EXISTENCE AND REGULARITY OF SOLUTIONS OF NON-NEWTONIAN FLOW

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Abstract. The existence and regularity of Young measure-valued solutions and weak solutions to non-Newtonian flows are considered. Galerkin approximation and an L^2 compactness theorem are main ingredients for the proof of the existence of Young measure-valued solutions. Under a certain convexity condition for the energy, we prove that Young measure-valued solutions are weak solutions. Also, for the limited cases, we prove a regularity theorem.

1. Introduction and statement of the result. Fluid dynamics has attracted the attention of many mathematicians and engineers. The Navier-Stokes equations are generally accepted as a right governing equations for the incompressible motion of viscous fluids. If the relation between the stress and the rate of strain is linear, then the fluid is called Newtonian. That is, Newtonian fluids satisfy the linear relation

$$\tau = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where τ is the stress and $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ is the rate of strain. The coefficient of proportionality μ is called the viscosity, and it is a characteristic material quantity for the fluid concerned, which in general depends on temperature and pressure. Air, other gases, water, motor oil, alcohols, simple hydrocarbon compounds and others tend to be Newtonian fluids. The governing equations of motion of them will be the Navier-Stokes equations. If the relation is not linear, the fluid is called non-Newtonian. Examples of non-Newtonian fluids are molten plastics, polymer solutions, dyes, varnishes, suspensions, adhesives, paints, greases, and biological fluids like blood. The simplest model of the stress-strain

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relation for such fluids is given by the power laws, which state that

$$\tau = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^q,$$

for $0 < q < 1$; see Böhme [3].

Ladyzhenskaya [5] proposed a new model to study some kinds of non-Newtonian fluids which are of interest to us. Ladyzhenskaya's model is the following:

$$\begin{cases} \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \Gamma_{ij}}{\partial x_j} + \rho f_i, \\ \frac{\partial u_j}{\partial x_j} = 0, \\ \Gamma_{ij} \stackrel{\text{def}}{=} (\mu_0 + \mu_1 |E(\nabla u)|^r) E_{ij}(\nabla u), \\ E_{ij}(\nabla u) \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{cases} \quad (1.1)$$

in $Q = \Omega \times (0, \infty)$, with the initial condition $u(x, 0) = u_0$ for $x \in \Omega$ and with the Dirichlet boundary condition, where $E = (E_{ij})$. Here, $\Omega \subset \mathbb{R}^3$ is bounded, $r > -1$, and $\mu_0, \mu_1 > 0$. These models are called

$$\begin{cases} \text{Newtonian} & \text{for } \mu_0 > 0, \mu_1 = 0, \\ \text{Rabinowitsch} & \text{for } \mu_0, \mu_1 > 0, \text{ and } r = 2, \\ \text{Ellis} & \text{for } \mu_0, \mu_1 > 0, \text{ and } r > 0, \\ \text{Ostwald-de Waele} & \text{for } \mu_0 = 0, \mu_1 > 0; \end{cases}$$

see Böhme [3].

In Ladyzhenskaya [5], for $\mu_0, \mu_1 > 0$ she obtained the existence of the weak solutions for $r > 2/5$ and their uniqueness for $r > 1/2$. Bellout, Bloom and Nečas [1] studied the non-Newtonian fluids for $\mu_0 = \mu_1 > 0$. They showed that there are Young measure-valued solutions if $r \in (-\frac{4}{5}, -\frac{1}{5}]$ and weak solutions if $r \in (-\frac{1}{5}, \infty)$. Moreover, the weak solution is unique if $r > \frac{1}{5}$. Here, we show the existence of Young measure-valued solutions for all $r \in (-1, \infty)$ (see Theorem 3.5), and even for the case $\mu_0 = 0$. Moreover, we show that the Young measure-valued solutions are weak solutions if a certain convexity condition for energy holds. (See Theorem 4.5.) We also show a regularity theorem of the weak solutions for $\mu_0 \geq 0$ on the space-periodic problem. Bellout, Bloom, and Nečas [1] showed the similar regularity for $\mu_0 > 0$; however, we allow the case $\mu_0 = 0$.

In Sec. 3, we consider (1.1) on a bounded domain. We show that there exist Young measure-valued solutions to (1.1) for any $r > -1$. This existence result works for Dirichlet, Neumann, and the periodic boundary conditions. Our existence result can be extended to the cases such as the Navier-Stokes equations, Ladyzhenskaya models, non-Newtonian fluids in Böhme [3]. In this point of view, our existence result in Theorem 3.5 is more general than those of others. It turns out that the Galerkin method is suitable for constructing the approximate smooth solutions. Then, a compactness lemma for L^2 -space guarantees that the weak limit of approximate solutions is indeed the Young measure-valued solution to our problem. We even show the existence of Young measure-valued solutions for the other extreme case $\mu_0 = 0$ and $\mu_1 > 0$.

In Sec. 4 we prove that Young measure-valued solutions are weak solutions if the L^{r+2} energy is convex in the class of weak solutions. Indeed, we find that the L^{r+2} energy could

concentrate only in a measure-zero set from the uniqueness of weakly strong solutions. Then, the higher integrability of the weak limit cancels the concentration in time.

In Sec. 5, we find a regularity criterion for the parameter r . If $r > 1/5$, and $\mu_1 > 0$ and $\mu_0 \geq 0$, we have a regularity; see Theorem 5.1. The proof is achieved by a careful application of the Sobolev inequality for parabolic norms. The boundedness of the L^{r+2} -norm of ∇u is essential in the proof.

2. Formulation of non-Newtonian flow. We consider the non-Newtonian flow on a bounded open domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \Gamma_{ij}}{\partial x_j} + \rho f_i, \\ \frac{\partial u_j}{\partial x_j} = 0, \\ \Gamma_{ij} = (\mu_0 + \mu_1 |E(\nabla u)|^r) E_{ij}(\partial u), \\ E_{ij}(\nabla u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{cases} \quad (2.1)$$

in $Q = \Omega \times (0, \infty)$, with the initial condition $u(x, 0) = u_0$ for $x \in \Omega$, and with Dirichlet boundary data $u = 0$ on $\partial\Omega$, where $r > -1$ and $\mu_0 \geq 0$, $\mu_1 > 0$. The constant ρ is the density; for simplicity, we let $\rho = 1$.

We define the usual Sobolev space $W^{1,q}$ by the set of all L^q functions whose j th-order derivatives, $j \leq 1$, are in L^q . Let

$$\mathcal{V} \stackrel{\text{def}}{=} \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\},$$

$$\mathbf{V} \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } W^{1,2}(\Omega),$$

$$\mathbf{V}_q \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } W^{1,q}(\Omega),$$

$$\mathbf{H} \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } L^2(\Omega).$$

Notice that the inclusion $\mathbf{V} \hookrightarrow \mathbf{H}$ is compact. The inner product of $\mathbf{V}$ is given by $\langle \nabla u, \nabla v \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product of $\mathbf{H}$. Let $\mathbf{V}'$ and $\mathbf{V}'_q$ be the dual spaces of $\mathbf{V}$ and $\mathbf{V}_q$, respectively, where $1/q + 1/q' = 1$. We will use the notation:

$$\|u\| \stackrel{\text{def}}{=} \|u\|_2 \stackrel{\text{def}}{=} \langle u, u \rangle^{1/2}, \quad \|u\|_q \stackrel{\text{def}}{=} \|u\|_{L^q} \stackrel{\text{def}}{=} \left(\int_\Omega |u|^q dx \right)^{1/q}. \quad (2.2)$$

If $q \leq 2$ then $\mathbf{V} \subset \mathbf{V}_q$, and if $q \geq 2$ then $\mathbf{V}_q \subset \mathbf{V}$. Denote

$$q \stackrel{\text{def}}{=} r + 2.$$

Then

$$q' \stackrel{\text{def}}{=} \frac{r+2}{r+1}$$

satisfies $1/q' + 1/q = 1$.

For the convection term we define a bilinear mapping B such that

$$\langle Bu, v \rangle \stackrel{\text{def}}{=} b(u, u, v),$$

where

$$b(u, v, w) \stackrel{\text{def}}{=} \sum_{i,j} \int u_j \partial_j v_i w_i dx.$$

We now present the weak formulation for the problem (2.1):

For given f, u_0 with

$$u_0 \in \mathbf{H} \quad \text{and} \quad \begin{cases} f \in L^2(0, T; \mathbf{V}'), & \text{for } \mu_0 > 0, \\ f \in L^{q'}(0, T; \mathbf{V}'_q), & \text{for } \mu_0 = 0, \end{cases}$$

find u with

$$u \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}) \cap L^q(0, T; \mathbf{V}_q), \quad (2.3)$$

and satisfying that, for all $v \in \mathbf{V}_q \cap \mathbf{V}$,

$$\frac{d}{dt} \langle u, v \rangle + b(u, u, v) + \mu_0 \Gamma_0(\nabla u, \nabla v) + \mu_1 \Gamma_1(\nabla u, \nabla v) = \langle f, v \rangle, \quad (2.4)$$

$$u(x, 0) = u_0(x),$$

where

$$\begin{aligned} \Gamma_0(\nabla u, \nabla v) &= \sum_{i,j} \int E_{ij}(\nabla u) E_{ij}(\nabla v) dx, \\ \Gamma_1(\nabla u, \nabla v) &= \sum_{i,j} \int |E(\nabla u)|^r E_{ij}(\nabla u) E_{ij}(\nabla v) dx. \end{aligned}$$

Here, Eq. (2.4) is defined in the distribution sense for the scalar functions of time t .

The constant C below depends only on the initial data u_0 and time T if there is no specific statement.

REMARK. If the boundary condition is Neumann or periodic, then we can redefine the solution spaces $\mathbf{H}, \mathbf{V}, \mathbf{V}_q$ corresponding to the boundary condition.

3. Existence of Young measure-valued solutions. Let $\mu_0 > 0$. We use the Galerkin method to get the existence. The energy estimate follows in a standard way. But the difficulty lies on showing compactness in L^2 -space. With the compactness we can prove the strong convergence of approximate convection terms. Consider (2.4) on $Q_T \stackrel{\text{def}}{=} \Omega \times [0, T]$, where $T > 0$ is fixed.

First we review the lemma below, which is easy to check.

LEMMA 3.1. There is a constant $C > 0$ such that for any $u \in \mathbf{V}$,

$$\begin{aligned} \|Bu\|_{\mathbf{V}'} &\leq C\|u\|_4^2 \leq C\|u\|^{1/2}\|\nabla u\|^{3/2}, \\ \|Bu\|_{\mathbf{V}'_q} &\leq C\|u\|_{2q'}^2, \end{aligned}$$

where $1/q' + 1/q = 1$.

Since $\mathcal{V}$ is dense in $\mathbf{V}$ and in $\mathbf{V}_q$, and $\mathbf{V}$ and $\mathbf{V}_q$ are separable, there exists an orthonormal subset $\{w_i: i = 1, 2, \dots\} \subset \mathcal{V}$ that spans $\mathbf{V} \cap \mathbf{V}_q$. For each m we define an approximate solution u^m of (2.4):

$$u^m \stackrel{\text{def}}{=} \sum_{\ell=1}^m g_\ell^m(t) w_\ell(x)$$

and

$$\langle u_t^m, w_k \rangle + \Gamma^m(E_{ij}(\nabla u^m), E_{ij}(\nabla w_k)) + b(u^m, u^m, w_k) = \langle f, w_k \rangle, \quad (3.1)$$

$$u^m(0) = u_0^m, \quad (3.2)$$

where

$$\Gamma^m(E_{ij}(\nabla u^m), E_{ij}(\nabla w_k)) = \mu_0 \Gamma_0(\nabla u^m, \nabla w_k) + \mu_1 \Gamma_1(\nabla u^m, \nabla w_k),$$

and u_0^m is the orthogonal projection in $\mathbf{H}$ of u_0 onto the space spanned by $\{w_1, \dots, w_m\}$. Here the initial data of g_ℓ^m satisfy $g_\ell^m(0) = \langle u_0, w_\ell \rangle$. For convenience, we ignore the superscript m of g . Then we have $u_t^m = \sum_{\ell=1}^m g_\ell'(t) w_\ell(x)$ and $\partial_j u^m = \sum_{\ell=1}^m g_\ell(t) \partial_j w_\ell(x)$. Equation (3.1) becomes

$$g_\ell' \langle w_\ell, w_k \rangle + g_\ell g_l \langle w_{\ell,j} \partial_j w_{l,i}, w_{k,i} \rangle + \langle \Gamma_{ij}^m, E_{ij}(\nabla w_k) \rangle = \langle f, w_k \rangle, \quad (3.3)$$

where

$$\langle \Gamma_{ij}^m, E_{ij}(\nabla w_k) \rangle = \mu_0 g_\ell \langle E_{ij}(\nabla w_\ell), E_{ij}(\nabla w_k) \rangle + \mu_1 g_\ell \langle |E^m|^r E_{ij}(\nabla w_\ell), E_{ij}(\nabla w_k) \rangle,$$

and $E^m = (E_{ij}(\nabla u^m))$. For summation, we use the Einstein notation, that is, the repeated indices mean that the summations are from $i, j = 1$ to $n = 3$, and from $\ell = 1$ to m . Notice that (3.3) is a system of nonlinear ODE's. For the existence and uniqueness of solution of (3.3), we need the Lipschitz property of the nonlinear terms of (3.3), which is easy to check. Thus, there exists a unique solution of (3.3) locally. In other words, for each $m \geq 1$, (3.3) has a maximal solution on some interval $[0, t_m)$. If $t_m < T$, then $\|u^m(t)\| \rightarrow \infty$ as $t \rightarrow t_m$. The following lemma shows that this does not happen; therefore $t_m = T$.

LEMMA 3.2. Assume that $\int \|f\|_{\mathbf{V}}^2 ds$ is bounded. Then we have

$$\begin{aligned} \sup_{0 < t < T} \|u^m(t)\|^2 + \mu_0 \int_0^T \|\nabla u^m\|^2 ds + \mu_1 \int_0^T \|\nabla u^m\|_{r+2}^{r+2} ds \\ \leq C \|u^m(0)\|^2 + C \int_0^T \|f\|_{\mathbf{V}}^2 ds. \end{aligned} \quad (3.4)$$

Proof. Multiply (3.3) by g_k , and add them for $k = 1, 2, \dots, m$. Then

$$\begin{aligned} \sum_{\ell,k=1}^m g_\ell' g_k \langle w_\ell, w_k \rangle + \mu_0 \sum_{\ell,k=1}^m g_\ell g_k \langle E_{ij}(\nabla w_\ell), E_{ij}(\nabla w_k) \rangle \\ + \mu_1 \sum_{\ell,k=1}^m g_\ell g_k \langle |E^m|^r E_{ij}(\nabla w_\ell), E_{ij}(\nabla w_k) \rangle = \langle f, u^m \rangle, \end{aligned} \quad (3.5)$$

since $b(u^m, u^m, u^m) = 0$. We recall Korn's inequality given in Nečas and Hlaváček [8] for $s = 2$, and in Mosolov and Mjasnikov [7] for $1 < s < \infty$:

$$\left(\int_{\Omega} |E_{ij}(\nabla v) E_{ij}(\nabla v)|^{s/2} dx \right)^{1/s} \geq C \|\nabla v\|_s, \quad \text{for } 1 < s < \infty.$$

Hence, considering the definition of u^m and Korn's inequality, we obtain

$$\frac{d}{dt} \|u^m\|^2 + \mu_0 C \|\nabla u^m\|^2 + \mu_1 C \|\nabla u^m\|_{r+2}^{r+2} \leq 2|\langle f, u^m \rangle|. \quad (3.6)$$

By Young's inequality we get

$$\begin{aligned} 2|\langle f, u^m \rangle| &\leq \varepsilon^{-1} \|f\|_{\mathbf{V}'}^2 + \varepsilon \|u^m\|_{\mathbf{V}}^2 \\ &\leq \varepsilon^{-1} \|f\|_{\mathbf{V}'}^2 + \varepsilon \|u^m\|^2 + \varepsilon \|\nabla u^m\|^2, \end{aligned}$$

and we have from (3.6), by taking ε small, that

$$\frac{d}{dt} \|u^m\|^2 + \mu_0 C \|\nabla u^m\|^2 + \mu_1 C \|\nabla u^m\|_{r+2}^{r+2} \leq \varepsilon^{-1} \|f\|_{\mathbf{V}'}^2 + \varepsilon \|u^m\|^2. \quad (3.7)$$

That is,

$$\frac{d}{dt} \|u^m\|^2 \leq \varepsilon^{-1} \|f\|_{\mathbf{V}'}^2 + \varepsilon \|u^m\|^2.$$

By Gronwall's inequality, we have

$$\|u^m(t)\|^2 \leq \|u^m(0)\|^2 + \varepsilon^{-1} \int_0^T \|f\|_{\mathbf{V}'}^2 ds.$$

Integrating (3.7) with respect to t from 0 to T , we also have

$$\begin{aligned} \|u^m(t)\|^2 + \mu_0 C \int_0^T \|\nabla u^m\|^2 ds + \mu_1 C \int_0^T \|\nabla u^m\|_{r+2}^{r+2} ds \\ \leq \varepsilon^{-1} \int_0^T \|f\|_{\mathbf{V}'}^2 ds + \varepsilon \int_0^T \|u^m(s)\|^2 ds \\ \leq \varepsilon^{-1} \int_0^T \|f\|_{\mathbf{V}'}^2 ds + \varepsilon T \sup_{0 \leq t \leq T} \|u^m(t)\|^2. \end{aligned}$$

Thus, we have (3.4). □

From the above theorem, we have

$$u^m \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}) \cap L^q(0, T; \mathbf{V}_q).$$

Thus, we have a subsequence, still denoted by u^m , that converges to u in $L^2(0, T; \mathbf{V})$ weakly, in $L^q(0, T; \mathbf{V}_q)$ weakly, and in $L^\infty(0, T; \mathbf{H})$ weak-star, as $m \rightarrow \infty$. For such a subsequence we need to take the limit to (3.1) in order to show that the limit u is a Young measure-valued solution. Owing to the above statement, we can take the limit to infinity for the first term of (3.1) and for the linear viscosity term Γ_0 . For the nonlinear third term of (3.1) we need the strong convergence of u^m in $L^2(0, T; \mathbf{H})$. For the nonlinear viscosity term Γ_1 we need to find the corresponding Young measure to the weak limit of ∇u^m .

For the strong convergence of the subsequence u^m in $L^2(0, T; \mathbf{H})$, we need a compactness lemma. We state the compact embedding lemma in Temam [9] (see Theorem 2.2 of Chapter III in [9]).

LEMMA 3.3. Assume that X_0, X, X_1 are Hilbert spaces with

$$X_0 \subset X \subset X_1,$$

the injection being continuous and the injection of X_0 into X being compact. Define the space

$$\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1) \stackrel{\text{def}}{=} \{v \in L^2(\mathbb{R}; X_0), \partial_t^\gamma v \in L^2(\mathbb{R}; X_1)\},$$

which is a Hilbert space for the norm

$$\|v\|_{\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1)} \stackrel{\text{def}}{=} \{\|v\|_{L^2(\mathbb{R}; X_0)}^2 + \| |\tau|^\gamma \hat{v}(\tau) \|_{L^2(\mathbb{R}; X_1)}^2\}^{1/2},$$

where $\hat{v}$ is the Fourier transform of v with respect to t ,

$$\hat{v}(\tau) = \int_{\mathbb{R}} e^{-2\pi i t \tau} v(t) dt.$$

We denote $i = \sqrt{-1}$ if there is no confusion with index i .

We also define a subspace $\mathcal{H}_K^\gamma$ of $\mathcal{H}^\gamma$ for any $K \subset \mathbb{R}$ defined as the set of functions u in $\mathbf{H}^\gamma$ with support contained in K :

$$\mathcal{H}_K^\gamma(\mathbb{R}; X_0, X_1) \stackrel{\text{def}}{=} \{u \in \mathcal{H}^\gamma(\mathbb{R}; X_0, X_1), \text{ support } u \in K\}.$$

Then for any bounded set K and any $\gamma > 0$, the injection of $\mathcal{H}_K^\gamma(\mathbb{R}; X_0, X_1)$ into $L^2(\mathbb{R}; X)$ is compact.

LEMMA 3.4. Suppose $\mu_0 > 0$ and $\mu_1 > 0$. Let $-1 < r$. If, for each m , u^m is a solution of (3.1) and (3.2), then $u^m \in \mathcal{H}_{[0,T]}^\gamma(\mathbb{R}; \mathbf{V}, \mathbf{H})$ for $0 < \gamma < 1/4$.

Proof. Let $\tilde{u}^m$ be the extension of u^m , which is defined to be u^m on $[0, T]$, and 0 outside of $[0, T]$. Then from (3.3) we have

$$\begin{aligned} \frac{d}{dt} \langle \tilde{u}^m, w_k \rangle + \mu_0 \Gamma_0(\nabla \tilde{u}^m, \nabla w_k) + \mu_1 \Gamma_1(\nabla \tilde{u}^m, \nabla w_k) + b(\tilde{u}^m, \tilde{u}^m, w_k) \\ = \langle f, w_k \rangle + \langle u_0^m, w_k \rangle \delta_0 - \langle u^m(T), w_k \rangle \delta_T, \end{aligned} \quad (3.8)$$

where δ_0 and δ_T are Dirac delta functions at 0 and T , respectively. Denote by $\tilde{\Gamma}_{1;ij}^m$ the nonlinear viscosity term $|\tilde{E}^m|^r \tilde{E}_{ij}^m$, where

$$\tilde{E}_{ij}^m \stackrel{\text{def}}{=} E_{ij}(\nabla \tilde{u}^m) = \frac{1}{2} \left(\frac{\partial \tilde{u}_i^m}{\partial x_j} + \frac{\partial \tilde{u}_j^m}{\partial x_i} \right).$$

We use the notation $E_{ij}^m \stackrel{\text{def}}{=} E_{ij}(\nabla u^m)$ and $\Gamma_{1;ij}^m \stackrel{\text{def}}{=} |E^m|^r E_{ij}^m$. Denote by $\hat{u}^m$, by $\hat{E}_{ij}^m$, and by $\hat{\Gamma}_{1;ij}^m$ the Fourier transforms of $\tilde{u}^m$, $E_{ij}(\tilde{u}^m)$, and $\tilde{\Gamma}_{1;ij}^m$ with respect to t , respectively. We claim that

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{u}^m(\tau)\|^2 d\tau \leq C$$

for $0 < \gamma < 1/4$.

First, for $|\tau| \leq 1$, by Plancherel's theorem we have

$$\begin{aligned}
 \int_{|\tau| \leq 1} |\tau|^{2\gamma} \|\hat{u}^m\|^2 d\tau &\leq \int_{\mathbb{R}} \|\hat{u}^m\|^2 d\tau \\
 &= \int_{\mathbb{R}} \int_{\Omega} |\hat{u}^m|^2 dx d\tau \\
 &= \int_{\Omega} \int_{\mathbb{R}} |\tilde{u}^m|^2 dt dx \\
 &= \int_0^T \|u^m\|^2 d\tau \leq C(T).
 \end{aligned}$$

Consider $|\tau| > 1$. Applying the Fourier transform to (3.8) with respect to t , we have

$$\begin{aligned}
 2\pi i\tau \langle \hat{u}^m, w_k \rangle + \mu_0 \langle \hat{E}_{ij}^m, E_{ij}(\nabla w_k) \rangle + \mu_1 \langle \hat{\Gamma}_{1:ij}^m, E_{ij}(\nabla w_k) \rangle + \langle \widehat{B\tilde{u}^m}, w_k \rangle \\
 = \langle \hat{f}, w_k \rangle + \langle u_0^m, w_k \rangle - \langle u^m(T), w_k \rangle e^{-2\pi i T \tau}.
 \end{aligned} \tag{3.9}$$

Multiplying (3.9) by the Fourier transform $\overline{\hat{g}_k(\tau)}$ of $\tilde{g}_k$ and adding the resulting equations for $k = 1, \dots, m$, we get

$$\begin{aligned}
 2\pi i\tau \|\hat{u}^m\|^2 + \mu_0 \|\hat{E}^m\|^2 + \mu_1 \langle \hat{\Gamma}_{1:ij}^m, \overline{\hat{E}_{ij}^m} \rangle + \langle \widehat{B\tilde{u}^m}, \overline{\hat{u}^m} \rangle \\
 = \langle \hat{f}, \overline{\hat{u}^m} \rangle + \langle u_0^m, \overline{\hat{u}^m} \rangle - \langle u^m(T), \overline{\hat{u}^m} \rangle e^{-2\pi i T \tau}.
 \end{aligned} \tag{3.10}$$

Notice that

$$\sup_{\tau} \|\widehat{B\tilde{u}^m}\|_{\mathbf{V}'} = \sup_{\tau} \sup_{\|v\|_{\mathbf{V}} \leq 1} \frac{\langle \widehat{B\tilde{u}^m}, v \rangle}{\|v\|_{\mathbf{V}}}.$$

According to Lemma 3.1, we have that

$$\begin{aligned}
 \int \|Bu^m\|_{\mathbf{V}'} dt &\leq \int_0^T \left(\int_{\Omega} |u^m|^4 dx \right)^{1/2} dt \\
 &\leq \int_0^T \|u^m\|^{1/2} \|\nabla u^m\|^{3/2} dt \\
 &\leq C \sup \|u^m\|^{1/2} \left(\int \int_{Q_T} |\nabla u^m|^2 dx dt \right)^{3/4} \leq C.
 \end{aligned}$$

For $\|v\|_{\mathbf{V}} \leq 1$,

$$\begin{aligned}
 \langle \widehat{B\tilde{u}^m}, v \rangle &= - \int_{\Omega} \left(\int_0^T u_j^m(x, s) u_i^m(x, s) e^{-2\pi i s \tau} ds \right) \partial_j v_i(x) dx \\
 &\leq \int_0^T \left(\int_{\Omega} |u^m|^4 dx \right)^{1/2} \|\nabla v\| ds \\
 &\leq C(T) \left(\int_0^T \int_{\Omega} |u^m|^4 dx ds \right)^{1/2} \leq C.
 \end{aligned}$$

Using the above inequalities we have

$$\sup_{\tau} \|\widehat{B\tilde{u}^m}\|_{\mathbf{V}'} \leq C(T).$$

Take imaginary parts of (3.10):

$$\begin{aligned} 2\pi\tau\|\hat{u}^m\|^2 + \mu_1 \operatorname{Im}\langle \widehat{\Gamma}_{1;ij}^m, \widehat{E}_{ij}^m \rangle + \operatorname{Im}\langle \widehat{B\hat{u}}^m, \widehat{u}^m \rangle \\ = \operatorname{Im}\langle \hat{f}, \widehat{u}^m \rangle + \operatorname{Im}\langle u_0^m, \widehat{u}^m \rangle - \operatorname{Im}\langle u^m(T), \widehat{u}^m \rangle e^{-2\pi i T \tau}. \end{aligned} \quad (3.11)$$

We first consider the second term of (3.11). We estimate $\tau^{2\gamma-1}\langle \widehat{\Gamma}_{1;ij}^m, \widehat{E}_{ij}^m \rangle$. Observe that

$$\begin{aligned} \langle \widehat{\Gamma}_{1;ij}^m, \widehat{E}_{ij}^m \rangle &= \int_{\Omega} \left(\int_0^T \tilde{\Gamma}_{1;ij}^m e^{-2\pi i t \tau} dt \right) \left(\int_0^T \tilde{E}_{ij}^m e^{2\pi i s \tau} ds \right) dx \\ &= \int_{\Omega} \int_0^T \int_0^T \Gamma_{1;ij}^m(x, t) E_{ij}^m(x, s) e^{-2\pi i(t-s)\tau} ds dt dx \\ &= \int_{\Omega} \int_0^T \int_0^T |E^m|^r E_{ij}^m(x, t) E_{ij}^m(x, s) e^{-2\pi i(t-s)\tau} ds dt dx \end{aligned}$$

and

$$\operatorname{Im}\langle \widehat{\Gamma}_{1;ij}^m, \widehat{E}_{ij}^m \rangle = - \iiint |E^m|^r E_{ij}^m(x, t) E_{ij}^m(x, s) \sin(2\pi(t-s)\tau) ds dt dx.$$

For $0 < \gamma < 1/4$ we want to estimate

$$\int_1^{\infty} \tau^{2\gamma-1} \int_{\Omega} \int_0^T \int_0^T \tilde{\Gamma}_{1;ij}^m(x, t) E_{ij}^m(x, s) \sin(2\pi(t-s)\tau) ds dt dx d\tau.$$

Set $\eta = 2\pi(t-s)\tau$. Then we get

$$\left| \int_1^{\infty} \tau^{2\gamma-1} \sin(2\pi(t-s)\tau) d\tau \right| = (2\pi|t-s|)^{-2\gamma} \left| \int_{2\pi|t-s|}^{\infty} \eta^{2\gamma-1} \sin \eta d\eta \right|.$$

Since we are assuming $\gamma \in (0, \frac{1}{4})$, $\eta^{2\gamma-1}$ decreases monotonically as η goes to infinity. Moreover, since $\sin \eta$ is oscillating, we have from the Leibnitz lemma for alternating series that

$$\left| \int_a^{\infty} \eta^{2\gamma-1} \sin \eta d\eta \right| < C_0$$

for all $a > 0$ and for some C_0 independent of a . With this reasoning, we conclude that

$$\left| \int_1^{\infty} \tau^{2\gamma-1} \sin(2\pi(t-s)\tau) d\tau \right| \leq C_0 (2\pi|t-s|)^{-2\gamma}.$$

Hence, integrating τ first, we obtain

$$\left| \operatorname{Im} \int_1^{\infty} \tau^{2\gamma-1} \langle \widehat{\Gamma}_{1;ij}^m, \widehat{E}_{ij}^m \rangle d\tau \right| \leq C \int_0^T \int_0^T \int_{\Omega} |\Gamma_{1;ij}^m E_{ij}^m| |t-s|^{-2\gamma} dx ds dt.$$

For $\tau < -1$ we have the same result. Setting $q \stackrel{\text{def}}{=} r + 2$ and $q' \stackrel{\text{def}}{=} (r + 2)/(r + 1)$, we have

$$\begin{aligned} & \left| \operatorname{Im} \int_{|\tau|>1} |\tau|^{2\gamma-1} \langle \widehat{\Gamma}_{1:ij}^m, \overline{\widehat{E}_{ij}^m} \rangle d\tau \right| \\ & \leq C \iiint |\nabla u^m(x, t)|^{r+1} |\nabla u^m(x, s)| |t - s|^{-2\gamma} dx ds dt \\ & \leq C \iint \left(\int |\nabla u^m(x, t)|^q dx \right)^{1/q'} \left(\int |\nabla u^m(x, s)|^q dx \right)^{1/q} |t - s|^{-2\gamma} ds dt \\ & \leq 2C \iint |t - s|^{-2\gamma} \int |\nabla u^m(t)|^{r+2} dx ds dt. \end{aligned}$$

From direct computations we obtain

$$\int_0^T |t - s|^{-2\gamma} ds = \frac{1}{1 - 2\gamma} ((T - t)^{1-2\gamma} + t^{1-2\gamma}),$$

and we have

$$\begin{aligned} & \int_0^T \int_0^T |t - s|^{-2\gamma} ds \int |\nabla u^m(t)|^{r+2} dx dt \\ & = \frac{1}{1 - 2\gamma} \iint |\nabla u^m(t)|^{r+2} ((T - t)^{1-2\gamma} + t^{1-2\gamma}) dx dt \\ & \leq C(T) \iint |\nabla u^m|^{r+2} dx dt \leq C(T). \end{aligned}$$

Therefore, we have

$$\left| \operatorname{Im} \int_{|\tau|>1} |\tau|^{2\gamma-1} \langle \widehat{\Gamma}_{1:ij}^m, \overline{\widehat{E}_{ij}^m} \rangle d\tau \right| \leq C.$$

Multiplying (3.11) by $|\tau|^{2\gamma-1}$, we have

$$\int_{|\tau|>1} |\tau|^{2\gamma} \|\hat{u}^m\|^2 d\tau \leq C \int_{|\tau|>1} |\tau|^{2\gamma-1} (\|\nabla \hat{u}^m\| + \|\hat{u}^m\|) d\tau + C(T). \quad (3.12)$$

Since we are assuming $0 < \gamma < 1/4$, by Young's inequality and Plancherel's theorem, we obtain

$$\begin{aligned} \int_{|\tau|>1} |\tau|^{2\gamma-1} \|\hat{u}^m\| d\tau & \leq \frac{1}{2} \int_{|\tau|>1} |\tau|^{4\gamma-2} d\tau + \frac{1}{2} \int_{|\tau|>1} \|\hat{u}^m\|^2 d\tau \\ & \leq \frac{1}{2} \frac{1}{1 - 4\gamma} + \frac{1}{2} \int \int_{Q_T} |u^m|^2 dx dt \leq C(T). \end{aligned}$$

For the term $\int_{|\tau|>1} |\tau|^{2\gamma-1} \|\nabla \hat{u}^m\| d\tau$ we can estimate in the same way. Therefore, combining all the above estimates, we conclude that

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{u}^m\|^2 d\tau \leq C,$$

for $0 < \gamma < 1/4$. □

Now we are ready to show our existence theorem of Young measure-valued solutions. This theorem extends the previously known results by [1]. Indeed, it was shown in [1] that there exists a measure-valued solution when $-\frac{4}{5} < r \leq -\frac{1}{5}$, a weak solution when $-\frac{1}{5} < r \leq \frac{1}{5}$, and a unique weakly regular solution when $r > \frac{1}{5}$. Here, we extend the existence of measure-valued solutions up to $r > -1$.

THEOREM 3.5 (Existence of Young measure-valued solutions). Let $\mu_0 > 0$ and $\mu_1 \geq 0$. Let $r > -1$. If $u_0 \in H$ and $f \in L^2(0, T; \mathbf{V}')$, then there is a measure-valued solution $(u, \nu_{x,t})$ of (2.4) satisfying

$$\begin{aligned} & - \int_{Q_T} u_i \partial_t \phi_i dx dt - \int_{Q_T} u_j u_i \nabla_j \phi_i dx dt + \mu_0 \Gamma_0(\nabla u, \nabla \phi) \\ & + \mu_1 \int_{Q_T} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \cdot E(\nabla \phi) dx dt = \int f_i \phi_i dx dt, \end{aligned}$$

for all $\phi \in C_0^\infty(Q_T)$ with $\nabla \cdot \phi = 0$, where

$$E_{ij}(\lambda) = \frac{1}{2}(\lambda_{ij} + \lambda_{ji}).$$

Proof. In order to use Lemma 3.3, we need the fact that the injection $X_0 \hookrightarrow X$ is compact. If we take $X_0 = \mathbf{V}$ and $X = \mathbf{H}$ then the injection is compact. Thus from Lemma 3.4, we have that the injection of $\mathcal{H}_{[0,T]}^\gamma(\mathbb{R}; \mathbf{V}, \mathbf{H})$ into $L^2(\mathbb{R}; \mathbf{H})$ is compact for $0 < \gamma < 1/4$.

As in the paragraph above Lemma 3.3, we have a subsequence, denoted by $u^{m'}$, that converges to u in $L^2(0, T; \mathbf{V})$ weakly, in $L^q(0, T; \mathbf{V}_q)$ weakly, and in $L^\infty(0, T; \mathbf{H})$ weak-star, as $m' \rightarrow \infty$, where $q = r + 2$. For such a subsequence we need to take the limit to (3.1) in order to show that the limit u is a weak solution. Owing to the above statement, we can take the limit to infinity for the first two terms of (3.1). By Lemma 3.4, we have the strong convergence of the subsequence $u^{m'}$ to u in $L^2(0, T; \mathbf{H})$; in other words, $u^{m'} \rightarrow u$ in $L^2(0, T; \mathbf{H})$ strongly, and this suffices to pass to the limit in (3.3). Let ψ be a continuously differentiable function on $[0, T]$ with $\psi(T) = 0$. We multiply (3.1) by $\psi(t)$, and then integrate by parts. This leads to the equation

$$\begin{aligned} & - \int_0^T \langle u^{m'}(t), \psi'(t) w_k \rangle dt + \mu_0 \int_0^T \Gamma_0(\nabla u^{m'}(t), \nabla w_k \psi(t)) dt \\ & + \mu_1 \int_0^T \Gamma_1(\nabla u^{m'}(t), \nabla w_k \psi(t)) dt + \int_0^T b(u^{m'}(t), u^{m'}(t), w_k \psi(t)) dt \\ & = - \langle u_0^{m'}, w_k \rangle \psi(0) + \int_0^T \langle f(t), w_k \psi(t) \rangle dt. \end{aligned}$$

The first two terms can pass to the limit. Consider the limit of the fourth term. Notice that

$$\int_0^T b(u^{m'}(t), u^{m'}(t), w_k \psi(t)) dt = - \int_0^T \int_\Omega u_j^{m'} \partial_j w_{k,i} \psi u_i^{m'} dx dt,$$

which converge to

$$- \int_0^T \int_\Omega u_j \partial_j w_{k,i} \psi u_i dx dt = \int b(u, u, w_k \psi) dt,$$

owing to the $L^2(0, T; \mathbf{V})$ weak and $L^2(0, T; L^2(\Omega))$ strong convergence of $u^{m'}$. Since Γ_1 is nonlinear with respect to E , we need to introduce a Young measure $\nu_{x,t}$ corresponding to the weak limit $\{\nabla u^m\}$. From the lower semi-continuity of the L^{r+2} -norm and Korn's inequality, we have a uniform bound such that

$$\int_0^T \int_{\Omega} |\nabla u^{m'}|^{r+2} \leq c$$

for some c independent of m' . Thus, as in the case of Bellout, Bloom, and Nečas [1], we can find a Young measure $\nu_{x,t}$ for the weak limit of $\{\nabla u^{m'}\}$ since $|\Gamma_1(u, u)| \leq C(1 + |\nabla u|^{r+2})$ (see Theorem 3.2 in Bellout, Bloom, and Nečas [1]). Therefore, we conclude that

$$\int \Gamma_1(\nabla u^{m'}(t), \nabla w_k \psi(t)) dt \rightarrow \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} \cdot E(\nabla w_k \phi_t) dx dt$$

as m' tends to infinity. □

We like to introduce pressure. To this purpose we define

$$\begin{aligned} U(t) &= \int_0^t u(\tau) d\tau, & B(t) &= \int_0^t u(\tau) \cdot \nabla u(\tau) d\tau, \\ F(t) &= \int_0^t f(\tau) d\tau, & Z(t) &= \int_0^t |E(\nabla u(\tau))|^r E(\nabla u(\tau)) d\tau. \end{aligned}$$

Thus, it is clear that $U \in C([0, T]; \mathbf{V})$, $F \in C([0, T]; \mathbf{V}')$, $Z \in C([0, T]; \mathbf{V}'_q)$. We take $v \in V$ in (2.4) and integrate from 0 to t . Hence we obtain

$$\langle u(t) - u_0, v \rangle + \mu_0 \langle E(\nabla U(t)), E(\nabla v) \rangle + \mu_1 \langle Z(t), \nabla v \rangle = \langle F(t), v \rangle$$

or

$$\langle u(t) - u_0 - \mu_0 \operatorname{div}(E(\nabla U(t))) - \mu_1 \operatorname{div}(Z(t)) - F(t), v \rangle = 0$$

for all $v \in \mathbf{V} \cap \mathbf{V}_q$ and for all $0 < t < T$. Consequently, if we consider a bounded domain D with smooth boundary, then $u(t) - u_0 - \mu_0 \operatorname{div}(E(\nabla U(t))) - \mu_1 \operatorname{div}(Z(t)) - F(t)$ is orthogonal to $\mathbf{V}_q(D)$ and, thus, we have $u(t) - u_0 - \mu_0 \operatorname{div}(E(\nabla U(t))) - \mu_1 \operatorname{div}(Z(t)) - F(t) = \nabla P_D(t)$ in the sense of distributions for some $P_D \in L^q(D)$. This fact follows from the unique decomposition of $L^q(D)$ -space by gradient fields and solenoidal vectors for a smooth domain D . Moreover, P_D is unique modulo a constant. Therefore, P_D is independent of the domain D and we prove that $u(t) - u_0 - \mu_0 \operatorname{div}(E(\nabla U(t))) - \mu_1 \operatorname{div}(Z(t)) - F(t) = \nabla P(t)$ for some $P(t) \in L^q(\Omega)$. Observing that $\mu_0 \operatorname{div}(E(\nabla U(t))) + \mu_1 \operatorname{div}(Z(t)) \in C(0, T; (W^{1,q})')$, we conclude

$$\nabla P(t) \in C(0, T; (W^{1,q})').$$

This enables us to differentiate $P(t)$ in the sense of distributions. Setting $p(t) = \frac{\partial P(t)}{\partial t}$ we get

$$L(u) \stackrel{\text{def}}{=} u_t - \mu_0 \operatorname{div}(E(\nabla u)) - \mu_1 \operatorname{div}(|E(\nabla u)|^r E(\nabla u)) + (u \cdot \nabla)u + \nabla p = f.$$

We now consider the non-Newtonian flow on a bounded domain Ω for $\mu_0 = 0$:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu_1 \frac{\partial}{\partial x_j} (|E(\nabla u)|^r E_{ij}(\nabla u)) + f_i, \quad \frac{\partial u_j}{\partial x_j} = 0 \quad (3.13)$$

in $Q = \Omega \times (0, \infty)$, with the initial condition $u(x, 0) = u_0$ for $x \in \Omega$, and with Dirichlet or Neumann or periodic boundary conditions, where $r > -1$ and $\mu_1 > 0$.

The symbols $\mathbf{H}, \mathbf{V}, \mathbf{V}_q$ denote the closure in $L^2(\Omega), W^{1,2}(\Omega), W^{1,q}(\Omega)$, respectively, of $\{v \in C^\infty : \operatorname{div} v = 0\}$ with the given boundary conditions.

LEMMA 3.6. Assume that $\int \|f\|_{\mathbf{V}_q}^2 ds$ is bounded. Then we have

$$\begin{aligned} \sup_{0 < t < T} \|u^m(t)\|^2 + \mu_1 \int_0^T \|\nabla u^m\|_{r+2}^{r+2} ds \\ \leq C \|u^m(0)\|^2 + C \int_0^T \|f\|_{\mathbf{V}_q}^2 ds. \end{aligned} \quad (3.14)$$

Proof. During the proof of Lemma 3.2, inequality (3.7) becomes

$$\frac{d}{dt} \|u^m\|^2 + \mu_1 C \|\nabla u^m\|_{r+2}^{r+2} \leq \varepsilon^{-1} \|f\|_{\mathbf{V}_{q'}}^{q'},$$

where $q = r + 2$ and $1/q + 1/q' = 1$. Integrating with respect to time t , we prove (3.14). $\square$

From the above theorem, we have

$$u^m \in L^\infty(0, T; \mathbf{H}) \cap L^q(0, T; \mathbf{V}_q).$$

Thus, we have a subsequence, still denoted by u^m , that converges to u in $L^q(0, T; \mathbf{V}_q)$ weakly, and in $L^\infty(0, T; \mathbf{H})$ weak-star, as $m \rightarrow \infty$. For such a subsequence we need to take the limit to (3.1) for $\mu_0 = 0$ in order to show that the limit u is a weak solution. Owing to the above statement, we can take the limit to infinity for the first two terms of (3.1). For the nonlinear third term of (3.1) we need the strong convergence of u^m in $L^2(0, T; \mathbf{H})$. Since $\mathbf{V}_q \subset \mathbf{V}$ on a bounded domain for $q \geq 2$, we can say $u^m \in L^2(0, T; \mathbf{V})$ if $r \geq 0$.

LEMMA 3.7. Let $\mu_0 = 0$ and $\mu_1 > 0$. Let $r \geq 0$. If, for each m , u^m is a solution of (3.1) and (3.2), then $u^m \in \mathcal{H}_{[0,T]}^\gamma(\mathbb{R}; \mathbf{V}, \mathbf{H})$ for $0 < \gamma < 1/4$.

We adapt the proof of Lemma 3.4 for $\mu_0 = 0$. With the same reasoning as that of Theorem 3.5, we have the following existence theorem for $\mu_0 = 0$, $\mu_1 > 0$, and $r \geq 0$.

THEOREM 3.8 (Existence). Let $\mu_0 = 0$, $\mu_1 > 0$ and $r \geq 0$. If $u_0 \in H$ and $f \in L^{q'}(0, T; \mathbf{V}_q')$, then there is a Young measure-valued solution u of (3.13) with

$$u \in L^\infty(0, T; \mathbf{H}) \cap L^q(0, T; \mathbf{V}_q),$$

where $q = r + 2$.

4. Measure-valued solutions are weak. In this section we consider the case $\mu_0 > 0$ for the periodic domain. We assume that our Young measure-valued solution satisfies a convexity condition for the L^{r+2} energy. When $r > -\frac{1}{5}$, Bellout, Bloom, and Nečas [1] showed that there is a weak solution without any assumption for the Young measure-valued solutions. We prove that the Young measure-valued solution $(u, \nu_{x,t})$ is weak for

all $r > -1$ if it satisfies

$$\int_{t_0}^{t_1} \int_{\Omega} \left[\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} - |E(\nabla v)|^r E(\nabla v) \right] (\nabla u - \nabla v) dx dt \geq 0 \quad (4.1)$$

for all $0 \leq t_0 \leq t_1 \leq T$ and for all weak solutions v . We note that any weak solution u satisfies the convexity condition (4.1). We define a weakly strong solution. A solution u is said to be *weakly strong* if it satisfies

$$\sup_{0 < t < T} \int_{\Omega} |\nabla u|^2 dx < \infty, \quad \int_0^T \int_{\Omega} |\nabla^2 u|^2 dx dt < \infty,$$

and

$$\int_0^T \left(\int_{\Omega} |\nabla u|^{3(r+2)} dx \right)^{1/3} dt < \infty.$$

We recall the generalized form of Korn's inequality in Bellout, Bloom, and Nečas [1], [2]:

$$\int_{\Omega} \left| \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \right|^{s/2} dx \geq C \|u\|_{2,s}^s \quad \text{for } 1 < s < \infty. \quad (4.2)$$

The following lemma will be useful for the proof of the regularity.

LEMMA 4.1.

$$\int_{\Omega} \partial_k (|E(\nabla u)|^r E_{ij}(\nabla u)) \partial_k E_{ij}(\nabla u) dx \geq C \|u\|_{1,3(r+2)}^{r+2}.$$

Proof. Observe that

$$\begin{aligned} \partial_k (|E|^r E_{ij}) \partial_k E_{ij} &= \partial_k ((E_{l\ell} E_{l\ell})^{r/2} E_{ij}) \partial_k E_{ij} \\ &= |E|^r \partial_k E_{ij} \partial_k E_{ij} + r |E|^{r-2} E_{l\ell} \partial_k E_{l\ell} E_{ij} \partial_k E_{ij}. \end{aligned}$$

We note that

$$\begin{aligned} \partial_k (|E|^{r/2} E_{ij}) \partial_k (|E|^{r/2} E_{ij}) &= \partial_k ((E_{l\ell} E_{l\ell})^{r/4} E_{ij}) \partial_k ((E_{l\ell} E_{l\ell})^{r/4} E_{ij}) \\ &= \left(\frac{r}{2} |E|^{r/2-2} E_{l\ell} \partial_k E_{l\ell} E_{ij} + |E|^{r/2} \partial_k E_{ij} \right)^2 \\ &= \frac{r^2}{4} |E|^{r-2} E_{l\ell} \partial_k E_{l\ell} E_{\alpha\beta} \partial_k E_{\alpha\beta} + r |E|^{r-2} E_{l\ell} \partial_k E_{l\ell} E_{ij} \partial_k E_{ij} \\ &\quad + |E|^r \partial_k E_{ij} \partial_k E_{ij} \\ &= \frac{r(r+4)}{4} |E|^{r-2} E_{l\ell} \partial_k E_{l\ell} E_{\alpha\beta} \partial_k E_{\alpha\beta} + |E|^r \partial_k E_{ij} \partial_k E_{ij}. \end{aligned}$$

Thus we have the equivalence of $\partial_k (|E|^r E_{ij}) \partial_k E_{ij}$ and $(\partial_k (|E|^{r/2} E_{ij}))^2$. In particular, we have

$$\frac{r+4}{4} \partial_k (|E|^r E_{ij}) \partial_k E_{ij} \geq (\partial_k (|E|^{r/2} E_{ij}))^2.$$

Thus we have by Sobolev's and Korn's inequalities that

$$\begin{aligned} \frac{r+4}{4} \int_{\Omega} \partial_k(|E|^r E_{ij}) \partial_k E_{ij} &\geq \int_{\Omega} (\partial_k(|E|^{r/2} E_{ij}))^2 \\ &\geq C \left(\int_{\Omega} (|E|^{r/2} E_{ij})^6 dx \right)^{1/3} \\ &\geq C \left(\int_{\Omega} |\nabla u|^{3(r+2)} dx \right)^{1/3}. \quad \square \end{aligned}$$

We now show the short time existence and the uniqueness of weakly strong solutions for regular initial data.

LEMMA 4.2 (Existence and uniqueness of weakly strong solution). Suppose that $u_0(0) \in H^1(\Omega)$ satisfies $\operatorname{div} u_0 = 0$. Then there is a positive number $\delta = \delta(\|u_0\|_{H^1}) > 0$ such that there is a unique solution u of (2.4) in $(t_0, t_0 + \delta)$ satisfying

$$\begin{aligned} \sup_{0 < t < \delta} \int |\nabla u|^2 dx + \mu_0 \int_0^\delta \int |\nabla^2 u|^2 dx dt &< \infty, \\ \mu_1 \int_0^\delta \left(\int |\nabla u|^{3(r+2)} dx \right)^{1/3} dt &< \infty. \end{aligned}$$

Proof. With the same argument as in the proof of Lemma 3.2, we find an estimate for the approximate Galerkin solution u^m :

$$\begin{aligned} &\int |\nabla u^m|^2 dx(\tau) + \mu_0 \int_0^\tau \int |\nabla^2 u^m|^2 dx dt \\ &\quad + \mu_1 \int_0^\tau \partial_k(|E|^r E(\nabla u^m)) \partial_k(E(\nabla u^m)) dx dt \\ &\leq \int_0^\tau \int |\nabla u^m|^3 dx dt + \int_0^\tau \int |f| |\nabla^2 u^m| dx dt. \end{aligned}$$

We note from Lemma 4.1 that

$$\int_0^\tau \int \partial_k(|E|^r E(\nabla u^m)) \partial_k(E(\nabla u^m)) dx dt \geq C \int_0^\tau \|u^m\|_{1,3(r+2)}^{r+2} dt.$$

So from Sobolev's inequality and Young's inequality, we have

$$\frac{d}{dt} \int |\nabla u^m|^2 dx + \mu_0 \int_0^\tau \int |\nabla^2 u^m|^2 dx dt \leq C \left(1 + \int |\nabla u^m|^2 \right)^3.$$

Thus there is $\delta = \delta(\int |\nabla u^m|^2 dx(0)) > 0$ such that

$$\int |\nabla^2 u^m|^2 dx(t) < \infty \quad \text{for all } t < \delta$$

and

$$\mu_0 \int_0^\delta \int |\nabla^2 u^m|^2 dx dt < \infty.$$

We also obtain

$$\mu_1 \int_0^\delta \left(\int |\nabla u^m|^{3(r+2)} dx \right)^{1/3} dt < \infty.$$

So from the compact embedding theorem (Lemma 3.1 in Bellout, Bloom, and Nečas [1]), we have that ∇u^m converges to ∇u strongly in $L^{r+2}((0, T) \times \Omega)$, if necessary choosing a different subsequence. Furthermore, from a standard comparison method, we find that the weak solution u is unique under the assumption of our Lemma. $\square$

Although we have shown that the weakly strong solution is unique, we still do not know if the uniqueness can be extended to the class of measure-valued solutions. This is the place where we need to assume the energy convexity.

LEMMA 4.3. Suppose that $(u, \nu_{x,t})$ is the measure-valued solution. If u satisfies

$$\sup_{t_1 \leq t < t_2} \int |\nabla u|^2 dx(t) < \infty,$$

then u is a weakly strong solution in $[t_1, t_2]$.

Proof. Here, we need only to prove that the measure-valued solution $(u, \nu_{x,t})$ is a weak solution. Then from Lemma 4.2, $\nu_{x,t}(\lambda) = \delta(\nabla u_{x,t} - \lambda)$ for the Dirac delta function δ and hence u is the weakly strong solution. Suppose that v is the weak solution of

$$v_t - \nabla \cdot \Gamma(u) + (v \cdot \nabla)v + \nabla p = f, \quad v(x, t_1) = u(x, t_1),$$

and let $w(x, t) = (u - v)(x, t)$. Then w satisfies

$$\begin{aligned} w_t - \mu_0 \operatorname{div}(E(\nabla w)) - \mu_1 \operatorname{div} \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} - |E(\nabla v)|^r E(\nabla v) \right) \\ + \nabla(p_u - p_v) + (w \cdot \nabla)w + (v \cdot \nabla)w + (w \cdot \nabla)v = 0. \end{aligned}$$

Thus, taking w as a test function, we have

$$\begin{aligned} \frac{d}{dt} \int |w|^2 dx(t) + \mu_0 \int |\nabla w|^2(t) \\ + \mu_1 \int \left[\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} - |E(\nabla v)|^r E(\nabla v) \right] (\nabla u - \nabla v) dx \\ \leq \int |\nabla v| |w|^2 dx(t) \end{aligned}$$

with

$$\int |w|^2 dx(0) = 0.$$

From Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
 & \int |\nabla v| |w|^2 dx \\
 &= \int |\nabla v| |w|^{3/2} |w|^{1/2} dx \\
 &\leq \left(\int |\nabla v|^6 dx \right)^{1/6} \left(\int |w|^2 dx \right)^{3/4} \left(\int |w|^6 dx \right)^{1/12} \\
 &\leq C \left(\int |\nabla^2 v|^2 dx \right)^{1/2} \left(\int |w|^2 dx \right)^{3/4} \left(\int |\nabla w|^2 dx \right)^{1/4} \\
 &\leq \frac{C}{\varepsilon} \left(\int |\nabla^2 v|^2 dx \right)^{2/3} \int |w|^2 dx + \varepsilon C \int |\nabla w|^2 dx.
 \end{aligned}$$

Choosing ε so small that $\varepsilon C < \frac{1}{2}$ and defining $h(t) \stackrel{\text{def}}{=} (\int \|\nabla^2 u\|^2 dx)^{2/3}$, we get

$$\begin{aligned}
 \frac{d}{dt} \int |w|^2 dx(t) &\leq h(t) \int |w|^2 dx(t) \\
 &\quad - \int \left[\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} - |E(\nabla v)|^r E(\nabla v) \right] \cdot [\nabla u - \nabla v] dx.
 \end{aligned} \tag{4.3}$$

Here, $h(t)$ is integrable on $[t_1, t_1 + \delta]$. From our energy convexity assumption (4.1) we have

$$\int \left[\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t} - |E(\nabla v)|^r E(\nabla v) \right] \cdot [\nabla u - \nabla v] dx \geq 0. \tag{4.4}$$

From (4.3) we have

$$\frac{d}{dt} \int |w|^2 dx(t) \leq h(t) \int |w|^2 dx(t),$$

and from Gronwall's inequality we conclude that $u \equiv v$ on $[t_1, t_1 + \delta]$. Changing the initial time if necessary, we complete the proof. $\square$

From now on we follow the idea of Foias, Guillopé, and Temam [4].

LEMMA 4.4. There is an open set $\mathcal{O} \subset [0, T]$ such that

$$|[0, T] \setminus \mathcal{O}| = 0$$

and

$$\int |\nabla u|^2 dx(t) < \infty \quad \text{for all } t \in \mathcal{O}.$$

Proof. Taking u as a test function to (2.4) and considering Lemma 3.2, we have an energy inequality

$$\sup_{0 \leq t \leq T} \int |u|^2 dx + \int_0^T \int |\nabla u|^2 dx dt \leq \int |u|^2 dx(0)$$

for any measure-valued solution. Since $\int |\nabla u|^2 dx(t)$ is integrable, $\int |\nabla u|^2 dx(t) < \infty$ for almost every $t \in [0, T]$. If $\int |\nabla u|^2 dx(t_0) < \infty$, then there is $\delta = \delta(\int |\nabla u|^2 dx(t_0)) > 0$

such that the weakly strong solution v to the non-Newtonian flow problem

$$\begin{cases} L(v) = f & \text{in } (t_0, t_0 + \delta), \\ v(t_0) = u(t_0) \end{cases}$$

exists. Moreover, from Lemma 4.3, our measure-valued solution $u \equiv v$ on $(t_0, t_0 + \delta)$ and thus u is weakly strong on $(t_0, t_0 + \delta)$.

We let $\mathcal{O}_1 = \{t: \int |\nabla u|^2 dx(t) < \infty\}$. Then for each $t \in \mathcal{O}_1$, there is a $\delta = \delta(t)$ such that $[t, t + \delta) \in \mathcal{O}_1$, that is, $\mathcal{O}_1$ is right semi-open. From elementary arguments we see that

$$\mathcal{O}_1 = \bigcup_{i=1}^{\infty} (a_i, b_i) \cup \bigcup_{j=1}^{\infty} \{c_j\},$$

where c_i is a left end point of an open interval (a_i, b_i) . If we set $\Sigma = \mathbb{R}^+ \setminus \mathcal{O}_1$, then $\Sigma = \{t: \int |\nabla u|^2 dx(t) = \infty\}$. Since $\int |\nabla u|^2 dx(t)$ is integrable, $|\Sigma| = 0$. Therefore, setting

$$\mathcal{O} = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

we conclude the proof. $\square$

THEOREM 4.5. Suppose that the energy convexity (4.1) for a measure-valued solution $(u, \nu_{x,t})$ holds. Then the measure-valued solution is a weak solution for all $r > -1$.

Proof. Let $\varepsilon > 0$ be fixed and let

$$\mathcal{O}_\varepsilon = \bigcup_{i=1}^{\infty} (a_i + 2^{-i}\varepsilon, b_i - 2^{-i}\varepsilon).$$

Then $|\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon| = \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon$. From Lemma 4.4 our measure-valued solution $(u, \nu_{x,t})$ is weakly strong in each interval $(a_i + 2^{-i}\varepsilon, b_i - 2^{-i}\varepsilon)$. Then for each $\phi \in C_0^\infty(\Omega \times \mathbb{R}^+)$ with $\operatorname{div} \phi = 0$, we have

$$\begin{aligned} 0 &= \int \int u \phi_t + (u \otimes u : \nabla \phi) + \mu_0 E(\nabla u) \cdot E(\nabla \phi) \\ &\quad + \mu_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \cdot E(\nabla \phi) dx dt \\ &= \int_{Q_\varepsilon} \int_{\Omega} \cdots + \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int_{\Omega} \cdots \\ &= \int_{Q_\varepsilon} \int_{\Omega} u \phi_t + (u \otimes u : \nabla \phi) + \mu_0 E(\nabla u) \cdot E(\nabla \phi) \\ &\quad + \mu_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \cdot E(\nabla \phi) \\ &\quad + \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int_{\Omega} u \phi_t + (u \otimes u : \nabla \phi) + \mu_0 E(\nabla u) \cdot E(\nabla \phi) \\ &\quad + \mu_1 \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int_{\Omega} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \cdot E(\nabla \phi) dx dt. \end{aligned}$$

Since $u \in L^2((0, \infty) \times \Omega)$ and $\nabla u \in L^2((0, \infty) \times \Omega)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int u \phi_t + (u \otimes u : \nabla \phi) + \mu_0 E(\nabla u) \cdot E(\nabla \phi) dx dt = 0.$$

Moreover, since each Galerkin approximate solution u^m satisfies

$$\int \int (|\nabla u^m|^{r+1})^{\frac{r+2}{r+1}} dx dt < C$$

independent of m , we have the weak limit

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \in L^{\frac{r+2}{r+1}}((0, \infty) \times \Omega).$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int_\Omega |E(\lambda)|^r E(\lambda) d\nu_{x,t}(\lambda) \cdot E(\nabla \phi) dx dt = 0$$

and

$$\begin{aligned} \int_0^\infty \int_\Omega u \phi_t + (u \otimes u : \nabla \phi) + \mu_0 \Gamma_0(\nabla u, \nabla \phi) \\ + \mu_1 \Gamma_1(u, u) dx dt = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^+ \setminus \mathcal{O}_\varepsilon} \int \cdots = 0. \quad \square \end{aligned}$$

5. Regularity. In this section we consider a regularity question on the periodic boundary value problem. We consider the equations (2.1) on $\Omega \times [0, T]$, where $\Omega = [0, 1]^3$. We impose the periodic boundary condition on this equation. The existence result does work for the equations with the periodic boundary condition. Bellout, Bloom, and Nečas [1] showed the regularity for the space periodic problem with

$$\Gamma_{ij} = \mu_0(1 + |E|^r)E_{ij} - 2\mu_2 \Delta E_{ij}$$

for $\mu_0 = \mu_1 > 0$, $\mu_2 \geq 0$ such that for $r > 1/5$,

$$u \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^{r+2}(0, T; W^{1,3(r+2)}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)).$$

Since the second derivative of the solution u does not belong to $L^{r+2}(Q_T)$, that is, $u \notin L^{r+2}(0, T; W^{2,r+2}(\Omega))$, we cannot expect that the weak solution u is strong.

We prove a similar regularity result to Bellout, Bloom, and Nečas [1] for the space periodic problem (2.1) for $\mu_0 \geq 0$ and $\mu_1 > 0$. But the method of proof is quite different and we even allow the case $\mu_0 = 0$. For the case $\mu_0 = 0$, we will obtain that

$$u \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^{r+2}(0, T; W^{1,3(r+2)}(\Omega)).$$

With the same reasoning, it is difficult to say that the weak solution u is strong.

We now obtain our regularity. We assume that our weak solutions are constructed from Galerkin approximated solutions. Since the approximate Galerkin solutions u^m are smooth, we can differentiate as many times as possible. Therefore, keeping the approximate Galerkin solutions in mind, we differentiate (2.1) with respect to x_k to get

$$\partial_t \partial_k u_i - \mu_0 \partial_k \partial_j E_{ij}(\nabla u) - \mu_1 \partial_j \partial_k (|E(\nabla u)|^r E_{ij}(\nabla u)) + \partial_i \partial_k p = \partial_k f_i - \partial_k (u_j \partial_j u_i).$$

Taking $\partial_k u_i$ as a test function and integrating with respect to x and t , we obtain by (4.2) and Lemma 4.1 that

$$\begin{aligned} & \int_{\Omega} |\nabla u(t)|^2 dx + C\mu_0 \int \int |\nabla^2 u|^2 dx dt + C\mu_1 \int \|\nabla u\|_{3(r+2)}^{r+2} dt \\ & \leq \int \int |\partial_k u_j \partial_j u_i \partial_k u_i| dx dt + C \int_0^T \left(\int_{\Omega} |\nabla f|^{\alpha} \right)^{1/\beta} dt + \int |\nabla u(0)|^2 dx, \end{aligned} \quad (5.1)$$

where

$$\alpha = \frac{3(r+2)}{3r+5}, \quad \beta = \frac{3(r+1)}{3r+5}.$$

Here, the fact that

$$\int \partial_i \partial_k p \partial_k u_i dx = - \int \partial_k p \partial_k \partial_i u_i dx = 0,$$

and

$$\begin{aligned} \int \int_{Q_T} |\nabla f| |\nabla u| & \leq \int_0^T \left(\int_{\Omega} |\nabla u|^{3(r+2)} dx \right)^{1/(3r+6)} \left(\int_{\Omega} |\nabla f|^{\alpha} dx \right)^{1/\alpha} dt \\ & \leq \varepsilon \int_0^T \left(\int_{\Omega} |\nabla u|^{3(r+2)} dx \right)^{1/3} dt + \varepsilon^{-1} \int_0^T \left(\int_{\Omega} |\nabla f|^{\alpha} dx \right)^{1/\beta} dt \end{aligned}$$

is used. Let C' denote the last two terms on the right side of (5.1). For $1 > r > -1$, by Hölder's inequality, we have

$$\begin{aligned} \int \int |\nabla u|^3 dx dt & \leq \int \int |\nabla u|^{1-r} |\nabla u|^{\alpha_1} |\nabla u|^{\alpha_2} dx dt \\ & \leq \int \left(\int |\nabla u|^2 dx \right)^{(1-r)/2} \left(\int |\nabla u|^{r+2} dx \right)^{(3r+1)/4} \\ & \quad \times \left(\int |\nabla u|^{3(r+2)} dx \right)^{(1-r)/4} dt, \end{aligned} \quad (5.2)$$

where $0 \leq \alpha_1 < 3$ and $0 \leq \alpha_2 < 3$,

$$\alpha_1 = \frac{(3r+1)(r+2)}{4}, \quad \alpha_2 = \frac{3(r+2)(1-r)}{4}.$$

Again by Hölder's inequality, we have

$$\begin{aligned}
 & \int \int_{Q_T} |\nabla u|^3 dx dt \\
 & \leq \sup_t \left(\int |\nabla u|^2 dx \right)^{(1-r)/2} \\
 & \quad \times \int \left(\int |\nabla u|^{r+2} dx \right)^{(3r+1)/4} \left(\int |\nabla u|^{3(r+2)} dx \right)^{(1-r)/4} dt \\
 & \leq \sup_t \left(\int |\nabla u|^2 dx \right)^{(1-r)/2} \\
 & \quad \times \left(\int \int |\nabla u|^{r+2} dx dt \right)^{(3r+1)/4} \left(\int \left(\int |\nabla u|^{3(r+2)} dx \right)^{1/3} dt \right)^{3(1-r)/4}.
 \end{aligned}$$

Since we already know that

$$\int \int |\nabla u|^{r+2} dx dt \leq C,$$

we have

$$\int \int |\nabla u|^3 dx dt \leq C \sup \left(\int |\nabla u|^2 dx \right)^{(1-r)/2} \left(\int \|\nabla u\|_{3(r+2)}^{r+2} dt \right)^{3(1-r)/4}.$$

By Young's inequality, we have

$$\int \int |\nabla u|^3 dx dt \leq C \sup \left(\int |\nabla u|^2 dx \right)^{5(1-r)/4} + C \left(\int \|\nabla u\|_{3(r+2)}^{r+2} dt \right)^{5(1-r)/4}.$$

Thus, if we denote by J ,

$$J = \sup_{0 \leq t \leq T} \|\nabla u\|^2 + \int_0^T \left(\int_{\Omega} |\nabla u|^{3(r+2)} dx \right)^{1/3} dt,$$

then we have

$$\int \int_{Q_T} |\nabla u|^3 dx dt \leq C J^{5(1-r)/4}. \quad (5.3)$$

Hence, by combining (5.1) and (5.3), we have by Young's inequality that for $1 > r > 1/5$,

$$\begin{aligned}
 & \int_{\Omega} |\nabla u(t)|^2 dx + C\mu_0 \int \int |\nabla^2 u|^2 dx dt + C \int \|\nabla u\|_{3(r+2)}^{r+2} dt \\
 & \leq C J^{5(1-r)/4} + C' \leq C + \varepsilon J + C'.
 \end{aligned}$$

Therefore, if we take ε small, then we have

$$\int_{\Omega} |\nabla u(t)|^2 dx + C\mu_0 \int \int |\nabla^2 u|^2 dx dt + C \int \|\nabla u\|_{3(r+2)}^{r+2} dt \leq C + C'.$$

On the other hand, the estimate of J for the case $r \geq 1$ follows from the fact that

$$\int \int_{Q_T} |\partial_k u_i \partial_i u_j \partial_j u_k| dx dt \leq C \left(\int \int_{Q_T} |\nabla u|^{r+2} dx dt \right)^{3/(r+2)} \leq C.$$

Finally, we have our regularity theorem for $\mu_0 \geq 0$.

THEOREM 5.1. Let $\mu_0 \geq 0$ and $\mu_1 > 0$. Suppose that u is a weak solution to (2.4). If r satisfies

$$r > \frac{1}{5},$$

then we have

$$\begin{aligned} \sup_t \int |\nabla u|^2 dx + C\mu_0 \int \int |\nabla^2 u|^2 dx dt + C \int \|\nabla u\|_{3(r+2)}^{r+2} dt \\ \leq \int |\nabla u_0|^2 dx + C \int_0^t \left(\int_{\Omega} |\nabla f|^\alpha \right)^{1/\beta} dt + C \end{aligned}$$

for some C depending only on r , where

$$\alpha = \frac{3(r+2)}{3r+5}, \quad \beta = \frac{3(r+1)}{3r+5}.$$

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