LINEAR STABILITY FOR A THERMOELECTROMAGNETIC MATERIAL WITH MEMORY

BY

GIOVAMBATTISTA AMENDOLA

*Dipartimento di Matematica Applicata "U. Dini", Facoltà di Ingegneria, University of Pisa, via
Diotisalvi 2, 56126-Pisa, Italy*

Abstract. In this paper we study the behaviour of a three-dimensional linear thermoelectromagnetic material, which has constitutive equations with memory effects for both the heat flux and the electric current density. We develop a linearized theory of thermodynamics, in which context we are able to introduce a maximal free energy defined in the frequency domain. Using this free energy, a domain of dependence is obtained. Moreover, we prove a theorem of uniqueness, existence, and asymptotic stability.

1. Introduction. Thermal effects in electromagnetism have been considered by Coleman and Dill [5], [6]. They derived the thermodynamic restrictions on the constitutive equations. The dissipative effects of heat conduction on the asymptotic decay of the solution have been studied in [10] for a dielectric nondissipative medium, but only for one-dimensional bodies.

In this paper we consider the linear theory of the thermodynamics of a homogeneous conductor. The material is characterized by two linearized constitutive equations for the heat flux and the electric current density. Both these constitutive functionals exhibit long-term memory.

In Sec. 2 we introduce the constitutive relations and the field equations, the latter consisting of Maxwell's equations and the energy equation. Then in Sec. 3 we consider the thermodynamics of simple materials [1], [2], which imposes restrictions on the constitutive equations. Moreover, we define a maximal free energy which allows us to derive an inequality, that is used in Sec. 4 to show a domain-of-dependence inequality. In the last section, theorems of existence, uniqueness, and asymptotic stability of solutions are proved for the three-dimensional model.

2. Constitutive equations. Let Ω be a region in the three-dimensional Euclidean space $\mathbf{R}^3$, that an electromagnetic solid $\mathcal{B}$ occupies at time t . We suppose that Ω is

Received January 5, 1999.

2000 *Mathematics Subject Classification.* Primary 35Q60, 78A25, 83C50.

Work performed under the support of C.N.R. and M.U.R.S.T..

a bounded and regular domain, that is simply-connected and has a smooth boundary, whose unit outward normal is $\mathbf{n}$.

We regard $\mathcal{B}$ as a homogeneous conductor with memory effects both for the electric current density $\mathbf{J}$ and for the heat flux $\mathbf{q}$ within the linear theory of thermoelectromagnetism. Thus, the following constitutive equations

$$\mathbf{D}(\mathbf{x}, t) = \varepsilon \mathbf{E}(\mathbf{x}, t) + \vartheta(\mathbf{x}, t) \mathbf{a}, \quad \mathbf{B}(\mathbf{x}, t) = \mu \mathbf{H}(\mathbf{x}, t), \quad (2.1)$$

$$\mathbf{J}(\mathbf{x}, t) = \int_0^{+\infty} \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds, \quad \mathbf{q}(\mathbf{x}, t) = - \int_0^{+\infty} k(s) \mathbf{g}^t(\mathbf{x}, s) ds, \quad (2.2)$$

$$h(\mathbf{x}, t) = c \dot{\vartheta}(\mathbf{x}, t) + \Theta_0 [\varepsilon^{-1} \mathbf{A}_1 \cdot \dot{\mathbf{D}}(\mathbf{x}, t) + \mu^{-1} \mathbf{A}_2 \cdot \dot{\mathbf{B}}(\mathbf{x}, t)] \quad (2.3)$$

are assumed for the electric displacement $\mathbf{D}$, the magnetic induction $\mathbf{B}$, the rate at which heat is absorbed per unit volume h , for $\mathbf{J}$ and $\mathbf{q}$. In these, $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic fields, respectively, ϑ is the temperature relative to the absolute reference temperature Θ_0 , uniform in Ω , $\mathbf{g} = \nabla \vartheta$ is the temperature gradient,

$$\mathbf{E}^t(s) = \mathbf{E}(t - s), \quad \mathbf{g}^t(s) = \mathbf{g}(t - s) \quad \forall s \in [0, +\infty) \quad (2.4)$$

are the histories of $\mathbf{E}$ and $\mathbf{g}$ up to time t . Moreover, the positive coefficients ε, μ , and c are, respectively, the dielectric constant, the permeability, and the specific heat, while $\mathbf{a}, \mathbf{A}_1$, and $\mathbf{A}_2$ are three constant vectors. Finally, both the electric conductivity $\sigma: \mathbf{R}^+ \rightarrow \mathbf{R}$ and the thermal conductivity $k: \mathbf{R}^+ \rightarrow \mathbf{R}$ are functions belonging to $L^1(\mathbf{R}^+) \cap H^1(\mathbf{R}^+)$.

It is useful to introduce the integrated histories of $\mathbf{E}$ and $\mathbf{g}$, that is, the functions $\bar{\mathbf{g}}^t(\mathbf{x}, \cdot): \mathbf{R}^+ \rightarrow \mathbf{R}^3$ and $\bar{\mathbf{E}}^t(\mathbf{x}, \cdot): \mathbf{R}^+ \rightarrow \mathbf{R}^3$ defined by

$$\bar{\mathbf{E}}^t(\mathbf{x}, s) = \int_{t-s}^t \mathbf{E}(\mathbf{x}, \tau) d\tau, \quad \bar{\mathbf{g}}^t(\mathbf{x}, s) = \int_{t-s}^t \mathbf{g}(\mathbf{x}, \tau) d\tau, \quad (2.5)$$

with which (2.2) can be rewritten [4] as

$$\mathbf{J}(\mathbf{x}, t) = - \int_0^{+\infty} \sigma'(s) \bar{\mathbf{E}}^t(\mathbf{x}, s) ds, \quad \mathbf{q}(\mathbf{x}, t) = \int_0^{+\infty} k'(s) \bar{\mathbf{g}}^t(\mathbf{x}, s) ds. \quad (2.6)$$

The field equations that we must consider are Maxwell's equations, whose local form is

$$\nabla \times \mathbf{H}(\mathbf{x}, t) = \dot{\mathbf{D}}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x}, t) + \mathbf{J}_f(\mathbf{x}, t), \quad \nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \quad (2.7)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\dot{\mathbf{B}}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{D}(\mathbf{x}, t) = \rho, \quad (2.8)$$

where $\mathbf{J}_f$ is a known function of $(\mathbf{x}, t)$, ρ is the free charge density, and the energy equation

$$h(\mathbf{x}, t) = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t), \quad (2.9)$$

r being the heat sources.

We observe that the free charge density ρ is determined by means of Eq. (2.8)₂.

Sometimes the dependence on $\mathbf{x}$ will be understood.

3. Thermodynamics of electromagnetic materials: its restrictions and a pseudo-free energy. The constitutive equations (2.1)–(2.3), that we have assumed, allow us to consider our electromagnetic solid as a simple material [1], [2], for which the laws of thermodynamics state that ([5], [6]) the following relations

$$\oint [h(t) + \mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t)] dt = 0, \quad (3.1)$$

$$\oint \{[\Theta_0 + \vartheta(t)]^{-1} h(t) + [\Theta_0 + \vartheta(t)]^{-2} \mathbf{q}(t) \cdot \mathbf{g}(t)\} dt \leq 0 \quad (3.2)$$

must hold for any cyclic process, the equality sign in (3.2) referring only to reversible processes. Here the mass density is supposed unitary.

It is well known that from these laws it is possible to prove the existence of the internal energy e and the entropy η and to derive, for smooth processes, the following relations [7]:

$$\dot{e}(t) = h(t) + \mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t), \quad (3.3)$$

$$\dot{\eta}(t) \geq [\Theta_0 + \vartheta(t)]^{-1} h(t) + [\Theta_0 + \vartheta(t)]^{-2} \mathbf{q}(t) \cdot \mathbf{g}(t). \quad (3.4)$$

We must consider the approximate expressions of (3.2) and (3.4), that is,

$$\Theta_0^{-2} \oint \{h(t)[\Theta_0 - \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)\} dt \leq 0, \quad (3.5)$$

$$\dot{\eta}(t) \geq \Theta_0^{-2} \{h(t)[\Theta_0 - \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)\}, \quad (3.6)$$

from which, on account of (3.1) and (3.3), we get

$$\Theta_0^{-2} \oint \{h(t)\vartheta(t) + \Theta_0[\mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t)] - \mathbf{q}(t) \cdot \mathbf{g}(t)\} dt \geq 0, \quad (3.7)$$

$$\Theta_0 \dot{\eta}(t) \geq \dot{e}(t) - [\mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t)] - \Theta_0^{-1} [\mathbf{q}(t) \cdot \mathbf{g}(t) - h(t)\vartheta(t)]. \quad (3.8)$$

If we introduce the approximate pseudo-free energy

$$\psi(\mathbf{x}, t) = e(\mathbf{x}, t) - \Theta_0 \eta(\mathbf{x}, t), \quad (3.9)$$

from (3.8) we derive

$$\dot{\psi}(t) \leq \mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t) + \Theta_0^{-1} [h(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t)]. \quad (3.10)$$

We remember that the potentials η and ψ are not uniquely determined in the presence of memory effects; we introduce the following definition.

DEFINITION 3.1. The potential ψ_M , such that (3.10) becomes the equality

$$\dot{\psi}_M(t) = \mathbf{E}(t) \cdot \dot{\mathbf{D}}(t) + \mathbf{H}(t) \cdot \dot{\mathbf{B}}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t) + \Theta_0^{-1} [h(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t)], \quad (3.11)$$

is called maximal free energy.

Let us consider the inequality (3.7), that, upon an integration on cycles, can be written as

$$\Theta_0^{-2} \oint \{h(t)\vartheta(t) - \Theta_0[\dot{\mathbf{E}}(t) \cdot \mathbf{D}(t) + \dot{\mathbf{H}}(t) \cdot \mathbf{B}(t) - \mathbf{E}(t) \cdot \mathbf{J}(t)] - \mathbf{q}(t) \cdot \mathbf{g}(t)\} dt \geq 0. \quad (3.12)$$

This expression, substituting

$$h(\mathbf{x}, t) = (c + \Theta_0 \varepsilon^{-1} \mathbf{A}_1 \cdot \mathbf{a})\dot{\vartheta}(\mathbf{x}, t) + \Theta_0[\mathbf{A}_1 \cdot \dot{\mathbf{E}}(\mathbf{x}, t) + \mathbf{A}_2 \cdot \dot{\mathbf{H}}(\mathbf{x}, t)], \quad (3.13)$$

derived by (2.3) taking account of (2.1)_{1,2}, becomes

$$\begin{aligned} \Theta_0^{-2} \oint \left\{ \Theta_0 \left([(\mathbf{A}_1 - \mathbf{a}) \cdot \dot{\mathbf{E}}(\mathbf{x}, t) + \mathbf{A}_2 \cdot \dot{\mathbf{H}}(\mathbf{x}, t)]\vartheta(\mathbf{x}, t) \right. \right. \\ \left. \left. + \int_0^{+\infty} \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds \cdot \mathbf{E}(\mathbf{x}, t) \right) + \int_0^{+\infty} k(s) \mathbf{g}^t(\mathbf{x}, s) ds \cdot \mathbf{g}(\mathbf{x}, t) \right\} dt \geq 0, \end{aligned} \quad (3.14)$$

since it holds for any cycle.

Taking account of the arbitrariness of ϑ , $\mathbf{E}$, and $\mathbf{g}$, from (3.14) we first obtain

$$\oint [(\mathbf{A}_1 - \mathbf{a}) \cdot \dot{\mathbf{E}}(\mathbf{x}, t) + \mathbf{A}_2 \cdot \dot{\mathbf{H}}(\mathbf{x}, t)]\vartheta(\mathbf{x}, t) dt \geq 0, \quad (3.15)$$

whence, since ϑ is independent of $\dot{\mathbf{E}}$ and $\dot{\mathbf{H}}$, it follows that

$$\mathbf{A}_1 = \mathbf{a}, \quad \mathbf{A}_2 = \mathbf{0}; \quad (3.16)$$

then, we also have

$$\begin{aligned} \oint \int_0^{+\infty} \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds \cdot \mathbf{E}(\mathbf{x}, t) &\geq 0, \\ \oint \int_0^{+\infty} k(s) \mathbf{g}^t(\mathbf{x}, s) ds \cdot \mathbf{g}(\mathbf{x}, t) &\geq 0. \end{aligned} \quad (3.17)$$

Thus, on account of (3.16), (3.13) becomes [10]

$$h(\mathbf{x}, t) = (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2)\dot{\vartheta}(\mathbf{x}, t) + \Theta_0 \mathbf{a} \cdot \dot{\mathbf{E}}(\mathbf{x}, t). \quad (3.18)$$

In (3.17) the equality signs correspond to constant histories; therefore, if we consider periodic processes of $\mathbf{g}(t)$ and of $\mathbf{E}(t)$ with the same frequency, we find that

$$\int_0^{+\infty} k(s) \cos(\omega s) ds > 0, \quad \int_0^{+\infty} \sigma(s) \cos(\omega s) ds > 0 \quad \forall \omega \neq 0. \quad (3.19)$$

If we introduce the Fourier transform of the function $f: \mathbf{R}^+ \rightarrow \mathbf{R}^n$, identified with its causal extension on $\mathbf{R}$,

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(s) \exp[-i\omega s] ds = f_c(\omega) - i f_s(\omega), \quad (3.20)$$

where we have considered the half-range Fourier cosine and sine transforms

$$f_c(\omega) = \int_0^{+\infty} f(s) \cos(\omega s) ds, \quad f_s(\omega) = \int_0^{+\infty} f(s) \sin(\omega s) ds, \quad (3.21)$$

relations (3.19) assume the following form:

$$k_c(\omega) > 0, \quad \sigma_c(\omega) > 0 \quad \forall \omega \neq 0. \quad (3.22)$$

We observe that, if f and $f' \in L^2(\mathbf{R}^+)$, then

$$\hat{f}'(\omega) = i\omega \hat{f}(\omega) - f(0), \quad f(0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) d\omega. \quad (3.23)$$

Moreover, we have

$$f'_s(\omega) = -\omega f_c(\omega), \quad f'_c(\omega) = \omega f_s(\omega) - f(0) \quad (3.24)$$

and their consequences

$$\omega k'_s(\omega) \leq 0, \quad k(0) = -\frac{2}{\pi} \int_0^{+\infty} \omega^{-1} k'_s(\omega) d\omega > 0, \quad (3.25)$$

$$\omega \sigma'_s(\omega) \leq 0, \quad \sigma(0) = -\frac{2}{\pi} \int_0^{+\infty} \omega^{-1} \sigma'_s(\omega) d\omega > 0. \quad (3.26)$$

If $k'' \in L^2(\mathbf{R}^+)$, $|k'(0)| < +\infty$ and $\sigma'' \in L^2(\mathbf{R}^+)$, $|\sigma'(0)| < +\infty$, it follows that

$$\sup_{\omega \in \mathbf{R}} |\omega k'_s(\omega)| < +\infty, \quad \lim_{\omega \rightarrow \infty} \omega k_s(\omega) = -\lim_{\omega \rightarrow \infty} \omega^2 k_c(\omega) = k'(0) \leq 0, \quad (3.27)$$

$$\sup_{\omega \in \mathbf{R}} |\omega \sigma'_s(\omega)| < +\infty, \quad \lim_{\omega \rightarrow \infty} \omega \sigma_s(\omega) = -\lim_{\omega \rightarrow \infty} \omega^2 \sigma_c(\omega) = \sigma'(0) \leq 0. \quad (3.28)$$

In the following we shall assume: $k_c(0) > 0$, $k'(0) < 0$ and $\sigma_c(0) > 0$, $\sigma'(0) < 0$.

Finally, if $\bar{\mathbf{g}}^t \in L^2(\mathbf{R}^+)$ and $\bar{\mathbf{E}}^t \in L^2(\mathbf{R}^+)$, we get

$$\dot{\bar{\mathbf{g}}}_c^t(\mathbf{x}, \omega) = -\omega \bar{\mathbf{g}}_s^t(\mathbf{x}, \omega), \quad \dot{\bar{\mathbf{g}}}_s^t(\mathbf{x}, \omega) = \omega^{-1} \mathbf{g}(\mathbf{x}, t) + \omega \bar{\mathbf{g}}_c^t(\mathbf{x}, \omega), \quad (3.29)$$

$$\dot{\bar{\mathbf{E}}}_c^t(\mathbf{x}, \omega) = -\omega \bar{\mathbf{E}}_s^t(\mathbf{x}, \omega), \quad \dot{\bar{\mathbf{E}}}_s^t(\mathbf{x}, \omega) = \omega^{-1} \mathbf{E}(\mathbf{x}, t) + \omega \bar{\mathbf{E}}_c^t(\mathbf{x}, \omega). \quad (3.30)$$

We can now prove the following theorem.

THEOREM 3.1. The potential ψ_M defined by

$$\begin{aligned} \psi_M(\mathbf{x}, t) = & \frac{1}{2} \left[\frac{c}{\Theta_0} \vartheta^2(\mathbf{x}, t) + \frac{1}{\varepsilon} \mathbf{D}^2(\mathbf{x}, t) + \frac{1}{\mu} \mathbf{B}^2(\mathbf{x}, t) \right] \\ & - \frac{1}{\pi} \int_0^{+\infty} \omega \sigma'_s(\omega) [\bar{\mathbf{E}}_s^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{E}}_s^t(\mathbf{x}, \omega) + \bar{\mathbf{E}}_c^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{E}}_c^t(\mathbf{x}, \omega)] d\omega \\ & - \frac{1}{\pi \Theta_0} \int_0^{+\infty} \omega k'_s(\omega) [\bar{\mathbf{g}}_s^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{g}}_s^t(\mathbf{x}, \omega) + \bar{\mathbf{g}}_c^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{g}}_c^t(\mathbf{x}, \omega)] d\omega \end{aligned} \quad (3.31)$$

is the maximal free energy; moreover, for any unit vector $\mathbf{u}$,

$$-[\Theta_0^{-1} \vartheta(\mathbf{x}, t) \mathbf{q}(\mathbf{x}, t) + \mathbf{E}(\mathbf{x}, t) \times \mathbf{H}(\mathbf{x}, t)] \cdot \mathbf{u} \leq \chi \psi_M(\mathbf{x}, t), \quad (3.32)$$

where

$$\chi = 2\{[k(0)/c]^{1/2} + 1/(\varepsilon\mu)[\varepsilon^{1/2} + |\mathbf{a}|(\Theta_0/c)^{1/2}]\mu^{1/2}\}. \quad (3.33)$$

Proof. On applying Plancherel's theorem to (2.6) and using (3.29) and (3.30), we get

$$\begin{aligned} \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) = & -\frac{d}{dt} \frac{1}{\pi} \int_0^{+\infty} \omega \sigma'_s(\omega) [\bar{\mathbf{E}}_s^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{E}}_s^t(\mathbf{x}, \omega) \\ & + \bar{\mathbf{E}}_c^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{E}}_c^t(\mathbf{x}, \omega)] d\omega, \end{aligned} \quad (3.34)$$

$$\begin{aligned} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t) = & \frac{d}{dt} \frac{1}{\pi} \int_0^{+\infty} \omega k'_s(\omega) [\bar{\mathbf{g}}_s^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{g}}_s^t(\mathbf{x}, \omega) \\ & + \bar{\mathbf{g}}_c^t(\mathbf{x}, \omega) \cdot \bar{\mathbf{g}}_c^t(\mathbf{x}, \omega)] d\omega, \end{aligned} \quad (3.35)$$

the latter of which has been derived in [10].

Then, the derivative of (3.31) with respect to time, taking into account (3.34), (3.35), (2.1)_{1.2} and (3.18), yields (3.11).

To establish the second assertion, we consider (3.31), from which, on account of (3.25)₁ and (3.26)₁, it follows that

$$|\vartheta| \leq (2\Theta_0 c^{-1} \psi_M)^{1/2}, \quad |\mathbf{D}| \leq (2\varepsilon \psi_M)^{1/2}, \quad |\mathbf{B}| \leq (2\mu \psi_M)^{1/2}, \quad (3.36)$$

whence (2.1)_{1.2} gives

$$|\mathbf{E}| \leq 2^{1/2} \varepsilon^{-1} [\varepsilon^{1/2} + |\mathbf{a}|(\Theta_0 c^{-1})^{1/2}] \psi_M^{1/2}, \quad |\mathbf{H}| \leq \mu^{-1} (2\mu \psi_M)^{1/2}; \quad (3.37)$$

moreover, (2.6)₂, as in [10], gives

$$|\mathbf{q}| \leq [2k(0)\Theta_0 \psi_M]^{1/2}; \quad (3.38)$$

thus, (3.32) follows. $\square$

4. Field equations and domain-of-dependence inequality. The system of field equations for the linear theory of thermoelectromagnetism consists of (2.7)–(2.9). Taking into account (2.1), (2.2), and (3.18), it becomes

$$\nabla \times \mathbf{H}(\mathbf{x}, t) = \varepsilon \dot{\mathbf{E}}(\mathbf{x}, t) + \dot{\vartheta}(\mathbf{x}, t) \mathbf{a} + \int_0^t \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds + \mathbf{J}_a(\mathbf{x}, t) + \mathbf{J}_f(\mathbf{x}, t), \quad (4.1)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\mu \dot{\mathbf{H}}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{H}(\mathbf{x}, t) = 0, \quad (4.2)$$

$$(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \dot{\vartheta}(\mathbf{x}, t) + \Theta_0 \mathbf{a} \cdot \dot{\mathbf{E}}(\mathbf{x}, t) = \nabla \cdot \int_0^t k(s) \mathbf{g}^t(\mathbf{x}, s) ds - \nabla \cdot \mathbf{q}_0(\mathbf{x}, t) + r(\mathbf{x}, t), \quad (4.3)$$

where

$$\mathbf{J}_a(\mathbf{x}, t) = \int_t^{+\infty} \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds = \int_0^{+\infty} \sigma(t + \tau) \mathbf{E}^0(\mathbf{x}, \tau) d\tau, \quad (4.4)$$

$$\mathbf{q}_0(\mathbf{x}, t) = - \int_t^{+\infty} k(s) \mathbf{g}^t(\mathbf{x}, s) ds = - \int_0^{+\infty} k(t + \tau) \mathbf{g}^0(\mathbf{x}, \tau) d\tau. \quad (4.5)$$

We must add to (4.1)–(4.3) the initial conditions

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}), \quad \mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \quad \vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}), \quad (4.6)$$

which are assumed consistent with the following boundary conditions:

$$\mathbf{E}(\mathbf{x}, t) \times \mathbf{n} = \mathbf{0}, \quad \vartheta(\mathbf{x}, t) = 0 \quad \forall (\mathbf{x}, t) \in \partial \Omega \times \mathbf{R}^+. \quad (4.7)$$

Moreover, we suppose $\mathbf{H}_0(\mathbf{x})$ is such that $\nabla \cdot \mathbf{H}_0(\mathbf{x}) = 0$ and $\mathbf{H}_0(\mathbf{x}) \cdot \mathbf{n} = 0$ on $\partial\Omega$. We observe that these conditions, by use of Maxwell's equation (2.8)₁ and (2.1)₂, imply that $\nabla \cdot \mathbf{H}(\mathbf{x}, t) = 0$ in Ω and $\mathbf{H}(\mathbf{x}, t) \cdot \mathbf{n} = 0$ on $\partial\Omega$ are satisfied for all t .

We can now prove that there exists a domain of dependence by introducing the total energy

$$E(D, t) = \int_D \psi_M(\mathbf{E}(\mathbf{x}, t), \mathbf{H}(\mathbf{x}, t), \vartheta(\mathbf{x}, t), \bar{\mathbf{g}}^t(\mathbf{x})) d\mathbf{x} \quad \forall D \subset \Omega. \quad (4.8)$$

THEOREM 4.1. Let $(\mathbf{E}, \mathbf{H}, \vartheta)$ be a solution of (4.1)–(4.3) and (4.6)–(4.7). Then

$$\begin{aligned} E(B(\mathbf{x}_0, \rho), T) &\leq E(B(\mathbf{x}_0, \rho + \chi T), 0) \\ &+ \int_0^T \int_{\Omega \cap B(\mathbf{x}_0, \rho + \chi(T-t))} [\Theta_0^{-1} r(\mathbf{x}, t) \vartheta(\mathbf{x}, t) - \mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t)] d\mathbf{x} dt \end{aligned} \quad (4.9)$$

for any fixed $(\mathbf{x}_0, T) \in \Omega \times \mathbf{R}^+$, where χ is given by (3.33) and $B(\mathbf{x}_0, \rho)$ is the ball of center $\mathbf{x}_0$ and radius ρ .

Proof. Let $(\mathbf{E}, \mathbf{H}, \vartheta)$ be a solution of our problem and $\phi(\mathbf{x}, t) \in C_0^\infty(\mathbf{R}^3, \mathbf{R}^+)$ be a nonnegative function. We define

$$E_\phi(\Omega, t) = \int_\Omega \psi_M(\mathbf{x}, t) \phi(\mathbf{x}, t) d\mathbf{x}, \quad (4.10)$$

whose derivative with respect to time, taking into account (3.11), (2.7)–(2.9), and (4.7), gives

$$\begin{aligned} \dot{E}_\phi(\Omega, t) &= \int_\Omega \{ [\Theta_0^{-1} r(\mathbf{x}, t) \vartheta(\mathbf{x}, t) - \mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t)] \phi(\mathbf{x}, t) \\ &+ [\Theta_0^{-1} \vartheta(\mathbf{x}, t) \mathbf{q}(\mathbf{x}, t) + \mathbf{E}(\mathbf{x}, t) \times \mathbf{H}(\mathbf{x}, t)] \cdot \nabla \phi(\mathbf{x}, t) + \psi_M(\mathbf{x}, t) \dot{\phi}(\mathbf{x}, t) \} d\mathbf{x}. \end{aligned} \quad (4.11)$$

If we put

$$\phi(\mathbf{x}, t) = \phi_\delta(\mathbf{x}, t) = \phi_\delta(|\mathbf{x} - \mathbf{x}_0| - \rho - \chi(T - t)) = \phi_\delta(y) = \begin{cases} 1 & \forall y \leq -\delta \\ 0 & \forall y > \delta \end{cases}, \quad (4.12)$$

where $\delta > 0$, $\rho > 0$ and χ is given by (3.33), for any fixed point $\mathbf{x}_0 \in \Omega$ and $T \in \mathbf{R}^+$, $t \in (0, T)$, $\phi_\delta \in C^\infty(\mathbf{R})$ and $\phi'_\delta(y) \leq 0$, then we have

$$\dot{\phi}_\delta(\mathbf{x}, t) = \phi'_\delta(y) \chi, \quad \nabla \phi_\delta(\mathbf{x}, t) = \phi'_\delta(y) \nabla |\mathbf{x} - \mathbf{x}_0| \equiv \phi'_\delta(y) \mathbf{u}(\mathbf{x}). \quad (4.13)$$

Substituting (4.13) into (4.11), the condition $\phi'_\delta(y) \leq 0$ and (3.32) allow us to derive the inequality

$$\dot{E}_\phi(\Omega, t) \leq \int_\Omega [\Theta_0^{-1} r(\mathbf{x}, t) \vartheta(\mathbf{x}, t) - \mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t)] \phi_\delta(\mathbf{x}, t) d\mathbf{x}, \quad (4.14)$$

which, with an integration over $(0, T)$, yields

$$E_{\phi_\delta}(\Omega, t) - E_{\phi_\delta}(\Omega, 0) \leq \int_0^T \int_\Omega [\Theta_0^{-1} r(\mathbf{x}, t) \vartheta(\mathbf{x}, t) - \mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t)] \phi_\delta(\mathbf{x}, t) d\mathbf{x} dt, \quad (4.15)$$

whence, the limit as $\delta \rightarrow 0$, since $\phi_\delta(\mathbf{x}, t)$ tends to the characteristic function of the ball of center $\mathbf{x}_0$ and radius $\rho + \chi(T - t)$, gives (4.9). $\square$

5. Existence, uniqueness, and asymptotic stability. The linear system of equations for the thermoelectromagnetic body is given by (4.1)–(4.3) together with the conditions (4.6) and (4.7). For simplicity, we can transform this problem into an equivalent one with zero initial data by putting in (4.1)–(4.3)

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \tilde{\mathbf{E}}(\mathbf{x}, t) + \mathbf{w}(\mathbf{x}, t), \\ \mathbf{H}(\mathbf{x}, t) &= \tilde{\mathbf{H}}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t), \\ \vartheta(\mathbf{x}, t) &= \tilde{\vartheta}(\mathbf{x}, t) + \alpha(\mathbf{x}, t),\end{aligned}$$

with $(\mathbf{w}, \mathbf{v}, \alpha)$ and $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\vartheta})$, that belong to the same function spaces of $(\mathbf{E}, \mathbf{H}, \vartheta)$, such that $\varepsilon \nabla \cdot \mathbf{w} = -\nabla \alpha \cdot \mathbf{a}$, $\nabla \cdot \mathbf{v} = 0$, and

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}), \quad \mathbf{w}_0(\mathbf{x}, \tau) = \mathbf{E}_0(\mathbf{x}, \tau), \quad \alpha_0(\mathbf{x}, \tau) = \vartheta_0(\mathbf{x}, \tau) \quad \forall (\mathbf{x}, \tau) \in \Omega \times (-\infty, 0],$$

but also changing the sources $\mathbf{J}_a + \mathbf{J}_f$ in (4.1) with

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{J}_f(\mathbf{x}, t) - \nabla \times \mathbf{v}(\mathbf{x}, t) + \varepsilon \dot{\mathbf{w}}(\mathbf{x}, t) + \dot{\alpha}(\mathbf{x}, t) \mathbf{a} + \int_0^{+\infty} \sigma(s) \mathbf{w}^t(\mathbf{x}, s) ds,$$

the sources $r - \nabla \cdot \mathbf{q}_0$ in (4.3) with

$$R(\mathbf{x}, t) = r(\mathbf{x}, t) + \nabla \cdot \int_0^{+\infty} k(s) \nabla \alpha^t(\mathbf{x}, s) ds - (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \dot{\alpha}(\mathbf{x}, t) - \Theta_0 \mathbf{a} \cdot \dot{\mathbf{w}}(\mathbf{x}, t)$$

and introducing a new source in $(4.2)_1$ given by

$$\mathbf{G}(\mathbf{x}, t) = -\nabla \times \mathbf{w}(\mathbf{x}, t) - \mu \dot{\mathbf{v}}(\mathbf{x}, t).$$

Thus, denoting again by $(\mathbf{E}, \mathbf{H}, \vartheta)$ the $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\vartheta})$, we can consider the following equivalent problem:

$$\nabla \times \mathbf{H}(\mathbf{x}, t) - \varepsilon \dot{\mathbf{E}}(\mathbf{x}, t) - \dot{\vartheta}(\mathbf{x}, t) \mathbf{a} - \int_0^t \sigma(s) \mathbf{E}^t(\mathbf{x}, s) ds = \mathbf{F}(\mathbf{x}, t), \quad (5.1)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) + \mu \dot{\mathbf{H}}(\mathbf{x}, t) = \mathbf{G}(\mathbf{x}, t), \quad (5.2)$$

$$(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \dot{\vartheta}(\mathbf{x}, t) + \Theta_0 \mathbf{a} \cdot \dot{\mathbf{E}}(\mathbf{x}, t) - \nabla \cdot \int_0^t k(s) \nabla \vartheta^t(\mathbf{x}, s) ds = R(\mathbf{x}, t) \quad (5.3)$$

for any $(\mathbf{x}, t) \in \Omega \times \mathbf{R}^+$, with

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{E}(\mathbf{x}, 0) = \mathbf{0}, \quad \vartheta(\mathbf{x}, 0) = 0, \quad (5.4)$$

$$\mathbf{E}(\mathbf{x}, t) \times \mathbf{n} = \mathbf{0}, \quad \vartheta(\mathbf{x}, t) = 0 \quad \forall (\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}^+. \quad (5.5)$$

We introduce the following function spaces:

$$\begin{aligned}
 \tilde{L}^2(\Omega) &= \{\mathbf{E}(\mathbf{x}) \in L^2(\Omega) : \int_{\Omega} \mathbf{E} \cdot \nabla \varphi \, d\mathbf{x} = 0 \quad \forall \varphi \in C_0^\infty(\Omega)\}, \\
 \tilde{H}^1(\Omega) &= \{\mathbf{H}(\mathbf{x}) \in H^1(\Omega) : \nabla \cdot \mathbf{H} = 0\}, \\
 H_E^1(\Omega) &= \{\mathbf{E}(\mathbf{x}) \in H^1(\Omega) : \mathbf{E}(\mathbf{x}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}\}, \\
 H_0^1(\Omega) &= \{\vartheta(\mathbf{x}) \in H^1(\Omega) : \vartheta(\mathbf{x})|_{\partial\Omega} = 0\}, \\
 H(\Omega, \mathbf{R}^+) &= L^2(\mathbf{R}^+; H_E^1(\Omega)) \times L^2(\mathbf{R}^+; \tilde{H}^1(\Omega)) \times L^2(\mathbf{R}^+; H_0^1(\Omega)), \\
 W(\Omega, \mathbf{R}^+) &= H^1(\mathbf{R}^+; L^2(\Omega)) \times H^1(\mathbf{R}^+; L^2(\Omega)) \\
 &\quad \times \{H^1(\mathbf{R}^+; L^2(\Omega)) \cap L^2(\mathbf{R}^+; H_0^1(\Omega))\}, \\
 V(\Omega, \mathbf{R}^+) &= L^2(\mathbf{R}^+; L^2(\Omega)) \times L^2(\mathbf{R}^+; \tilde{L}^2(\Omega)) \times L^2(\mathbf{R}^+; L^2(\Omega)).
 \end{aligned}$$

They are Hilbert spaces with the usual scalar products.

We suppose that the new sources $(\mathbf{F}, \mathbf{G}, R) \in V(\Omega, \mathbf{R}^+)$ are such that

$$\frac{\partial^{n+1}}{\partial t^{n+1}}(\mathbf{F}, \mathbf{G}, R) \in V(\Omega, \mathbf{R}^+), \quad \left[\frac{\partial^n}{\partial t^n}(\mathbf{F}, \mathbf{G}, R) \right]_{t=0} = 0 \quad (n = 0, 1, 2, 3), \quad (5.6)$$

the last of which can be obtained by choosing the corresponding derivatives of $\mathbf{w}, \mathbf{v}$, and α opportunely.

We denote by $\hat{H}(\Omega, \mathbf{R})$, $\hat{W}(\Omega, \mathbf{R})$, and $\hat{V}(\Omega, \mathbf{R})$ the spaces of the Fourier transforms with respect to time of the functions of $H(\Omega, \mathbf{R}^+)$, $W(\Omega, \mathbf{R}^+)$, and $V(\Omega, \mathbf{R}^+)$. Each of these spaces is isomorphic to the corresponding one by virtue of the Plancherel theorem that allows us to define in a natural way the scalar products by use of those introduced for $H(\Omega, \mathbf{R}^+)$, $W(\Omega, \mathbf{R}^+)$, and $V(\Omega, \mathbf{R}^+)$.

Thus, for example, we have the norm

$$\begin{aligned}
 \|(\hat{\mathbf{E}}(\mathbf{x}, \omega), \hat{\mathbf{H}}(\mathbf{x}, \omega), \hat{\vartheta}(\mathbf{x}, \omega))\|_{\hat{H}}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\Omega} (|\nabla \times \hat{\mathbf{E}}|^2 + |\nabla \times \hat{\mathbf{H}}|^2 \\
 &\quad + |\hat{\mathbf{E}}|^2 + |\hat{\mathbf{H}}|^2 + |\nabla \hat{\vartheta}|^2 + |\hat{\vartheta}|^2) \, d\mathbf{x} \, d\omega.
 \end{aligned} \quad (5.7)$$

We shall use $*$ to denote the complex conjugate.

DEFINITION 5.1. A triplet $(\mathbf{E}, \mathbf{H}, \vartheta) \in H(\Omega, \mathbf{R}^+)$ is called a weak solution of (5.1)–(5.5), with sources $(\mathbf{F}, \mathbf{G}, R) \in V(\Omega, \mathbf{R}^+)$ that satisfy (5.6), if

$$\begin{aligned}
 &\int_0^{+\infty} \int_{\Omega} \{-[\varepsilon \mathbf{E}(\mathbf{x}, t) + \vartheta(\mathbf{x}, t) \mathbf{a}] \cdot \dot{\mathbf{e}}(\mathbf{x}, t) + \mu \mathbf{H}(\mathbf{x}, t) \cdot \dot{\mathbf{h}}(\mathbf{x}, t) \\
 &\quad + [(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \vartheta(\mathbf{x}, t) + \Theta_0 \mathbf{a} \cdot \mathbf{E}(\mathbf{x}, t)] \dot{\beta}(\mathbf{x}, t) - \nabla \times \mathbf{H}(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{x}, t) \\
 &\quad - \nabla \times \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t) + \int_0^t \sigma(s) \mathbf{E}^t(\mathbf{x}, s) \, ds \cdot \mathbf{e}(\mathbf{x}, t) \\
 &\quad - \int_0^t k(s) \nabla \vartheta^t(\mathbf{x}, s) \, ds \cdot \nabla \beta(\mathbf{x}, t)\} \, d\mathbf{x} \, dt \\
 &\quad + \int_0^{+\infty} \int_{\Omega} [\mathbf{F}(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t) + R(\mathbf{x}, t) \beta(\mathbf{x}, t)] \, d\mathbf{x} \, dt = 0
 \end{aligned} \quad (5.8)$$

for any $(\mathbf{e}, \mathbf{h}, \beta) \in W(\Omega, \mathbf{R}^+)$ such that $\mathbf{e}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{h}(\mathbf{x}, 0) = \mathbf{0}$, $\beta(\mathbf{x}, 0) = 0$.

Plancherel's theorem allows us to transform our problem. Equation (5.8), on account of (3.23)₁ with zero initial data, becomes

$$\begin{aligned}
& \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} \{ (-i\omega[\varepsilon \widehat{\mathbf{E}}(\mathbf{x}, \omega) + \hat{\vartheta}(\mathbf{x}, \omega) \mathbf{a}] + \nabla \times \widehat{\mathbf{H}}(\mathbf{x}, \omega) - \hat{\sigma}(\omega) \widehat{\mathbf{E}}(\mathbf{x}, \omega)) \\
& \quad \cdot \hat{\mathbf{e}}^*(\mathbf{x}, \omega) + [i\omega\mu \widehat{\mathbf{H}}(\mathbf{x}, \omega) + \nabla \times \widehat{\mathbf{E}}(\mathbf{x}, \omega)] \cdot \hat{\mathbf{h}}^*(\mathbf{x}, \omega) \\
& \quad + i\omega[(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \hat{\vartheta}(\mathbf{x}, \omega) + \Theta_0 \mathbf{a} \cdot \widehat{\mathbf{E}}(\mathbf{x}, \omega)] \hat{\beta}^*(\mathbf{x}, \omega) \\
& \quad + \hat{k}(\omega) \nabla \hat{\vartheta}(\mathbf{x}, \omega) \cdot \nabla \hat{\beta}^*(\mathbf{x}, \omega) \} d\mathbf{x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} [\widehat{\mathbf{F}}(\mathbf{x}, \omega) \cdot \hat{\mathbf{e}}^*(\mathbf{x}, \omega) \\
& \quad + \widehat{\mathbf{G}}(\mathbf{x}, \omega) \cdot \hat{\mathbf{h}}^*(\mathbf{x}, \omega) + \widehat{R}(\mathbf{x}, \omega) \hat{\beta}^*(\mathbf{x}, \omega)] d\mathbf{x} d\omega
\end{aligned} \tag{5.9}$$

for any $(\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta}) \in \widehat{W}(\Omega, \mathbf{R})$.

Denoting by $A[(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \hat{\vartheta}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta})]$ the expression in the left-hand side of (5.9) and by $\langle (\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta}) \rangle$ the last integral of the same (5.9), we show the following theorem.

THEOREM 5.1. There exists one and only one solution $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \hat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R})$ for any $(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}) \in \widehat{V}(\Omega, \mathbf{R})$ such that

$$A[(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \hat{\vartheta}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta})] = \frac{1}{2\pi} \langle (\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta}) \rangle \quad \forall (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\beta}) \in \widehat{W}(\Omega, \mathbf{R}). \tag{5.10}$$

Proof. We first observe that, if $(\mathbf{E}, \mathbf{H}, \vartheta) \in H(\Omega, \mathbf{R}^+)$ is a weak solution of our problem, its transform $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \hat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R})$ is a weak solution of the following problem:

$$-i\omega[\varepsilon \widehat{\mathbf{E}}(\mathbf{x}, \omega) + \hat{\vartheta}(\mathbf{x}, \omega) \mathbf{a}] - \hat{\sigma}(\omega) \widehat{\mathbf{E}}(\mathbf{x}, \omega) + \nabla \times \widehat{\mathbf{H}}(\mathbf{x}, \omega) = \widehat{\mathbf{F}}(\mathbf{x}, \omega), \tag{5.11}$$

$$+i\omega\mu \widehat{\mathbf{H}}(\mathbf{x}, \omega) + \nabla \times \widehat{\mathbf{E}}(\mathbf{x}, \omega) = \widehat{\mathbf{G}}(\mathbf{x}, \omega), \tag{5.12}$$

$$i\omega[(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \hat{\vartheta}(\mathbf{x}, \omega) + \Theta_0 \mathbf{a} \cdot \widehat{\mathbf{E}}(\mathbf{x}, \omega)] - \nabla \cdot [\hat{k}(\omega) \nabla \hat{\vartheta}(\mathbf{x}, \omega)] = \widehat{R}(\mathbf{x}, \omega), \tag{5.13}$$

for almost all $\omega \in \mathbf{R}$, with

$$\widehat{\mathbf{E}}(\mathbf{x}, \omega) \times \mathbf{n} = \mathbf{0}, \quad \hat{\vartheta}(\mathbf{x}, \omega) = 0 \quad \forall (\mathbf{x}, \omega) \in \partial\Omega \times \mathbf{R}. \tag{5.14}$$

(*Uniqueness*). We must show because of linearity that the system with zero data has only the solution $\widehat{\mathbf{E}}^*(\mathbf{x}, \omega) = \mathbf{0}$, $\widehat{\mathbf{H}}^*(\mathbf{x}, \omega) = \mathbf{0}$, and $\hat{\vartheta}^*(\mathbf{x}, \omega) = 0$ for almost all $\omega \in \mathbf{R}$.

Upon taking the inner products of the modified (5.11)–(5.13) with $\widehat{\mathbf{E}}^*$, $\widehat{\mathbf{H}}^*$, and $\hat{\vartheta}^*$, respectively, we obtain

$$\begin{aligned}
& \int_{\Omega} \{ -[i\omega\varepsilon + \hat{\sigma}(\omega)] |\widehat{\mathbf{E}}(\mathbf{x}, \omega)|^2 - i\omega \hat{\vartheta}(\mathbf{x}, \omega) \mathbf{a} \cdot \widehat{\mathbf{E}}^*(\mathbf{x}, \omega) \\
& \quad + \nabla \times \widehat{\mathbf{H}}(\mathbf{x}, \omega) \cdot \widehat{\mathbf{E}}^*(\mathbf{x}, \omega) \} d\mathbf{x} = 0,
\end{aligned} \tag{5.15}$$

$$\int_{\Omega} [i\omega\mu |\widehat{\mathbf{H}}(\mathbf{x}, \omega)|^2 + \nabla \times \widehat{\mathbf{E}}(\mathbf{x}, \omega) \cdot \widehat{\mathbf{H}}^*(\mathbf{x}, \omega)] d\mathbf{x} = 0, \tag{5.16}$$

$$\begin{aligned}
& \int_{\Omega} \{ i\omega[(c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) |\hat{\vartheta}(\mathbf{x}, \omega)|^2 + \Theta_0 \mathbf{a} \cdot \widehat{\mathbf{E}}(\mathbf{x}, \omega) \hat{\vartheta}^*(\mathbf{x}, \omega)] \\
& \quad + \hat{k}(\omega) |\nabla \hat{\vartheta}(\mathbf{x}, \omega)|^2 \} d\mathbf{x} = 0.
\end{aligned} \tag{5.17}$$

Hence, the real part of (5.16) gives

$$\operatorname{Re} \int_{\Omega} \nabla \times \widehat{\mathbf{E}}(\mathbf{x}, \omega) \cdot \widehat{\mathbf{H}}^*(\mathbf{x}, \omega) d\mathbf{x} \equiv \operatorname{Re} \int_{\Omega} \widehat{\mathbf{E}}^*(\mathbf{x}, \omega) \cdot \nabla \times \widehat{\mathbf{H}}(\mathbf{x}, \omega) d\mathbf{x} = 0, \quad (5.18)$$

on account of (5.14)₁ too.

Moreover, the real parts of (5.15) and (5.17), by use of (5.18)₂, yield

$$\begin{aligned} \int_{\Omega} \sigma_c(\omega) |\widehat{\mathbf{E}}(\mathbf{x}, \omega)|^2 d\mathbf{x} &= -\operatorname{Im} \int_{\Omega} \omega \mathbf{a} \cdot \widehat{\mathbf{E}}(\mathbf{x}, \omega) \hat{\vartheta}^*(\mathbf{x}, \omega) d\mathbf{x} \\ &= -\Theta_0^{-1} \int_{\Omega} k_c(\omega) |\nabla \hat{\vartheta}(\mathbf{x}, \omega)|^2 d\mathbf{x}, \end{aligned} \quad (5.19)$$

that is,

$$\int_{\Omega} [\Theta_0 \sigma_c(\omega) |\widehat{\mathbf{E}}(\mathbf{x}, \omega)|^2 + k_c(\omega) |\nabla \hat{\vartheta}(\mathbf{x}, \omega)|^2] d\mathbf{x} = 0. \quad (5.20)$$

Thus, the conditions (3.22) allow us to write

$$\int_{\Omega} |\widehat{\mathbf{E}}(\mathbf{x}, \omega)|^2 d\mathbf{x} = 0, \quad \int_{\Omega} |\nabla \hat{\vartheta}(\mathbf{x}, \omega)|^2 d\mathbf{x} = 0, \quad (5.21)$$

where $\hat{\vartheta}(\mathbf{x}, \omega) \in H_0^1(\Omega)$. Therefore, it follows that

$$\int_{\Omega} |\hat{\vartheta}(\mathbf{x}, \omega)|^2 d\mathbf{x} = 0. \quad (5.22)$$

Hence a similar relation for $\widehat{\mathbf{H}}(\mathbf{x}, \omega)$ can be derived from (5.15) and (5.16), or directly from the modified (5.12), which reduces to

$$i\omega \widehat{\mathbf{H}}(\mathbf{x}, \omega) = \mathbf{0}. \quad (5.23)$$

(*Existence*). The proof is based on the following lemmas.

LEMMA 5.1. Let

$$\mathcal{I}(\omega) = \int_{\Omega} (|\nabla \times \widehat{\mathbf{H}}|^2 + |\nabla \times \widehat{\mathbf{E}}|^2 + |\widehat{\mathbf{H}}|^2 + |\widehat{\mathbf{E}}|^2 + |\nabla \hat{\vartheta}|^2 + |\hat{\vartheta}|^2) d\mathbf{x}. \quad (5.24)$$

Then

$$\mathcal{I}(\omega) \leq \eta^2(\omega) \int_{\Omega} (|\widehat{\mathbf{F}}|^2 + |\widehat{\mathbf{G}}|^2 + |\widehat{\mathbf{R}}|^2) d\mathbf{x}, \quad (5.25)$$

where $\eta(\omega)$ is a positive function of ω , the material constants and Ω .

Proof. Let us consider the system (5.11)–(5.13) and the relations

$$\begin{aligned} -[i\omega\varepsilon + \hat{\sigma}(\omega)] \int_{\Omega} |\widehat{\mathbf{E}}|^2 d\mathbf{x} - i\omega \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \widehat{\mathbf{E}}^* d\mathbf{x} \\ + \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} = \int_{\Omega} \widehat{\mathbf{F}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x}, \end{aligned} \quad (5.26)$$

$$\begin{aligned} -[i\omega\varepsilon + \hat{\sigma}(\omega)] \mathbf{a} \cdot \int_{\Omega} \widehat{\mathbf{E}} \hat{\vartheta}^* d\mathbf{x} - i\omega \mathbf{a}^2 \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} \\ + \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \mathbf{a} \hat{\vartheta}^* d\mathbf{x} = \int_{\Omega} \widehat{\mathbf{F}} \cdot \mathbf{a} \hat{\vartheta}^* d\mathbf{x}, \end{aligned} \quad (5.27)$$

$$\begin{aligned}
& -[i\omega\varepsilon + \hat{\sigma}(\omega)] \int_{\Omega} \widehat{\mathbf{E}} \cdot \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} - i\omega \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} \\
& + \int_{\Omega} |\nabla \times \widehat{\mathbf{H}}|^2 d\mathbf{x} = \int_{\Omega} \widehat{\mathbf{F}} \cdot \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x},
\end{aligned} \tag{5.28}$$

derived by integrating (5.11) after taking the inner products with $\widehat{\mathbf{E}}^*$, $\hat{\vartheta}^* \mathbf{a}$, and $\nabla \times \widehat{\mathbf{H}}^*$. Moreover,

$$i\omega\mu \int_{\Omega} |\widehat{\mathbf{H}}|^2 d\mathbf{x} + \int_{\Omega} \nabla \times \widehat{\mathbf{E}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} = \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x}, \tag{5.29}$$

$$i\omega\mu \int_{\Omega} \widehat{\mathbf{H}} \cdot \nabla \times \widehat{\mathbf{E}}^* d\mathbf{x} + \int_{\Omega} |\nabla \times \widehat{\mathbf{E}}|^2 d\mathbf{x} = \int_{\Omega} \widehat{\mathbf{G}} \cdot \nabla \times \widehat{\mathbf{E}}^* d\mathbf{x}, \tag{5.30}$$

which again follow by taking the inner products of (5.12) with $\widehat{\mathbf{H}}^*$ and $\nabla \times \widehat{\mathbf{E}}^*$, and finally

$$i\omega(c + \Theta_0\varepsilon^{-1}\mathbf{a}^2) \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} + i\omega\Theta_0\mathbf{a} \cdot \int_{\Omega} \widehat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} + \hat{k}(\omega) \int_{\Omega} |\nabla\hat{\vartheta}|^2 d\mathbf{x} = \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x}, \tag{5.31}$$

obtained by multiplying (5.13) by $\hat{\vartheta}^*$.

We first observe that the real parts of (5.28) and (5.30) yield

$$\begin{aligned}
& \int_{\Omega} |\nabla \times \widehat{\mathbf{H}}|^2 d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} - \omega \operatorname{Im} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} \\
& + \sigma_c(\omega) \operatorname{Re} \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} - [\sigma_s(\omega) - \varepsilon\omega] \operatorname{Im} \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x},
\end{aligned} \tag{5.32}$$

$$\int_{\Omega} |\nabla \times \widehat{\mathbf{E}}|^2 d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \nabla \times \widehat{\mathbf{E}}^* d\mathbf{x} + \mu\omega \operatorname{Im} \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x}, \tag{5.33}$$

while the real part of (5.31) gives

$$k_c(\omega) \int_{\Omega} |\nabla\hat{\vartheta}|^2 d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x} - \Theta_0\omega \operatorname{Im} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \widehat{\mathbf{E}}^* d\mathbf{x}. \tag{5.34}$$

From (5.29), its real part, given by

$$\operatorname{Re} \int_{\Omega} \nabla \times \widehat{\mathbf{H}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} \tag{5.35}$$

allows us to derive from the real part of (5.26)

$$\omega \operatorname{Im} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \widehat{\mathbf{E}}^* d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} - \operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} + \sigma_c(\omega) \int_{\Omega} |\widehat{\mathbf{E}}|^2 d\mathbf{x}. \tag{5.36}$$

Thus, from (5.34), using (5.36), we get

$$\begin{aligned}
& k_c(\omega) \int_{\Omega} |\nabla\hat{\vartheta}|^2 d\mathbf{x} + \Theta_0\sigma_c(\omega) \int_{\Omega} |\widehat{\mathbf{E}}|^2 d\mathbf{x} = \operatorname{Re} \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x} \\
& + \Theta_0 \left(\operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} - \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} \right).
\end{aligned} \tag{5.37}$$

Let us consider the imaginary part of (5.31):

$$\begin{aligned} \omega \operatorname{Re} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \hat{\mathbf{E}}^* d\mathbf{x} &= \Theta_0^{-1} \left[\operatorname{Im} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} + k_s(\omega) \int_{\Omega} |\nabla \hat{\vartheta}|^2 d\mathbf{x} \right. \\ &\quad \left. - (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \omega \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} \right], \end{aligned} \quad (5.38)$$

which, again from the same (5.34), allows us to get the imaginary part of (5.27) in the form

$$\begin{aligned} \operatorname{Im} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \nabla \times \hat{\mathbf{H}}^* d\mathbf{x} &= -\operatorname{Im} \int_{\Omega} \hat{\mathbf{F}} \cdot \mathbf{a} \hat{\vartheta}^* d\mathbf{x} \\ &\quad + \frac{1}{\omega \Theta_0} \left\{ \sigma_c(\omega) \operatorname{Re} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} + [\sigma_s(\omega) - \varepsilon \omega] \operatorname{Im} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} \right\} \\ &\quad + \Theta_0^{-1} \left\{ \frac{1}{\omega} (k_s(\omega) [\sigma_s(\omega) - \varepsilon \omega] - k_c(\omega) \sigma_c(\omega)) \int_{\Omega} |\nabla \hat{\vartheta}|^2 d\mathbf{x} \right. \\ &\quad \left. + [c \varepsilon \omega - (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \sigma_s(\omega)] \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} \right\}. \end{aligned} \quad (5.39)$$

From the real part of (5.27), on account of (5.36) and (5.38), we have

$$\begin{aligned} \omega \operatorname{Re} \mathbf{a} \cdot \int_{\Omega} \hat{\vartheta} \nabla \times \hat{\mathbf{H}}^* d\mathbf{x} &= \omega \operatorname{Re} \int_{\Omega} \hat{\mathbf{F}} \cdot \mathbf{a} \hat{\vartheta}^* d\mathbf{x} \\ &\quad + \Theta_0^{-1} \left\{ \sigma_c(\omega) \operatorname{Im} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} - [\sigma_s(\omega) - \varepsilon \omega] \operatorname{Re} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} \right\} \\ &\quad + \Theta_0^{-1} \left\{ (k_c(\omega) [\sigma_s(\omega) - \varepsilon \omega] + k_s(\omega) \sigma_c(\omega)) \int_{\Omega} |\nabla \hat{\vartheta}|^2 d\mathbf{x} \right. \\ &\quad \left. - (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \omega \sigma_c(\omega) \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} \right\}, \end{aligned} \quad (5.40)$$

which, together with (5.35), allows us to obtain from the imaginary part of (3.28) the last term, which we had to derive for (5.32) and (5.33),

$$\begin{aligned} \operatorname{Im} \int_{\Omega} \nabla \times \hat{\mathbf{H}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} &= \frac{1}{\sigma_c(\omega)} \left\{ \operatorname{Im} \int_{\Omega} \hat{\mathbf{F}} \cdot \nabla \times \hat{\mathbf{H}}^* d\mathbf{x} + \omega \operatorname{Re} \int_{\Omega} \hat{\mathbf{F}} \cdot \mathbf{a} \hat{\vartheta}^* d\mathbf{x} \right. \\ &\quad \left. - [\sigma_s(\omega) - \varepsilon \omega] \operatorname{Re} \int_{\Omega} \hat{\mathbf{G}} \cdot \hat{\mathbf{H}}^* d\mathbf{x} \right\} \\ &\quad + \Theta_0^{-1} \left\{ \operatorname{Im} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} - \frac{1}{\sigma_c(\omega)} [\sigma_s(\omega) - \varepsilon \omega] \operatorname{Re} \int_{\Omega} \hat{R} \hat{\vartheta}^* d\mathbf{x} \right\} \\ &\quad + \Theta_0^{-1} \left\{ \left(\frac{k_c(\omega)}{\sigma_c(\omega)} [\sigma_s(\omega) - \varepsilon \omega] + k_s(\omega) \right) \int_{\Omega} |\nabla \hat{\vartheta}|^2 d\mathbf{x} \right. \\ &\quad \left. - (c + \Theta_0 \varepsilon^{-1} \mathbf{a}^2) \omega \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} \right\}. \end{aligned} \quad (5.41)$$

We next observe that the boundary conditions (5.14) assure that there exist two positive constants, depending on Ω , such that

$$\begin{aligned} \int_{\Omega} |\hat{\vartheta}|^2 d\mathbf{x} &\leq \alpha_{,,}(\Omega) \int_{\Omega} |\nabla \hat{\vartheta}|^2 d\mathbf{x}, \\ \int_{\Omega} |\hat{\mathbf{H}}|^2 d\mathbf{x} &\leq \alpha_*(\Omega) \int_{\Omega} |\nabla \times \hat{\mathbf{H}}|^2 d\mathbf{x}, \end{aligned} \quad (5.42)$$

the second of which follows from (5.14)₁ taking into account that $\mathbf{H} \in \tilde{H}^1(\Omega)$ and that Ω is simply connected [9]. Therefore, from (5.24) it follows that

$$\mathcal{I}(\omega) \leq \int_{\Omega} \{[1 + \alpha_*(\Omega)]|\nabla \times \hat{\mathbf{H}}|^2 + |\nabla \times \hat{\mathbf{E}}|^2 + [1 + \alpha_{,,}(\Omega)](|\nabla \hat{\vartheta}|^2 + |\hat{\mathbf{E}}|^2)\} d\mathbf{x}. \quad (5.43)$$

Then, the positiveness of $k_c(\omega)$, $\sigma_c(\omega) \forall \omega \in \mathbf{R}$ and of Θ_0 allows us to derive from (5.43) the following inequality:

$$\begin{aligned} \mathcal{I}(\omega) &\leq \int_{\Omega} \left\{ \alpha_2 |\nabla \times \hat{\mathbf{H}}|^2 + |\nabla \times \hat{\mathbf{E}}|^2 \right. \\ &\quad \left. + \alpha_1 \left[\frac{1}{k_c(\omega)} + \frac{1}{\Theta_0 \sigma_c(\omega)} \right] [k_c(\omega) |\nabla \hat{\vartheta}|^2 + \Theta_0 \sigma_c(\omega) |\hat{\mathbf{E}}|^2] \right\} d\mathbf{x}, \end{aligned} \quad (5.44)$$

where we have put

$$\begin{aligned} \alpha_1 &= 1 + \alpha_*(\Omega), \\ \alpha_2 &= 1 + \alpha_{,,}(\Omega) \end{aligned} \quad (5.45)$$

and introduced the expression in the left-hand side of (5.37).

Substituting (5.37) and the relations derived from (5.32) and (5.33), on account of (5.39), (5.35) and (5.41), (5.44) yields

$$\begin{aligned}
 \mathcal{I}(\omega) \leq & \left(\alpha_1 \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} + \frac{1}{\sigma_c(\omega)} [(\varepsilon\alpha_1 + \mu)\omega - \alpha_1\sigma_s(\omega)] \operatorname{Im} \int_{\Omega} \widehat{\mathbf{F}} \cdot \nabla \times \widehat{\mathbf{H}}^* d\mathbf{x} \right) + \left(\operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \nabla \times \widehat{\mathbf{E}}^* d\mathbf{x} \right) \\
 & + \left(\left\{ \frac{1}{\sigma_c(\omega)} \{ \alpha_1(\sigma_c^2(\omega) + [\varepsilon\omega - \sigma_s(\omega)]^2) + \mu\omega[\varepsilon\omega - \sigma_s(\omega)] \} \right. \right. \\
 & \quad \left. \left. + \alpha_2 \left[\frac{\Theta_0}{k_c(\omega)} + \frac{1}{\sigma_c(\omega)} \right] \right\} \operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} \right) \\
 & + \left(\frac{1}{\sigma_c(\omega)} [(\varepsilon\alpha_1 + \mu)\omega - \alpha_1\sigma_s(\omega)] \omega \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \mathbf{a}\hat{\vartheta}^* d\mathbf{x} + \alpha_1\omega \operatorname{Im} \int_{\Omega} \widehat{\mathbf{F}} \cdot \mathbf{a}\hat{\vartheta}^* d\mathbf{x} \right) \\
 & + \left(\Theta_0^{-1} \left\{ \frac{1}{\sigma_c(\omega)} \{ \alpha_1([\varepsilon\omega - \sigma_s(\omega)]^2 - \sigma_c^2(\omega)) + \mu\omega[\varepsilon\omega - \sigma_s(\omega)] \} \right. \right. \\
 & \quad \left. \left. + \alpha_2 \left[\frac{\Theta_0}{k_c(\omega)} + \frac{1}{\sigma_c(\omega)} \right] \right\} \operatorname{Re} \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x} \right. \\
 & \quad \left. + \Theta_0^{-1} \{ 2\alpha_1[\varepsilon\omega - \sigma_s(\omega)] + \mu\omega \} \operatorname{Im} \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x} \right) \\
 & + \left(-\alpha_2 \left[\frac{\Theta_0}{k_c(\omega)} + \frac{1}{\sigma_c(\omega)} \right] \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} \right) \\
 & + \Theta_0^{-1} \left\{ \alpha_1 \left\{ \frac{1}{\sigma_c(\omega)} (\sigma_c^2(\omega) - [\varepsilon\omega - \sigma_s(\omega)]^2) + 2\frac{k_s(\omega)}{k_c(\omega)} [\varepsilon\omega - \sigma_s(\omega)] \right\} \right. \\
 & \quad \left. + \mu\omega \left\{ \frac{k_s(\omega)}{k_c(\omega)} - \frac{1}{\sigma_c(\omega)} [\varepsilon\omega - \sigma_s(\omega)] \right\} \right\} \int_{\Omega} k_c(\omega) |\nabla\hat{\vartheta}|^2 d\mathbf{x} \\
 & + \omega \frac{1}{k_c(\omega)} \left\{ \alpha_1 \left[2 \left(\frac{c}{\Theta_0} + \frac{\mathbf{a}^2}{\varepsilon} \right) \sigma_s(\omega) - \varepsilon \left(\frac{2c}{\Theta_0} + \frac{\mathbf{a}^2}{\varepsilon} \right) \omega \right] \right. \\
 & \quad \left. - \mu \left(\frac{c}{\Theta_0} + \frac{\mathbf{a}^2}{\varepsilon} \right) \omega \right\} \int_{\Omega} k_c(\omega) |\hat{\vartheta}|^2 d\mathbf{x}.
 \end{aligned} \tag{5.46}$$

Using (5.42)₁ and (5.37), the last two integrals of this inequality, denoting by $\xi_{\nabla\vartheta}(\omega)$ and $\xi_{\vartheta}(\omega)$ the corresponding coefficients, can be transformed in the following manner:

$$\begin{aligned}
 & \xi_{\nabla\vartheta}(\omega) \int_{\Omega} k_c(\omega) |\nabla\hat{\vartheta}|^2 d\mathbf{x} + \xi_{\vartheta}(\omega) \int_{\Omega} k_c(\omega) |\hat{\vartheta}|^2 d\mathbf{x} \\
 & \leq [|\xi_{\nabla\vartheta}(\omega)| + \alpha_{,,}(\Omega)|\xi_{\vartheta}(\omega)|] \int_{\Omega} [k_c(\omega) |\nabla\hat{\vartheta}|^2 + \Theta_0\sigma_c(\omega) |\widehat{\mathbf{E}}|^2] d\mathbf{x} \\
 & \leq [|\xi_{\nabla\vartheta}(\omega)| + \alpha_{,,}(\Omega)|\xi_{\vartheta}(\omega)|] \left[\operatorname{Re} \int_{\Omega} \widehat{R}\hat{\vartheta}^* d\mathbf{x} \right. \\
 & \quad \left. + \Theta_0 \left(\operatorname{Re} \int_{\Omega} \widehat{\mathbf{G}} \cdot \widehat{\mathbf{H}}^* d\mathbf{x} - \operatorname{Re} \int_{\Omega} \widehat{\mathbf{F}} \cdot \widehat{\mathbf{E}}^* d\mathbf{x} \right) \right].
 \end{aligned} \tag{5.47}$$

We now consider

$$\xi(\omega) = \max\{|\gamma_{i1}(\omega)| + |\gamma_{i2}(\omega)| + |\gamma_i(\omega)| \quad (i = 1, 2, \dots, 6)\}, \quad (5.48)$$

where $\gamma_{ij}(\omega)$ ($i = 1, 2, \dots, 6; j = 1, 2$) denote the coefficients of the real ($j = 1$) and imaginary ($j = 2$) parts of the six ordered integrals in round brackets in (5.46), and

$$\gamma_i(\omega) = 0 \quad (i = 1, 2, 4), \quad \gamma_3(\omega) = -\gamma_6(\omega) = \Theta_0 \gamma_5(\omega) = \Theta_0[|\xi_{\nabla \vartheta}(\omega)| + \alpha_{,,}(\Omega)|\xi_{\vartheta}(\omega)|]. \quad (5.49)$$

Thus, the inequality (5.46), on account of (5.47), (5.48), and (5.49), becomes

$$\begin{aligned} \mathcal{I}(\omega) \leq \xi(\omega) & \left[\left(\int_{\Omega} |\widehat{\mathbf{F}}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\nabla \times \widehat{\mathbf{H}}|^2 d\mathbf{x} \right)^{1/2} + \left(\int_{\Omega} |\widehat{\mathbf{G}}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\nabla \times \widehat{\mathbf{E}}|^2 d\mathbf{x} \right)^{1/2} \right. \\ & + \left(\int_{\Omega} |\widehat{\mathbf{G}}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\widehat{\mathbf{H}}|^2 d\mathbf{x} \right)^{1/2} + |\mathbf{a}| \left(\int_{\Omega} |\widehat{\mathbf{F}}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\widehat{\vartheta}|^2 d\mathbf{x} \right)^{1/2} \\ & \left. + \left(\int_{\Omega} |\widehat{R}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\widehat{\vartheta}|^2 d\mathbf{x} \right)^{1/2} + \left(\int_{\Omega} |\widehat{\mathbf{F}}|^2 d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\widehat{\mathbf{E}}|^2 d\mathbf{x} \right)^{1/2} \right]. \end{aligned} \quad (5.50)$$

Then, taking $\max\{1, |\mathbf{a}|\}$, it follows that there exists $\eta(\omega)$ such that we have the inequality

$$\mathcal{I}(\omega) \leq \eta(\omega) \left[\int_{\Omega} (|\widehat{\mathbf{F}}|^2 + |\widehat{\mathbf{G}}|^2 + |\widehat{R}|^2) d\mathbf{x} \right]^{1/2} \mathcal{I}^{1/2}(\omega), \quad (5.51)$$

from which (5.25) follows. $\square$

LEMMA 5.2. If the sources $(\mathbf{F}, \mathbf{G}, R) \in V(\Omega, \mathbf{R}^+)$ satisfy (5.6) and the hypotheses on $\sigma(s)$ and $k(s)$ hold, then the inverse Fourier transforms of $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R})$ exist and are L^2 -functions with zero initial conditions (5.4).

Proof. The hypotheses assumed for $(\mathbf{F}, \mathbf{G}, R)$ with (5.6), and the properties of $\sigma(s)$ and $k(s)$, stated in Sec. 3, in particular, those given by (3.27) and (3.28), assure that $\eta(\omega)$ is bounded for any $\omega \in \mathbf{R}$. Therefore, the right-hand side of (5.25) is integrable over $\mathbf{R}$, that is,

$$\int_{-\infty}^{+\infty} \int_{\Omega} \eta^2(\omega) (|\widehat{\mathbf{F}}|^2 + |\widehat{\mathbf{G}}|^2 + |\widehat{R}|^2) d\mathbf{x} d\omega < +\infty. \quad (5.52)$$

Comparison of (5.52) and (5.25) gives

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{\Omega} (|\nabla \times \widehat{\mathbf{H}}|^2 + |\nabla \times \widehat{\mathbf{E}}|^2 + |\widehat{\mathbf{H}}|^2 + |\widehat{\mathbf{E}}|^2 + |\nabla \widehat{\vartheta}|^2 + |\widehat{\vartheta}|^2) d\mathbf{x} d\omega \\ \leq \int_{-\infty}^{+\infty} \int_{\Omega} \eta^2(\omega) (|\widehat{\mathbf{F}}|^2 + |\widehat{\mathbf{G}}|^2 + |\widehat{R}|^2) d\mathbf{x} d\omega. \end{aligned} \quad (5.53)$$

Therefore, application of Plancherel's theorem to (5.53) allows us to prove the existence of the inverse Fourier transforms of $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta})$. $\square$

COROLLARY 5.1. If we consider two source fields $(\widehat{\mathbf{F}}^{(j)}, \widehat{\mathbf{G}}^{(j)}, \widehat{R}^{(j)})$ ($j = 1, 2$) and the corresponding solutions $(\widehat{\mathbf{E}}^{(j)}, \widehat{\mathbf{H}}^{(j)}, \widehat{\vartheta}^{(j)})$ ($j = 1, 2$), then

$$\begin{aligned} & \|(\widehat{\mathbf{E}}^{(1)} - \widehat{\mathbf{E}}^{(2)}, \widehat{\mathbf{H}}^{(1)} - \widehat{\mathbf{H}}^{(2)}, \widehat{\vartheta}^{(1)} - \widehat{\vartheta}^{(2)})\|_{\widehat{H}}^2 \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} \eta^2(\omega) (|\widehat{\mathbf{F}}^{(1)} - \widehat{\mathbf{F}}^{(2)}|^2 + |\widehat{\mathbf{G}}^{(1)} - \widehat{\mathbf{G}}^{(2)}|^2 + |\widehat{R}^{(1)} - \widehat{R}^{(2)}|^2) d\mathbf{x} d\omega. \end{aligned} \quad (5.54)$$

In order to complete the proof of existence we have to show the following lemma.

LEMMA 5.3. The subset

$$\begin{aligned} S = \{(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}) \in \widehat{V}(\Omega, \mathbf{R}) : \exists(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R}) \\ \text{solution of (5.10) } \forall(\widehat{\mathbf{e}}, \widehat{\mathbf{h}}, \widehat{\beta}) \in \widehat{W}(\Omega, \mathbf{R})\} \end{aligned} \quad (5.55)$$

is dense and closed in $\widehat{V}(\Omega, \mathbf{R})$.

Proof. We denote by $\overline{S}$ the closure of S in $\widehat{V}(\Omega, \mathbf{R})$. Let us suppose that there exists $(\widehat{\mathbf{F}}^0, \widehat{\mathbf{G}}^0, \widehat{R}^0) \neq 0$ such that $(\widehat{\mathbf{F}}^0, \widehat{\mathbf{G}}^0, \widehat{R}^0) \in \widehat{V}(\Omega, \mathbf{R}) \setminus \overline{S}$. Application of the Hahn-Banach theorem assures the existence of $(\widehat{\mathbf{e}}^0, \widehat{\mathbf{h}}^0, \widehat{\beta}^0) \in \widehat{W}(\Omega, \mathbf{R})$ such that

$$\langle(\widehat{\mathbf{F}}^0, \widehat{\mathbf{G}}^0, \widehat{R}^0), (\widehat{\mathbf{e}}^0, \widehat{\mathbf{h}}^0, \widehat{\beta}^0)\rangle \neq 0, \quad \langle(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}), (\widehat{\mathbf{e}}^0, \widehat{\mathbf{h}}^0, \widehat{\beta}^0)\rangle = 0 \quad \forall(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}) \in S \quad (5.56)$$

hold. The same proof of the uniqueness theorem can be used for the condition

$$A[(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta}), (\widehat{\mathbf{e}}^0, \widehat{\mathbf{h}}^0, \widehat{\beta}^0)] = 0, \quad \forall(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R}), \quad (5.57)$$

that, on account of (5.10), is equivalent to (5.56)₂ and yields

$$(\widehat{\mathbf{e}}^0, \widehat{\mathbf{h}}^0, \widehat{\beta}^0) = 0, \quad (5.58)$$

contrary to (5.56)₁. Therefore, S is dense.

To prove that S is closed, let $(\widehat{\mathbf{F}}^{(n)}, \widehat{\mathbf{G}}^{(n)}, \widehat{R}^{(n)}) \in S$, $n = 1, 2, \dots$, be a sequence of sources with $\lim_{n \rightarrow +\infty} (\widehat{\mathbf{F}}^{(n)}, \widehat{\mathbf{G}}^{(n)}, \widehat{R}^{(n)}) = (\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}) \in \widehat{V}(\Omega, \mathbf{R})$ and $(\widehat{\mathbf{E}}^{(n)}, \widehat{\mathbf{H}}^{(n)}, \widehat{\vartheta}^{(n)}) \in \widehat{H}(\Omega, \mathbf{R})$, $n = 1, 2, \dots$, be the corresponding solutions.

Application of Corollary 5.1 yields

$$\begin{aligned} & \|(\widehat{\mathbf{E}}^{(n)} - \widehat{\mathbf{E}}^{(m)}, \widehat{\mathbf{H}}^{(n)} - \widehat{\mathbf{H}}^{(m)}, \widehat{\vartheta}^{(n)} - \widehat{\vartheta}^{(m)})\|_{\widehat{H}}^2 \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} \eta^2(\omega) (|\widehat{\mathbf{F}}^{(n)} - \widehat{\mathbf{F}}^{(m)}|^2 + |\widehat{\mathbf{G}}^{(n)} - \widehat{\mathbf{G}}^{(m)}|^2 + |\widehat{R}^{(n)} - \widehat{R}^{(m)}|^2) d\mathbf{x} d\omega. \end{aligned} \quad (5.59)$$

Therefore, $(\widehat{\mathbf{E}}^{(n)}, \widehat{\mathbf{H}}^{(n)}, \widehat{\vartheta}^{(n)})$, $n = 1, 2, \dots$, is a Cauchy sequence and, since the space is complete, we have

$$\lim_{n \rightarrow +\infty} (\widehat{\mathbf{E}}^{(n)}, \widehat{\mathbf{H}}^{(n)}, \widehat{\vartheta}^{(n)}) = (\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta}) \in \widehat{H}(\Omega, \mathbf{R}). \quad (5.60)$$

Substitution of the corresponding solutions $(\widehat{\mathbf{E}}^{(n)}, \widehat{\mathbf{H}}^{(n)}, \widehat{\vartheta}^{(n)})$ and $(\widehat{\mathbf{F}}^{(n)}, \widehat{\mathbf{G}}^{(n)}, \widehat{R}^{(n)})$, $n = 1, 2, \dots$, gives a sequence of identities, whose limit as $n \rightarrow +\infty$ is an analogous identity expressed in terms of the limits $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}, \widehat{\vartheta})$ and $(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R})$. Therefore it follows that $(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}, \widehat{R}) \in S$.

Application of Plancherel's theorem leads to the existence of $(\mathbf{E}, \mathbf{H}, \vartheta) \in H(\Omega, \mathbf{R}^+)$, the solution of our problem; therefore, the proof of Theorem 5.1 is complete. $\square$

REFERENCES

- [1] W. Noll, *A new mathematical theory of simple materials*, Arch. Rational Mech. Anal. **48**, 1–50 (1972)
- [2] B. D. Coleman and D. R. Owen, *A mathematical foundation of thermodynamics*, Arch. Rational Mech. Anal. **54**, 1–104 (1974)
- [3] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31**, 113–126 (1968)
- [4] M. McCarthy, *Constitutive equations for thermodynamical materials with memory*, Internat. J. Engrg. Sci. **8**, 467–474 (1970)
- [5] B. D. Coleman and E. H. Dill, *Thermodynamic restrictions on the constitutive equations of electromagnetic theory*, ZAMP **22**, 691–702 (1971)
- [6] B. D. Coleman and E. H. Dill, *On the thermodynamics of electromagnetic fields in materials with memory*, Arch. Rational Mech. Anal. **41**, 132–162 (1971)
- [7] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*, SIAM Studies in Applied Mathematics, Philadelphia, 1992
- [8] M. Fabrizio and A. Morro, *A boundary condition with memory in electromagnetism*, Arch. Rational Mech. Anal. **136**, 359–381 (1996)
- [9] E. B. Bykhovskiy and N. V. Smirnov, *On the orthogonal decomposition of the space of vector functions square summable in a given domain*, Trudy Mat. Inst. Steklov **59**, 6–36 (1960)
- [10] G. Amendola, *On thermodynamic conditions for the stability of a thermoelectromagnetic system*, Math. Methods Appl. Sci. **23**, 17–39 (2000)
- [11] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Monographs and Studies in Mathematics, Vol. 1, Pitman, London, 1977
- [12] F. Trèves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975
- [13] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford, 1960
- [14] C. Müller, *Foundation of Mathematical Theory of Electromagnetic Waves*, Springer, Berlin, 1969