independent of ε . So,

$$\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^4 = \int_a^b \varphi(\xi) \left[\lim_{\varepsilon \rightarrow 0^+} \int_a^\xi U_\varepsilon(\eta) d\eta \right] d\xi. \quad (2.17)$$

In order to simplify (2.17) we calculate the function

$$\lim_{\varepsilon \rightarrow 0^+} \int_a^\xi U_\varepsilon(\eta) d\eta; \quad \text{a.e. } \xi \in [a, b].$$

Let $\xi \in [a, b]$ be fixed. If $a < \xi < \sigma_j$ then, in view of the (uniform) convergence $U_\varepsilon \rightarrow V$ on $[a, \xi]$ and of the equality $F(V(\eta)) = \eta \dot{V}(\eta)$, $\eta \in [a, \sigma_j]$, we have

$$\lim_{\varepsilon \rightarrow 0^+} \int_a^\xi U_\varepsilon(\eta) d\eta = \int_a^\xi V(\eta) d\eta = -[F(V(\eta)) - \eta V(\eta)]_a^\xi.$$

On the other hand, if $\xi > \sigma_j$, let $\alpha > 0$ be such that $a + \alpha < \sigma_j < \xi - \alpha$ and let $\psi \in C_0^\infty(\mathbf{R})$ be a test function satisfying $\text{supp } \psi \subset (a, \xi)$ and $\psi(\eta) = 1$ if $\eta \in [a + \alpha, \xi - \alpha]$.

Then

$$\begin{aligned} \int_a^\xi (U_\varepsilon - V) d\eta &= \int_a^\xi (U_\varepsilon - V)(1 - \psi) d\eta + \int_a^\xi (U_\varepsilon - V)\psi d\eta \\ &= \left(\int_a^{a+\alpha} + \int_{\xi-\alpha}^\xi \right) (U_\varepsilon - V)(1 - \psi) d\eta + \int_{-\infty}^{+\infty} (U_\varepsilon - V)\psi d\eta. \end{aligned}$$

Since $U_\varepsilon \rightarrow V$ uniformly on $[a, a + \alpha]$ we have

$$\lim_{\varepsilon \rightarrow 0^+} \left\| \int_a^{a+\alpha} (U_\varepsilon - V)(1 - \psi) d\eta \right\| \leq \|1 - \psi\|_\infty \lim_{\varepsilon \rightarrow 0^+} \int_a^{a+\alpha} \|U_\varepsilon - V\| d\eta = 0.$$

Similarly, we obtain $\lim_{\varepsilon \rightarrow 0^+} \left\| \int_{\xi-\alpha}^\xi (U_\varepsilon - V)(1 - \psi) d\eta \right\| = 0$.

So, $\lim_{\varepsilon \rightarrow 0^+} \int_a^\xi [U_\varepsilon - V] d\eta = \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} (U_\varepsilon - V)\psi d\eta = C_j \psi(\sigma_j) = C_j$, if $\xi \in (\sigma_j, b]$. (We have used Theorem 2.4 to obtain the second equality.)

Therefore, integrating by parts the equality $\dot{F}(V(\eta)) = \eta \dot{V}(\eta)$ satisfied for $\eta \in (a, \sigma_j) \cup (\sigma_j, b]$, we can verify that $\int_a^\xi V(\eta) d\eta + C_j = -[F(V(\eta)) - \eta V(\eta)]_a^\xi$, if $\xi \in (\sigma_j, b]$.

We have concluded that $\lim_{\varepsilon \rightarrow 0^+} \int_a^\xi U_\varepsilon(\eta) d\eta = -[F(V(\eta)) - \eta V(\eta)]_a^\xi$, for all $\xi \in [a, \sigma_j) \cup (\sigma_j, b]$ and hence, $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^4 = - \int_a^b \varphi(\xi) [F(V(\eta)) - \eta V(\eta)]_a^\xi d\xi$.

Now, from (2.16) we have $\lim_{\varepsilon \rightarrow 0^+} \int_a^b F(U_\varepsilon) \varphi d\xi = \int_a^b F(V) \varphi d\xi + C_j \sigma_j \varphi(\sigma_j)$, i.e.,

$$\lim_{\varepsilon \rightarrow 0^+} \langle F(U_\varepsilon), \varphi \rangle = \langle F(D), \varphi \rangle \quad (2.18)$$

for any test function φ with support contained in a neighbourhood of σ_j . If ϕ is an arbitrary test function, then by using a decomposition as (2.15) we prove (2.18) for ϕ . $\square$

REMARK 2.2. In view of the Dominated Convergence Theorem we can replace the requirement of the uniform convergence of U_ε in Theorems 2.4 and 2.5 by the pointwise convergence $U_\varepsilon(\xi) \rightarrow V(\xi)$ a.e. $\xi \in I_j$ together with the uniform boundedness of U_ε in $L_{\text{loc}}^\infty(I_j)$.

At last, we show that the distribution D obtained as the weak limit of viscosity wavefans in Theorem 2.4 is a weak solution of (1.4)–(1.5) provided that the distribution $F(D)$ is defined by (2.7).

THEOREM 2.6. Let D and $F(D)$ be defined by (2.6) and (2.7) respectively. Then

$$\langle \dot{F}(D) - \xi \dot{D}, \phi \rangle = 0; \quad \phi \in C_0^\infty(\mathbf{R}). \quad (2.19)$$

Proof. Let $\sigma_j \in \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ be fixed and let $\varphi \in C_0^\infty(\mathbf{R})$ be such that $\text{supp } \varphi \subset I = [a, b]$, with $\sigma_j \in I$ and $\sigma_i \notin [a, b]$ if $i \neq j$.

Since $\langle \dot{F}(D), \varphi \rangle = -\langle F(D), \dot{\varphi} \rangle$ and $\dot{F}(V) = \xi \dot{V}$ on $[a, \sigma_j) \cup (\sigma_j, b]$, we obtain

$$\langle \dot{F}(D), \varphi \rangle = \varphi(\sigma_j) [F(V(\sigma_j^+)) - F(V(\sigma_j^-))] + \int_a^b \xi \varphi \dot{V} d\xi - C_j \sigma_j \dot{\varphi}(\sigma_j).$$

On the other hand, since $\langle \xi \dot{D}, \varphi \rangle = \langle \dot{D}, \xi \varphi \rangle = -\langle D, \frac{d}{d\xi} [\xi \varphi] \rangle$, we have

$$\begin{aligned} -\langle \xi \dot{D}, \varphi \rangle &= [\xi V \varphi]_a^{\sigma_j} + [\xi V \varphi]_{\sigma_j}^b - \int_a^b \xi \varphi \dot{V} d\xi + C_j \frac{d}{d\xi} [\xi \varphi] \Big|_{\xi=\sigma_j} \\ &= -\varphi(\sigma_j) [F(V(\sigma_j^+)) - F(V(\sigma_j^-))] - \int_a^b \xi \varphi \dot{V} d\xi + C_j \sigma_j \dot{\varphi}(\sigma_j) \\ &= -\langle \dot{F}(D), \varphi \rangle. \end{aligned}$$

We have proved (2.19) for φ . However, if ϕ is an arbitrary function in $C_0^\infty(\mathbf{R})$ then, using a decomposition as (2.15) we prove (2.19) for ϕ . $\square$

3. Solution of the Riemann problem. In this section we present the weak solution of (1.6)–(1.7) under the conditions (1.8). This solution is in self-similar form $(u(\frac{x}{t}), v(\frac{x}{t}))$ where $(u(\xi), v(\xi))$ is a weak solution of the boundary value problem:

$$\begin{cases} \dot{f}(u) = \xi \dot{u}, \\ \frac{d}{d\xi} [vg(u)] = \xi \dot{v} \quad ; \quad \xi \in (-\infty, +\infty), \end{cases} \quad (3.1)$$

$$\begin{cases} (u(-\infty), v(-\infty)) = (u_l, v_l), \\ (u(+\infty), v(+\infty)) = (u_r, v_r). \end{cases} \quad (3.2)$$

The eigenvalues and the corresponding right eigenvectors of (1.6) are

$$\lambda_1 = f'(u), \quad r_1 = \left(1, \frac{vg'(u)}{f'(u) - g(u)}\right)^T, \quad \lambda_2 = g(u), \quad r_2 = (0, 1)^T. \quad (3.3)$$

In view of (1.8) the system (1.6) is strictly hyperbolic ($\lambda_1 < \lambda_2$), the 1-characteristic family is genuinely nonlinear ($\nabla \lambda_1 \cdot r_1 > 0$), and the 2-characteristic family is linearly degenerate ($\nabla \lambda_2 \cdot r_2 = 0$).

For each left state (u_l, v_l) the Hugoniot locus $H(u_l, v_l)$ is the curve passing at (u_l, v_l) formed by all points (u, v) satisfying

$$\begin{cases} s(u - u_l) = f(u) - f(u_l), \\ s(v - v_l) = vg(u) - v_l g(u_l), \end{cases}$$

for some $s \in \mathbf{R}$.

The portion of $H(u_l, v_l)$ used to build the classical weak solutions of (1.6)–(1.7) involving discontinuities is the subset $S_1(u_l, v_l) \cup D_2(u_l, v_l)$ where

$$D_2(u_l, v_l) = \{(u_l, v) : v \in \mathbf{R}\}$$

is the 2-contact discontinuity curve and

$$S_1(u_l, v_l) = \left\{ (u, v) : v = v_l \frac{s(u) - \lambda_2(u_l)}{s(u) - \lambda_2(u)}, \quad u \leq u_l, \quad s(u) < \lambda_2(u) \right\}$$

is the 1-shock curve. Here $s(u) = \frac{f(u) - f(u_l)}{u - u_l}$ is the shock velocity.

REMARK 3.1. Since $s(u_l) < \lambda_2(u_l)$, the function $s(u) - \lambda_2(u)$ is negative on an interval that contains u_l . However, if $-\infty < u_l^* < u_l$ is the first zero of this function to the left of u_l , then $S_1(u_l, v_l)$ tends to the line $u = u_l^*$ when $u \rightarrow (u_l^*)^-$, i.e., its v -coordinate becomes unbounded.

The 1-rarefaction curve $R_1(u_l, v_l)$ is the portion of the integral curve in the direction of r_1 in which λ_1 increases. Thus,

$$R_1(u_l, v_l) = \left\{ (u, v) : v = v_l \exp \left(- \int_{u_l}^u \frac{g'(z)}{\lambda_2(z) - \lambda_1(z)} dz \right), u \geq u_l \right\}.$$

The rarefaction curve corresponding to the eigenvalue λ_2 does not exist since λ_2 is constant on the integral curve in the direction of r_2 , which coincides with $D_2(u_l, v_l)$.

The classical solution of (1.6)–(1.7) is obtained by using the curves $S_1(u_l, v_l)$, $R_1(u_l, v_l)$, and $D_2(u_r, v_r)$.

If $u_r > u_l$ the solution of (1.6)–(1.7) is the composition $R_1 + D_2$ that consists of the rarefaction wave R_1 connecting (u_l, v_l) to (u_r, v_m) followed by the contact discontinuity D_2 connecting (u_r, v_m) to (u_r, v_r) . The *intermediate state* (u_r, v_m) is the unique point in $R_1(u_l, v_l) \cap D_2(u_r, v_r)$. This solution can be written in the following self-similar form:

$$u(\xi) = \begin{cases} u_l; & \xi \leq \lambda_1(u_l), \\ \lambda_1^{-1}(\xi); & \lambda_1(u_l) \leq \xi \leq \lambda_1(u_r), \\ u_r; & \xi \geq \lambda_1(u_r), \end{cases} \quad (3.4)$$

$$v(\xi) = \begin{cases} v_l \exp \left(- \int_{u_l}^{u(\xi)} \frac{g'(z)}{\lambda_2(z) - \lambda_1(z)} dz \right); & \xi < \lambda_2(u_r), \\ v_r; & \xi > \lambda_2(u_r). \end{cases} \quad (3.5)$$

If $u_r < u_l$ and $s(u_r) < \lambda_2(u_r)$ the solution of (1.6)–(1.7) is the composition $S_1 + D_2$ that consists of the shock wave S_1 connecting (u_l, v_l) to (u_r, v_m) followed by the contact discontinuity D_2 connecting (u_r, v_m) to (u_r, v_r) , where the intermediate state (u_r, v_m) is the unique point in $S_1(u_l, v_l) \cap D_2(u_r, v_r)$. In the self-similar form this solution becomes:

$$u_s(\xi) = \begin{cases} u_l; & \xi < s, \\ u_r; & \xi > s, \end{cases} \quad (3.6)$$

$$v(\xi) = \begin{cases} v_l; & \xi < s, \\ v_m; & s < \xi < \lambda_2(u_r), \\ u_r; & \xi > \lambda_2(u_r), \end{cases}$$

where $s = s(u_r)$ and $v_m = v_l \left(\frac{s(u_r) - \lambda_2(u_l)}{s(u_r) - \lambda_2(u_r)} \right)$.

If $u_r < u_l$ and $s(u_r) \geq \lambda_2(u_r)$ then, according to Remark 3.1 above, the right state (u_r, v_r) cannot belong to $S_1(u_l, v_l)$ or $R_1(u_l, v_l)$. Henceforth, in this case there exists no standard weak solution for (1.6)–(1.7). So, we solve the problem as in [13] using the delta-shock wave S_δ which we present below as being a distribution of the self-similar form (2.6) satisfying a condition of overcompressivity. With this condition we obtain uniqueness of solutions in the class of the simple waves and delta-shock waves.

DEFINITION 3.1. Let S_δ be a distribution of the form:

$$S_\delta = (u_s(\xi), v_s(\xi) + c\delta_s), \quad (3.7)$$

where s is a real number, u_s is a function as in (3.6),

$$v_s(\xi) = \begin{cases} v_l; & \xi < s, \\ v_r; & \xi > s \end{cases} \quad (3.8)$$

and

$$c = s[v_r - v_l] - [v_r g(u_r) - v_l g(u_l)]. \quad (3.9)$$

Then S_δ is called a *delta-shock wave* for (3.1)–(3.2) if the following two conditions are satisfied:

(i) the S_δ -Rankine-Hugoniot condition

$$s[u_r - u_l] = [f(u_r) - f(u_l)], \quad (3.10)$$

(ii) the S_δ -entropy condition

$$\lambda_2(u_r) \leq s \leq \lambda_1(u_l). \quad (3.11)$$

Some remarks are in order.

1. According to [13] we define u_s at s such that $\lambda_2(u_s(s)) = s$. So, from Theorem 2.6, a delta-shock S_δ is a solution of (3.1)–(3.2) in the sense of distributions if the product $c\delta_s \cdot g(u_s)$ is defined by

$$\langle c\delta_s \cdot g(u_s), \phi \rangle := cg(u_s(s))\phi(s) = cs\phi(s) = \langle sc\delta_s, \phi \rangle; \quad \phi \in C_0^\infty(\mathbf{R}),$$

which is a Radon measure.

2. In view of (1.8) and of (3.3) a delta-shock S_δ is an overcompressive wave, i.e., all characteristics enter on the discontinuity.

3. The inequality $s \leq \lambda_1(u_l)$ in (3.11) is equivalent to $u_l > u_r$, because of the convexity condition on f in (1.8). So, the following two conditions characterize a delta-shock S_δ :

$$u_l > u_r \quad \text{and} \quad \lambda_2(u_r) \leq s.$$

4. Without (3.11), the distribution S_δ could be another weak solution than $S_1 + R_1$ when $s(u_r) < \lambda_2(u_r)$.

5. Taking $\xi = x/t$ and denoting S_δ by $S_{\delta,x,t}$ we can write

$$S_{\delta,x,t} = \left(u_s\left(\frac{x}{t}\right), v_s\left(\frac{x}{t}\right) + v_\delta \right), \quad (3.12)$$

where the distribution v_δ is defined by

$$\langle v_\delta, \phi \rangle = c \int_{-\infty}^{+\infty} t\phi(st, t) dt; \quad \phi \in C_0^\infty(\mathbf{R}_+^2).$$

So, v_δ is the Dirac-delta with *weight* c supported on the discontinuity $x = st$.

4. Existence, uniqueness and estimates. In this section we show existence, uniqueness and estimates in L^∞ and L^1_{loc} for the smooth solutions of the boundary value problem:

$$\begin{cases} e\ddot{u} = \dot{f}(u) - \xi\dot{u}; \\ e\ddot{v} = \frac{d}{d\xi}[vg(u)] - \xi\dot{v}, \end{cases} \quad \xi \in (-\infty, +\infty), \quad (4.1)$$

$$\begin{cases} (u(-\infty), v(-\infty)) = (u_l, v_l), \\ (u(+\infty), v(+\infty)) = (u_r, v_r). \end{cases} \quad (4.2)$$

We assume that $f(u)$ and $g(u)$ are smooth functions with $g'(u) > 0$ for all u on the closed interval with extremes u_l and u_r , which we denote by $I[u_l, u_r]$.

REMARK 4.1. All results of this section remain true, with small changes in the proofs, if we replace the second equation in (4.1) by the following nonlinear equation:

$$e\ddot{v} = \frac{d}{d\xi}[vg(u, v) + h(v)] - \xi\dot{v},$$

where $h(v)$ and $g(u, v)$ are smooth functions with $g_u > 0$ on $I[u_l, v_l]$, and $g(J, \mathbf{R})$ is a bounded set for each closed interval J .

THEOREM 4.1. Let $(u_1(\xi), v_1(\xi))$ and $(u_2(\xi), v_2(\xi))$ be two smooth solutions of (4.1)–(4.2). Then $u_1 = u_2$ and $v_1 = v_2$.

Proof. First we prove that $u_1 = u_2$ by showing that $w := u_2 - u_1$ is null everywhere. We can verify that w is a smooth solution of the boundary value problem

$$\begin{cases} \varepsilon\ddot{w} = \frac{d}{d\xi}[wh] - \xi\dot{w}; & \xi \in (-\infty, +\infty), \\ w(-\infty) = 0, & w(+\infty) = 0, \end{cases} \quad (4.3)$$

where $h(\xi) = \int_0^1 f'(u_2(\xi)\theta + (1-\theta)u_1(\xi)) d\theta$. (We observe that h is a bounded function.)

Let us suppose that w is not the null function. Let α and β be consecutive zeros of w with $-\infty \leq \alpha < \beta \leq +\infty$. So, integrating (4.3) by parts on (α, β) we find

$$\varepsilon(\dot{w}(\beta) - \dot{w}(\alpha)) = \int_\alpha^\beta w d\xi. \quad (4.4)$$

(If $\alpha = -\infty$ and/or $\beta = +\infty$ then we turn to Remark 2.1 to verify that $\lim_{\xi \rightarrow \alpha^-} \xi w(\xi) = 0 = \lim_{\xi \rightarrow \beta^+} \xi w(\xi)$.)

Now, if $w > 0$ on (α, β) , then $\dot{w}(\beta) \leq 0 \leq \dot{w}(\alpha)$ and $\int_\alpha^\beta w d\xi > 0$; and if $w < 0$ on (α, β) , then $\dot{w}(\beta) \geq 0 \geq \dot{w}(\alpha)$ and $\int_\alpha^\beta w d\xi < 0$. In both cases we have a contradiction with (4.4). So, we must have $w \equiv 0$.

Similarly we prove that $v := v_2 - v_1$ is null everywhere since v is a smooth solution of the boundary value problem (4.3) with $h = g(u_1) = g(u_2)$. $\square$

To prove existence of a solution to (4.1)–(4.2) using Dafermos' Theorem stated in Sec. 2 (Theorem 2.2) we need to show that all smooth solutions $(u(\xi), v(\xi))$ of the following two-parameter boundary value problem

$$\begin{cases} \varepsilon\ddot{u} = \mu\dot{f}(u) - \xi\dot{u}, \\ \varepsilon\ddot{v} = \mu\frac{d}{d\xi}[vg(u)] - \xi\dot{v}, \end{cases} \quad \xi \in (-L, L), \quad (4.5)$$

$$\begin{cases} (u(-L), v(-L)) = \mu(u_l, v_l), \\ (u(L), v(L)) = \mu(u_r, v_r), \end{cases} \quad (4.6)$$

satisfy the *a priori* L^∞ -estimate:

$$\sup_{\xi \in (-L, L)} \|(u(\xi), v(\xi))\| \leq M, \quad (4.7)$$

where the positive constant M must be independent of the parameters $L \geq 1$ and $\mu \in [0, 1]$.

REMARK 4.2. The solutions of the first equation in (4.5) are monotonic. In fact, if u is a smooth solution of $\varepsilon \ddot{u} = \mu f'(u) - \xi \dot{u}$ on an interval I then, by integration, we can verify that

$$\dot{u}(\xi) = \dot{u}(\xi_0) \exp \left(-\frac{1}{\varepsilon} \int_{\xi_0}^{\xi} [\mu f'(u(\eta)) - \eta] d\eta \right),$$

for any $\xi_0 \in I$.

As a consequence of Remark 4.2, the $u(\xi)$ -component of any smooth solution of (4.5)–(4.6) satisfies

$$\sup_{\xi \in (-L, L)} |u(\xi)| \leq \max\{|\mu u_l|, |\mu u_r|\} \leq \bar{u}, \quad (4.8)$$

where $\bar{u} = \max\{|u_l|, |u_r|\}$, which is a constant that does not depend on $L \geq 1$ and $\mu \in [0, 1]$.

Therefore, we need only prove that there exists a constant C that is independent of L and μ such that

$$\sup_{\xi \in (-L, L)} |v(\xi)| \leq C, \quad (4.9)$$

for all solutions $v(\xi)$ of the following linear (considering $g(u(\xi))$ as a coefficient) boundary value problem:

$$e\ddot{v} = \mu \frac{d}{d\xi} [vg(u)] - \xi \dot{v}; \quad \xi \in (-L, L), \quad (4.10)$$

$$v(-L) = \mu v_l, \quad v(L) = \mu v_r. \quad (4.11)$$

LEMMA 4.2. Let $v(\xi)$ be a smooth solution of (4.10) on $(-L, L)$. We claim that

- (i) if $v(\alpha) = v(\beta) = 0$ for any $\alpha, \beta \in (-L, L)$, then $v(\xi) = 0$ for all $\xi \in (-L, L)$;
- (ii) if $u(-L) < u(L)$, then a critical point $\xi_0 \in (-L, L)$ of v is a maximum (or minimum) point if and only if $v(\xi_0) < 0$ (or $v(\xi_0) > 0$), and furthermore,

$$|v(\xi)| \leq \max\{|v(-L)|, |v(L)|\}; \quad (4.12)$$

- (iii) if $u(-L) > u(L)$ then a critical point $\xi_0 \in (-L, L)$ of v is a maximum (or minimum) point if and only if $v(\xi_0) > 0$ (or $v(\xi_0) < 0$).

Proof. The proof of (i) is based on the same arguments that we have used to prove the uniqueness above. We only observe that if $v(\xi) = 0$ for all $\xi \in (\alpha, \beta)$, then $v \equiv 0$ on $(-L, L)$ in view of the classical uniqueness theorems about initial-value problems for

linear ordinary differential equations. In order to verify (ii) and (iii) we evaluate (4.10) at $\xi = \xi_0$ to get

$$e\ddot{v}(\xi_0) = \mu v(\xi_0)g'(u(\xi_0))\dot{u}(\xi_0). \quad (4.13)$$

Since $g' > 0$ and u is a strictly monotonic function, we conclude the proof of statements (ii) and (iii) directly from (4.13).

We observe from Lemma 4.2 that if $u(-L) \neq u(L)$ and v assumes three local extremes, then v vanishes twice, which implies from (i) that $v \equiv 0$. Furthermore, if $u(-L) < u(L)$ and $v(-L), v(L) \leq 0$ (or $v(-L), v(L) \geq 0$) then in view of (i), v cannot assume a local minimum (or maximum). Also, if $v(-L)v(L) < 0$ then v is strictly monotonic. At last, if $u(-L) > u(L)$ and v assumes two local extremes ξ_1 and ξ_2 , then $v(\xi_1)v(\xi_2) < 0$ and v vanishes only at one $\bar{\xi} \in (\xi_1, \xi_2)$.

To prove estimates in $L^\infty(\mathbf{R})$ and $L^1_{\text{loc}}(\mathbf{R})$ on v we need another lemma.

LEMMA 4.3. Let $v(\xi)$ be a smooth solution of (4.10) on $[a, b] \subset (-L, L)$ with $u(b) > u(a)$. Then

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |v(\eta)| d\eta \leq c \left(1 + \frac{k}{\beta - \alpha} \right) \quad (4.14)$$

and

$$\max_{\xi \in [a, b]} |v(\xi)| \leq ce^k \left(1 + \frac{k}{\varepsilon} \right), \quad (4.15)$$

where $c = \max\{|v(a)|, |v(b)|\}$ and $k = (u(a) - u(b)) \max\{g'(u) : u \in [u(b), u(a)]\}$.

Proof. We can assume that v is not monotonic. Otherwise, the lemma is trivial. First we prove (4.14). The proof is based on arguments encountered in [2] or in [15] in a more general situation. Initially we suppose that $0 \leq v(a) < v(b)$ or $0 \leq v(b) < v(a)$. So it follows from Lemma 4.2 that $v > 0$ on (a, b) and that there exists $\xi_0 \in (a, b)$ such that v is strictly increasing on (a, ξ_0) , it assumes the maximum value in $\xi = \xi_0$ and it is strictly decreasing on (ξ_0, b) .

Let $a \leq \bar{a} < \xi_0 < \bar{b} \leq b$ be such that

$$0 < v(\bar{a}) = v(\bar{b}) = c < v(\eta); \quad \eta \in (\bar{a}, \bar{b}). \quad (4.16)$$

We observe that if $0 \leq v(a) < v(b)$, then $a < \bar{a}$ and $\bar{b} = b$, and if $0 \leq v(b) < v(a)$, then $a = \bar{a}$ and $\bar{b} < b$. By considering these two possibilities we obtain

$$\int_{\alpha}^{\beta} (v(\eta) - c) d\eta \leq \int_{\bar{a}}^{\bar{b}} (v(\eta) - c) d\eta; \quad a \leq \alpha < \beta \leq b. \quad (4.17)$$

Now, for each $\xi \in [\bar{a}, \xi_0]$ let $\xi' \in (\xi_0, \bar{b}]$ be the unique number satisfying

$$0 < v(\xi) = v(\xi') < v(\eta); \quad \eta \in (\xi, \xi'). \quad (4.18)$$

Thus, integrating (4.10) by parts on $[\xi, \xi']$ we get

$$\varepsilon(\dot{v}(\xi') - \dot{v}(\xi)) = \mu v(\xi) |g(u)|_{\xi}^{\xi'} + \int_{\xi}^{\xi'} [v(\eta) - v(\xi)] d\eta.$$

Since $\dot{v}(\xi') \leq 0 \leq \dot{v}(\xi)$ we have

$$\int_{\xi}^{\xi'} [v(\eta) - v(\xi)] d\eta \leq \mu v(\xi)(g(u(\xi)) - g(u(\xi'))) \leq kv(\xi).$$

So, if $\xi = \bar{a}$ then $\xi' = \bar{b}$ and $\int_{\bar{a}}^{\bar{b}} [v(\eta) - c] d\eta \leq kc$ (because $v(\bar{a}) = v(\bar{b}) = c$). Therefore, it follows from (4.17) that

$$\int_{\alpha}^{\beta} (v(\eta) - c) d\eta \leq kc; \quad a \leq \alpha < \beta \leq b.$$

Dividing this last inequality by $\beta - \alpha$ we obtain immediately (4.14) since $v \geq 0$.

On the other hand, if $\xi \in (\bar{a}, \xi_0)$ we have

$$\varepsilon \dot{v}(\xi) \leq \varepsilon \dot{v}(\xi') - \mu v(\xi)[g(u)]_{\xi}^{\xi'} \leq \mu v(\xi)[g(u)]_{\xi'}^{\xi} \leq kv(\xi),$$

since $\int_{\xi}^{\xi'} [v(\eta) - v(\xi)] \geq 0$.

We have concluded that v satisfies the differential inequality $\dot{v} \leq \frac{kv}{\varepsilon}$ on $(\bar{a}, \xi_0)$, which after an integration gives us

$$v(\xi_0) < v(\xi) \exp\left(\frac{k}{\varepsilon}(\xi_0 - \xi)\right); \quad \xi \in (\bar{a}, \xi_0). \quad (4.19)$$

If $\xi_0 - \bar{a} \leq \varepsilon$ we evaluate (4.19) at $\xi = \bar{a}$ to find

$$\max_{\xi \in [a, b]} |v(\xi)| = v(\xi_0) \leq ce^k \leq ce^k \left(1 + \frac{k}{\varepsilon}\right).$$

Otherwise, if $\xi_0 - \bar{a} > \varepsilon$ then we take $\tilde{\xi} \in (\bar{a}, \xi_0)$ such that $\xi_0 - \tilde{\xi} = \varepsilon$. Thus, evaluating (4.19) at $\xi = \tilde{\xi}$ we have

$$\max_{\xi \in [a, b]} |v(\xi)| = v(\xi_0) < v(\tilde{\xi}) \exp\left(\frac{k}{\varepsilon}(\xi_0 - \tilde{\xi})\right) \leq v(\tilde{\xi})e^k.$$

Since v is strictly increasing on $[\tilde{\xi}, \xi_0)$ we get

$$\max_{\xi \in [a, b]} |v(\xi)| < \frac{e^k}{\xi_0 - \tilde{\xi}} \int_{\tilde{\xi}}^{\xi_0} v(\eta) d\eta \leq ce^k \left(1 + \frac{k}{\varepsilon}\right).$$

We have finished the proof when $v(a)$ and $v(b)$ are both nonnegative and consequently, in view of the linearity of (4.10), also when $v(a)$ and $v(b)$ are both nonpositive. Now, if $v(a)v(b) < 0$ then there exists $\tilde{\xi} \in (a, b)$ such that $v(\tilde{\xi}) = 0$. By using (4.14) and (4.15) on intervals $[a, \tilde{\xi}]$ and $[\tilde{\xi}, b]$ we complete this proof. $\square$

Next, assuming that $u_l > u_r$ we prove the existence of viscosity wavefans and give two estimates for these functions. For the case $u_l \leq u_r$ the existence of smooth solutions of (4.1)–(4.2) follows directly from Theorem 2.2 (Sec. 2) in view of (4.9) and (ii) from Lemma 4.2. Furthermore, these solutions are uniformly bounded in the sup norm and in total variation.

THEOREM 4.4. For each $\varepsilon > 0$ there exists a unique smooth solution $(u_{\varepsilon}, v_{\varepsilon})$ of the boundary value problem (4.1)–(4.2) with $u_l > u_r$. Furthermore,

(i) there exists a positive constant M , independent of ε , such that

$$\|u_\varepsilon\|_\infty \leq M, \quad \|v_\varepsilon\|_\infty \leq \frac{M}{\varepsilon}; \quad (4.20)$$

(ii) for each closed interval I , there exists a positive constant K_I , independent of ε , such that

$$\int_I |v_\varepsilon(\xi)| d\xi \leq K_I. \quad (4.21)$$

Proof. The uniqueness was proved in Theorem 4.1. In order to prove the existence through Theorem 2.2 we need to verify (4.7) for any solution $(u(\xi), v(\xi))$ of (4.5)–(4.6). In view of (4.8) we need only to prove (4.9) for a constant C that must be independent of L and μ . But, taking $a \rightarrow -L$ and $b \rightarrow L$ in Lemma 4.3 we have

$$\sup_{\xi \in (-L, L)} |v(\xi)| \leq c e^k \left(1 + \frac{k}{\varepsilon}\right),$$

where $c = \mu \max\{|v_l|, |v_r|\}$ and $k = \mu(u_l - u_r) \max\{g'(u) : u \in [\mu u_r, \mu u_l]\}$.

By taking $\bar{v} = \max\{|v_l|, |v_r|\}$ and $\bar{u} = \max\{|u_l|, |u_r|\}$ we have $c \leq \bar{v}$ and $k \leq \bar{k} := (u_l - u_r) \max\{g'(u) : |u| \leq \bar{u}\}$. Therefore, we obtain (4.9) with $C = \varepsilon^{-1} e^{\bar{k}} \bar{v} (1 + \bar{k})$ and conclude, via Theorem 2.2, that there exists a smooth solution $(u_\varepsilon, v_\varepsilon)$ of (4.1)–(4.2) satisfying the same bounds (4.8) and (4.9) with $L = \infty$. We also obtain (4.20) by taking $M = \max\{\bar{u}, \bar{v} e^{\bar{k}} (1 + \bar{k})\}$.

To prove (4.21) we fix a closed interval $I = [\alpha, \beta]$ and take $L > 0$ such that $I \subset (-L, L)$. Taking $\mu = 1$ in Lemma 4.3 and noting that $u_\varepsilon(-L) > u_\varepsilon(L)$ we have

$$\int_I |v_\varepsilon(\eta)| d\eta \leq c_\varepsilon(L) (\beta - \alpha + k_\varepsilon(L))$$

where $k_\varepsilon(L) = (u_\varepsilon(-L), u_\varepsilon(L)) \max\{g'(u) : u \in [u_\varepsilon(L), u_\varepsilon(-L)]\}$ and $c_\varepsilon(L) = \max\{|v_\varepsilon(-L)|, |v_\varepsilon(L)|\}$.

Now, when $L \rightarrow \infty$ we have that $c_\varepsilon(L) \rightarrow \bar{v}$ and

$$k_\varepsilon(L) \rightarrow (u_l - u_r) \max\{g'(u) : u \in [u_r, u_l]\} \leq \bar{k}.$$

Therefore, we have proved (4.21) with $K_I = \bar{v}(\beta - \alpha + \bar{k})$. $\square$

5. Convergence. In this section we prove the weak convergence of the viscosity wavefans associated with the Riemann problem (1.6)–(1.7) for $u_l > u_r$. We assume the conditions in (1.8) on f and g .

In the following $(u_\varepsilon, v_\varepsilon)$ denotes the viscosity wavefans in the self-similar form, i.e., the solutions of (4.1)–(4.2) that, in conforming with Sec. 4, are unique for each $\varepsilon > 0$. Also, u_s and v_s denote respectively the discontinuities (3.6) and (3.8) at $\xi = s$ where

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \quad (5.1)$$

We begin by stating a result that can be found in [13].

LEMMA 5.1. For each $\varepsilon > 0$ there exists a unique ξ_ε such that $\xi_\varepsilon = \lambda_1(u_\varepsilon(\xi_\varepsilon))$ and $\lim_{\varepsilon \rightarrow 0} \xi_\varepsilon = s$. Furthermore

(a) there exists a constant M_1 , independent of ε , such that

$$|\dot{u}_\varepsilon(\xi)| \leq \frac{M_1}{\varepsilon} \exp\left(-\frac{1}{2\varepsilon}(\xi - \xi_\varepsilon)^2\right); \quad \xi \in (-\infty, +\infty); \quad (5.2)$$

(b) u_ε converges to u_s uniformly on $(-\infty, s - \alpha] \cup [s + \alpha, \infty)$ for any $\alpha > 0$.

In view of Lemma 5.1 we can apply Theorem 2.4 (Sec. 2) to verify the weak convergence $u_\varepsilon \rightharpoonup u_s$.

Next we pursue the weak convergence of v_ε . For this we will return to Lemma 5.1 later.

PROPOSITION 5.2. For each $\varepsilon > 0$ there exists a unique η_ε such that

$$\eta_\varepsilon = \lambda_2(u_\varepsilon(\eta_\varepsilon)). \quad (5.3)$$

Furthermore, $\xi_\varepsilon < \eta_\varepsilon$ and

$$\lim_{\varepsilon \rightarrow 0} \eta_\varepsilon = \max\{s, \lambda_2(u_r)\}. \quad (5.4)$$

Proof. Since $\lambda_2(u_\varepsilon)$ is a decreasing smooth function in $(-\infty, +\infty)$ (because $g' > 0$ and $\dot{u}_\varepsilon < 0$) it has a unique fixed point η_ε . From (1.8) it follows that $\xi_\varepsilon < \eta_\varepsilon$.

Since η_ε is a bounded sequence, it suffices to prove that all convergent subsequences of η_ε converge to $\max\{s, \lambda_2(u_r)\}$. So, let η_{ε_n} be a subsequence converging to some η_0 . Taking $n \rightarrow \infty$ on inequality $\xi_{\varepsilon_n} < \eta_{\varepsilon_n}$ we have from Lemma 5.1 that $s \leq \eta_0$.

If $s < \eta_0$ then from Lemma 5.1 (item (b)) it follows that $u_{\varepsilon_n}(\eta_{\varepsilon_n}) \rightarrow u_r$ and so from (5.3) we have $s < \eta_0 = \lambda_2(u_r)$. Therefore, in this case $\eta_0 = \max\{s, \lambda_2(u_r)\}$.

If $s = \eta_0$ we must have again $\eta_0 = \max\{s, \lambda_2(u_r)\}$ because, otherwise, if $s < \lambda_2(u_r)$ then $s < \lambda_2(u_r) < \lambda_2(u_{\varepsilon_n}(\eta_{\varepsilon_n})) = \eta_{\varepsilon_n}$ and taking $n \rightarrow \infty$ we could arrive at the contradiction $s < \lambda_2(u_r) \leq \eta_0$.

Now we prove an estimate for $\dot{v}_\varepsilon$ in terms of the auxiliary function

$$E_\varepsilon(\xi) := \max\left\{\exp\left(-\frac{1}{2\varepsilon}(\xi - \xi_\varepsilon)^2\right), \exp\left(-\frac{1}{2\varepsilon}(\xi - \eta_\varepsilon)^2\right)\right\}; \quad \xi \in (-\infty, +\infty).$$

PROPOSITION 5.3. There exists a constant C , independent of ε , such that

$$|\dot{v}_\varepsilon(\xi)| \leq \frac{C}{\varepsilon^3} E_\varepsilon(\xi)(1 + |\xi - \eta_\varepsilon|), \quad \xi \in \mathbf{R}. \quad (5.5)$$

Proof. Integrating the second equation in (4.1) we obtain

$$\dot{v}_\varepsilon(\xi) = a_\varepsilon(\xi) + b_\varepsilon(\xi); \quad \xi \in (-\infty, +\infty), \quad (5.6)$$

where $a_\varepsilon(\xi) = \dot{v}_\varepsilon(\eta_\varepsilon) \exp\left(\frac{1}{\varepsilon} \int_{\eta_\varepsilon}^\xi [\lambda_2(u_\varepsilon(\eta)) - \eta] d\eta\right)$ and

$$b_\varepsilon(\xi) = \frac{1}{\varepsilon} \int_{\eta_\varepsilon}^\xi v_\varepsilon(\tau) \dot{\lambda}_2(u_\varepsilon(\tau)) \exp\left(\frac{1}{\varepsilon} \int_\tau^\xi [\lambda_2(u_\varepsilon(\eta)) - \eta] d\eta\right) d\tau.$$

From the monotonicity of $\lambda_2(u_\varepsilon)$ and from the definition of η_ε it follows that

$$\exp\left(\frac{1}{\varepsilon} \int_{\eta_\varepsilon}^{\xi} [\lambda_2(u_\varepsilon(\eta)) - \eta] d\eta\right) \leq \exp\left(-\frac{1}{2\varepsilon}(\xi - \xi_\varepsilon)^2\right) \leq E_\varepsilon(\xi) \quad (5.7)$$

for all $\xi \in (-\infty, +\infty)$.

From (4.20) we have $\|(u_\varepsilon, v_\varepsilon)\|_\infty = O(1/\varepsilon)$ and $\|(f(u_\varepsilon), v_\varepsilon \lambda_2(u_\varepsilon))\|_\infty = O(1/\varepsilon)$. Hence, taking $I = [\lambda_2(u_r), \lambda_2(u_l)]$ and $p = 1$ in Remark 2.1 it follows that $|\dot{v}_\varepsilon(\xi)| \leq C_1/\varepsilon^2$ for all $\xi \in I$, where C_1 depends only on $\lambda_2(u_l)$ and $\lambda_2(u_r)$. Since $\eta_\varepsilon \in I$, we have $|\dot{v}_\varepsilon(\eta_\varepsilon)| \leq C_1/\varepsilon^2$ and so

$$|a_\varepsilon(\xi)| \leq \frac{C_1}{\varepsilon^2} E_\varepsilon(\xi); \quad \xi \in (-\infty, +\infty). \quad (5.8)$$

On the other hand, from Lemma 5.1 (item (a)) and (4.20) we get

$$\begin{aligned} |b_\varepsilon(\xi)| &\leq \frac{KMM_1}{\varepsilon^3} \left| \int_{\eta_\varepsilon}^{\xi} \exp\left(\frac{1}{\varepsilon} \int_{\tau}^{\xi} [\lambda_2(u_\varepsilon(\eta)) - \eta] d\eta - \frac{1}{2\varepsilon}(\tau - \xi_\varepsilon)^2\right) d\tau \right| \\ &= \frac{C_2}{\varepsilon^3} \left| \int_{\eta_\varepsilon}^{\xi} F_\varepsilon(\xi, \tau) d\tau \right|, \end{aligned}$$

where $K = \max\{g'(u) : u \in [u_r, u_l]\}$, $C_2 = KMM_1$, and

$$0 < F_\varepsilon(\xi, \tau) := \exp\left(\frac{1}{\varepsilon} \int_{\tau}^{\xi} [\lambda_2(u_\varepsilon(\eta)) - \eta] d\eta - \frac{1}{2\varepsilon}(\tau - \xi_\varepsilon)^2\right).$$

Therefore, $|b_\varepsilon(\xi)| \leq (C_2/\varepsilon^3)|\xi - \eta_\varepsilon| \max\{F_\varepsilon(\xi, \tau) : \tau \in I[\xi, \eta_\varepsilon]\}$, where $I[\xi, \eta_\varepsilon]$ denotes the closed interval with extremes ξ and η_ε .

If τ_0 is a critical point of $F_\varepsilon(\xi, \cdot)$ for each fixed ξ , then we can check that

$$\frac{d^2}{d\tau^2} F_\varepsilon(\xi, \tau_0) = -\frac{1}{\varepsilon} \dot{\lambda}_2(u_\varepsilon(\tau_0)) F_\varepsilon(\xi, \tau_0) > 0$$

to conclude that $F_\varepsilon(\xi, \cdot)$ cannot have a local maximum point.

Therefore, $\max\{F_\varepsilon(\xi, \tau) : \tau \in I[\xi, \eta_\varepsilon]\} = \max\{F_\varepsilon(\xi, \eta_\varepsilon), F_\varepsilon(\xi, \xi)\} \leq E_\varepsilon(\xi)$ and so

$$|b_\varepsilon(\xi)| \leq \frac{C_2}{\varepsilon^3} |\xi - \eta_\varepsilon| E_\varepsilon(\xi). \quad (5.9)$$

From (5.6), (5.8), and (5.9) we obtain (5.5) with $C = \max\{C_1, C_2\}$. $\square$

The next result is an immediate consequence of Proposition 5.3. We omit the proof.

COROLLARY 5.4. There exists a constant C that is independent of ε such that

$$|\dot{v}_\varepsilon(\xi)| \leq \frac{C}{\varepsilon^3} \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \xi)^2\right) [1 + \xi_\varepsilon - \xi]; \quad \xi \leq \xi_\varepsilon, \quad (5.10)$$

$$|\dot{v}_\varepsilon(\xi)| \leq \frac{C}{\varepsilon^3}; \quad \xi_\varepsilon \leq \xi \leq \eta_\varepsilon,$$

$$|\dot{v}_\varepsilon(\xi)| \leq \frac{C}{\varepsilon^3} \exp\left(-\frac{1}{2\varepsilon}(\xi - \eta_\varepsilon)^2\right) [1 + \xi - \eta_\varepsilon]; \quad \eta_\varepsilon \leq \xi. \quad (5.11)$$

In the following, $s_r = \max\{s, \lambda_2(u_r)\}$.

COROLLARY 5.5. For any $0 < \alpha < L$ we have

- (a) $\dot{v}_\varepsilon \rightarrow 0$ uniformly on $[s - L, s - \alpha] \cup [s_r + \alpha, s_r + L]$;
- (b) if $s < \lambda_2(u_r)$ then $\dot{v}_\varepsilon \rightarrow 0$ uniformly on $[s - \alpha, \lambda_2(u_r) + \alpha]$ (here $2\alpha < \lambda_2(u_r) - s$);
- (c) $v_\varepsilon \rightarrow v_s$ uniformly on $(-\infty, s - \alpha] \cup [s_r + \alpha, +\infty)$.

Proof. Since $\xi_\varepsilon \rightarrow s$ and $\eta_\varepsilon \rightarrow s_r$ (Proposition 5.2) we conclude from Corollary 5.4 that there exists a constant C and $\varepsilon_0 > 0$ such that

$$|\dot{v}_\varepsilon(\xi)| \leq \frac{C}{\varepsilon^3} \exp\left(-\frac{\alpha^2}{8\varepsilon}\right); \quad \varepsilon < \varepsilon_0,$$

if $\xi \in [s - L, s - \alpha] \cup [s_r + \alpha, s_r + L]$ (in this case C depends only on α and L) or if $\xi \in [s + \alpha, \lambda_2(u_r) - \alpha]$ when $s < \lambda_2(u_r) = s_r$ (in this case C depends only on α).

The conclusions of (a) and (b) follow immediately by the inequality above when $\varepsilon \rightarrow 0^+$.

Now if $\eta < \xi \leq s - \alpha$, then from (5.10) we have

$$|v_\varepsilon(\eta) - v_\varepsilon(\xi)| \leq \int_\eta^\xi |\dot{v}_\varepsilon(\theta)| d\theta \leq C[I_\varepsilon^1(\xi, \eta) + I_\varepsilon^2(\xi, \eta)], \quad (5.12)$$

where

$$\begin{aligned} I_\varepsilon^1(\xi, \eta) &= \frac{1}{\varepsilon^3} \int_\eta^\xi \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \theta)^2\right) d\theta, \\ I_\varepsilon^2(\xi, \eta) &= \frac{1}{\varepsilon^3} \int_\eta^\xi (\xi_\varepsilon - \theta) \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \theta)^2\right) d\theta \\ &= \frac{1}{\varepsilon^2} \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \xi)^2\right) - \frac{1}{\varepsilon^2} \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \eta)^2\right), \end{aligned}$$

and C is a constant that does not depend on ξ, η , and ε .

Taking $\sigma = \frac{\xi_\varepsilon - \theta}{\sqrt{2\varepsilon}}$ and $\eta \rightarrow -\infty$ we obtain

$$I_\varepsilon^1(\xi, -\infty) = \frac{\sqrt{2\varepsilon}}{\varepsilon^3} \int_{\frac{\xi_\varepsilon - \xi}{\sqrt{2\varepsilon}}}^{+\infty} \exp(-\sigma^2) d\sigma.$$

Since $\xi_\varepsilon \rightarrow s$ there exists $\varepsilon_0 > 0$ such that $\xi_\varepsilon - \xi > \frac{\alpha}{2}$ if $\varepsilon < \varepsilon_0$. Consequently, for $\varepsilon < \varepsilon_0$ we have $I_\varepsilon^1(\xi, -\infty) \leq \frac{\sqrt{2}}{\varepsilon^2 \sqrt{\varepsilon}} \int_{\frac{\alpha}{2\sqrt{2\varepsilon}}}^{+\infty} \exp(-\sigma^2) d\sigma$ and

$$I_\varepsilon^2(\xi, -\infty) = \frac{1}{\varepsilon^2} \exp\left(-\frac{1}{2\varepsilon}(\xi_\varepsilon - \xi)^2\right) \leq \frac{1}{\varepsilon^2} \exp\left(-\frac{\alpha^2}{8\varepsilon}\right).$$

Making $\eta \rightarrow -\infty$ in (5.12) we have

$$|v_l - v_\varepsilon(\xi)| \leq \frac{\sqrt{2}}{\varepsilon^2 \sqrt{\varepsilon}} \int_{\frac{\alpha}{2\sqrt{2\varepsilon}}}^{+\infty} \exp(-\sigma^2) d\sigma + \frac{1}{\varepsilon^2} \exp\left(-\frac{\alpha^2}{8\varepsilon}\right); \quad \varepsilon < \varepsilon_0.$$

Hence we obtain (c) on $(-\infty, s - \alpha]$ since

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 \sqrt{\varepsilon}} \int_{\frac{\alpha}{2\sqrt{2\varepsilon}}}^{+\infty} \exp(-\sigma^2) d\sigma = 0 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \exp\left(-\frac{\alpha^2}{8\varepsilon}\right).$$

Similarly we prove (c) on $[s_r + \alpha, +\infty)$. □

Now we can prove the weak convergence of the viscosity wavefans to the delta-shock S_δ .

THEOREM 5.6. Suppose $u_l > u_r$ and $\lambda_2(u_r) \leq s$. Then $(u_\varepsilon, v_\varepsilon) \rightharpoonup S_\delta$ as $\varepsilon \rightarrow 0^+$.

Proof. As a consequence of Corollary 5.5 (item (c)) and from Lemma 5.1 (item (b)) we have the uniform convergence $(u_\varepsilon, v_\varepsilon) \rightarrow (u_s, v_s)$ on each closed interval contained on $(-\infty, s) \cup (s, +\infty)$, since $s_r = \lambda_2(u_r)$. Putting together this fact with (ii) from Theorem 4.4 of Sec. 4 we can apply Theorem 2.4 of Sec. 2 to obtain immediately the result. $\square$

Next we give a sketch of the proof of the weak convergence $(u_\varepsilon, v_\varepsilon) \rightharpoonup (u_s, v_s)$ when $u_l > u_r$ and $s < \lambda_2(u_r)$.

First, we observe that the family $\{v_\varepsilon\}$ is uniformly bounded on each closed interval $I \subset (s, \lambda_2(u_r))$. In fact, denoting by $|I|$ the length of I and writing $|v_\varepsilon(\xi_0)| = \frac{1}{|I|} \int_I |v_\varepsilon(\xi_0) - v_\varepsilon(\eta)| d\eta$ we have, for each $\xi_0 \in I$:

$$|v_\varepsilon(\xi_0)| \leq \frac{1}{|I|} (\|\dot{v}_\varepsilon\|_{L^\infty(I)} \int_I |\xi_0 - \eta| d\eta + \frac{1}{|I|} \int_I |v_\varepsilon(\eta)| d\eta) \leq C_1 + \frac{K_I}{|I|},$$

where K_I and C_I are constants that depend only I . K_I is obtained from (4.21) and C_I is such that $|I| \|\dot{v}_\varepsilon\|_{L^\infty(I)} \leq C_I$ since $\dot{v}_\varepsilon \rightarrow 0$ uniformly on I (Corollary 5.5 item (b)).

Second, in view of the boundedness above and the uniform convergence $\dot{v}_\varepsilon \rightarrow 0$ on closed intervals contained in $(s, \lambda_2(u_r))$ we can prove that there exists a constant v_m such that $v_\varepsilon \rightarrow v_m$ uniformly on $[\lambda_2(u_r) - \alpha, \lambda_2(u_r) + \alpha]$ for each $\alpha > 0$. So, from Lemma 5.1 and Corollary 5.5 (item (c)) we can conclude that $(u_\varepsilon, v_\varepsilon) \rightarrow (u_s, v)$ uniformly on each closed interval contained in $(-\infty, s) \cup (s, \lambda_2(u_r)) \cup (\lambda_2(u_r), +\infty)$ where

$$v(\xi) = \begin{cases} v_l; & \xi < s, \\ v_m; & s < \xi < \lambda_2(u_r), \\ v_r; & \lambda_2(u_r) < \xi. \end{cases} \quad (5.13)$$

Third, we apply Theorem 2.4 to conclude that $(u_\varepsilon, v_\varepsilon) \rightharpoonup (u_s, v + c\delta_s)$ where

$$c = s[v_m - v_l] - [v_m\lambda_2(u_r) - v_l\lambda_2(u_l)].$$

The constant that could appear multiplying $\delta_{\lambda_2(u_r)}$ is trivially zero.

At last, we prove that $c = 0$, which is equivalent to proving that (u_r, v_m) belongs to the 1-shock curve $S_1(u_l, v_l)$ if $v_l \neq 0$. In view of the linearity of the second equation in (4.1) we need only to prove that $c = 0$ for the following two cases.

Case 1: $0 < v_l$ and $v_r = \bar{v}_l + 1$ where $\bar{v}_l$ is such that

$$s[\bar{v}_l - v_l] = \lambda_2(u_r)\bar{v}_l - \lambda_2(u_l)v_l, \quad (5.14)$$

i.e., the state $(u_r, \bar{v}_l)$ belongs to the 1-shock curve $S_1(u_l, v_l)$. We can verify easily that $\bar{v}_l > v_l$.

In this case we claim that v_ε is uniformly bounded on an interval that contains s . In fact, if v_ε is not monotonic then, since $v_r > \bar{v}_l > v_l > 0$ there exists a unique maximum point τ_ε of v_ε and a unique point α_ε satisfying $v_l < v_\varepsilon(\xi) < v_\varepsilon(\alpha_\varepsilon) = v_r$ if $\xi \in (-\infty, \alpha_\varepsilon)$ and $v_\varepsilon(\eta) > v_r$ if $\eta \in (\alpha_\varepsilon, \infty)$. Therefore, from Theorem 2.3 we obtain

$$0 < \int_{-\infty}^{\alpha_\varepsilon} [v_\varepsilon(\xi) - v_l] d\xi + \int_{\alpha_\varepsilon}^{+\infty} [v_\varepsilon(\xi) - v_r] d\xi = \alpha_\varepsilon[v_r - v_l] - [v_r\lambda_2(u_r) - v_l\lambda_2(u_l)].$$

So, from (5.14) we have

$$\alpha_\varepsilon[v_r - v_l] > [v_r \lambda_2(u_r) - v_l \lambda_2(u_l)] = s[v_r - v_l] + \lambda_2(u_r) - s,$$

i.e., $\alpha_\varepsilon - s > k := \frac{\lambda_2(u_r) - s}{v_r - v_l}$. We observe that the positive constant k depends only on u_l, u_r, v_l , and v_r .

Thus, $v_l < v_\varepsilon(\xi) < v_r; \xi \in (-\infty, s + k]$, since $(-\infty, s + k] \subset (-\infty, \alpha_\varepsilon)$. The last inequalities remain valid if v_ε is monotonic.

Hence, $\lim_{\varepsilon \rightarrow 0} \int_{s-\beta}^{s+\beta} v_\varepsilon(\xi) d\xi = \beta[v_m + v_l]$, where we have applied the Dominated Convergence Theorem.

On the other hand, integrating the second equation in (4.1) on the set $[s - \beta, s) \cup (s, s + \beta] \subset [-\infty, s + k]$ and taking $\varepsilon \rightarrow 0$ we find

$$\lim_{\varepsilon \rightarrow 0} \int_{s-\beta}^{s+\beta} v_\varepsilon d\xi = \lim_{\varepsilon \rightarrow 0} [\varepsilon \dot{v}_\varepsilon(\xi) + \xi v_\varepsilon(\xi) - v_\varepsilon(\xi) g(u_\varepsilon(\xi))]_{s-\beta}^{s+\beta} = c + \beta[v_m + v_l].$$

Therefore $c = 0$ and v_ε converges weakly to v in (5.13) with $v_m = v_l \frac{s - \lambda_2(u_l)}{s - \lambda_2(u_r)}$.

Case 2: $v_l = 0 < v_r$. We prove that $c = 0$ for this case using the same arguments as in Case 1 above. So, v_ε converges weakly to v in (5.13) with $v_m = 0 = v_l$.

Now, for the general case we prove weak convergence by using Cases 1 and 2 above and the linearity of (4.1) with respect to v . For example, if $v_r \leq 0 < v_l$, then the (unique) solution of (4.1)–(4.2) is $(u_\varepsilon, v_\varepsilon^1 - v_\varepsilon^2, -v_\varepsilon^3)$ where $(u_\varepsilon, v_\varepsilon^1)$, $(u_\varepsilon, v_\varepsilon^2)$, and $(u_\varepsilon, v_\varepsilon^3)$ are the solutions of (4.1) satisfying $u_\varepsilon(-\infty) = u_l$, $u_\varepsilon(+\infty) = u_r$, $v_\varepsilon^1(-\infty) = v_l$, $v_\varepsilon^1(+\infty) = \bar{v}_l + 1$, $v_\varepsilon^2(-\infty) = 0$, $v_\varepsilon^2(+\infty) = \bar{v}_l + 1$, $v_\varepsilon^3(-\infty) = 0$, and $v_\varepsilon^3(+\infty) = -v_r$. We observe that $(u_\varepsilon, v_\varepsilon^1)$ corresponds to Case 1 and that $(u_\varepsilon, v_\varepsilon^2)$ and $(u_\varepsilon, v_\varepsilon^3)$ correspond to Case 2.

So, in this example we obtain $(u_\varepsilon, v_\varepsilon^1 - v_\varepsilon^2, -v_\varepsilon^3) \rightarrow (u_s, v)$ where v is the function in (5.13) with $v_m = v_l \frac{s - \lambda_2(u_l)}{s - \lambda_2(u_r)}$.

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