SLOW DECAY IN LINEAR THERMOELASTICITY

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1. Summary. Energy estimates show that linearized thermoelasticity defines a contraction semigroup on a Hilbert space. We show that under a geometric condition this contraction is not strict, or, more precisely, the norm of the semigroup is 1 for all $t \geq 0$. Convex domains always satisfy the geometric condition.

2. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ . We consider the problem

$$\begin{aligned} \rho \partial_{tt}^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) + \alpha \nabla \theta &= 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ \rho c_p \partial_t \theta - \kappa \Delta \theta + \alpha \partial_t \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \times \mathbb{R}^+, \end{aligned} \quad (2.1)$$

where $\mathbf{u}$ is a time-dependent deformation vector field, θ is the normalized temperature, ρ, μ , and c_p are positive constants, $2\mu + \lambda > 0$ (ρ the density, μ and λ are Lamé's constants, and c_p is the specific heat), α is the coupling constant between pressure waves and heat, $\nabla(\nabla \cdot \mathbf{u}) = \text{grad div } \mathbf{u}$, and $\Delta \mathbf{u}$ denotes the component-wise Laplacian. This system is complemented by initial conditions and the boundary condition

$$\begin{aligned} \mathbf{u} &= 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ \theta &= 0 \quad \text{on } \Gamma \times \mathbb{R}^+. \end{aligned} \quad (2.2)$$

Dafermos [2] showed that (2.1)–(2.2) defines a contraction semigroup

$$\begin{aligned} (H_0^1)^n \times (L^2)^n \times L^2 \ni (\mathbf{u}_0, \mathbf{u}_1, \theta_0) \\ \rightarrow S(t)(\mathbf{u}_0, \mathbf{u}_1, \theta_0) = (\mathbf{u}(t), \partial_t \mathbf{u}(t), \theta(t)) \in (H_0^1)^n \times (L^2)^n \times L^2 \end{aligned} \quad (2.3)$$

where $(\mathbf{u}, \theta)$ is the solution with initial values

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \mathbf{u}_t(\cdot, 0) = \mathbf{u}_1, \quad \theta(\cdot, 0) = \theta_0.$$

He also showed that

$$(\mathbf{u}, \partial_t \mathbf{u}, \theta) \rightarrow (0, 0, 0) \quad \text{as } t \rightarrow \infty \quad \text{in } (H_0^1)^n \times (L^2)^n \times L^2$$

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if there is no nontrivial periodic solution to (2.1)–(2.2), which then necessarily satisfies $\theta = 0$ and $\nabla \cdot \mathbf{u} = 0$ and the existence of which is equivalent to the existence of a nontrivial eigenfunction that satisfies

$$\begin{aligned}\Delta \mathbf{u} &= \lambda \mathbf{u} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega.\end{aligned}\tag{2.4}$$

There are infinitely many linearly independent solutions in balls; see [2]. This is an overdetermined system which, for generic domains, admits only trivial solutions, and hence it is natural to ask whether the decay is uniform, at least for generic domains. This question has been answered affirmatively for the case $n = 1$ by Henry, Lopes, and Perissinotto [1] and Muñoz Rivera [7]. Moreover, it has been shown (Jiang, Muñoz Rivera, and Racke [6]) that radially symmetric solutions to radially symmetric problems decay uniformly for $n \geq 2$.

For $n \geq 3$ and very special domains, Henry [3] showed that $|S(t)| = 1$, i.e., there is no uniform decay. He conjectures this to be true for arbitrary convex domains. This note confirms his conjecture for $n \geq 2$.

2.1. Assumption (A). *There exists a two-periodic orbit of the billiard in Ω .*

We recall that a (regular) orbit of the billiard consists of lines that are reflected transversally at the boundary. A two-periodic orbit consists of a single line that intersects the boundary perpendicularly. Such an orbit clearly exists if the boundary is convex. There is an example of a two-dimensional domain without two-periodic orbits, which is due to D. Hulin; see [5].

2.2. Theorem. *Suppose that (A) holds. Then $|S(t)| = 1$ for all $t \geq 0$.*

2.3. Remark. The statement of the theorem is incomplete: It is the operator norm with respect to an energy which will be defined in the next section.

2.4. Corollary (Slow decay). *Let $\omega(t)$ be a positive function with $\omega(t) \rightarrow 0$ as $t \rightarrow \infty$. Then there exists a solution with*

$$\limsup_{t \rightarrow \infty} \omega^{-1}(t) \|S(t)[\mathbf{u}_0, \mathbf{u}_1, \theta_0]\|_E \rightarrow \infty.$$

Proof. This is an immediate consequence of Theorem 2.2 and the uniform boundedness principle. $\square$

Henry [3] proved Theorem 2.2 in the special case that the two-periodic orbit hits the boundary at flat spots and $n \geq 3$. Minor changes extend his result to $n = 2$ and the case that the second fundamental form of the boundary is zero where the two-periodic orbit hits the boundary. The general case cannot be settled by this ingenious and simple argument.

Dissipation occurs through the temperature, which is coupled to changes in the density, hence to pressure waves. Shear waves that reflect perpendicularly transfer little energy to pressure waves. Hence they are only weakly damped. Theorem 2.2 is proved by constructing shear waves whose energy is concentrated near the two-periodic orbit

of the billiard. This is reduced to a similar problem for the scalar wave equation. The construction requires standard propagation of singularities results for broken bicharacteristics.

It is worth observing that propagation of singularities for systems is not sufficient for our purpose. It is very likely that, in general, singularities in the shear wave lead to singularities in the pressure wave even when the reflection is perpendicular, in contrast to what happens with plane wave solutions. Let us be more precise.

The interesting and beautiful idea of Lopes, Henry, and Perissinotto [1] reduces the problem of exponential decay to a hyperbolic system with dissipation for pressure waves. A different proof for this can be found in Zuazua [11].

Let $P \in \Gamma$. It follows from Taylor [9] that there is no micro-locally incoming energy singularity if there is no outgoing energy singularity. We refer to [9] for the notation. A stronger statement would imply the desired conclusion:

There is no singularity in the perpendicularly incoming pressure wave if there is no perpendicularly outgoing singularity in the pressure wave.

This is presumably false in general, but it is likely to be true for energy singularities, i.e., the incoming pressure singularity is weaker than the outgoing singularity if the reflection is perpendicular and if there is no singularity in the outgoing pressure wave.

Theorem 2.2 has been announced independently by Lebeau and Zuazua, [5]. They work directly with the system. In contrast we pursue the simpler approach of Henry, complemented by the basic results about propagation of singularities, i.e., propagation of singularities in the interior and for transversally reflected waves at the boundary for scalar equations.

Our arguments can be modified for nonhomogeneous material or other boundary conditions. For reasons of simplicity we only consider Dirichlet boundary conditions.

Notation. Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary Γ . Let ν be the outer unit normal. The cylinders $(0, T) \times \Omega$ and $(0, T) \times \Gamma$ are denoted by Ω_T and Γ_T . Similarly, if J is an interval, we define the cylinders Ω_J and Γ_J . For vectors and vector fields we use bold letters. $\mathbf{A} \cdot \mathbf{B}$ is the standard inner product. This notation is extended to differential operators: $\nabla \cdot \mathbf{u}$ is the divergence of $\mathbf{u}$. We slightly abuse the notation by understanding by $\nabla \mathbf{u}$ and $\Delta \mathbf{u}$ the component-wise gradient, resp. Laplacian. We will suppress arguments when they are clear from the context. For example, $\mathbf{u}(t)$ is the vector field $x \rightarrow \mathbf{u}(t, x)$. Differentiation with respect to time is sometimes denoted by a dot. The tangential gradient with respect to Γ is denoted by ∇_Γ and the one with respect to $\Gamma_{\mathbb{R}}$ by $\nabla_{\Gamma_{\mathbb{R}}}$.

Generic constants are denoted by c , which may change from line to line. We write $c(\varepsilon)$, for example, if we want to stress that it depends on parameters.

We omit the exponent n in the notation of spaces like $(H^1)^n$ where no confusion should arise.

3. Proof of Theorem 2.2. We define the energy norm

$$\|(\mathbf{u}_0, \mathbf{u}_1, \theta_0)\|_E^2 = \frac{1}{2} \int_{\Omega} \mu |\nabla_0|^2 + (\mu + \lambda) |\nabla \cdot \mathbf{u}_0|^2 + \rho |\mathbf{u}_1|^2 + c_p \rho |\theta|^2 dx.$$

Here $|\nabla \mathbf{u}| = \sqrt{\sum (\partial_i u^j)^2}$. An energy estimate shows that this energy is nonincreasing. Theorem 2.2 is a consequence of the following equivalent statement.

3.1. Proposition. *Suppose that Assumption (A) holds. Let $\varepsilon > 0$ and $t > 0$. There exists a nontrivial solution $\mathbf{u}$ that satisfies*

$$\|(\mathbf{u}(t), \mathbf{u}_t(t), \theta(t))\|_E \geq (1 - \varepsilon) \|(\mathbf{u}(0), \mathbf{u}_t(0), \theta(0))\|_E.$$

Proposition 3.1 will be a simple consequence of the following two lemmata. Consider the inhomogeneous problem

$$\begin{aligned} \rho \partial_{tt}^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) + \alpha \nabla \theta &= 0 && \text{in } \Omega_{\mathbb{R}^+}, \\ \rho c_p \dot{\theta} - \kappa \Delta \theta + \alpha \partial_t \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega_{\mathbb{R}^+}, \\ \mathbf{u} &= \phi && \text{on } \Gamma_{\mathbb{R}^+}, \\ \theta &= 0 && \text{on } \Gamma_{\mathbb{R}^+}, \\ \mathbf{u}(0) &= \mathbf{u}_0, \\ \mathbf{u}_t(0) &= \mathbf{u}_1, \\ \theta(0) &= \theta_0, \end{aligned} \tag{3.1}$$

where $\mathbf{u}_0 \in H^1(\Omega)$, $\mathbf{u}_1 \in L^2(\Omega)$, $\theta \in L^2(\Omega)$, $\phi \in H^1(\Gamma_T)$, and $\mathbf{u}_0|_{\Gamma} = \phi|_{t=0}$.

3.2. Lemma. *There exists a unique solution to problem (3.1) and a constant c such that*

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \|\mathbf{u}(\tau), \mathbf{u}_t(\tau), \theta(\tau)\|_E^2 + \frac{1}{1+t} \|\partial_\nu \mathbf{u}\|_{L^2(\Omega_t)}^2 \\ \leq c \|\mathbf{u}(0, \cdot), \mathbf{u}_t(0, \cdot), \theta(0, \cdot)\|_E^2 + c(1+t) \|\phi\|_{H^1(\Gamma_t)}^2. \end{aligned}$$

Now suppose that (A) holds. Rotating, translating, and scaling our coordinates, we may and do assume that there is a vertical two-periodic orbit of the billiard that hits the boundary at $P = (0, \dots, 1)$ and $Q = (0, \dots, -1)$.

3.3. Lemma. *Let $\varepsilon > 0$ and $T > 0$. There exists a closed infinite-dimensional subspace $V \subset C(0, T; H^2) \cap C^1(0, T; H^1)$ of functions v satisfying $\partial_{tt} v - \mu \Delta v = 0$, $\partial_\nu v = 0$ at Γ_T and*

$$\begin{aligned} \|\partial_1 v\|_{H^1(\Gamma_T)}^2 + \|\partial_n v\|_{H^1(\Gamma_T)}^2 \\ \leq \varepsilon (\|\partial_n v(0)\|_{H^1}^2 + \|\partial_1 v(0)\|_{H^1}^2 + \|\partial_{tn}^2 v(0)\|_{L^2}^2 + \|\partial_{1t} v(0)\|_{L^2}^2). \end{aligned} \tag{3.2}$$

Proof of Proposition 3.1. We choose $v \in V$ with V as in Lemma 3.3 (after some affine changes of coordinates). We make the ansatz $\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^1$ where

$$\mathbf{u}^0(t, x) = (\partial_n v(t, x), 0, \dots, 0, -\partial_1 v(t, x)).$$

If

$$\begin{aligned}
 \rho \partial_{tt}^2 \mathbf{u}^1 - \mu \Delta \mathbf{u}^1 - (\mu + \lambda) \nabla \nabla \cdot \mathbf{u}^1 + \alpha \nabla \theta &= 0 && \text{in } \Omega_{\mathbb{R}^+}, \\
 \rho c_p \dot{\theta} - \kappa \Delta \theta + \alpha \partial_t \nabla \cdot \mathbf{u}^1 &= 0 && \text{in } \Omega_{\mathbb{R}^+}, \\
 \mathbf{u}^1 &= -\mathbf{u}^0 && \text{on } \Gamma_{\mathbb{R}^+}, \\
 \theta &= 0 && \text{on } \Gamma_{\mathbb{R}^+}, \\
 \mathbf{u}^1(0) &= 0, \\
 \mathbf{u}_t^1(0) &= 0, \\
 \theta(0) &= 0,
 \end{aligned} \tag{3.3}$$

then $\mathbf{u}$ satisfies (2.1) with initial data $(\mathbf{u}^0(0), \partial_t \mathbf{u}^0(0), 0)$. Now, by Lemma 3.2,

$$\begin{aligned}
 \|\mathbf{u}^1(t), \mathbf{u}_t^1(t), \theta(t)\|_E^2 &\leq c(1+t) \|\mathbf{u}^0|_{\Gamma_t}\|_{H^1(\Gamma_t)}^2 \\
 &\leq c\varepsilon(1+t) (\|\mathbf{u}^0(0)\|_{H^1}^2 + \|\mathbf{u}_t^0(0)\|_{L^2}^2)
 \end{aligned}$$

where the second inequality is a consequence of Lemma 3.3. Thus

$$\|(\mathbf{u}(t), \mathbf{u}_t(t), \theta(t))\|_E \geq (1 - \sqrt{c\varepsilon(1+t)}) \|(\mathbf{u}(0), \mathbf{u}_t(0), \theta(0))\|_E,$$

which completes the proof by choosing ε properly. $\square$

4. Proof of the lemmata.

Proof of Lemma 3.2. Suppose that $(\mathbf{u}, \theta)$ satisfy (3.1). Then

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \mu |\nabla \mathbf{u}|^2 + (\mu + \lambda) (\nabla \cdot \mathbf{u})^2 + \rho \mathbf{u}_t^2 dx \right) \\
 = \int_{\Omega} -\alpha \nabla \theta \cdot \mathbf{u}_t - \int_{\Gamma} \mu \partial_{\nu} \mathbf{u} \cdot \phi_t + (\mu + \lambda) \nabla \cdot \mathbf{u} \nu \cdot \phi_t
 \end{aligned} \tag{4.1}$$

and

$$\frac{\rho c_p}{2} \partial_t \left(\int_{\Omega} \theta^2 \right) + \kappa \int_{\Omega} |\nabla \theta|^2 = \alpha \int_{\Omega} \nabla \theta \cdot \mathbf{u}_t. \tag{4.2}$$

Hence

$$\begin{aligned}
 \frac{d}{dt} \|\mathbf{u}(t), \mathbf{u}_t(t), \theta(t)\|_E^2 &= -\kappa \|\nabla \theta\|_2^2 \\
 &\quad - \int_{\Gamma} \mu \partial_{\nu} \mathbf{u} \cdot \phi_t + (\mu + \lambda) \nabla \cdot \mathbf{u} \nu \cdot \phi_t.
 \end{aligned} \tag{4.3}$$

Let $\mathbf{X} = (\xi_i)_{1 \leq i \leq n}$ be a smooth vector field that is equal to the outer unit normal ν at the boundary. Multiplying (2.1) by $\mathbf{u}_\mathbf{X}$ and integrating we get

$$\begin{aligned}
 0 &= \int_{\Omega} (\rho \mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \nabla \cdot \mathbf{u} - \alpha \nabla \theta) \cdot \mathbf{u}_\mathbf{X} \\
 &= \frac{d}{dt} \rho \int_{\Omega} \mathbf{u}_t \cdot \mathbf{u}_\mathbf{X} + \frac{\rho}{2} \int_{\Omega} (\nabla \cdot \mathbf{X}) |\mathbf{u}_t|^2 - \frac{\rho}{2} \int_{\Gamma} |\mathbf{u}_t|^2 \\
 &\quad + \mu \int_{\Omega} \sum_{i,j,k} \partial_i u^j (\partial_i \xi_k) \partial_k u^j - \frac{\mu}{2} \int_{\Omega} (\nabla \cdot \mathbf{X}) |\nabla \mathbf{u}|^2 - \mu \int_{\Gamma} |\partial_\nu \mathbf{u}|^2 + \frac{\mu}{2} \int_{\Gamma} |\nabla \mathbf{u}|^2 \\
 &\quad - (\mu + \lambda) \int_{\Gamma} \nabla \cdot \mathbf{u} \nu \partial_\nu \mathbf{u} + \frac{\mu + \lambda}{2} \int_{\Gamma} (\nabla \cdot \mathbf{u})^2 - \frac{\mu + \lambda}{2} \int_{\Omega} (\nabla \cdot \mathbf{X}) (\nabla \cdot \mathbf{u})^2 \\
 &\quad + (\mu + \lambda) \int_{\Omega} \nabla \cdot \mathbf{u} \sum_{i,j} (\partial_i \xi_j) \partial_j u^i - \alpha \int_{\Omega} \nabla \theta \cdot \mathbf{u}_\mathbf{X}.
 \end{aligned} \tag{4.4}$$

Hence, with all terms evaluated at time t and a small constant ε ,

$$\begin{aligned}
 \frac{\mu - \varepsilon}{2} \int_{\Gamma} |\partial_\nu \mathbf{u}|^2 + \frac{\rho}{2} \int_{\Gamma} |\partial_t \phi|^2 + \frac{\mu + \lambda}{2} \int_{\Gamma} ((\nu \cdot \partial_\nu \mathbf{u}))^2 + \rho \frac{d}{dt} \int_{\Omega} \mathbf{u}_t \mathbf{u}_\mathbf{X} \\
 \leq c(\varepsilon) (\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \|\mathbf{u}_t\|_{L^2(\Omega)}^2 + \|\nabla \theta\|_{L^2(\Omega)}^2 + \|\phi\|_{H^1(\Gamma)}^2)
 \end{aligned}$$

and hence, again with everything evaluated at time t ,

$$\begin{aligned}
 \|\partial_\nu \mathbf{u}(t)\|_{L^2(\Gamma)}^2 + \frac{2\rho}{\mu - \varepsilon} \frac{d}{dt} \int_{\Omega} \mathbf{u}_t \mathbf{u}_\mathbf{X} \\
 \leq c(\|(\mathbf{u}(t), \mathbf{u}_t(t), \theta(t))\|_E^2 + \|\phi\|_{H^1(\Gamma)}^2 + \|\nabla \theta\|_2^2).
 \end{aligned} \tag{4.5}$$

Thus, using (4.3) and (4.5),

$$\begin{aligned}
 \frac{d}{dt} \left(e^{-kt} \left(\|\mathbf{u}, \mathbf{u}_t, \theta\|_E^2 + \frac{2\delta}{\mu - \varepsilon} \int_{\Omega} \mathbf{u}_t \mathbf{u}_\mathbf{X} \right) \right) + \frac{\delta}{2} e^{-kt} \|\partial_\nu \mathbf{u}\|_{L^2(\Gamma)}^2 \\
 \leq e^{-kt} \left(-k \|\mathbf{u}, \mathbf{u}_t, \theta\|_E^2 - \frac{2k\delta}{\mu - \varepsilon} \int_{\Omega} \mathbf{u}_t \mathbf{u}_\mathbf{X} \right. \\
 \quad \left. - (\kappa - k\delta) \int_{\Omega} |\nabla \theta|^2 - \frac{\delta}{2} \int_{\Gamma} |\partial_\nu \mathbf{u}|^2 - \int_{\Gamma} \mu \partial_\nu \mathbf{u} \cdot \phi_t + (\mu + \lambda) \nabla \cdot \mathbf{u} \nu \cdot \phi_t \right. \\
 \quad \left. + c\delta (\|(\mathbf{u}, \mathbf{u}_t, \theta)\|_E^2 + \|\phi\|_{H^1(\Gamma)}^2) \right) \\
 \leq c e^{-kt} (\|\phi(t)\|_{H^1(\Gamma)}^2 + \|\partial_t \phi(t)\|_{L^2(\Gamma)}^2)
 \end{aligned} \tag{4.6}$$

provided that k is large and δ is small. Integration over $(0, T)$ implies Lemma 3.2, up to a justification of the t -dependence of the constants.

Arguments of this type have been used intensively. They seem to go back to Rellich [8].

The control of the t -dependence is interesting, but not needed for other results in this paper. We may assume that t is a positive integer. Let $\psi \in C_0^\infty([-1, 1])$ satisfy

$$\sum_{k \in \mathbb{Z}} \psi(t + k) = 1$$

for all t . We set $\phi_i(\tau, x) = \psi(\tau - i)\phi(\tau, x)$. Then $\mathbf{u} = \sum \mathbf{u}_i$ where $\mathbf{u}_i$ solves the inhomogeneous problem (3.1) with data ϕ_i for $i > 0$, resp. ϕ_0 and initial data for $i = 0$. The functions $\mathbf{u}_i$ are estimated in $i - 1 \leq \tau \leq i + 1$ by (4.6):

$$\sup_{i-1 \leq \tau \leq i+1} \|(\mathbf{u}_i, \partial_t \mathbf{u}_i, \theta_i)\|_E^2 + \|\partial_\nu \mathbf{u}_i\|_{L^2(\Gamma_{(i-1, i+1)})}^2 \leq c \int_{i-1}^{i+1} \|\nabla_{\Gamma_{\mathbb{R}}} \phi_i\|_{L^2(\Gamma)}^2$$

and, by the usual energy estimate, also

$$\sup_{0 \leq \tau \leq t} \|(\mathbf{u}_i, \partial_t \mathbf{u}_i, \theta_i)\|_E^2 + \delta \|\partial_\nu \mathbf{u}_i\|_{L^2(\Gamma_{(i-1, i+1)})}^2 \leq c \int_{i-1}^{i+1} \|\nabla_{\Gamma_{\mathbb{R}}} \phi_i\|_{L^2(\Gamma)}^2.$$

Now we use again the integrated form of (4.6) to conclude

$$\|\partial_\nu \mathbf{u}_i\|_{L^2(\Gamma_{j, j+1})}^2 \leq c \|(\mathbf{u}(j), \mathbf{u}_t(j), \theta(j))\|_E^2 \leq c \int_{i-1}^{i+1} \|\nabla_{\Gamma_{\mathbb{R}}} \phi_i\|_{L^2(\Gamma)}^2 \quad (4.7)$$

for $i + 1 \leq j \leq t - 1$. This implies

$$\sup_{0 \leq \tau \leq t} \|(\mathbf{u}_i, \partial_t \mathbf{u}_i, \theta_i)\|_E^2 + \frac{c}{1+t} \|\partial_\nu \mathbf{u}_i\|_{L^2(\Gamma)}^2 \leq c \|\phi_i\|_{H^1(\Gamma_t)}^2.$$

We sum with respect to i on both sides and add the solution with initial data $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$ and homogeneous boundary data:

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \|(\mathbf{u}, \partial_t \mathbf{u}, \theta)\|_E^2 &\leq \sup_{0 \leq \tau \leq t} \left(\sum_i \|(\mathbf{u}_i, \partial_t \mathbf{u}_i, \theta_i)\|_E \right)^2 \\ &\leq c \left(\|(\mathbf{u}(0), \mathbf{u}_t(0), \theta(0))\|_E^2 + \sum_{i=0}^t \|\phi\|_{H^1(\Gamma_{(i, i+1)})}^2 \right)^2 \\ &\leq c(1+t) \|\phi\|_{H^1(\Gamma_t)}^2 + c \|(\mathbf{u}(0), \mathbf{u}_t(0), \theta(0))\|_E^2. \end{aligned} \quad (4.8)$$

The estimate of the normal derivative is obtained in exactly the same way. This gives the desired a priori estimate assuming that $\mathbf{u}$ is sufficiently regular. On the other hand, it is not difficult to obtain an existence result using the a priori estimates. $\square$

Proof of Lemma 3.3. We begin with constructing the main objects. Then the strategy of the proof is explained before we turn to details. We may and do assume for notational simplicity that $\mu = 1$. It suffices to prove Lemma 3.3 for complex-valued functions. Taking real parts in the construction below implies then the result for real-valued functions.

We recall that the boundary is given as the graph of two functions $\psi_\pm$ in a neighborhood of $(0, \dots, 1)$, resp. $(0, \dots, -1)$, and that the gradient of $\psi_\pm$ is zero at 0. The proof of Henry [3] could be generalized, with small changes, to the case when the second derivative of $\psi_\pm$ vanishes at zero.

This assumption could also be avoided by much more work, using the usual asymptotic expansion for high frequency solutions. We choose, however, a different, less elementary, but, as we hope, more transparent proof.

We consider the problem

$$\begin{aligned} v_{tt} - \Delta v &= 0 \quad \text{in } \Omega, \\ \partial_\nu v &= 0 \quad \text{on } \Gamma. \end{aligned} \quad (4.9)$$

The wave front set of v is a conical set in the cotangent space. We identify a ray in the cotangent space with any of its points. Suppose that we have a solution v with wave front set contained in $((t; 0, \dots, 0, t), \pm(e_0 - e_n))$ for $-\delta < t < \delta$ where e_0 is the unit cotangent vector pointing into the t -direction and e_n is the cotangent vector pointing into the x_n -direction. Then the wave front set of the solution is contained in the broken bicharacteristic γ starting at $((0, 0), \pm(e_0 - e_n))$. The wave front set S of the restriction to the boundary of v is contained in the union of the points

$$((4k+1, 0, \dots, 1), (\pm 1, 0, \dots, 0)), \quad ((4k-1, 0, \dots, -1), (\pm 1, 0, \dots, 0)),$$

where $k \in \mathbb{Z}$, the projection of the restriction of the wavefront set in $T_{\partial\Omega}^* \mathbb{R}^{n+1}$ to the cotangent space of $\partial\Omega \times \mathbb{R}$ along the conormal vector. This follows, for example, from Theorem 24.2.1 of [4]. Let Q be the singular support of $v|_{\Gamma_{\mathbb{R}}}$. It is contained in the union of the points $(4k+1, 0, \dots, 1)$ and $(4k-1, 0, \dots, 1)$.

The strategy is the following. We construct a closed subspace of $H^2(\Omega_T)$ of functions satisfying (4.9) with wave front set as described above. We pick any of these solutions. The wave front set of the restriction to the boundary is contained in S . The symbol of the tangential gradient at Γ vanishes in the wave front set. This indicates that tangential derivatives are small in some sense. Most of the proof is devoted to quantifying this statement, which then gives the desired assertion.

Locally the boundary Γ is a graph over $\mathbb{R}^{n-1}$ near $(0, \dots, \pm 1)$. We parameterize Γ in a neighborhood of $(0, \dots, \pm 1)$ by the inverse of the projection to a neighborhood U of 0 in $\mathbb{R}^{n-1}$. In the sequel we use these local coordinates. For a moment we restrict ourselves to the upper part. We define the symbol

$$\begin{aligned} a(t, x, s, y, \tau, \xi) &:= \eta_\delta((s-t)^2) \eta_\delta(d^2((t, x), Q)) \eta_\delta(d^2((s, y), Q)) \\ &\quad \times \eta_\delta\left(\frac{|\xi|^2}{|\xi|^2 + \tau^2}\right) (1 - \eta(|\xi|^2 + \tau^2)) \end{aligned}$$

where $\eta \in C_0^\infty$ is 1 near zero and 0 for arguments larger than 1, $\eta_\delta(s) := \eta(s/\delta^2)$, δ is a small number, and $d(x, t, Q)$ is the distance between (x, t) and Q . Here $x, y, \xi \in \mathbb{R}^{n-1}$.

For $f \in C^\infty(U \times \mathbb{R})$ we define the operator P_δ using the parametrization of the boundary by

$$P_\delta f(x, t) = (2\pi)^{-n} \int_{(\mathbb{R} \times \mathbb{R}^{n-1})^2} a(t, x, s, y, \tau, \xi) e^{i((x-y)\cdot\xi + (t-s)\tau)} f(y, s) d\tau d\xi ds dy$$

whose principal symbol is given by a evaluated at $x = y$ and $t = s$. The relation between a and standard symbols (i.e., those that do not depend on y and s) is described, for example, in Proposition 7.3.1 of Taylor [10]. This is a symmetric pseudo-differential operator with symbol a supported in a conical δ -neighborhood of S , up to a smoothing operator. Moreover, the symbol is 1 in a conical δ -neighborhood of S intersected with the complement of the ball of radius 1 in the cotangent variables, again modulo smoothing operators.

Let m be a positive integer, $J = (1/2, 2m - 1/2)$, $K = (0, 2m)$. We will later construct an infinite-dimensional closed subspace V in $H^2(\Omega_K)$ of solutions to (4.9) whose elements have wave front set in S . Suppose that we have such a subspace V . We may extend every element v to a solution to $\mathbb{R}$. For $v \in V$ we set $u = \partial_t v$ or $u = \partial_i v$ for $1 \leq i \leq n-1$.

As above one sees by extending the normal to the interior and testing the wave equation with $\partial_X u$ that the quadratic form

$$B_0(u) = \int_{\Gamma_J} (\partial_\nu u)^2 + u_t^2 - |\nabla_\Gamma u|^2 \quad (4.10)$$

is bounded in terms of the H^2 norm of v . Here ∇_Γ denotes the tangential part of the gradient. Hence

$$B(u) = \int_{\Gamma_J} u_t^2 - |\nabla_\Gamma u|^2 \quad (4.11)$$

is bounded from above by a constant times the $H^2(\Omega_J)$ norm of v which in turn is bounded by the corresponding norm of the initial data for v . We write $u|_{\Gamma_{\mathbb{R}}} = P_\delta u|_{\Gamma_{\mathbb{R}}} + (1 - P_\delta)u|_{\Gamma_{\mathbb{R}}}$. The map

$$V \ni v \rightarrow (1 - P_\delta)u|_{\Gamma_{\mathbb{R}}} \in C^k(\Gamma_{\mathbb{R}}) \quad (4.12)$$

is well defined because the wave front set is contained in S and a is identically 1 in a neighborhood of S .

The map

$$V \ni v \rightarrow (1 - P_\delta)u|_{\Gamma_{\mathbb{R}}} \in L^2(\Gamma_K)$$

is continuous by standard trace results. Let D_ε be a family of continuous operators from $L^2_{\text{unif}}(\Gamma_{\mathbb{R}})$, the subspace of L^2_{loc} with uniform bounds on time intervals of length 1, to $C^\infty(\Gamma_K)$ with $D_\varepsilon f \rightarrow f$ in $C^k(\Gamma_K)$ for all $f \in C^k(\Gamma_K)$. Then

$$D_\varepsilon(1 - P_\delta)u|_{\Gamma_{\mathbb{R}}} \rightarrow (1 - P_\delta)u|_{\Gamma_{\mathbb{R}}}$$

in C^k for all $v \in V$. Thus, by the theorem of Banach-Steinhaus, the map defined in (4.12) is a bounded continuous linear operator. Moreover, it is compact in compact time intervals because it factors over compact imbeddings.

We claim that the map

$$V \ni v \rightarrow P_\delta u|_{\Gamma_{\mathbb{R}}} \in H^1(\Gamma_J) \quad (4.13)$$

is bounded. The symbol a is supported in a conical δ neighborhood of the τ direction. Let $P_{R\delta}$ be the operator defined by the symbol a_R where we use $R\delta$ instead of δ . Then the symbol a_R is 1 in a neighborhood of the support of a in the complement of a ball provided that R is sufficiently large, but independent of δ . Thus by standard symbol calculus, $P_{R\delta}P_\delta - P_\delta$ has a smooth kernel. Moreover, its kernel is supported in $|t-s| \leq R\delta$ and $d((x,t), Q) \leq R\delta$, $d(y,t) \leq R\delta$. Thus, if $R\delta \leq 1/2$,

$$\|P_{R\delta}P_\delta w - P_\delta w\|_{C^k(\Gamma_J)} \leq c\|w\|_{L^2(\Gamma_K)} \quad (4.14)$$

and hence

$$\begin{aligned} \|\nabla_\Gamma P_\delta u|_{\Gamma_{\mathbb{R}}}\|_{L^2(\Gamma_J)} &\leq \|\nabla_\Gamma P_{R\delta}P_\delta u|_{\Gamma_{\mathbb{R}}}\|_{L^2(\Gamma_J)} + c(R, \delta)\|u|_{\Gamma_{\mathbb{R}}}\|_{L^2(\Gamma_K)} \\ &\leq 2R\delta\|P_\delta u|_{\Gamma_{\mathbb{R}}}\|_{H^1(\Gamma_J)} + c(R, \delta)\|u|_{\Gamma_{\mathbb{R}}}\|_{L^2(\Gamma_K)}, \end{aligned} \quad (4.15)$$

where the first inequality is an immediate consequence of (4.14) and the second estimate holds because $\nabla_\Gamma P_{R\delta}$ is a pseudo-differential operator of order 1 with symbol bounded

by $R\delta\sqrt{|\tau|^2 + |\xi|^2}$. This is most easily seen for the gradient in local coordinates, where the principal symbol is

$$\xi a_R(t, x, t, x, \tau, \xi) \quad (4.16)$$

for $|\xi|^2 + \tau^2 \geq 1$. The principal symbol is supported in $|\xi| = R\delta\sqrt{\tau^2 + |\xi|^2}$ and hence $|\xi| \leq R\delta\sqrt{|\tau|^2 + |\xi|^2}$ there. Thus

$$\begin{aligned} \|\nabla_\Gamma P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 &\leq 4(R\delta)^2(\|\partial_t P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 + \|\nabla_\Gamma P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Omega_J)}^2) \\ &\quad + c(R, \delta)\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_K)}^2 \end{aligned} \quad (4.17)$$

and, if $R\delta \leq 1/10$,

$$\|\nabla_\Gamma P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)} \leq 4R\delta\|\partial_t P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)} + c(R, \delta)\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_K)}. \quad (4.18)$$

Now

$$\begin{aligned} \|\partial_t P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 &\leq B(P_\delta u) + \|\nabla_{\Gamma_{\bar{z}}} P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 \\ &\leq B(P_\delta u) + 20(R\delta)^2\|\partial_t P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 + c\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_K)}^2, \end{aligned} \quad (4.19)$$

which implies

$$\|\partial_t P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 \leq 2(B(P_\delta u)) + c\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_K)}^2, \quad (4.20)$$

and, if we combine it again with (4.18),

$$\|\nabla_\Gamma P_\delta u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_J)}^2 \leq 100R^2\delta^2(B(P_\delta u|_{\Gamma_{\bar{z}}})) + c(R, \delta)\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_K)}^2. \quad (4.21)$$

We complete the proof of claim (4.13) by estimating the quadratic form B . We recall that

$$B(u) \leq B_0(u) \leq c(\|v(0)\|_{H^2}^2 + \|v_t(0)\|_{H^1(\Gamma)}^2) \quad (4.22)$$

and

$$\|(1 - P_\delta)u|_{\Gamma_{\bar{z}}}\|_{C^k} \leq c(\|v(0)\|_{H^2} + \|v_t(0)\|_{H^1(\Omega)^2}). \quad (4.23)$$

Hence

$$\begin{aligned} B(P_\delta u|_{\Gamma_{\bar{z}}}) &= B(P_\delta u|_{\Gamma_{\bar{z}}} - u|_{\Gamma_{\bar{z}}}) + B(u|_{\Gamma_{\bar{z}}}) \\ &\quad - 2 \int_{\Gamma_J} (u|_{\Gamma_{\bar{z}}})_t ((P_\delta - 1)u|_{\Gamma_{\bar{z}}})_t - \nabla_\Gamma(u|_{\Gamma_{\bar{z}}}) \cdot \nabla_\Gamma((P_\delta - 1)u|_{\Gamma_{\bar{z}}}) \\ &\leq c(\|v(0)\|_{H^2(\Omega)}^2 + \|v_t(0)\|_{H^1(\Omega)}^2), \end{aligned} \quad (4.24)$$

where we slightly abused the notation by plugging in functions defined only on the boundary. The first identity is an expansion of the quadratic form. The arguments above yield a priori estimates, which can be turned into regularity statements by an approximation. We combine (4.21) and (4.24) to get

$$\|\nabla_\Gamma P_\delta u|_{\Gamma_{\bar{z}}}\|_{\Gamma_K} \leq c\delta(\|v(0)\|_{H^2(\Omega)} + \|v_t(0)\|_{H^1(\Omega)}) + c(\delta)\|u|_{\Gamma_{\bar{z}}}\|_{L^2(\Gamma_R)}. \quad (4.25)$$

We need the following obvious lemma.

4.1. Lemma. *Let H_1 and H_2 be Hilbert spaces, $T : H_1 \rightarrow H_2$ a compact linear operator and h_i a basis in H_1 . For $\varepsilon > 0$ there exists N such that the restriction $T : \langle h_i \rangle_{i \geq N} \rightarrow H_2$ has norm smaller than ε .*

By Lemma 4.1 we may choose a subspace of finite codimension of V such that the L^2 -norm in (4.25) is controlled by the first term on the right-hand side. Hence

$$V \ni v \rightarrow \nabla_{\Gamma} u|_{\Gamma_T} \in L^2(\Gamma_{\mathbb{R}}) \quad (4.26)$$

is compact because it can be written as the sum of an operator of size $O(\delta)$, a compact operator, and an operator of finite range.

This immediately implies the statement of Lemma 3.3 because $\partial_\nu v = 0$ provided we prove existence of the subspace V such that, in addition,

$$\|v(0)\|_{H^2}^2 + \|v_t(0)\|_{H^1}^2 \leq 4\|\partial_n v(0)\|_{H^1}^2. \quad (4.27)$$

We simply take a suitable subspace of V of finite codimension to ensure smallness of $\nabla_{\Gamma} u|_{\Gamma_J}$ in L^2 .

We begin with the construction of a closed subspace W of H_0^2 with wave front set in $(0, e_n) \in (\mathbb{R}^n)^2$. Suppose we have constructed W . Then we set $v_0 = f \in W$ and $v_1 = i\psi\sqrt{\mu}(-\Delta_{\mathbb{R}^n})^{1/2}v_0$ where $\psi \in C_0^\infty(\Omega)$ is identically 1 in a neighborhood of zero. We define v as the solution to the homogeneous Neumann problem with these initial data. Then we define V as the space of these solutions.

Let $\phi \in C_0^\infty$ satisfy $\|\phi\|_2 = 1$. We set for $k \in \mathbb{N}$

$$f_k(x) = k^{n/2-8} e^{ik^4 x_n} \phi(kx).$$

Clearly

$$\begin{aligned} \|f_k\|_{L^2} &= k^{-8}, \\ \|f_k\|_{H^2} &= 1 + O(k^{-2}) = \|\partial_n^2 f_k\|_{L^2} + O(k^{-2}) \end{aligned}$$

and

$$\langle f_k, f_j \rangle_{H^2} = O((|k| + |j|)^{-2})$$

if $k \neq j$ by an integration by parts. Now let $a_i \in l^2$. Then $f = \sum a_i f_i \in H^2$ with wave front set contained in $(0, e_n)$ and

$$c^{-1}\|a_i\|_{l^2} \leq \|f\|_{H^2} \leq c\|a_i\|_{l^2}.$$

This completes the proof, up to a standard verification of (4.27), which holds at least if $a_i = 0$ for $i \leq N$ and some large number N . $\square$

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