ELEMENTARY EVALUATION
OF CERTAIN INFINITE INTEGRALS
INVOLVING BESSEL FUNCTIONS

BY

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Abstract. Although it is known theoretically that certain infinite integrals of Bessel functions can be expressed in terms of elementary functions, the practical evaluation of such integrals was quite difficult due to the algebraic complexity of the expressions involved. A simple and elegant algebra is introduced here which allows these integrals to be calculated in an elementary way in terms of elementary functions. Some relationships are shown between the integrals involving Bessel functions and two-dimensional integrals over a circle of elementary functions involving distances between points. A comparison is made with existing results, and some of them were found in error (or were misprints).

1. Introduction. The following infinite integrals are encountered in various engineering problems (see, e.g., Elliott [4]):

$$\begin{Bmatrix} C_\nu^\mu \\ S_\nu^\mu \end{Bmatrix} = \int_0^\infty e^{-zx} x^\mu \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (ax) J_\nu(\rho x) dx. \quad (1.1)$$

In the case of integer $\mu = m$ and integer $\nu = n$, the integrals (1.1) can be calculated in closed form in terms of elementary functions. In practice, though, only a limited number of such integrals is given in the tables. For example, in the table of Gradshteyn and

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Ryzhik [5, No. 6.751 3, 6.752 1, 6.752 2], one can find

$$\int_0^\infty e^{-zx} \cos(ax) J_0(\rho x) dx = \frac{\sqrt{\sqrt{(z^2 + \rho^2 - a^2)^2 + 4a^2 z^2} + z^2 + \rho^2 - a^2}}{\sqrt{2}\sqrt{(z^2 + \rho^2 - a^2)^2 + 4a^2 z^2}} \quad (1.2)$$

[Re $z > |\operatorname{Im} a| + |\operatorname{Im} \rho|$],

$$\int_0^\infty e^{-zx} \sin(ax) J_0(\rho x) \frac{dx}{x} = \arcsin \left(\frac{2a}{\sqrt{z^2 + (a + \rho)^2} + \sqrt{z^2 + (a - \rho)^2}} \right) \quad (1.3)$$

[Re $z > |\operatorname{Im} a| + |\operatorname{Im} \rho|$],

$$\int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x} = \frac{1}{\rho} \left(a - \frac{1}{\sqrt{2}} \sqrt{\sqrt{(z^2 + \rho^2 - a^2)^2 + 4a^2 z^2} - (z^2 + \rho^2 - a^2)} \right) \quad (1.4)$$

[Re $z > |\operatorname{Im} a| + |\operatorname{Im} \rho|$],

where the latter formula is given in an implicit form.

In principle, when $\nu = 0$, the integrals (1.1) can be evaluated from (1.2) and (1.3) just by direct differentiation with respect to z for positive m and by integration for negative m . Practical implementation of these ideas, however, is quite difficult: differentiation would lead to unwieldy expressions, while integration seems to be too formidable to perform. Some elegant algebra is introduced below, which allows us to simplify these calculations significantly. A number of integrals of type (1.1) for integer parameters $\mu = m$ and $\nu = n$ are given in Fabrikant [6, p. 254].

2. Evaluation of the integrals. In order to overcome these difficulties, the following notation (Fabrikant [6, p. 12]) is introduced:

$$l_1(x) \equiv l_1(x, \rho, z) = \frac{1}{2} [\sqrt{(x + \rho)^2 + z^2} - \sqrt{(x - \rho)^2 + z^2}], \quad (2.1)$$

$$l_2(x) \equiv l_2(x, \rho, z) = \frac{1}{2} [\sqrt{(x + \rho)^2 + z^2} + \sqrt{(x - \rho)^2 + z^2}] \quad (2.2)$$

[$x \geq 0, \rho \geq 0, z \geq 0$].

If x, ρ, z are complex, the square roots in these formulae are to be taken with positive real parts. The geometric interpretation of l_1 and l_2 is quite obvious if one considers a circle of radius x in the plane $z = 0$ and a point with the polar cylindrical coordinates (ρ, ϕ, z) ; l_1 and l_2 then represent half of the difference and the sum, respectively, of the longest and the shortest distance from the point to the circle.

The following properties of l_1 and l_2 can be verified directly:

$$\begin{aligned} \lim_{z \rightarrow 0} l_1(x) &= \min(x, \rho), & l_1 &\leq \min(x, \rho) \quad \text{for any } z, \\ \lim_{z \rightarrow 0} l_2(x) &= \max(x, \rho), & l_2 &\geq \max(x, \rho) \quad \text{for any } z, \end{aligned} \quad (2.3)$$

$$\begin{aligned}
l_2^2(x) - l_1^2(x) &= \sqrt{(x+\rho)^2 + z^2} \sqrt{(x-\rho)^2 + z^2}, \\
l_1^2(x) + l_2^2(x) &= x^2 + \rho^2 + z^2, \\
l_1(x)l_2(x) &= x\rho,
\end{aligned} \tag{2.4}$$

$$\sqrt{x^2 - l_1^2(x)} \sqrt{l_2^2(x) - x^2} = xz, \quad \sqrt{\rho^2 - l_1^2(x)} \sqrt{l_2^2(x) - \rho^2} = \rho z, \tag{2.5}$$

$$\sqrt{x^2 - l_1^2(x)} \sqrt{\rho^2 - l_1^2(x)} = z l_1(x), \quad \sqrt{l_2^2(x) - x^2} \sqrt{l_2^2(x) - \rho^2} = z l_2(x). \tag{2.6}$$

In the first equation of (2.5), the square roots $\sqrt{x^2 - l_1^2(x)}$ and $\sqrt{l_2^2(x) - x^2}$ are to be taken as odd and even functions of x , respectively. The determination of the other square roots follows from the other equations (2.5) and (2.6).

The differentiation can be performed by using the main formulae:

$$\begin{aligned}
\frac{\partial}{\partial \rho} l_1(x) &= \frac{x l_2(x) - \rho l_1(x)}{l_2^2(x) - l_1^2(x)} = \frac{\rho[x^2 - l_1^2(x)]}{l_1(x)[l_2^2(x) - l_1^2(x)]} \\
&= \frac{x[l_2^2(x) - \rho^2]}{l_2(x)[l_2^2(x) - l_1^2(x)]} = \frac{\partial}{\partial x} l_2(x),
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\frac{\partial}{\partial \rho} l_2(x) &= \frac{\rho l_2(x) - x l_1(x)}{l_2^2(x) - l_1^2(x)} = \frac{\rho[l_2^2(x) - x^2]}{l_2(x)[l_2^2(x) - l_1^2(x)]} \\
&= \frac{x[\rho^2 - l_1^2(x)]}{l_1(x)[l_2^2(x) - l_1^2(x)]} = \frac{\partial}{\partial x} l_1(x),
\end{aligned} \tag{2.8}$$

$$\frac{\partial}{\partial z} l_1(x) = -\frac{z l_1(x)}{l_2^2(x) - l_1^2(x)}, \quad \frac{\partial}{\partial z} l_2(x) = \frac{z l_2(x)}{l_2^2(x) - l_1^2(x)}. \tag{2.9}$$

A number of additional formulae can be found in Fabrikant [7, p. 338].

Using l_1 and l_2 in (1.2)–(1.4) allows us to simplify these formulae significantly. With the abbreviations $l_1 \equiv l_1(a, \rho, z)$ and $l_2 \equiv l_2(a, \rho, z)$ we have

$$\int_0^\infty e^{-zx} \cos(ax) J_0(\rho x) dx = \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}, \tag{2.10}$$

$$\int_0^\infty e^{-zx} \sin(ax) J_0(\rho x) \frac{dx}{x} = \arcsin\left(\frac{a}{l_2}\right) = \arcsin\left(\frac{l_1}{\rho}\right), \tag{2.11}$$

$$\int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x} = \frac{a - \sqrt{a^2 - l_1^2}}{\rho} \tag{2.12}$$

$$[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|].$$

The simplicity of (2.10)–(2.12) makes it possible to differentiate or integrate whenever necessary. For example, integration with respect to z of both sides of (2.12) yields

$$\int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x^2} = \frac{\sqrt{l_2^2 - a^2}(a - \sqrt{a^2 - l_1^2})^2}{2a\rho} + \frac{\rho}{2} \arcsin\left(\frac{a}{l_2}\right) \tag{2.13}$$

$$[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|].$$

To illustrate the method, we show this in some detail. Indefinite integration by parts of the negative right-hand side in (2.12) with respect to z gives

$$-\int dz \frac{a - \sqrt{a^2 - l_1^2}}{\rho} = -z \frac{a - \sqrt{a^2 - l_1^2}}{\rho} + \frac{1}{\rho} \int z \frac{l_1 dl_1}{\sqrt{a^2 - l_1^2}}.$$

Using (2.6), we obtain for the integral on the right-hand side

$$\frac{1}{\rho} \int \sqrt{\rho^2 - l_1^2} dl_1 = \frac{1}{2\rho} \left[l_1 \sqrt{\rho^2 - l_1^2} + \rho^2 \arcsin \left(\frac{l_1}{\rho} \right) \right];$$

hence,

$$- \int dz \frac{a - \sqrt{a^2 - l_1^2}}{\rho} = \frac{z}{2\rho} \left(-2a + 2\sqrt{a^2 - l_1^2} + \frac{l_1}{z} \sqrt{\rho^2 - l_1^2} \right) + \frac{\rho}{2} \arcsin \left(\frac{l_1}{\rho} \right).$$

Using (2.6) again, we finally obtain

$$\begin{aligned} - \int dz \frac{a - \sqrt{a^2 - l_1^2}}{\rho} &= \frac{z}{2\rho} \frac{1}{\sqrt{a^2 - l_1^2}} \left[-2a\sqrt{a^2 - l_1^2} + 2(a^2 - l_1^2) + l_1^2 \right] + \frac{\rho}{2} \arcsin \left(\frac{l_1}{\rho} \right) \\ &= \frac{1}{2\rho} \frac{z}{\sqrt{a^2 - l_1^2}} \left(a - \sqrt{a^2 - l_1^2} \right)^2 + \frac{\rho}{2} \arcsin \left(\frac{l_1}{\rho} \right). \end{aligned}$$

We note here that, by going from the indefinite to the definite integrals, the integration constant must vanish: as $z \rightarrow \infty$, the integral in (2.13) vanishes, and also $l_2 \rightarrow z$ and $l_1 = a\rho/l_2 \approx a\rho/z \rightarrow 0$. Applying (2.4) and (2.5) finally yields (2.13). (Note that the arcsin-term in the corresponding formula [6, No. (5.3.16)] has the wrong sign.)

In the table of Prudnikov et al [11, No. 3.12.11.16], the integral (2.13) reads (after correction, i.e., multiplication of the right-hand side by a^2),

$$\begin{aligned} \int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x^2} &= \frac{1}{2\rho} [R\sqrt{a^2 + z^2} \sin(\vartheta + \eta) - (a^2 + z^2) \sin(2\vartheta)] \\ &\quad + \frac{\rho}{2} \arctan \left(\frac{R \sin \eta + \sqrt{a^2 + z^2} \sin \vartheta}{R \cos \eta + \sqrt{a^2 + z^2} \cos \vartheta} \right) \\ &\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \end{aligned}$$

where

$$\begin{aligned} R^4 &= (z^2 + \rho^2 - a^2)^2 + 4a^2 z^2, \quad \text{i.e.,} \quad R^2 = l_2^2 - l_1^2; \\ \tan \vartheta &= a/z, \quad 2az \cot(2\eta) = z^2 + \rho^2 - a^2. \end{aligned}$$

Evidently, this formula is more complicated than (2.13). Note that the corresponding formula in Prudnikov et al [9, No. 2.12.25.6] is also incorrect: for ρ^2 read ρ in the denominator of the first term.

Quite a number of other integrals can be calculated by using the well-known recurrence relation for the Bessel functions

$$J_{n+1}(z) = 2nJ_n(z)/z - J_{n-1}(z). \quad (2.14)$$

For example, for $n = 1$ we obtain

$$\begin{aligned} \int_0^\infty e^{-zx} \sin(ax) J_2(\rho x) \frac{dx}{x} &= \int_0^\infty e^{-zx} \sin(ax) \left(\frac{2}{\rho x} J_1(\rho x) - J_0(\rho x) \right) \frac{dx}{x} \\ &= \frac{(2a^2 - l_1^2) \sqrt{l_2^2 - a^2} - 2a^2 z}{a \rho^2} \\ &= \frac{\sqrt{l_2^2 - a^2} (a - \sqrt{a^2 - l_1^2})^2}{a \rho^2} \\ &\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \end{aligned} \quad (2.15)$$

$$\begin{aligned} \int_0^\infty e^{-zx} \sin(ax) J_2(\rho x) dx &= \int_0^\infty e^{-zx} \sin(ax) \left(\frac{2}{\rho x} J_1(\rho x) - J_0(\rho x) \right) dx \\ &= \frac{2(a - \sqrt{a^2 - l_1^2})}{\rho^2} - \frac{\sqrt{a^2 - l_1^2}}{l_2^2 - l_1^2} \\ &\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]. \end{aligned} \quad (2.16)$$

The integrals of higher-order Bessel functions can be obtained in a similar way. Integrals of this type can be found in Prudnikov et al [9]. For example, the integral (2.15) corresponds to I_0 in [9, No. 2.12.25.3] for $\alpha = 0$ and $\nu = 2$:

$$\begin{aligned} \int_0^\infty e^{-zx} \sin(ax) J_2(\rho x) \frac{dx}{x} &= \frac{1}{2} \frac{\rho^2}{(l_2 + \sqrt{l_2^2 - \rho^2})^2} \sin \left(2 \arcsin \left(\frac{a}{l_2} \right) \right) \\ &\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \end{aligned}$$

which is another form of (2.15).

These integrals of Bessel functions can be related to two-dimensional integrals over a circle. For example, the following integral was given in Fabrikant [7, No. (A5.41)]:

$$\int_0^{2\pi} \int_0^a \frac{\rho_0 d\rho_0 d\varphi_0}{R_0 \sqrt{a^2 - \rho_0^2}} = 2\pi \arcsin \left(\frac{a}{l_2} \right), \quad (2.17)$$

where $R_0^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0) + z^2$. Comparison of (2.11) with (2.17) allows us to write

$$2\pi \int_0^\infty e^{-zx} \sin(ax) J_0(\rho x) \frac{dx}{x} = \int_0^{2\pi} \int_0^a \frac{\rho_0 d\rho_0 d\varphi_0}{R_0 \sqrt{a^2 - \rho_0^2}}. \quad (2.18)$$

Many other relations can be obtained by differentiation and integration of (2.18).

For a nonnegative integer $\nu = n$, the following formula was given by Fabrikant [6, p. 348] without a direct proof:

$$\begin{aligned} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} &= \sqrt{\frac{2}{\pi a}} (a\rho)^{-\nu} \int_0^{l_1} \frac{x^{2\nu} dx}{\sqrt{\rho^2 - x^2}} \\ &\quad \left[\nu > -\frac{1}{2}, \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho| \right]. \end{aligned} \quad (2.19)$$

We prove it here by induction. For $\nu = 0$, this formula is correct by comparison with (2.11). Assuming that (2.19) is correct for a given ν , we show that it is then also correct

for $\nu + 1$. Thus we shall prove that

$$\int_0^\infty e^{-zx} J_{\nu+3/2}(ax) J_{\nu+1}(\rho x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi a}} (a\rho)^{-(\nu+1)} \int_0^{l_1} \frac{x^{2\nu+2} dx}{\sqrt{\rho^2 - x^2}}. \quad (2.20)$$

Using the well-known properties of Bessel functions (e.g., Gradshteyn and Ryzhik [5, No. 8.472]),

$$\frac{\partial}{\partial \rho} \left[\frac{J_\nu(\rho x)}{\rho^\nu} \right] = -\frac{x J_{\nu+1}(\rho x)}{\rho^\nu}, \quad (2.21)$$

$$\int_0^a a^{\nu+3/2} J_{\nu+1/2}(ax) da = a^{\nu+3/2} J_{\nu+3/2}(ax)/x, \quad (2.22)$$

and substituting $y = x/\rho$ in (2.19) yields

$$\begin{aligned} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} &= \sqrt{\frac{2}{\pi a}} \left(\frac{\rho}{a} \right)^\nu \int_0^{l_1/\rho} \frac{y^{2\nu} dy}{\sqrt{1-y^2}} \\ &= \sqrt{\frac{2}{\pi a}} \left(\frac{\rho}{a} \right)^\nu \int_0^{a/l_2} \frac{y^{2\nu} dy}{\sqrt{1-y^2}} \end{aligned} \quad (2.23)$$

from (2.4). Dividing both sides of (2.23) by ρ^ν , differentiating with respect to ρ , and taking account of (2.4)–(2.8) leads to

$$\begin{aligned} \int_0^\infty \sqrt{x} e^{-zx} J_{\nu+1/2}(ax) J_{\nu+1}(\rho x) dx &= \sqrt{\frac{2}{\pi a}} \left(\frac{\rho}{a} \right)^\nu \left(\frac{a}{l_2} \right)^{2\nu} \frac{(l_2^2 - a^2)a\rho}{\sqrt{l_2^2 - a^2}(l_2^2 - l_1^2)l_2^2} \\ &= \sqrt{\frac{2}{\pi a}} \left(\frac{l_1}{l_2} \right)^{\nu+1} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}. \end{aligned} \quad (2.24)$$

These expressions, based on a/l_2 , will be referred to later. For the present purpose, it is more straightforward to use l_1/ρ , which yields

$$\begin{aligned} \int_0^\infty \sqrt{x} e^{-zx} J_{\nu+1/2}(ax) J_{\nu+1}(\rho x) dx &= \sqrt{\frac{2}{\pi a}} \left(\frac{\rho}{a} \right)^\nu \left(\frac{l_1}{\rho} \right)^{2\nu+1} \frac{\rho}{\sqrt{\rho^2 - l_1^2}} \left[-\frac{\rho(a^2 - l_1^2)}{l_1^2(l_2^2 - l_1^2)} + \frac{1}{\rho} \right] \\ &= \sqrt{\frac{2}{\pi a}} \frac{1}{a^\nu \rho^{\nu+1}} \frac{l_1^{2\nu+1}}{\sqrt{\rho^2 - l_1^2}} \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2}. \end{aligned}$$

Applying (2.22) to both sides of this equation gives

$$a^{\nu+3/2} \int_0^\infty e^{-zx} J_{\nu+3/2}(ax) J_{\nu+1}(\rho x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} \frac{1}{\rho^{\nu+1}} \int_0^a \frac{l_1^{2\nu+1}}{\sqrt{\rho^2 - l_1^2}} \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} a da. \quad (2.25)$$

Using (2.8),

$$\frac{\rho^2 - l_1^2}{l_2^2 - l_1^2} a da = l_1 dl_1 \quad (2.26)$$

yields, after division of both sides by $a^{\nu+3/2}$, formula (2.20). The proof is thus complete for integer values $\nu = n$.

Formula (2.19) is, however, valid for any ν . Indeed, when multiplied by $\cos(\nu\varphi)$, both sides of (2.19) represent harmonic functions (this can be verified explicitly for the right-hand side; it is done in the Appendix), satisfying the same boundary conditions; they are thus equal. These boundary conditions are [3, No. 8.11(9)]:

$$\begin{aligned} \text{at } z = 0 : \quad V(\rho < a) &= \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{2}\Gamma(\nu + 1)a^{\nu+1/2}} \rho^\nu \cos(\nu\varphi) \quad [\nu > -\frac{1}{2}], \\ \partial V / \partial z &= 0 \quad [\rho > a]; \end{aligned}$$

$$\text{at } z = \infty \text{ or } \rho = \infty : \quad V = 0;$$

$$\text{at } \varphi = 0 \text{ and } \varphi = \pi/\nu < 2\pi : \quad \partial V / \partial \varphi = 0.$$

The condition $\varphi < 2\pi$ implies $\nu > \frac{1}{2}$, whereas the integrals in (2.19) are convergent for $\nu > -\frac{1}{2}$. In order to prove the remaining case $-\frac{1}{2} < \nu \leq \frac{1}{2}$, we shall now show that if (2.20) holds, then (2.19) is also true. First, we note that

$$a^{-\mu-1} \frac{\partial}{\partial a} [a^{\mu+1} J_{\mu+1}(ax)] = x J_\mu(ax), \quad (2.27)$$

$$-\rho^\nu \int_\infty^\rho \rho^{\nu+1} J_{\nu+1}(\rho x) d\rho = \frac{1}{x} J_\nu(\rho x) \quad [\nu > -\frac{1}{2}]. \quad (2.28)$$

Taking $\mu = \nu + \frac{1}{2}$ in (2.27), we multiply both sides of (2.20) by $a^{\mu+1}$, calculate the derivative with respect to a , then multiply by $a^{-\mu-1}$. Using (2.8) and then (2.4)–(2.6), this leads to

$$\begin{aligned} \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_{\nu+1}(\rho x) \sqrt{x} dx &= a^{-\nu-3/2} \rho^{-\nu-1} \frac{l_1^{2\nu+2}}{\sqrt{\rho^2 - l_1^2}} \frac{a(\rho^2 - l_1^2)}{l_1(l_2^2 - l_1^2)} \\ &= a^{\nu+1/2} \frac{\rho^{\nu+1}}{l_2^{2\nu+2}} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}. \end{aligned}$$

In order to apply (2.28), we now multiply both sides of this equation by $\rho^{-\nu}$, integrate with respect to ρ (a being a constant), then multiply by $-\rho^\nu$. This yields

$$\sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} = -\rho^\nu a^{\nu+1/2} \int_\infty^\rho \frac{1}{l_2^{2\nu+2}} \frac{1}{\sqrt{l_2^2 - a^2}} \frac{l_2^2 - a^2}{l_2^2 - l_1^2} \rho d\rho.$$

Because of (2.8),

$$\frac{l_2^2 - a^2}{l_2^2 - l_1^2} \rho d\rho = l_2 dl_2; \quad (2.29)$$

hence,

$$\begin{aligned} \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} &= \rho^\nu a^{\nu+1/2} \int_0^{1/l_2} \frac{1}{l_2^{2\nu}} \frac{1}{\sqrt{1 - a^2/l_2^2}} d\left(\frac{1}{l_2}\right) \\ &= a^{-\nu-1/2} \rho^\nu \int_0^{a/l_2} x^{2\nu} \frac{dx}{\sqrt{1 - x^2}}. \end{aligned}$$

By noting that $a/l_2 = l_1/\rho$, we see that the last formula is the same as (2.19), i.e., (2.20) where $\nu + 1$ is replaced by ν . This completes the proof.

We note further that

$$\int_0^x \frac{x^{2\nu} dx}{\sqrt{1-x^2}} = \frac{x^{2\nu+1}}{2\nu+1} {}_2F_1\left(\frac{1}{2}, \nu + \frac{1}{2}; \nu + \frac{3}{2}; x^2\right). \quad (2.30)$$

In the case of integer $\nu = n$, the integral on the right-hand side of (2.23) can be expressed in terms of elementary functions (see Prudnikov et al [8, No. 1.5.2.2]):

$$\begin{aligned} & \int_0^\infty e^{-zx} J_{n+1/2}(ax) J_n(\rho x) \frac{dx}{\sqrt{x}} \\ &= \sqrt{\frac{2}{\pi a}} \left(\frac{\rho}{a}\right)^n \left\{ \frac{(\frac{1}{2})_n}{(1)_n} \arcsin\left(\frac{a}{l_2}\right) \right. \\ & \quad \left. - \left(\frac{a}{l_2}\right)^{2n} \frac{\sqrt{l_2^2 - a^2}}{2na} \left[1 + \sum_{k=1}^{n-1} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n-k)}{\Gamma(n-k + \frac{1}{2}) \Gamma(n)} \left(\frac{a}{l_2}\right)^{-2k} \right] \right\} \quad [n > 0] \\ &= \sqrt{\frac{2}{\pi a}} \frac{(\frac{1}{2})_n}{(1)_n} \left(\frac{\rho}{a}\right)^n \left[\arcsin\left(\frac{a}{l_2}\right) - \sqrt{1 - \left(\frac{a}{l_2}\right)^2} \sum_{k=0}^{n-1} \frac{(1)_k}{(\frac{3}{2})_k} \left(\frac{a}{l_2}\right)^{2k+1} \right] \quad [n \geq 0] \\ & \quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \end{aligned} \quad (2.31)$$

where we have used $l_1 l_2 = a\rho$. Note that the last series over k consists of the first n terms of the hypergeometric function

$$x {}_2F_1\left(1, 1; \frac{3}{2}; x^2\right) = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

The integral (2.31) belongs to the class described in (1.1), since $J_{n+1/2}$ is a combination of sine and cosine functions, and of negative powers of x .

Our aim is now to generalize formula (2.19). We can apply the formula given by Fabrikant [6, No. 1.4.18], [7, No. 1.3.23] for integer values of ν , i.e.,

$$V(\rho, \varphi, z) = \frac{2}{\pi} \int_0^{t=a} \frac{dl_1(t)}{\sqrt{\rho^2 - l_1^2(t)}} \left(\frac{l_1^2(t)}{\rho t^2}\right)^\nu \frac{d}{dt} \int_0^t \frac{\rho_0^{\nu+1} d\rho_0}{\sqrt{t^2 - \rho_0^2}} V(\rho_0, \varphi, 0) \quad (2.32)$$

with a variation $\cos(\nu\varphi)$. The potential V at $z = 0$ is supposed to be known for $\rho < a$, with the boundary condition $\partial V/\partial z = 0$ for $\rho > a$, and vanishing at infinity. Following Erdélyi et al [3, No. 8.11(9)], a potential at $z = 0$ and $\rho < a$ may be taken as

$$\begin{aligned} V(\rho, \varphi, 0) &= \int_0^\infty x^{\nu-\mu+2n} J_\mu(ax) J_\nu(\rho x) dx \times \cos(\nu\varphi) \\ &= 2^{\nu-\mu+2n} a^{\mu-2\nu-2n-1} \rho^\nu \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu+1)\Gamma(\mu-\nu-n+\frac{1}{2})} \\ & \quad \times {}_2F_1\left(\nu+n+\frac{1}{2}, \nu-\mu+n+\frac{1}{2}; \nu+1; \frac{\rho^2}{a^2}\right) \times \cos(\nu\varphi) \\ & \quad [\mu > \nu+2n-1, \nu+n+\frac{1}{2} > 0]. \end{aligned}$$

Formula (2.32) allows us to generalize formula (2.19) or formula (2.23) to

$$\begin{aligned}
 \int_0^\infty e^{-zx} x^{\nu-\mu+2n} J_\mu(ax) J_\nu(\rho x) dx &= \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^\nu \\
 &\times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \sum_{q=0}^\infty \frac{\Gamma(\nu+n+q+\frac{1}{2})(\nu-\mu+n+\frac{1}{2})_q}{q! \Gamma(\nu+q+\frac{1}{2})} \\
 &\times a^{-2q} \int_0^{l_1/\rho} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q \\
 &\quad \left[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|, \mu > \nu + 2n, n = 0, 1, 2, \dots, \nu > -\frac{1}{2} \right].
 \end{aligned} \tag{2.33}$$

The series terminates when $\mu = \nu + \frac{1}{2} + 2n + m$ ($m = 0, 1, 2, \dots$), and the integer n ensures that $\partial V/\partial z = 0$ for $\rho > a$ in the left-hand side. The condition $\nu > -\frac{1}{2}$ ensures that the integral in the right-hand side converges at $x = 0$, and that the potential vanishes at infinity. In this integral, the q -power (including x^{2q}) is the power of

$$t^2 = l_1^2(t) \left(1 + \frac{z^2}{\rho^2 - l_1^2(t)} \right), \quad x = l_1(t)/\rho, \tag{2.34}$$

where ρ and z are constant.

Formula (2.33), which is obtained for integer ν , can be proved to be valid for any ν by using the same argument as for proving formula (2.19). For $\nu > \frac{1}{2}$, a rather lengthy calculation, given in the Appendix, reveals that each q -term in the series, after multiplication by $\cos(\nu\varphi)$, is a harmonic function. For $-\frac{1}{2} < \nu \leq \frac{1}{2}$, it can be shown that from the assumption that (2.33) is correct for $(\mu+1, \nu+1)$ it follows that it is also correct for (μ, ν) . A derivation is given in the Appendix.

The series over q in (2.33) can be summed in closed form. We rewrite this relation as

$$\begin{aligned}
 \int_0^\infty e^{-zx} x^{\nu-\mu+2n} J_\mu(ax) J_\nu(\rho x) dx \\
 = \frac{1}{\sqrt{\pi}} \frac{1}{2^{\mu-\nu-2n-1}} \frac{a^{\mu-2\nu-2n-1}}{\rho^\nu} \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\mu-\nu-n+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \\
 \times \int_0^{l_1} \frac{dy}{\sqrt{\rho^2-y^2}} y^{2\nu} \sum_{q=0}^\infty \frac{(\nu+n+\frac{1}{2})_q (\nu-\mu+n+\frac{1}{2})_q}{q! (\nu+\frac{1}{2})_q} \left(\frac{t^2}{a^2}\right)^q,
 \end{aligned}$$

where the sum now represents a Gaussian hypergeometric function. Using [10, Nos. 7.3.1.4, 7.3.1.22], we obtain

$$\begin{aligned}
 {}_2F_1\left(\nu+n+\frac{1}{2}, \nu-\mu+n+\frac{1}{2}; \nu+\frac{1}{2}; \frac{t^2}{a^2}\right) \\
 = \left(1 - \frac{t^2}{a^2}\right)^{\mu-\nu-2n-1/2} {}_2F_1\left(-n, \mu-n; \nu+\frac{1}{2}; \frac{t^2}{a^2}\right) \\
 = \frac{(\nu-\mu+n+\frac{1}{2})_n}{(\nu+\frac{1}{2})_n} \left(1 - \frac{t^2}{a^2}\right)^{\mu-\nu-2n-1/2} {}_2F_1\left(-n, \mu-n; \mu-\nu-2n+\frac{1}{2}; 1 - \frac{t^2}{a^2}\right) \\
 = (-1)^n \frac{(\mu-\nu-2n+\frac{1}{2})_n}{(\nu+\frac{1}{2})_n} \sum_{q=0}^n \frac{(-n)_q (\mu-n)_q}{q! (\mu-\nu-2n+\frac{1}{2})_q} \left(1 - \frac{t^2}{a^2}\right)^{\mu-\nu-2n+q-1/2}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_0^\infty e^{-zx} x^{\nu-\mu+2n} J_\mu(ax) J_\nu(\rho x) dx \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{2^{\mu-\nu-2n-1}} \frac{1}{a^\mu \rho^\nu} \sum_{q=0}^n (-1)^{n-q} \binom{n}{q} \frac{(\mu-n)_q}{\Gamma(\mu-\nu-2n+q+\frac{1}{2})} a^{2n-2q} \\
&\quad \times \int_0^{l_1} \frac{dy}{\sqrt{\rho^2-y^2}} y^{2\nu} (a^2-t^2)^{\mu-\nu-2n+q-1/2} \\
&\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|, \mu > \nu + 2n - \frac{1}{2}, n = 0, 1, 2, \dots, \nu > -\frac{1}{2}]. \quad (2.35)
\end{aligned}$$

Taking the derivative with respect to z , we obtain

$$\begin{aligned}
& \int_0^\infty e^{-zx} x^{\nu-\mu+2n+1} J_\mu(ax) J_\nu(\rho x) dx \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{2^{\mu-\nu-2n-1}} \frac{1}{a^\mu \rho^\nu} \sum_{q=0}^n (-1)^{n-q} \binom{n}{q} \frac{(\mu-n)_q}{\Gamma(\mu-\nu-2n+q-\frac{1}{2})} a^{2n-2q} \\
&\quad \times 2z \int_0^{l_1} \frac{dy}{(\rho^2-y^2)^{3/2}} y^{2\nu+2} (a^2-t^2)^{\mu-\nu-2n+q-3/2} \\
&\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|, \mu > \nu + 2n + \frac{1}{2}, n = 0, 1, 2, \dots, \nu > -1]. \quad (2.36)
\end{aligned}$$

For $n = 0$ and $\mu = \nu + \frac{1}{2}$, formula (2.35) immediately reproduces (2.19). For the evaluation of the integrals in (2.35) and (2.36), with (2.34) the following factorisation might be useful:

$$a^2 - t^2 = a^2 - y^2 \left(1 + \frac{z^2}{\rho^2 - y^2} \right) = \frac{(l_1^2 - y^2)(l_2^2 - y^2)}{\rho^2 - y^2}, \quad y = l_1(t).$$

3. Discussion. It is of interest to compare (2.19) with existing formulae in the tables. For example, Erdélyi et al [3, No. 8.11(21)] give

$$\begin{aligned}
& \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} \\
&= \frac{\rho^\nu a^{\nu+1/2} \Gamma(\nu + \frac{1}{2})}{\sqrt{2\pi} \Gamma(\nu + 1)} \int_0^\pi \frac{(\sin \varphi)^{2\nu+1} d\varphi}{[(z + ia \cos \varphi)^2 + \rho^2]^{\nu+1/2}} \\
&\quad [\operatorname{Re} \nu > -\frac{1}{2}, \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]. \quad (3.1)
\end{aligned}$$

It is not difficult to show that the case $\nu = 0$ agrees with (2.19):

$$\begin{aligned}
& \int_0^\infty e^{-zx} J_{1/2}(ax) J_0(\rho x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{1}{2\pi a}} \left[\arcsin \left(\frac{a-iz}{\rho} \right) + \arcsin \left(\frac{a+iz}{\rho} \right) \right] \\
&\quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \quad (3.2)
\end{aligned}$$

and that the right-hand side agrees with (2.11) or (2.19), because

$$\arcsin \left(\frac{a-iz}{\rho} \right) + \arcsin \left(\frac{a+iz}{\rho} \right) = 2 \arcsin \left(\frac{a}{l_2} \right). \quad (3.3)$$

For $\nu \neq 0$, the integration becomes quite complicated.

Another formula is (Erdélyi et al [2, No. 4.14(24)])

$$\begin{aligned} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} \\ = \frac{2^{-2\nu-1/2} \Gamma(2\nu+1)}{\Gamma(\nu+\frac{3}{2}) \Gamma(\nu+1)} a^{\nu+1/2} \rho^\nu [z + i(a+\rho)]^{-2\nu-1} \\ \times F_2 \left(2\nu+1, \nu+1, \nu+\frac{1}{2}; 2\nu+2, 2\nu+1; \frac{2ia}{z+i(a+\rho)}, \frac{2i\rho}{z+i(a+\rho)} \right) \\ [\operatorname{Re} \nu > -\frac{1}{2}; \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \quad (3.4) \end{aligned}$$

where (Gradshteyn and Ryzhik [5, No. 9.180 2])

$$F_2(a, b, b'; c, c'; w, z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{j+k} (b)_j (b')_k}{(c)_j (c')_k} \frac{w^j z^k}{j! k!} \quad [|w| + |z| < 1]$$

is a hypergeometric function of two variables. From (2.31) and (3.4) it follows that F_2 can be expressed in this case in terms of elementary functions.

Prudnikov et al [11, No. 3.12.15.31] give the following integral:

$$\begin{aligned} \int_0^\infty \sqrt{x} e^{-zx} J_\nu(ax) J_{\nu+1/2}(\rho x) dx = \sqrt{\frac{2}{\pi}} \frac{a^\nu \rho^{\nu+1/2} l_2^{-2\nu-1} \sqrt{l_2^2 - a^2}}{\sqrt{(\rho+a)^2 + z^2} \sqrt{(\rho-a)^2 + z^2}} \\ [\operatorname{Re} \nu > -1, \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|], \end{aligned}$$

which is equivalent to (2.24) when ν is replaced by $\nu + \frac{1}{2}$, or to the z -derivative of (2.19).

A very general integral can be found in Prudnikov et al [9, No. 2.12.38.2] or in Erdélyi et al [3, No. 8.11(22)], where in the latter the first factor given in (3.5) is missing. In [9], two expressions for this integral are given in terms of hypergeometric functions of two variables F_2 and F_4 [5, No. 9.180 4] respectively, and a third one as an infinite series of Gaussian hypergeometric functions ${}_2F_1$. Expressed in terms of ${}_2F_1$, the integral reads

$$\begin{aligned} \int_0^\infty x^{\alpha-1} e^{-px} J_\mu(bx) J_\nu(cx) dx = \frac{b^\mu c^\nu}{2^{\mu+\nu} p^{\alpha+\mu+\nu} \Gamma(\nu+1)} \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+\mu+\nu+2k)}{k! \Gamma(\mu+k+1)} {}_2F_1 \left(-k, -\mu-k; \nu+1; \frac{c^2}{b^2} \right) \left(-\frac{b^2}{4p^2} \right)^k \\ [\operatorname{Re}(\alpha+\mu+\nu) > 0, \operatorname{Re} p > |\operatorname{Im} b| + |\operatorname{Im} c|]. \quad (3.5) \end{aligned}$$

This formula can also be deduced from Watson [12, Sect. 5.41(2)]. Watson [12, p. 399] gives also another form of (3.5), which in our notation reads

$$\begin{aligned} \int_0^\infty x^{\alpha-1} e^{-px} J_\mu(bx) J_\nu(cx) dx \\ = \frac{(\frac{1}{2}b)^\mu (\frac{1}{2}c)^\nu}{\Gamma(\nu+1)(c^2+p^2)^{(\alpha+\mu+\nu)/2}} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+\mu+\nu+2m)}{m! \Gamma(\mu+m+1)} \\ \times {}_2F_1 \left(\frac{\alpha+\mu+\nu}{2} + m, -\frac{\alpha+\mu-\nu-1}{2} - m; \nu+1; \frac{c^2}{c^2+p^2} \right) \left(-\frac{b^2}{4(c^2+p^2)} \right)^m. \quad (3.6) \end{aligned}$$

We observe that the hypergeometric function ${}_2F_1$ in this equation may be written as

$$\left(\frac{c^2 + p^2}{p^2}\right)^{(\alpha + \mu + \nu)/2 + m} {}_2F_1\left(\frac{\alpha + \mu + \nu}{2} + m, \frac{\alpha + \mu + \nu + 1}{2} + m; \nu + 1; -\frac{c^2}{p^2}\right).$$

Thus, by writing ${}_2F_1$ in (3.5) explicitly as a series over n and letting $k - n = m \geq 0$, it is easy to see that (3.5) and (3.6) are equivalent.

Prudnikov et al [9] give quite a number of useful special cases for the integral (3.5), whereas the general formula seems to be of little use in numerical applications. In particular, when p is small, difficulties of convergence are likely to occur in (3.5) if $p < b$.

By comparing (2.24) with (3.5), we find an interesting summation formula:

$$\begin{aligned} & \frac{a^{\nu-1/2} \rho^\nu}{\Gamma(\nu+1)} 2^{-2\nu+1/2} z^{-2\nu-1} \sum_{k=0}^{\infty} \frac{\Gamma(2\nu+2k+1)}{k! \Gamma(\nu+k+\frac{1}{2})} \\ & \times {}_2F_1\left(-k, \frac{1}{2} - \nu - k; \nu + 1; \frac{\rho^2}{a^2}\right) \left(-\frac{a^2}{4z^2}\right)^k = \sqrt{\frac{2}{\pi a}} \left(\frac{l_1}{l_2}\right)^\nu \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2}. \end{aligned}$$

The following notation is used in Prudnikov et al [11, No. 3.12.11]:

$$r = \frac{1}{4} [\sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2}]^2, \quad (3.7)$$

$$R^4 = (z^2 + \rho^2 - a^2)^2 + 4a^2 z^2, \quad (3.8)$$

$$\tan \vartheta = a/z, \quad (3.9)$$

$$2az \cot 2\eta = z^2 + \rho^2 - a^2, \quad (3.10)$$

$$u_{\pm} = 1/\sqrt{(z \pm ia)^2 + \rho^2}, \quad (3.11)$$

$$v_{\pm} = \rho^{-1} [z \pm ia + \sqrt{(z \pm ia)^2 + \rho^2}], \quad (3.12)$$

$$w_{\pm} = \frac{1}{\sqrt{2a}} \sqrt{\sqrt{(z^2 + \rho^2 - a^2)^2 + 4a^2 z^2} \pm (z^2 + \rho^2 - a^2)}, \quad (3.13)$$

$$\varphi = \arg(z^2 + \rho^2 - a^2 - 2iaz) \quad (-\pi < \varphi < \pi), \quad (3.14)$$

$$\psi = \nu \arcsin(2a/r). \quad (3.15)$$

Note that $2\eta = -\varphi$; hence, one of the definitions (3.10) or (3.14) is superfluous in any case. Note also that (3.15) for ψ is incorrect. The notation r in (3.7) corresponds to l_2^2 ; so the correct formula for ψ should read

$$\psi = \nu \arcsin(a/\sqrt{r}), \quad (3.16)$$

which corresponds to $\nu \arcsin(a/l_2)$. Thus, introducing just two parameters l_1 and l_2 simplifies the notation (3.7)–(3.16) considerably and effectively eliminates the need for them in all cases. For example, using l_1 and l_2 in (3.8) leads to a complete square, i.e., $R^4 = (l_2^2 - l_1^2)^2$.

Prudnikov et al [11] also use

$$z_{\pm} = (z \pm ia)^{-\mu-\nu-1} {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1; -\frac{\rho^2}{(z \pm ia)^2}\right). \quad (3.17)$$

Defining x, y by the two equations

$$\sinh(x + iy) = \frac{z + ia}{\rho}, \quad \sinh(x - iy) = \frac{z - ia}{\rho} \quad (3.18)$$

for complex z, a, ρ implies

$$\sinh x \cos y = z/\rho, \quad \cosh x \sin y = a/\rho. \quad (3.19)$$

The simplest solution is given by

$$\cosh x = l_2/\rho = a/l_1, \quad \sin y = a/l_2 = l_1/\rho \quad [\operatorname{Re} x > 0]. \quad (3.20)$$

Let now $u = x \pm iy$ be such that

$$\sinh u = \frac{z \pm ia}{\rho} \quad [\operatorname{Re} u > 0], \quad (3.21)$$

where *one* sign is chosen. This equation has two solutions: u and $(-u \pm i\pi)$ in e^u or e^{-u} . Imposing the condition $\operatorname{Re} u > 0$ makes e^u and e^{-u} uniquely defined. To enforce the condition $\operatorname{Re} u > 0$, take $\operatorname{Re} x > 0$ and $\operatorname{Re}(\pm iy) > 0$ in (3.20). This makes $e^x, e^{\pm iy}$, and e^u uniquely defined. If $\operatorname{Re}(\pm iy) = 0$, i.e., if y is real, (3.20) cannot distinguish between y and $\pi - y$; the choice is made by the sign of $\cos y$ in (3.19) when $\operatorname{Re} x > 0$.

Using (3.21) and a quadratic transformation of the hypergeometric function [10, No. 7.3.1.56]:

$$\begin{aligned} {}_2F_1\left(\frac{\mu + \nu + 1}{2}, \frac{\mu + \nu + 2}{2}; \nu + 1; -\frac{\rho^2}{(z \pm ia)^2}\right) \\ = (1 - e^{-2u})^{\mu + \nu + 1} {}_2F_1(\mu + \nu + 1, \mu + 1; \nu + 1; -e^{-2u}) \quad [\operatorname{Re} u > 0], \end{aligned} \quad (3.22)$$

the relation (3.17) can be written as

$$z_{\pm} = \left(\frac{2e^{-u}}{\rho}\right)^{\mu + \nu + 1} {}_2F_1(\mu + \nu + 1, \mu + 1; \nu + 1; -e^{-2u}).$$

When $z > 0, a > 0, \rho > 0$ are real, then

$$\begin{aligned} u &= x \pm iy, \\ \cosh x &= l_2/\rho = a/l_1 \geq 1 \quad [x \geq 0]; \\ \sin y &= a/l_2 = l_1/\rho \leq 1 \quad [\cos y > 0, \text{ i.e., } 0 \leq y \leq \pi/2]. \end{aligned} \quad (3.23)$$

Using l_1 and l_2 with (2.4) and (2.5) in (3.11)–(3.12) allows us to separate the real and imaginary parts in a simple manner:

$$\begin{aligned} u_{\pm} &= [\sqrt{(z \pm ia)^2 + \rho^2}]^{-1} = \left[\sqrt{l_2^2 - a^2 \pm i\sqrt{a^2 - l_1^2}}\right]^{-1} \\ &= [\operatorname{Re} e^{\mp i\varphi/2}]^{-1} = (\rho \cosh u)^{-1}, \\ v_{\pm} &= \frac{1}{a\rho} \left(a + \sqrt{a^2 - l_1^2}\right) \left(\sqrt{l_2^2 - a^2 \pm ia}\right) \\ &= \frac{a\rho}{(a - \sqrt{a^2 - l_1^2})\sqrt{l_2^2 - a^2 \mp ia}} = e^{x \pm iy} = e^u. \end{aligned} \quad (3.24)$$

The expression for $w_{\pm}$ in (3.13) can also be simplified:

$$w_+ = \frac{\sqrt{l_2^2 - a^2}}{a}, \quad w_- = \frac{\sqrt{a^2 - l_1^2}}{a}.$$

The argument of φ in (3.14) is transformed into a complete square:

$$\begin{aligned} \varphi &= \arg \left(\sqrt{l_2^2 - a^2} - i\sqrt{a^2 - l_1^2} \right)^2, \\ \frac{\varphi}{2} &= \arg \left(\sqrt{l_2^2 - a^2} - i\sqrt{a^2 - l_1^2} \right). \end{aligned} \quad (3.25)$$

In (3.25), the two square roots are even and odd functions of a , respectively.

Apelblat [1, p. 246] had introduced quantities A and B which are the same as w_+ and w_- defined later as (3.13) by Prudnikov et al. The notation R, φ is also used in Gradshteyn and Ryzhik [5, No. 6.753]. Several examples are considered below in order to show how the new notation allows us to simplify results given in the tables.

The following integral is given in Prudnikov et al [11, No. 3.12.11.12] (without the factor a^3 in the denominator):

$$\int_0^\infty x e^{-zx} \sin ax J_0(\rho x) dx = \frac{3w_+w_-(zw_+ + aw_-) - aw_+^3 - zw_-^3}{a^3(w_+^2 + w_-^2)^3} [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]. \quad (3.26)$$

The complexity of (3.13) obscures the obvious fact that $w_+w_- = z/a$. The same integral in our notation reads

$$\int_0^\infty x e^{-zx} \sin ax J_0(\rho x) dx = \frac{a^2 \sqrt{l_2^2 - a^2} [2a^2(a^2 + z^2) - l_1^2(a^2 + \rho^2 + z^2)]}{a(l_2^2 - l_1^2)^3} [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|].$$

Yet another example is taken from Apelblat [1, No. 12.4.72], which in a modified notation reads

$$\begin{aligned} \int_0^\infty e^{-zx} (1 - \cos ax) J_1(\rho x) \frac{dx}{x^2} &= \frac{a^2(1 - B) + zaA - z\sqrt{z^2 + \rho^2}}{2\rho} \\ &+ \frac{\rho}{4} \left\{ \ln \left[\left(A + \frac{z}{a} \right)^2 + (B + 1)^2 \right] - \ln \left(\frac{z + \sqrt{z^2 + \rho^2}}{a} \right)^2 \right\} \\ &[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]. \end{aligned} \quad (3.27)$$

The first logarithm in (3.27) is a complete square which can be combined with the second logarithm, and the whole expression can be simplified significantly to read

$$\begin{aligned} \int_0^\infty e^{-zx} (1 - \cos ax) J_1(\rho x) \frac{dx}{x^2} &= \frac{1}{2\rho} \left[a \left(a - \sqrt{a^2 - l_1^2} \right) + z \left(\sqrt{l_2^2 - a^2} - \sqrt{\rho^2 + z^2} \right) \right] + \frac{\rho}{2} \ln \frac{l_2 + \sqrt{l_2^2 - \rho^2}}{z + \sqrt{z^2 + \rho^2}} \\ &[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]. \end{aligned}$$

The integrals in (1.1) can be represented as the real and imaginary parts of the integral

$$I_\nu^\mu = \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^\mu dx. \quad (3.28)$$

For $\mu = 0$, Gradshteyn and Ryzhik [5, No. 6.611 1] give

$$\begin{aligned} \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) dx &= \frac{[\sqrt{(z-ia)^2 + \rho^2} - (z-ia)]^\nu}{\rho^\nu \sqrt{(z-ia)^2 + \rho^2}} \\ &= \frac{\exp\{-\nu \operatorname{arcsinh}(\frac{z-ia}{\rho})\}}{\sqrt{(z-ia)^2 + \rho^2}} \\ &\quad [\operatorname{Re} \nu > -1, \operatorname{Re}(z-ia) > |\operatorname{Im} \rho|]. \end{aligned} \quad (3.29)$$

The arguments of the numerator $e^{-\nu u}$ and of the denominator in (3.29) are given by (3.21) or (3.23) and $\varphi/2$ in (3.25), respectively.

Using the notation l_1 and l_2 in (3.29) allows us to separate the real and imaginary parts in a simple way:

$$\begin{aligned} \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) dx &= \frac{(a - \sqrt{a^2 - l_1^2})^\nu (\sqrt{l_2^2 - a^2} + ia)^\nu}{a^\nu \rho^\nu (\sqrt{l_2^2 - a^2} - i\sqrt{a^2 - l_1^2})} \\ &\quad [\operatorname{Re} \nu > -1, z > 0, a > 0, \rho > 0]. \end{aligned} \quad (3.30)$$

From (3.23) and (3.25), the arguments of the numerator and of the denominator are νy and $\varphi/2$, respectively.

The case of a positive integer $\mu = m$ in (3.28) can be calculated by differentiation of (3.30) with respect to z . The case of a negative integer $\mu = -m$ requires integration with respect to z . Both operations can be performed in a simple way by noting that

$$\frac{\partial}{\partial z} \left[\left(a \pm \sqrt{a^2 - l_1^2} \right) \left(\sqrt{l_2^2 - a^2} \mp ia \right) \right] = \pm \frac{(a \pm \sqrt{a^2 - l_1^2})(\sqrt{l_2^2 - a^2} \mp ia)}{\sqrt{l_2^2 - a^2} - i\sqrt{a^2 - l_1^2}}.$$

Integration of both sides of (3.30) with respect to z yields

$$\begin{aligned} \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) \frac{dx}{x} &= \frac{(a - \sqrt{a^2 - l_1^2})^\nu (\sqrt{l_2^2 - a^2} + ia)^\nu}{\nu a^\nu \rho^\nu} \\ &\quad [\operatorname{Re} \nu > 0, z > 0, a > 0, \rho > 0]. \end{aligned} \quad (3.31)$$

Formula (3.31) agrees with Gradshteyn and Ryzhik [5, No. 6.623 3]. A similar result in Prudnikov et al [11, No. 3.12.11.10] has a misprint: the parameter ϑ (as defined by (3.9)) should read ψ (see (3.16)).

Formula (3.31) can be integrated again with respect to z :

$$\begin{aligned} \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) \frac{dx}{x^2} &= \frac{\rho}{2\nu} \left[\frac{f^{\nu-1}}{\nu-1} + \frac{f^{\nu+1}}{\nu+1} \right] \\ &\quad [\operatorname{Re} \nu > 1, z > 0, a > 0, \rho > 0], \end{aligned} \quad (3.32)$$

where

$$f = \frac{(a - \sqrt{a^2 - l_1^2})(\sqrt{l_2^2 - a^2} + ia)}{a\rho} = e^{-u} \quad (3.33)$$

and where $\sinh u = (z - ia)/\rho$, $\operatorname{Re} u > 0$.

Formula (3.32) agrees with Prudnikov et al [11, No. 3.12.11.11]. From the structure of (3.32) it is clear that (3.31) can be integrated with respect to z as many times as necessary. For example, a third integration gives

$$\int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) \frac{dx}{x^3} = \frac{\rho^2}{4\nu} \left[\frac{1}{\nu-1} \left(\frac{f^{\nu-2}}{\nu-2} + \frac{f^\nu}{\nu} \right) + \frac{1}{\nu+1} \left(\frac{f^\nu}{\nu} + \frac{f^{\nu+2}}{\nu+2} \right) \right] \\ [\operatorname{Re} \nu > 2, z > 0, a > 0, \rho > 0]. \quad (3.34)$$

There seems to be no equivalent of this formula in the tables. The imaginary part of (3.34) remains finite when $\nu = 2$, but the result should be calculated as a limit for $\nu \rightarrow 2$.

There exists a general result in Gradshteyn and Ryzhik [5, No. 6.621 1] and in Prudnikov et al [11, No. 3.12.11.5]:

$$\int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^\mu dx \\ = \frac{\rho^\nu \Gamma(\mu + \nu + 1)}{2^\nu (z - ia)^{\mu + \nu + 1} \Gamma(\nu + 1)} {}_2F_1 \left(\frac{\mu + \nu + 1}{2}, \frac{\mu + \nu + 2}{2}; \nu + 1; -\frac{\rho^2}{(z - ia)^2} \right) \\ [\operatorname{Re}(\mu + \nu + 1) > 0, \operatorname{Re}(z - ia) > |\operatorname{Im} \rho|]. \quad (3.35)$$

Using (3.22) with $\sinh u = (z - ia)/\rho$, $\operatorname{Re} u > 0$, we obtain

$$\int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^\mu dx \quad (3.36) \\ = \frac{\rho^\nu \Gamma(\mu + \nu + 1)}{2^\nu (z - ia)^{\mu + \nu + 1} \Gamma(\nu + 1)} (1 - e^{-2u})^{\mu + \nu + 1} {}_2F_1(\mu + \nu + 1, \mu + 1; \nu + 1; -e^{-2u}) \\ = \left(\frac{\rho}{2} \right)^{-\mu-1} \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + 1)} f^{\mu + \nu + 1} {}_2F_1(\mu + \nu + 1, \mu + 1; \nu + 1; -f^2) \\ = \left(\frac{\rho}{2} \right)^{-\mu-1} \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + 1)} f^{\mu + \nu + 1} (1 + f^2)^{-2\mu-1} {}_2F_1(-\mu, \nu - \mu; \nu + 1; -f^2) \\ [\operatorname{Re}(\mu + \nu + 1) > 0, \operatorname{Re}(z - ia) > |\operatorname{Im} \rho|],$$

where $e^{-2u} = f^2$. The hypergeometric functions in the last two lines of (3.36) are polynomials when $\mu = -n - 1$, $n = 0, 1, 2, \dots$ and $\mu = n$ or $\mu = \nu + n$, respectively.

One may verify from the comparison of the previous results with (3.36) that the hypergeometric functions in (3.36) can be expressed in terms of elementary functions for any positive or negative integer μ . Thus (3.31), (3.32), and (3.34) are special cases of (3.36) for $\mu = -1$, $\mu = -2$, and $\mu = -3$, respectively. It will also turn into elementary functions for any $\mu = \nu + n$, where $n \geq 0$ is an integer.

For example, for $\mu = \nu + 1$, the last line of (3.36) gives

$$\int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^{\nu+1} dx \\ = \frac{1}{\sqrt{\pi}} 2^{2\nu+1} \Gamma\left(\nu + \frac{3}{2}\right) \left(\frac{\rho}{2}\right)^{-\nu-2} f^{2\nu+2} (1 + f^2)^{-2\nu-3} (1 - f^2) \\ = \frac{1}{\sqrt{\pi}} 2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right) \rho^\nu (\rho \cosh u)^{-2\nu-3} (\rho \sinh u),$$

because of $f = e^{-u}$. With (3.21) and (3.24), it may be written as

$$\int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^{\nu+1} dx = \frac{2^{\nu+1} \Gamma(\nu + \frac{3}{2})(z-ia)\rho^\nu}{\sqrt{\pi}(\sqrt{l_2^2 - a^2} - i\sqrt{a^2 - l_1^2})^{2\nu+3}} \quad [\operatorname{Re} \nu > -1, \operatorname{Re}(z-ia) > |\operatorname{Im} \rho|]. \quad (3.37)$$

In this particular case, the same result can also be obtained directly from (3.35) when using (3.24). It is in agreement with Gradshteyn and Ryzhik [5, Nos. 6.753 3, 6.753 4], apart from a misprint (read J_ν instead of I_ν), and with Prudnikov et al [11, No. 3.12.11.8].

The separation into real and imaginary parts in (3.37) can be achieved by rewriting this formula as

$$\begin{aligned} & \int_0^\infty e^{-(z-ia)x} J_\nu(\rho x) x^{\nu+1} dx \\ &= \frac{2(2\rho)^\nu \Gamma(\nu + \frac{3}{2})(z-ia)}{\sqrt{\pi}(l_2^2 - l_1^2)^{\nu+3/2}} \exp \left[i(2\nu + 3) \arctan \sqrt{\frac{a^2 - l_1^2}{l_2^2 - a^2}} \right] \\ &= \frac{2(2\rho)^\nu \Gamma(\nu + \frac{3}{2})(z-ia)}{\sqrt{\pi}(l_2^2 - l_1^2)^{\nu+3/2}} \exp \left[i(2\nu + 3) \arcsin \sqrt{\frac{a^2 - l_1^2}{l_2^2 - l_1^2}} \right] \\ & \quad [\operatorname{Re} \nu > -1, z > 0, a > 0, \rho > 0], \end{aligned}$$

where

$$\arcsin \sqrt{\frac{a^2 - l_1^2}{l_2^2 - l_1^2}} = -\frac{\varphi}{2}$$

is defined in (3.25). Numerous other examples can be considered. This is left to the interested reader.

4. Conclusions. The introduction of the parameters l_1 and l_2 according to (2.1) and (2.2) allows us to simplify significantly the calculation of various infinite integrals involving Bessel functions. It eliminates the need for numerous parameters (Eqs. (3.7) to (3.14)), which are used in various tables of integrals. It seems appropriate to rewrite the integrals in the tables in a unified form, utilizing the parameters l_1 and l_2 . In the case of complex integrals, the introduction of these parameters allows a simple separation of the real and imaginary parts. Some of the integrals given in the tables are found to be incorrect, and the corrected version is presented. The simplicity of integration and differentiation of expressions involving the parameters l_1 and l_2 allows us to calculate various integrals not given in the tables, for example, the integrals (2.31) and (3.34). An analogy has been pointed out between the integrals of type (1.1) and two-dimensional integrals over a circle, with the integrand involving distances between two points.

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Appendix. In this Appendix, we shall prove that (2.33) is valid for any $\nu > -\frac{1}{2}$. We start with the case $\nu > \frac{1}{2}$. Here we shall prove that each q -term of (2.33), when multiplied by $\cos(\nu\varphi)$, is a harmonic function. In cylindrical coordinates, a function $V(\rho, \varphi, z)$ is harmonic if

$$\Delta V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

In particular, for a function $V(\rho, \varphi, z) = \rho^\nu f(\rho, z) \cos(\nu\varphi)$, we obtain

$$\begin{aligned} \frac{\partial V}{\partial \rho} &= \left(\nu \rho^{\nu-1} f + \rho^\nu \frac{\partial f}{\partial \rho} \right) \cos(\nu\varphi), \\ \Delta V &= \left\{ \frac{1}{\rho} \left[\nu^2 \rho^{\nu-1} f + [\nu \rho^\nu + (\nu+1) \rho^\nu] \frac{\partial f}{\partial \rho} + \rho^{\nu+1} \frac{\partial^2 f}{\partial \rho^2} \right] - \nu^2 \rho^{\nu-2} f + \rho^\nu \frac{\partial^2 f}{\partial z^2} \right\} \cos(\nu\varphi) \\ &= \rho^\nu \left[\frac{2\nu+1}{\rho} \frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} + \frac{\partial^2 f}{\partial z^2} \right] \cos(\nu\varphi). \end{aligned}$$

The condition $\Delta V = 0$ is thus equivalent to

$$\mathcal{L}(f) \equiv \frac{2\nu+1}{\rho} \frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} + \frac{\partial^2 f}{\partial z^2} = 0. \quad (\text{A.1})$$

We shall show that each q -term in the series (2.33) satisfies (A.1). Therefore, we take

$$f_q(\rho, z) = a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q,$$

where the upper limit of integration $x = a/l_2$ is a function of (ρ, z) . We obtain

$$\begin{aligned} \frac{\partial f_q}{\partial x} &= a^{-2q} \frac{x^{2\nu+2q}}{\sqrt{1-x^2}} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q \\ &= \frac{l_2}{\sqrt{l_2^2 - a^2}} a^{-2q} \left(\frac{a}{l_2} \right)^{2\nu+2q} \left(\rho^2 + \frac{z^2 l_2^2}{l_2^2 - a^2} \right)^q = \frac{l_2}{\sqrt{l_2^2 - a^2}} \left(\frac{a}{l_2} \right)^{2\nu}, \end{aligned}$$

where from (2.6), $\rho^2 + z^2 l_2^2 / (l_2^2 - a^2) = l_2^2$. Thus

$$\begin{aligned} \frac{1}{\rho} \frac{\partial f_q}{\partial \rho} &= \frac{l_2^{-2\nu}}{\sqrt{l_2^2 - a^2}} a^{2\nu+1} \left[-\frac{1}{l_2^2} \frac{l_2^2 - a^2}{l_2^2 - l_1^2} \right] + a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} 2q \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \\ &= -a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} + a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} 2q \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1}, \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_q}{\partial \rho^2} &= \frac{\partial}{\partial \rho} \left[\rho \left(\frac{1}{\rho} \frac{\partial f_q}{\partial \rho} \right) \right] \\
&= \frac{1}{\rho} \frac{\partial f_q}{\partial \rho} + \rho \left\{ -a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} \right. \\
&\quad \times \left[-\frac{2\nu+2}{l_2^2} \frac{\rho(l_2^2 - a^2)}{l_2^2 - l_1^2} + \frac{\rho}{l_2^2 - l_1^2} - \frac{2\rho}{(l_2^2 - l_1^2)^2} (l_2^2 + l_1^2 - 2a^2) \right] \\
&\quad - 2qa^{2\nu+1} l_2^{-2\nu-4} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} \rho \\
&\quad \left. + 4q(q-1)a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-2} \rho \right\};
\end{aligned}$$

hence,

$$\begin{aligned}
&\frac{2\nu+1}{\rho} \frac{\partial f_q}{\partial \rho} + \frac{\partial^2 f_q}{\partial \rho^2} \\
&= (2\nu+2) \left\{ -a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} + 2qa^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \right\} \\
&\quad + \rho^2 \left\{ -a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} \right. \\
&\quad \times \left[-\frac{2\nu+2}{l_2^2} \frac{l_2^2 - a^2}{l_2^2 - l_1^2} + \frac{1}{l_2^2 - l_1^2} - \frac{2}{(l_2^2 - l_1^2)^2} (l_2^2 + l_1^2 - 2a^2) + \frac{2q}{l_2^2} \right] \\
&\quad \left. + 4q(q-1) \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-2} \right\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f_q}{\partial z} &= \frac{l_2^{-2\nu}}{\sqrt{l_2^2 - a^2}} a^{2\nu+1} \left(-\frac{z}{l_2^2 - l_1^2} \right) \\
&\quad + a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} 2q \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \frac{z}{1-x^2} \\
&= -a^{2\nu} l_2^{-2\nu} \frac{\sqrt{a^2 - l_1^2}}{l_2^2 - l_1^2} + 2qa^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \frac{z}{1-x^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_q}{\partial z^2} &= -a^{2\nu} l_2^{-2\nu} \frac{\sqrt{a^2 - l_1^2}}{l_2^2 - l_1^2} \left[-2\nu \frac{z}{l_2^2 - l_1^2} + \frac{1}{a^2 - l_1^2} \frac{z l_1^2}{l_2^2 - l_1^2} - \frac{2z(l_2^2 + l_1^2)}{(l_2^2 - l_1^2)^2} \right] \\
&\quad + 2qz \frac{l_2^{-2\nu}}{\sqrt{l_2^2 - a^2}} \frac{a^{2\nu+1}}{l_2^2 - a^2} \left(-\frac{z}{l_2^2 - l_1^2} \right) + 2qa^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \\
&\quad \times \left[2(q-1) \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-2} \frac{z^2}{(1-x^2)^2} + \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \frac{1}{1-x^2} \right].
\end{aligned}$$

By noting that from (2.5), $\sqrt{a^2 - l_1^2} = az/\sqrt{l_2^2 - a^2}$, we obtain

$$\begin{aligned} \mathcal{L}(f_q) &= a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{l_2^2 - l_1^2} \\ &\times \left[-(2\nu+2) + (2\nu+2) \frac{\rho^2}{l_2^2} \frac{l_2^2 - a^2}{l_2^2 - l_1^2} - \frac{\rho^2}{l_2^2 - l_1^2} + \frac{2\rho^2}{(l_2^2 - l_1^2)^2} (l_2^2 + l_1^2 - 2a^2) - 2q \frac{\rho^2}{l_2^2} \right] \\ &+ a^{2\nu+1} l_2^{-2\nu-2} \frac{z^2}{\sqrt{l_2^2 - a^2}} \frac{l_2^2}{l_2^2 - l_1^2} \left[\frac{2\nu}{l_2^2 - l_1^2} - \frac{\rho^2 - l_1^2}{z^2} \frac{1}{l_2^2 - l_1^2} + \frac{2(l_2^2 + l_1^2)}{(l_2^2 - l_1^2)^2} - \frac{2q}{l_2^2 - a^2} \right] \\ &+ (2\nu+2) 2q a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \\ &+ 4q(q-1) a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-2} \rho^2 \\ &+ 2q a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \frac{1}{1-x^2} \\ &+ 4q(q-1) a^{-2q} \int_0^x \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-2} \frac{z^2}{(1-x^2)^2}. \end{aligned}$$

We gather now all terms independent of q (they correspond to $q = 0$):

$$\begin{aligned} A &= a^{2\nu+1} l_2^{-2\nu-2} \frac{1}{\sqrt{l_2^2 - a^2} (l_2^2 - l_1^2)^2} \\ &\times \left\{ -(2\nu+2)(l_2^2 - a^2)(l_2^2 - l_1^2) + (2\nu+2) \frac{\rho^2}{l_2^2} (l_2^2 - a^2)^2 - \rho^2 (l_2^2 - a^2) \right. \\ &\quad \left. + \left[\frac{2\rho^2}{l_2^2 - l_1^2} (l_2^2 - a^2)(l_2^2 + l_1^2 - 2a^2) \right] + 2\nu z^2 l_2^2 - l_2^2 (\rho^2 - l_1^2) + \left[\frac{2z^2 l_2^2}{l_2^2 - l_1^2} (l_2^2 + l_1^2) \right] \right\}. \end{aligned}$$

We note that $l_2^2(\rho^2 - l_1^2) = l_2^2 \rho^2 - a^2 \rho^2$ and combine the terms in the two square brackets in A :

$$\begin{aligned} &\frac{2\rho^2}{l_2^2 - l_1^2} (l_2^2 - a^2)(l_2^2 + l_1^2 - 2a^2) + \frac{2(l_2^2 - \rho^2)(l_2^2 - a^2)}{l_2^2 - l_1^2} (l_2^2 + l_1^2) \\ &= 2 \frac{l_2^2 - a^2}{l_2^2 - l_1^2} [\rho^2 (l_2^2 + l_1^2 - 2a^2) + (l_2^2 - \rho^2)(l_2^2 + l_1^2)] \\ &= 2 \frac{l_2^2 - a^2}{l_2^2 - l_1^2} [l_2^2 (l_2^2 + l_1^2) - 2l_1^2 l_2^2] = 2l_2^2 (l_2^2 - a^2). \end{aligned}$$

We obtain

$$\begin{aligned} A &= a^{2\nu+1} l_2^{-2\nu-2} \frac{\sqrt{l_2^2 - a^2}}{(l_2^2 - l_1^2)^2} \\ &\times \left\{ (2\nu+2) \left[-(l_2^2 - l_1^2) + \frac{\rho^2}{l_2^2} (l_2^2 - a^2) \right] - \rho^2 + 2\nu(l_2^2 - \rho^2) - \rho^2 + 2l_2^2 \right\}. \end{aligned}$$

By noting that $-(l_2^2 - l_1^2) + (l_2^2 - a^2)\rho^2/l_2^2 = -l_2^2 + \rho^2$, the terms in the curly bracket reduce to $\{-2(l_2^2 - \rho^2) - 2\rho^2 + 2l_2^2\} = 0$. Thus the term for $q = 0$ yields a harmonic function, what was stated for the integral (2.19) or (2.23). We are left with

$$\begin{aligned}
\mathcal{L}(f_q) = & -2qa^{2\nu+1}l_2^{-2\nu-2}\frac{\sqrt{l_2^2-a^2}}{l_2^2-l_1^2}\left[\frac{\rho^2}{l_2^2}+\frac{l_2^2-\rho^2}{l_2^2-a^2}\right] \\
& +4q(\nu+1)a^{-2q}\int_0^x\frac{dx}{\sqrt{1-x^2}}x^{2\nu+2q}\left(\rho^2+\frac{z^2}{1-x^2}\right)^{q-1} \\
& +4q(q-1)a^{-2q}\rho^2\int_0^x\frac{dx}{(1-x^2)^{3/2}}x^{2\nu+2q}\left(\rho^2+\frac{z^2}{1-x^2}\right)^{q-2}(1-x^2) \\
& +4q\left(q-\frac{1}{2}\right)a^{-2q}\int_0^x\frac{dx}{(1-x^2)^{3/2}}x^{2\nu+2q}\left(\rho^2+\frac{z^2}{1-x^2}\right)^{q-1} \\
& -4q(q-1)a^{-2q}\rho^2\int_0^x\frac{dx}{(1-x^2)^{3/2}}x^{2\nu+2q}\left(\rho^2+\frac{z^2}{1-x^2}\right)^{q-2}.
\end{aligned}$$

Using the binomial theorem, we get

$$\begin{aligned}
\mathcal{L}(f_q) = & -2qa^{2\nu+1}l_2^{-2\nu-2}\frac{\sqrt{l_2^2-a^2}}{l_2^2-l_1^2}\frac{l_2^4-l_1^2l_2^2}{l_2^2(l_2^2-a^2)} \\
& +4q(\nu+1)\sum_{m=0}^{q-1}\binom{q-1}{m}a^{-2q}\rho^{2(q-1-m)}z^{2m}\int_0^x\frac{dx}{(1-x^2)^{m+1/2}}x^{2\nu+2q} \\
& -4q(q-1)\sum_{m=0}^{q-1}\binom{q-2}{m}a^{-2q}\rho^{2(q-1-m)}z^{2m}\int_0^x\frac{dx}{(1-x^2)^{m+3/2}}x^{2\nu+2q+2} \\
& +4q\left(q-\frac{1}{2}\right)\sum_{m=0}^{q-1}\binom{q-1}{m}a^{-2q}\rho^{2(q-1-m)}z^{2m}\int_0^x\frac{dx}{(1-x^2)^{m+3/2}}x^{2\nu+2q}.
\end{aligned}$$

The last two integrals can be transformed into the first one by using

$$\begin{aligned}
2\int_0^x\frac{dx}{(1-x^2)^{m+3/2}}x^{2\nu+2q+2} \\
= \frac{1}{m+\frac{1}{2}}(1-x^2)^{-m-1/2}x^{2\nu+2q+1} - \frac{2(\nu+q+\frac{1}{2})}{m+\frac{1}{2}}\int_0^x\frac{dx}{(1-x^2)^{m+1/2}}x^{2\nu+2q}
\end{aligned}$$

and

$$\begin{aligned}
2\int_0^x\frac{dx}{(1-x^2)^{m+3/2}}x^{2\nu+2q} &= 2\int_0^x\frac{dx}{(1-x^2)^{m+1/2}}\left(\frac{x^2}{1-x^2}+1\right)x^{2\nu+2q} \\
&= \frac{1}{m+\frac{1}{2}}(1-x^2)^{-m-1/2}x^{2\nu+2q+1} - \left[\frac{\nu+q+\frac{1}{2}}{m+\frac{1}{2}}-1\right]2\int_0^x\frac{dx}{(1-x^2)^{m+1/2}}x^{2\nu+2q}.
\end{aligned}$$

Noting that

$$(q-1)\binom{q-2}{m} = (q-1-m)\binom{q-1}{m},$$

we see that the terms containing the last remaining integral combine into

$$\begin{aligned}
& 4q\binom{q-1}{m}a^{-2q}\rho^{2(q-1-m)}z^{2m}\int_0^x\frac{dx}{(1-x^2)^{m+1/2}}x^{2\nu+2q} \\
& \times \left[(\nu+1)+(q-1-m)\frac{\nu+q+\frac{1}{2}}{m+\frac{1}{2}} - \left(q-\frac{1}{2}\right)\left(\frac{\nu+q+\frac{1}{2}}{m+\frac{1}{2}}-1\right)\right] = 0.
\end{aligned}$$

We are finally left with

$$\mathcal{L}(f_q) = -2qa^{2\nu+1}l_2^{-2\nu-2} \frac{1}{\sqrt{l_2^2 - a^2}} + B,$$

where

$$\begin{aligned} B &= 2q \sum_{m=0}^{q-1} \binom{q-1}{m} \left[-(q-1-m) + \left(q - \frac{1}{2} \right) \right] \\ &\quad \times \frac{1}{m + \frac{1}{2}} (1-x^2)^{-m-1/2} x^{2\nu+2q+1} a^{-2q} \rho^{2(q-1-m)} z^{2m} \\ &= 2qx^{2\nu+2q+1} a^{-2q} \frac{1}{\sqrt{1-x^2}} \sum_{m=0}^{q-1} \binom{q-1}{m} \rho^{2(q-1-m)} \left(\frac{z^2}{1-x^2} \right)^m \\ &= 2qx^{2\nu+2q+1} a^{-2q} \frac{1}{\sqrt{1-x^2}} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q-1} \quad (\text{where } x = a/l_2) \\ &= 2qa^{2\nu+1} \frac{1}{\sqrt{l_2^2 - a^2}} l_2^{-2\nu-2}, \end{aligned}$$

and hence $\mathcal{L}(f_q) = 0$.

We now treat the case $-\frac{1}{2} < \nu \leq \frac{1}{2}$. We assume that (2.33) is valid for $(\mu+1, \nu+1)$, i.e.,

$$\begin{aligned} \int_0^\infty e^{-zx} x^{\nu-\mu+2n} J_{\mu+1}(ax) J_{\nu+1}(\rho x) dx &= \frac{1}{\sqrt{\pi}} \left(\frac{a}{2} \right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a} \right)^{\nu+1} \\ &\times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \sum_{q=0}^\infty \frac{\Gamma(\nu+n+\frac{1}{2}+q+1)(\nu+n+\frac{1}{2}-\mu)_q}{q! \Gamma(\nu+\frac{1}{2}+q+1)} \\ &\times a^{-2q} \int_0^{a/l_2} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q. \end{aligned} \quad (\text{A.2})$$

We apply (2.27) and obtain

$$\begin{aligned} \int_0^\infty e^{-zx} x^{\nu-\mu+2n+1} J_\mu(ax) J_{\nu+1}(\rho x) dx &= \frac{1}{\sqrt{\pi}} \frac{a^{-\mu-1}}{2^{\mu-\nu-2n-1}} \\ &\times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \sum_{q=0}^\infty \frac{\Gamma(\nu+n+\frac{1}{2}+q+1)(\nu+n+\frac{1}{2}-\mu)_q}{q! \Gamma(\nu+\frac{1}{2}+q+1)} \\ &\times \left\{ \rho^{\nu+1} 2 \left(\mu-\nu-n-\frac{1}{2}-q \right) a^{2\mu-2\nu-2n-2q-2} \int_0^{a/l_2} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q \right. \\ &\quad \left. + \rho^{\nu+1} a^{2\mu-2\nu-2n-2q-1} \frac{1}{\sqrt{1-(a/l_2)^2}} l_2^{2q} \left(\frac{a}{l_2} \right)^{2\nu+2q+3} \frac{1}{a} \left[1 - \frac{a^2 - l_1^2}{l_2^2 - l_1^2} \right] \right\}. \end{aligned} \quad (\text{A.3})$$

The curly bracket in (A.3), which contains all the relevant variables, can be written as

$$\begin{aligned} &\left\{ -2\rho^{\nu+1} \left(\nu+n+\frac{1}{2}-\mu+q \right) a^{2\mu-2\nu-2n-2q-2} \int_0^{a/l_2} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q \right. \\ &\quad \left. + \rho^{\nu+1} a^{2\mu-2n+1} \frac{1}{\sqrt{l_2^2 - a^2}} \frac{1}{l_2^{2\nu+2}} \frac{l_2^2 - a^2}{l_2^2 - l_1^2} \right\}. \end{aligned}$$

We now apply (2.28) to this bracket and obtain

$$\begin{aligned}
& -\rho^\nu \int_{\infty}^{\rho} d\rho \rho^{-\nu} \{\dots\} \\
& = \rho^\nu \left\{ \int_{\infty}^{\rho} 2\rho d\rho \left(\nu + n + \frac{1}{2} - \mu + q \right) a^{2\mu-2\nu-2n-2q-2} \right. \\
& \quad \times \int_0^{\frac{a}{l_2}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^q \\
& \quad \left. - \int_{\infty}^{\rho} a^{2\mu-2n+1} \frac{1}{\sqrt{l_2^2-a^2}} \frac{1}{l_2^{2\nu+2}} \frac{l_2^2-a^2}{l_2^2-l_1^2} \rho d\rho \right\} \\
& = \rho^\nu \left\{ \left(\nu + n + \frac{1}{2} - \mu + q \right) a^{2\mu-2\nu-2n-2q-2} \frac{1}{q+1} \right. \\
& \quad \times \left[\int_0^{\frac{a}{l_2}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q+1} - \int_0^{\frac{a}{l_2}} \frac{d(a/l_2)}{\sqrt{1-(a/l_2)^2}} \left(\frac{a}{l_2} \right)^{2\nu+2q+2} l_2^{2q+2} \right] \\
& \quad \left. + a^{2\mu-2\nu-2n} \int_0^{\frac{a}{l_2}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu} \right\},
\end{aligned}$$

where we have used (2.29). In the first integral, the variable is a (with ρ and z constant), whereas in the last two integrals the variable is ρ (with a and z constant). We note that, when $\rho \rightarrow \infty, l_2 \rightarrow \infty$ such that $l_2/\rho \rightarrow 1$; therefore, the first integral is a definite integral that vanishes when $\rho \rightarrow \infty$. In the last two integrals, the limits of integration correspond to ρ and $\rho = \infty$. Thus

$$\begin{aligned}
& -\rho^\nu \int_{\infty}^{\rho} d\rho \rho^{-\nu} \{\dots\} \\
& = \rho^\nu a^{2\mu-2\nu-2n} \left\{ \frac{\nu + n + \frac{1}{2} - \mu + q}{q+1} a^{-2q-2} \int_0^{\frac{a}{l_2}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q+2} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q+1} \right. \\
& \quad \left. - \frac{\nu + n + \frac{1}{2} - \mu - 1}{q+1} \int_0^{\frac{a}{l_2}} \frac{x^{2\nu} dx}{\sqrt{1-x^2}} \right\}.
\end{aligned}$$

Applying (2.28) to (A.3) gives

$$\begin{aligned}
& \int_0^{\infty} e^{-zx} x^{\nu-\mu+2n} J_{\mu}(ax) J_{\nu}(\rho x) dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{2} \right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a} \right)^{\nu} \\
& \quad \times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \left\{ \sum_{q=0}^{\infty} \frac{\Gamma(\nu+n+\frac{1}{2}+q+1)(\nu+n+\frac{1}{2}-\mu)_{q+1}}{(q+1)! \Gamma(\nu+\frac{1}{2}+q+1)} \right. \\
& \quad \times a^{-2(q+1)} \int_0^{\frac{a}{l_2}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2(q+1)} \left(\rho^2 + \frac{z^2}{1-x^2} \right)^{q+1} \\
& \quad \left. - \sum_{q=0}^{\infty} \frac{\Gamma(\nu+n+\frac{1}{2}+q+1)(\nu+n-\frac{1}{2}-\mu)_{q+1}}{(q+1)! \Gamma(\nu+\frac{1}{2}+q+1)} \int_0^{\frac{a}{l_2}} \frac{x^{2\nu} dx}{\sqrt{1-x^2}} \right\},
\end{aligned}$$

where the last integral does not depend on q . We see that the last series over q can be rewritten as

$$\begin{aligned}
 & \sum_{q=1}^{\infty} \frac{\Gamma(\nu + n + \frac{1}{2} + q)(\nu + n - \frac{1}{2} - \mu)_q}{q! \Gamma(\nu + \frac{1}{2} + q)} \\
 &= \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} [{}_2F_1(\nu + n + \frac{1}{2}, \nu + n - \frac{1}{2} - \mu; \nu + \frac{1}{2}; 1) - 1] \\
 &= \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(n + \frac{1}{2})} \left[\frac{\Gamma(\nu + \frac{1}{2})\Gamma(\mu - \nu - 2n + \frac{1}{2})}{\Gamma(-n)\Gamma(\mu - n + 1)} - 1 \right] \\
 &= -\frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})}.
 \end{aligned}$$

Thus we obtain

$$\begin{aligned}
 & \int_0^{\infty} e^{-zx} x^{\nu-\mu+2n} J_{\mu}(ax) J_{\nu}(\rho x) dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^{\nu} \frac{1}{\Gamma(\mu - \nu - n + \frac{1}{2})} \\
 & \times \left\{ \sum_{q=0}^{\infty} \frac{\Gamma(\nu + n + \frac{1}{2} + q)(\nu + n + \frac{1}{2} - \mu)_q}{q! \Gamma(\nu + \frac{1}{2} + q)} a^{-2q} \int_0^{\frac{a}{\rho}} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q \right\},
 \end{aligned}$$

which is the same as (A.2) where $(\mu + 1, \nu + 1)$ is replaced by (μ, ν) . This completes the proof.

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