ABOUT THIN FILM MICROPOLAR ASYMPTOTIC EQUATIONS

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Abstract. We study from a mathematical point of view the asymptotic behaviour of micropolar fluids in lubrication theory. Different assumptions are made on the dependence of the micropolar characteristic numbers and of the Dirichlet boundary conditions for the velocity and the rotation, with respect to the small gap parameter. Various generalized Reynolds equations are rigorously obtained and second-order estimates are also given.

1. Introduction. The micropolar Stokes or Navier-Stokes model is of common use to describe the behavior of a flow in which some additives are included. Micropolar fluid is a Newtonian fluid with a large number of small spherical particles uniformly dispersed through it. The particles may rotate with an angular velocity not exactly equal to half the fluid vorticity. Assuming that these particles are rigid, a theory was proposed in [1] and [6] in which the angular velocity is a new unknown. This theory has been extensively examined from various points of view, both mechanical [5], [12], [13] and mathematical [9], [10], [11].

Lubrication is mostly concerned with the behavior of a lubricant through a narrow gap. Fluid is either forced through the gap under a pressure gradient or else the flow is driven by the motion of one of the boundaries. This is the situation that is considered here. For the classical Newtonian fluid, it is well known that the asymptotic (when the gap is small) Navier-Stokes system reduces to the Reynolds equation [3], [7]. Taking micropolar effects into account seems of particular interest for a very small gap [5] because in that case, microstructure effects can be more sensitive. From a mathematical aspect, a rigorous basis for the asymptotic Stokes and Navier-Stokes system can be found in [4], [11]. It is

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shown how to derive the micropolar Reynolds model for the pressure:

$$\nabla \left\{ \frac{h^3(x)}{1-N^2} \Phi(h(x), N) \nabla p(x) \right\} = \frac{s}{2} \frac{\partial h(x)}{\partial x}, \quad (1.1)$$

where N ($0 < N < 1$) is a characteristic number for the micropolar fluid, $h(x)$ represents the gap of the fluid and

$$\Phi(h, N) = \frac{1}{12} + \frac{1}{4h^2(1-N^2)} - \frac{1}{4h} \sqrt{\frac{N^2}{1-N^2}} \coth(N\sqrt{1-N^2}h) \quad (1.2)$$

from the micropolar Stokes system:

$$-(\nu + \nu_\gamma) \Delta \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + \nabla p = 2\nu_\gamma \operatorname{rot} \vec{\omega}, \quad (1.3)$$

$$\operatorname{div} \vec{u} = 0, \quad (1.4)$$

$$-(c_a + c_d) \Delta \vec{\omega} + I(\vec{u} \cdot \nabla) \vec{\omega} - (c_o + c_d + c_a) \nabla \operatorname{div} \vec{\omega} + 4\nu_\gamma \vec{\omega} = 2\nu_\gamma \operatorname{rot} \vec{u}. \quad (1.5)$$

The functions $\vec{u}(x)$, $p(x)$, and $\vec{\omega}(x)$ are the velocity vector, the pressure, and the angular velocity of rotation of particles of the fluid. Positive constants $\nu, \nu_\gamma, c_o, c_a, c_d$ characterize the isotropic properties of the fluid: ν is the usual kinematic Newtonian viscosity, $\nu_\gamma, c_o, c_a, c_d$ are new viscosities connected with the asymmetric properties of the stress tensor and, consequently, with the appearance of the internal rotation field $\vec{\omega}(x)$ in the present model; $I > 0$ and $c_o + c_d > c_a$.

In the preceding studies a more or less implicit condition is that the ratio $R_M = \frac{c_a + c_d}{\nu + \nu_\gamma}$ varies like ε^2 where ε is the small parameter describing the thin film aspect of the flow and zero boundary conditions are generally assumed for the microrotation.

In the present paper, we restrict ourselves to a linearized problem issued from (1.3)–(1.5) by neglecting $(\vec{u} \cdot \nabla) \vec{u}$, $I(\vec{u} \cdot \nabla) \vec{\omega}$ as well as $(c_o + c_d - c_a) \nabla \operatorname{div} \vec{\omega}$. Moreover, the internal rotation field is assumed to be reduced to a unique nonzero component $(0, 0, \omega)$. We introduce various behaviors of R_M with respect to ε and deduce different corresponding asymptotic Reynolds equations. Moreover, attention is focused on the importance of nonzero boundary conditions for the rotation, leading to a wide range of different variation of the boundary conditions with respect to ε .

The second chapter is devoted to the adimensional procedure in order to obtain from (1.3)–(1.5) a new system with only two characteristic parameters. An heuristic discussion leads to three possible asymptotic systems. The rigorous determination of these behaviors is the topic of the paper.

In the third chapter, we first get the estimates for the velocity field and the micro rotation and then for the pressure. They are different according to the behavior of R_M as a function of ε . These estimates are based on an assumption that links the maximal value of the gap and the value of one of the non-Newtonian characteristic parameters of the fluid.

In the fourth part, we deduce the various Reynolds equations and the related boundary conditions for the asymptotic pressure which are explicitly expressed in terms of the initial boundary conditions for the rotation and the velocity.

We give in Sec. 5, second-order estimates that allow us to obtain in Sec. 6 the following term for the pressure in the asymptotic expansion.

For simplicity of the proofs, we restrict ourselves to a two-dimensional problem. However, all the results can easily be extended to a three-dimensional domain ([8]).

2. Notation and mechanical aspect. In this section, we will introduce rescaled functions and variables in order to show the various available asymptotic systems that will be obtained by rigorous proofs in Sections 3 and 4.

Let us define a 2-dimensional thin domain by

$$x = (x_1, x_2), \quad 0 < x_1 < R, \quad 0 < x_2 < ch(x_1),$$

where h is a smooth strictly positive function whose values are normalized ($O(1)$) and $\varepsilon = c/R$ is small. The fluid is mainly driven by the tangential velocity of the upper or lower surface whose characteristic value is denoted by s . We introduce the new normalized variables and functions:

$$\begin{aligned} x &= \frac{x_1}{R}, \quad y = \frac{x_2}{c}, \quad \bar{u}_1 = \frac{u_1}{s}, \quad \bar{u}_2 = \frac{u_2}{s}, \\ \bar{\omega} &= \frac{R\varepsilon}{s}\omega, \quad \bar{p} = \frac{\varepsilon^2 R}{s(\nu + \nu_\gamma)}p, \\ N^2 &= \frac{\nu_\gamma}{\nu + \nu_\gamma}, \quad R_M = \frac{c_a + c_d}{\nu + \nu_\gamma} \frac{1}{R^2}. \end{aligned} \quad (2.1)$$

Denoting $\Delta_\varepsilon = \varepsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, the linearized system corresponding to (1.3)–(1.5) becomes:

$$\Delta_\varepsilon \bar{u}_1 + 2N^2 \frac{\partial \bar{\omega}}{\partial y} = \frac{\partial \bar{p}}{\partial x}, \quad (2.2)$$

$$\Delta_\varepsilon \bar{u}_2 - 2N^2 \varepsilon \frac{\partial \bar{\omega}}{\partial x} = \frac{1}{\varepsilon} \frac{\partial \bar{p}}{\partial y}, \quad (2.3)$$

$$-R_M \Delta_\varepsilon \bar{\omega} + 4\varepsilon^2 N^2 \bar{\omega} = 2N^2 \left(\varepsilon^3 \frac{\partial \bar{u}_2}{\partial x} - \varepsilon^2 \frac{\partial \bar{u}_1}{\partial y} \right), \quad (2.4)$$

$$\varepsilon \frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{u}_2}{\partial y} = 0. \quad (2.5)$$

The physical meaning of this rescaling is to introduce functions whose significant values as well as their variations are near one, or at most $O(1)$ when ε tends to zero. If this assumption seems reasonable for the velocity, it seems more hazardous for the rotation and pressure. Nevertheless, it will enable us to discuss the asymptotic situation and is in accordance with the rigorous estimates of Chapter 3. The leading equation is (2.4) in which the term for which the order of magnitude is not given is R_M since N^2 lies between zero and one.

For a given physical situation, R_M is a positive dimensionless number which can be compared with ε in various ways:

Assume first that there exists γ such that $R_M = \varepsilon^\gamma$, with $0 < \gamma < 2$. Equation (2.4) reduces to

$$\frac{\partial^2 \bar{\omega}}{\partial y^2} = 0, \quad (2.6)$$

which can be integrated knowing boundary values for the rotation $\bar{\omega}$. The system (2.2), (2.3) reduces now to the following “thin film system” [2], [3]:

$$\frac{\partial^2 \bar{u}_1}{\partial y^2} = \frac{\partial \bar{p}}{\partial x} - 2N^2 \frac{\partial \bar{\omega}}{\partial y}, \quad \frac{\partial \bar{p}}{\partial y} = 0, \quad (2.7)$$

where $2N^2 \frac{\partial \bar{\omega}}{\partial y}$ acts as a given volumic force.

A second situation occurs when $R_M = \varepsilon^\gamma$ with $\gamma > 2$. Equation (2.4) reduces to

$$\bar{\omega} = -\frac{1}{2} \frac{\partial \bar{u}_1}{\partial y}, \quad (2.8)$$

and equations (2.2), (2.3) become

$$\frac{\partial^2 \bar{u}_1}{\partial y^2} (1 - N^2) = \frac{\partial \bar{p}}{\partial x}, \quad \frac{\partial \bar{p}}{\partial y} = 0. \quad (2.9)$$

This leads to a near classical Reynolds equation in which the pressure is modified by the factor $\frac{1}{1-N^2}$.

At last, if $R_M = O(\varepsilon^2)$, namely $R_M = \lambda \varepsilon^2$, Eq. (2.4) shows a strong coupling between velocity and rotation:

$$\lambda \frac{\partial^2 \bar{\omega}}{\partial y^2} + 4N^2 \bar{\omega} = -2N^2 \frac{\partial \bar{u}_1}{\partial y}. \quad (2.10)$$

This last case is the situation that is commonly introduced to study the micropolar fluid and it leads to an asymptotic equation like (1.1).

As a last remark, let us mention that Eq. (2.7) demonstrates that the asymptotic value of $\bar{\omega}$ is obtained without taking account of the given boundary conditions, contrary to (2.6). This situation is linked to the appearance of a boundary layer for $\bar{\omega}$ near these surfaces and will lead in the mathematical study to estimations for $\bar{\omega}$ in the L^2 -norm only.

REMARK. Coming back to the real pressure and data, and taking into account the fact that

$$\bar{p} = \frac{\varepsilon^2 R}{s(\nu + \nu_\gamma)} p = \frac{\varepsilon^2 R(1 - N^2)}{s\nu} p,$$

we get

$$\begin{aligned} \text{for } 0 < \gamma < 2 : \quad & \frac{d}{dx_1} \left(\frac{(1 - N^2)(ch)^3}{12\nu} \frac{dp}{dx_1} \right) = \frac{s}{2} \frac{d(ch)}{dx_1}, \\ \text{for } \gamma > 2 : \quad & \frac{d}{dx_1} \left(\frac{(ch)^3}{12\nu} \frac{dp}{dx_1} \right) = \frac{s}{2} \frac{d(ch)}{dx_1}, \end{aligned} \quad (2.11)$$

so that the effects of microrotation disappear for $\gamma > 2$ and are included in the equation for $\gamma < 2$ by way of a single coefficient $(1 - N^2)$.

3. Assumptions and estimates. We denote by Ω^ε the thin domain and by Ω the rescaled one:

$$\begin{aligned} \Omega^\varepsilon &= \{(x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < R, 0 < x_2 < \varepsilon R h(x_1)\}, \\ \Omega &= \{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < h(x)\}. \end{aligned}$$

Let Γ^s be the upper boundary of Ω , Γ^i the lower boundary of Ω and Γ_L the lateral one. The variational formulation of the problem (2.2)–(2.5) is

Find u in $(H^1(\Omega^\varepsilon))^2$, ω in $H^1(\Omega^\varepsilon)$, and p in $L^2(\Omega^\varepsilon)$ such that

$$\begin{aligned} (\nu + \nu_\gamma) \int_{\Omega^\varepsilon} \nabla u_1 \nabla \varphi_1 \, dx_1 \, dx_2 - \int_{\Omega^\varepsilon} p \frac{\partial \varphi_1}{\partial x_1} \, dx_1 \, dx_2 &= 2\nu_\gamma \int_{\Omega^\varepsilon} \frac{\partial \omega}{\partial x_2} \varphi_1 \, dx_1 \, dx_2, \\ (\nu + \nu_\gamma) \int_{\Omega^\varepsilon} \nabla u_2 \nabla \varphi_2 \, dx_1 \, dx_2 - \int_{\Omega^\varepsilon} p \frac{\partial \varphi_2}{\partial x_2} \, dx_1 \, dx_2 &= -2\nu_\gamma \int_{\Omega^\varepsilon} \frac{\partial \omega}{\partial x_1} \varphi_2 \, dx_1 \, dx_2, \\ (c_a + c_d) \int_{\Omega^\varepsilon} \nabla \omega \nabla \psi \, dx_1 \, dx_2 + 4\nu_\gamma \int_{\Omega^\varepsilon} \omega \psi \, dx_1 \, dx_2 &= 2\nu_\gamma \int_{\Omega^\varepsilon} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \psi \, dx_1 \, dx_2, \\ \int_{\Omega^\varepsilon} q \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \, dx_1 \, dx_2 &= 0 \end{aligned} \quad (3.1)$$

for any functions $\varphi_1, \varphi_2, \psi \in H_0^1(\Omega^\varepsilon)$, $q \in L^2(\Omega^\varepsilon)$, with given Dirichlet boundary conditions for the velocity and the rotation on $\partial\Omega^\varepsilon$, such that $\int_{\partial\Omega^\varepsilon} u^\varepsilon \cdot n \, d\sigma = 0$.

The existence of solutions to problem (3.1) is known [4], [10]:

THEOREM 3.1. There exists a unique solution (u, ω, p) of (3.1) such that $\int_{\Omega^\varepsilon} p \, dx_1 \, dx_2 = 0$.

After the change of variables (2.1) and using the notation defined in Sec. 2, we define the rescaled weak system, and the corresponding normalized unknowns are denoted by $u^\varepsilon, \omega^\varepsilon, p^\varepsilon$ instead of $\bar{u}, \bar{\omega}, \bar{p}$. Setting

$$\nabla_\varepsilon = \left(\varepsilon \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \quad \operatorname{div}_\varepsilon \varphi = \varepsilon \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y},$$

we have

$$\begin{aligned} \int_{\Omega} \nabla_\varepsilon u_1^\varepsilon \nabla_\varepsilon \varphi_1 \, dx \, dy - \int_{\Omega} p^\varepsilon \frac{\partial \varphi_1}{\partial x} \, dx \, dy &= 2N^2 \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial y} \varphi_1 \, dx \, dy, \\ \int_{\Omega} \nabla_\varepsilon u_2^\varepsilon \nabla_\varepsilon \varphi_2 \, dx \, dy - \frac{1}{\varepsilon} \int_{\Omega} p^\varepsilon \frac{\partial \varphi_2}{\partial y} \, dx \, dy &= -2N^2 \varepsilon \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial x} \varphi_2 \, dx \, dy, \\ R_L \int_{\Omega} \nabla_\varepsilon \omega^\varepsilon \nabla_\varepsilon \psi \, dx \, dy + 4N^2 \varepsilon^2 \int_{\Omega} \omega^\varepsilon \psi \, dx \, dy &= 2N^2 \varepsilon^2 \int_{\Omega} \left(\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} - \frac{\partial u_1^\varepsilon}{\partial y} \right) \psi \, dx \, dy, \\ \int_{\Omega} q \operatorname{div}_\varepsilon u^\varepsilon \, dx \, dy &= 0. \end{aligned} \quad (3.2)$$

HYPOTHESES. Let us introduce the following hypotheses for the parameters and the boundary conditions; they will be assumed through the whole paper.

- (H1): There exists $\gamma > 0$ and $\lambda > 0$ such that $R_L = \lambda \varepsilon^\gamma$.
- (H2): The boundary value f^ε for the rotation is such that there exists a function $F(x, y)$ in $H^1(\Omega)$ constant with respect to ε and δ such that

$$\omega^\varepsilon = f^\varepsilon = \frac{F}{\varepsilon^\delta} \quad \text{on } \partial\Omega.$$

• (H3): The boundary values g_i^ε for the velocities are such that there exist functions J_1 and J_2 in $H^1(\Omega)$, constant with respect to ε and δ , with

$$\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} = 0 \quad \text{in } \Omega, \quad J_1 = g_1^\varepsilon \quad \text{and} \quad J_2 = \frac{g_2^\varepsilon}{\varepsilon} \quad \text{on } \partial\Omega.$$

The maximum value of the gap $t = \max_{x \in]0,1[} h(x)$ and rheological parameters N and λ satisfy the inequalities:

• (H4)

$$\begin{cases} \sqrt{\lambda} - tN^2 > 0 & \text{if } \gamma < 2, \\ 2N^2 < 1 & \text{if } \gamma > 2, \\ 2N^2 < 1 \text{ or } F(\lambda, N^2, t) > 0 & \text{if } \gamma = 2, \end{cases}$$

where $F(\lambda, N^2, t) = \lambda(4 - 2N^2) - N^4 t^2(4 - N^2)$.

REMARK. Assumption (H4) is given in a normalized form. Coming back to real data allows us to discuss the physical related meaning especially with respect to the maximum real gap $T = ct$.

For $\gamma = 2$, then $\lambda = \frac{c_a + c_d}{\nu + \nu_\gamma} \frac{1}{c^2}$ and $F(\lambda, N^2, t) > 0$ may be rewritten as

$$T^2 \leq \frac{4 - 2N^2}{(4 - N^2)N^4} \frac{c_a + c_d}{\nu + \nu_\gamma}. \quad (i)$$

So for a sufficiently small gap, this assumption is satisfied.

For $\gamma < 2$, let us remark that for a physical given R_M , there are an infinity of ways to choose λ and γ such that $R_M = \lambda \varepsilon^\gamma$. The related assumption is that it is possible to have simultaneously $\lambda = O(1)$ and $\gamma < 2$ for this R_M .

Assumption $\sqrt{\lambda} - tN^2 > 0$ is rewritten as

$$\left(\frac{c}{R}\right)^{2-\gamma} > N^4 T^2 \frac{\nu + \nu_\gamma}{c_a + c_d}. \quad (ii)$$

If

$$N^4 T^2 \frac{\nu + \nu_\gamma}{c_a + c_d} < 1,$$

then it is always possible to choose γ such that $0 < \gamma < \gamma^* < 2$, with

$$\gamma^* = \ln^{-1} \left(\frac{c}{R} \right) \left[2 - \ln \left(\frac{N^4 T^2 (\nu + \nu_\gamma)}{c_a + c_d} \right) \right]$$

such that the inequality (ii) is satisfied. For γ close from γ^* we choose the related λ by

$$\lambda = R_M \varepsilon^{-\gamma},$$

which is of the order of $R_M \varepsilon^{-\gamma^*} = c^2 T^{-2} N^4 = O(1)$ for given N .

As a conclusion, assumptions (H4) are fulfilled for $\gamma \leq 2$ as soon as the gap is sufficiently small.

In the following, $\|\cdot\|$ will denote the L^2 -norm in Ω . Other norms will be made precise in the subscript.

PROPOSITION 3.2. For $\gamma \leq 2$, we get the estimates:

$$\varepsilon \left(\left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial u_2^\varepsilon}{\partial x} \right\| \right) + \left\| \frac{\partial u_1^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial u_2^\varepsilon}{\partial y} \right\| + \varepsilon \left\| \frac{\partial \omega^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial \omega^\varepsilon}{\partial y} \right\| + \varepsilon^{1-\frac{\gamma}{2}} \|\omega^\varepsilon\| \leq K_\delta, \quad (3.3)$$

where $K_\delta \leq O(1)$ if $\delta \leq 0$ and $K_\delta \leq O(\varepsilon^{-\delta})$ if $\delta > 0$.

Proof. Adding (3.2.1) and (3.2.2) with (3.2.3) multiplied by $\varepsilon^{-\gamma}$, we get for any test function φ such that $\frac{\partial \varphi_1}{\partial x} + \frac{1}{\varepsilon} \frac{\partial \varphi_2}{\partial y} = 0$:

$$\begin{aligned} & \int_{\Omega} \nabla_\varepsilon u^\varepsilon \nabla_\varepsilon \varphi \, dx \, dy + \lambda \int_{\Omega} \nabla_\varepsilon \omega^\varepsilon \nabla_\varepsilon \psi \, dx \, dy + 4N^2 \varepsilon^{2-\gamma} \int_{\Omega} \omega^\varepsilon \psi \, dx \, dy \\ &= 2N^2 \int_{\Omega} \left(\frac{\partial \omega^\varepsilon}{\partial y} \varphi_1 - \varepsilon \frac{\partial \omega^\varepsilon}{\partial x} \varphi_2 \right) \, dx \, dy + 2N^2 \varepsilon^{2-\gamma} \int_{\Omega} \left(\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} - \frac{\partial u_1^\varepsilon}{\partial y} \right) \psi \, dx \, dy. \end{aligned} \quad (3.4)$$

We choose $\varphi = (u_1^\varepsilon - J_1, u_2^\varepsilon - \varepsilon J_2)$ and $\psi = \omega^\varepsilon - F/\varepsilon^\delta$ in the preceding equation and we define

$$\begin{aligned} A^\varepsilon &= \left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\|, & B^\varepsilon &= \left\| \frac{\partial u_1^\varepsilon}{\partial y} \right\|, & C^\varepsilon &= \left\| \frac{\partial u_2^\varepsilon}{\partial x} \right\|, \\ D^\varepsilon &= \left\| \frac{\partial u_2^\varepsilon}{\partial y} \right\|, & E^\varepsilon &= \left\| \frac{\partial \omega^\varepsilon}{\partial x} \right\|, & G^\varepsilon &= \left\| \frac{\partial \omega^\varepsilon}{\partial y} \right\|. \end{aligned}$$

Applying Cauchy-Schwarz's inequality, using Poincaré-Friedrich's inequality in the y -direction ([2]), the classical inequality $2ab \leq a^2 + b^2$, and Green's formula yield

$$\begin{aligned} \varepsilon^2 \left| \int_{\Omega} \frac{\partial u_1^\varepsilon}{\partial x} \frac{\partial J_1}{\partial x} \, dx \, dy \right| &\leq \varepsilon^2 \left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\| \left\| \frac{\partial J_1}{\partial x} \right\| \leq \frac{\varepsilon^2}{2} ((A^\varepsilon)^2 + k), \\ \varepsilon^3 \left| \int_{\Omega} \frac{\partial u_2^\varepsilon}{\partial x} \frac{\partial J_2}{\partial x} \, dx \, dy \right| &\leq \varepsilon^3 \frac{1}{\sqrt{\rho_1}} \left\| \frac{\partial u_2^\varepsilon}{\partial x} \right\| \sqrt{\rho_1} \left\| \frac{\partial J_2}{\partial x} \right\| \leq \frac{\varepsilon^2}{2} \left(\frac{(C^\varepsilon)^2}{\rho_1} + k_{\rho_1} \varepsilon^2 \right). \end{aligned}$$

By the same procedure, we obtain

$$\begin{aligned}
\left| \int_{\Omega} \frac{\partial u_1^\varepsilon}{\partial y} \frac{\partial J_1}{\partial y} dx dy \right| &\leq \frac{1}{2} \left(\frac{(B^\varepsilon)^2}{\rho^2} + k\rho_2 \right), \\
\varepsilon \left| \int_{\Omega} \frac{\partial u_2^\varepsilon}{\partial y} \frac{\partial J_2}{\partial y} dx dy \right| &\leq \frac{1}{2} \left(\frac{(D^\varepsilon)^2}{\rho_3} + k\rho_3\varepsilon^2 \right), \\
\lambda\varepsilon^{2-\delta} \left| \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial x} \frac{\partial F}{\partial x} dx dy \right| &\leq \frac{\lambda\varepsilon^2}{2} \left(\frac{(E^\varepsilon)^2}{\rho_4} + \rho_4 k\varepsilon^{-2\delta} \right), \\
\lambda\varepsilon^{-\delta} \left| \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial y} \frac{\partial F}{\partial y} dx dy \right| &\leq \frac{\lambda}{2} \left(\frac{(G^\varepsilon)^2}{\rho_5} + \rho_5 k\varepsilon^{-2\delta} \right), \\
4N^2\varepsilon^{2-\gamma-\delta} \left| \int_{\Omega} \omega^\varepsilon F dx dy \right| &\leq 2N^2\varepsilon^{2-\gamma} \left(\frac{1}{\rho_6} \|\omega^\varepsilon\|^2 + \rho_6 k\varepsilon^{-2\delta} \right), \\
2N^2 \left| \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial y} (u_1^\varepsilon - J_1) dx dy \right| &\leq N^2 \left(\frac{(G^\varepsilon)^2}{\rho_7} + t^7 \rho_7 ((B^\varepsilon)^2 + k) \right), \\
2N^2\varepsilon \left| \int_{\Omega} \frac{\partial \omega^\varepsilon}{\partial x} (u_2^\varepsilon - \varepsilon J_2) dx dy \right| &\leq N^2 \left[\frac{\varepsilon^2 (E^\varepsilon)^2}{\rho_8} + t^2 \rho_8 ((D^\varepsilon)^2 + k\varepsilon^2) \right], \\
2N^2\varepsilon^{3-\gamma} \left| \int_{\Omega} \frac{\partial u_2^\varepsilon}{\partial x} \omega^\varepsilon dx dy \right| &\leq N^2 \left(\frac{\varepsilon^2 (C^\varepsilon)^2}{\rho_9} + \rho_9 \varepsilon^{4-2\gamma} \|\omega^\varepsilon\|^2 \right), \\
2N^2\varepsilon^{2-\gamma} \left| \int_{\Omega} \frac{\partial u_1^\varepsilon}{\partial y} \omega^\varepsilon dx dy \right| &\leq N^2 \left(\frac{(B^\varepsilon)^2}{\rho_{10}} + \rho_{10} \varepsilon^{4-2\gamma} \|\omega^\varepsilon\|^2 \right), \\
2N^2\varepsilon^{3-\gamma-\delta} \left| \int_{\Omega} \frac{\partial u_2^\varepsilon}{\partial x} F dx dy \right| &\leq N^2 \left(\frac{\varepsilon^2 (C^\varepsilon)^2}{\rho_{11}} + \rho_{11} k\varepsilon^{2(2-\gamma-\delta)} \right), \\
2N^2\varepsilon^{2-\gamma-\delta} \left| \int_{\Omega} \frac{\partial u_1^\varepsilon}{\partial y} F dx dy \right| &\leq N^2 \left(\frac{(B^\varepsilon)^2}{\rho_{12}} + \rho_{12} k\varepsilon^{2(2-\gamma-\delta)} \right),
\end{aligned}$$

where ρ_i , $i = 1, 12$ are any positive constants and $k = \max\{\|J\|_{H^1(\Omega)}^2, \|F\|_{H^1(\Omega)}^2\}$. With all these inequalities, we obtain

$$\begin{aligned}
&\frac{\varepsilon^2 (A^\varepsilon)^2}{2} + (B^\varepsilon)^2 \left[1 - \frac{1}{2\rho_2} - N^2 \left(\rho_7 t^2 + \frac{1}{\rho_{12}} + \frac{1}{\rho_{10}} \right) \right] \\
&+ \varepsilon^2 (C^\varepsilon)^2 \left[1 - \frac{1}{2\rho_1} - N^2 \left(\frac{1}{\rho_4} + \frac{1}{\rho_{11}} \right) \right] + (D^\varepsilon)^2 \left(1 - \frac{1}{2\rho_3} - \rho_8 N^2 t^2 \right) \\
&+ \varepsilon^2 (E^\varepsilon)^2 \left(\lambda - \frac{\lambda}{2\rho_4} - \frac{N^2}{\rho_8} \right) + (G^\varepsilon)^2 \left(\lambda - \frac{\lambda}{2\rho_5} - \frac{N^2}{\rho_7} \right) \\
&+ N^2 \varepsilon^{2-\gamma} \|\omega^\varepsilon\|^2 \left[4 - \frac{2}{\rho_6} - \varepsilon^{2-\gamma} (\rho_4 + \rho_{10}) \right] \leq K_\delta,
\end{aligned}$$

where $K_\delta = C(1 + \varepsilon^{-2\delta})$, $C > 0$. All the coefficients of the square terms in the left-hand side must be positive in order to get estimates.

Let us consider first the case $\gamma < 2$; the last coefficient becomes $4 - 2/\rho_6$. Obvious choices of $\rho_1, \rho_4, \rho_{11}$, and ρ_6 lead to

$$4 - \frac{2}{\rho_6} > 0 \quad \text{and} \quad 1 - \frac{1}{2\rho_1} - N^2 \left(\frac{1}{\rho_4} + \frac{1}{\rho_{11}} \right) > 0.$$

To get the four other coefficients positive, we need $\sqrt{\lambda} - tN^2 > 0$.

For $\gamma = 2$, the last coefficient is $4 - 2/\rho_6 - \rho_4 - \rho_{10}$ and a supplementary condition has to be assumed, taking account of the coupling with the conditions:

$$1 - \frac{1}{2\rho_2} - N^2 \left(\rho_7 t^2 + \frac{1}{\rho_{12}} + \frac{1}{\rho_{10}} \right) > 0 \quad \text{and} \quad 1 - \frac{1}{2\rho_1} - N^2 \left(\frac{1}{\rho_4} + \frac{1}{\rho_{11}} \right) > 0.$$

Letting $\rho_1, \rho_2, \rho_5, \rho_6, \rho_{11}, \rho_{12}$ be chosen large enough, we obtain

$$4 - N^2 > \rho_{10} > \frac{\lambda N^2}{\lambda - t^2 N^4} > 0,$$

which gives the necessary condition $\mathcal{F}(\lambda, N^2, t) > 0$. $\square$

PROPOSITION 3.3. For $\gamma \geq 2$, we get the estimates:

$$\varepsilon \left(\left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial u_2^\varepsilon}{\partial x} \right\| \right) + \left\| \frac{\partial u_1^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial u_2^\varepsilon}{\partial y} \right\| + \varepsilon^{\frac{\gamma}{2}} \left\| \frac{\partial \omega^\varepsilon}{\partial x} \right\| + \varepsilon^{\frac{\gamma}{2}-1} \left\| \frac{\partial \omega^\varepsilon}{\partial y} \right\| + \|\omega^\varepsilon\| \leq K_\delta, \quad (3.5)$$

where $K_\delta \leq O(1)$ if $\delta \leq 0$ and $K_\delta \leq O(\varepsilon^{-\delta})$ if $\delta > 0$.

Proof. The proof is driven exactly as the one of Prop. 3.2 and we omit it. $\square$

Let us define the Sobolev spaces ([2])

$$V_y = \left\{ \varphi \in L^2(\Omega); \frac{\partial \varphi}{\partial y} \in L^2(\Omega) \right\},$$

$$L_0^2(\Omega) = \left\{ \varphi \in L^2(\Omega); \int_\Omega \varphi \, dx \, dy = 0 \right\}.$$

From Propositions 3.2 and 3.3, we directly deduce

LEMMA 3.4 (Estimates for the velocity and rotation).

For $\delta \leq 0$,

$$\|u^\varepsilon\|_{V_y} \leq C, \quad (3.6)$$

$$\|\omega^\varepsilon\|_{V_y} \leq C \quad \text{for } \gamma \leq 2, \quad (3.7)$$

$$\varepsilon^{\frac{\gamma}{2}-1} \|\omega^\varepsilon\|_{V_y} \leq C, \quad \|\omega^\varepsilon\|_{L^2(\Omega)} \leq C \quad \text{for } \gamma > 2. \quad (3.8)$$

For $\delta > 0$,

$$\varepsilon^\delta \|u^\varepsilon\|_{V_y} \leq C, \quad (3.9)$$

$$\varepsilon^\delta \|\omega^\varepsilon\|_{V_y} \leq C \quad \text{for } \gamma \leq 2, \quad (3.10)$$

$$\varepsilon^{\frac{\gamma}{2}-1+\delta} \|\omega^\varepsilon\|_{V_y} \leq C, \quad \varepsilon^\delta \|\omega^\varepsilon\|_{L^2(\Omega)} \leq C \quad \text{for } \gamma > 2. \quad (3.11)$$

LEMMA 3.5 (Estimates for the pressure).

For $\delta \leq 0$,

$$\left\| \frac{\partial p^\varepsilon}{\partial x} \right\|_{H^{-1}(\Omega)} \leq C, \quad \left\| \frac{\partial p^\varepsilon}{\partial y} \right\|_{H^{-1}(\Omega)} \leq C\varepsilon, \quad \|p^\varepsilon\|_{L_0^2(\Omega)} \leq C. \quad (3.12)$$

For $\delta > 0$,

$$\varepsilon^\delta \left\| \frac{\partial p^\varepsilon}{\partial x} \right\|_{H^{-1}(\Omega)} \leq C, \quad \varepsilon^\delta \left\| \frac{\partial p^\varepsilon}{\partial y} \right\|_{H^{-1}(\Omega)} \leq C\varepsilon, \quad \varepsilon^\delta \|p^\varepsilon\|_{L_0^2(\Omega)} \leq C, \quad (3.13)$$

where C is constant with respect to ε and δ .

Proof. From (3.2.1), we get

$$\prec \frac{\partial p^\varepsilon}{\partial x}, \varphi_1 \succ_{(H^{-1}, H_0^1)} \leq \left(2N^2 \|\omega^\varepsilon\|_{L^2(\Omega)} + \varepsilon^2 \left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\|_{L^2(\Omega)} + \left\| \frac{\partial u_1^\varepsilon}{\partial y} \right\|_{L^2(\Omega)} \right) \|\varphi_1\|_{H_0^1(\Omega)}, \quad (3.14)$$

or, alternatively,

$$\prec \frac{\partial p^\varepsilon}{\partial x}, \varphi_1 \succ_{(H^{-1}, H_0^1)} \leq \left(2N^2 \left\| \frac{\partial \omega^\varepsilon}{\partial y} \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \frac{\partial u_1^\varepsilon}{\partial x} \right\|_{L^2(\Omega)} + \left\| \frac{\partial u_1^\varepsilon}{\partial y} \right\|_{L^2(\Omega)} \right) \|\varphi_1\|_{H_0^1(\Omega)}. \quad (3.15)$$

Applying estimates of Propositions 3.2 and 3.3 to (3.14) if $\delta \leq 0$ and $\gamma \leq 2$, and to (3.15) if $\delta \leq 0$ and $\gamma > 2$, we get $\left\| \frac{\partial p^\varepsilon}{\partial x} \right\|_{H^{-1}} \leq C$. If $\delta > 0$, the same procedure gives the required result after multiplying both sides of (3.3) and (3.5) by ε^δ .

Estimates for $\left\| \frac{\partial p^\varepsilon}{\partial y} \right\|_{H^{-1}}$ are obtained in the same way by using (3.2.2) instead of (3.2.1).

At last, we deduce the estimates on the pressure using the inequality [14]:

$$\|p^\varepsilon\|_{L_0^2(\Omega)} \leq C(\Omega) \left(\left\| \frac{\partial p^\varepsilon}{\partial x} \right\|_{H^{-1}(\Omega)} + \left\| \frac{\partial p^\varepsilon}{\partial y} \right\|_{H^{-1}(\Omega)} \right). \quad \square \quad (3.16)$$

4. Asymptotic problems. Now we are able to define the weak convergences of the velocities and pressure, depending on the values of δ and γ . Let $\Lambda =]0, 1[$ and let n_1 denote the unit external normal to $\partial\Lambda$.

In the following it will be necessary to distinguish the known boundary values on the upper and lower boundaries of Ω denoted by

$$\begin{aligned} \Gamma^s &= \{(x, y) \in \mathbb{R}^2; 0 < x < 1, y = h(x)\}, \\ \Gamma^i &= \{(x, y) \in \mathbb{R}^2; 0 < x < 1, y = 0\}. \end{aligned}$$

Set

$$J^s = J|_{\Gamma^s}, \quad J^i = J|_{\Gamma^i}; \quad F^s = F|_{\Gamma^s}, \quad F^i = F|_{\Gamma^i}. \quad (4.1)$$

PROPOSITION 4.1. There exist $u^* \in V_y^2, p^* \in L_0^2(\Omega), \omega_* \in L^2(\Omega), \omega^* \in V_y$ such that

- i) If $\delta \leq 0$,
  - u^ε tends to u^* weakly in V_y^2 , p^ε tends to p^* weakly in $L_0^2(\Omega)$;
  - if $\gamma \leq 2$, ω^ε tends to ω^* weakly in V_y ;
  - if $\gamma > 2$, $\varepsilon^{\gamma/2-1}\omega^\varepsilon$ tends to 0 weakly in V_y and ω^ε tends to ω_* weakly in $L^2(\Omega)$.
- ii) If $\delta > 0$,
  - $\varepsilon^\delta u^\varepsilon$ tends to u^* weakly in V_y^2 , $\varepsilon^\delta p^\varepsilon$ tends to p^* weakly in $L_0^2(\Omega)$;
  - if $\gamma \leq 2$, $\varepsilon^\delta \omega^\varepsilon$ tends to ω^* weakly in V_y ;
  - if $\gamma > 2$, $\varepsilon^{\gamma/2-1+\delta}\omega^\varepsilon$ tends to 0 weakly in V_y and $\varepsilon^\delta \omega^\varepsilon$ tends to ω_* weakly in $L^2(\Omega)$.
- iii) $u_2^* = 0$ and $\frac{\partial p^*}{\partial y} = 0$ in Ω ,
  - $u_1^* = J_1^s$ (resp. J_1^i) on Γ^s (resp. Γ^i) if $\delta \leq 0$;
  - $u_1^* = 0$ on $\Gamma_s \cup \Gamma_i$ if $\delta > 0$;

- $\omega^* = 0$ on $\Gamma_s \cup \Gamma_i$ if $\delta < 0$;
- $\omega^* = F^s$ (resp. F^i) on Γ^s (resp. Γ^i) if $\delta = 0$;
- $\omega^* = F^s$ (resp. F^i) on Γ^s (resp. Γ^i) if $\delta > 0$ and $\gamma \leq 2$.

Proof. The convergence results i), ii) are direct consequences of the estimates of the preceding section (Propositions 3.2, 3.3 and Lemma 3.4). The continuity of the application from V_y to $L^2(\Gamma^s)$ and to $L^2(\Gamma^i)$ ([2]) allows us to deduce the values of ω^* and u^* on $\Gamma^s \cup \Gamma^i$ in iii) from the values of u^ε and ω^ε on $\Gamma^s \cup \Gamma^i$. The comparison between the estimates of $\left\| \frac{\partial p^\varepsilon}{\partial x} \right\|_{H^{-1}(\Omega)}$ and $\left\| \frac{\partial p^\varepsilon}{\partial y} \right\|_{H^{-1}(\Omega)}$ leads to $\frac{\partial p^*}{\partial y} = 0$.

To get $u_2^* = 0$ on Ω , we use the fact that u_i^ε tends towards u_i^* in L^2 weak, so that $\frac{\partial u_i^\varepsilon}{\partial x}$ tends towards $\frac{\partial u_i^*}{\partial x}$ in $\mathcal{D}'(\Omega)$. From the incompressibility property (2.5), we get

$$\frac{\partial u_2^*}{\partial y} = - \lim_{\varepsilon \rightarrow 0} \varepsilon \frac{\partial u_1^\varepsilon}{\partial x} = 0.$$

Taking into account the boundary values of u_2^ε , then $u_2^* = 0$ on $\Gamma^s \cup \Gamma^i$. $\square$

PROPOSITION 4.2.

For $\gamma < 2$, then u_1^*, ω^*, p^* are such that

$$\begin{aligned} \frac{\partial^2 u_1^*}{\partial y^2} + 2N^2 \frac{\partial \omega^*}{\partial y} &= \frac{\partial p^*}{\partial x} \quad \text{in } H^{-1}(\Omega), \\ \frac{\partial^2 \omega^*}{\partial y^2} &= 0 \quad \text{in } L^2(\Omega). \end{aligned} \quad (4.2)$$

For $\gamma = 2$, then $u_1^*, \omega_* = \omega^*, p^*$ are such that

$$\begin{aligned} \frac{\partial^2 u_1^*}{\partial y^2} + 2N^2 \frac{\partial \omega^*}{\partial y} &= \frac{\partial p^*}{\partial x} \quad \text{in } H^{-1}(\Omega), \\ \lambda \frac{\partial^2 \omega^*}{\partial y^2} - 4N^2 \omega^* &= 2N^2 \frac{\partial u_1^*}{\partial y} \quad \text{in } H^{-1}(\Omega). \end{aligned} \quad (4.3)$$

For $\gamma > 2$, then u_1^*, ω_*, p^* are such that

$$\begin{aligned} \frac{\partial^2 u_1^*}{\partial y^2} + 2N^2 \frac{\partial \omega^*}{\partial y} &= \frac{\partial p^*}{\partial x} \quad \text{in } H^{-1}(\Omega), \\ \omega_* &= -\frac{1}{2} \frac{\partial u_1^*}{\partial y} \quad \text{in } L^2(\Omega). \end{aligned} \quad (4.4)$$

Proof. These systems are directly deduced from (3.2), taking Proposition 4.1 into account and choosing successively $\psi = 0$ and $\varphi = 0$. $\square$

In the next proposition, we will prove that p^* lies not only in $L^2(\Omega)$ but also in $H^1(\Omega)$. We introduce the following notation:

$$\text{for any function } \theta \text{ in } L^2(\Omega), \quad \tilde{\theta}(x) = \frac{1}{h(x)} \int_0^{h(x)} \theta(x, y) dy.$$

PROPOSITION 4.3. The limit pressure p^* lies in $H^1(\Lambda)$ and satisfies the equation

$$\frac{h^3}{6} \frac{\partial p^*}{\partial x} = (-2\tilde{u}_1^* + u_1^s + u_1^i)h - 2N^2 \int_0^{h(x)} \tilde{\omega}(h - 2y) dy, \quad (4.5)$$

where u^i and u^s are the values of the limit u^* on Γ^i and Γ^s , and $\hat{\omega}$ will denote ω^* if $\gamma \leq 2$ or ω_* if $\gamma > 2$.

Proof. We rewrite (3.2.1) with any test function ϕ in $H_0^1(H^\varepsilon)$. Passing to the limit using Proposition 4.1, we get the following equation which is valid for any γ :

$$\int_{\Omega} \frac{\partial u_1^*}{\partial y} \frac{\partial \phi}{\partial y} dx dy - \int_{\Omega} p^* \frac{\partial \phi}{\partial x} dx dy = -2N^2 \int_{\Omega} \hat{\omega} \frac{\partial \phi}{\partial y} dx dy, \quad \forall \phi \in H_0^1(\Omega).$$

Let θ be in $H_0^1(\Lambda)$ and choose $\phi = \theta(x)y(y-h)$. We get

$$\int_{\Omega} \frac{\partial u_1^*}{\partial y} \theta(2y-h) dx dy + 2N^2 \int_{\Omega} \hat{\omega} \theta(2y-h) dx dy = - \int_{\Lambda} p^* \frac{\partial}{\partial x} \left(\frac{h^3}{6} \theta \right) dx.$$

Then

$$\int_{\Lambda} \theta \frac{h^3}{6} \frac{\partial p^*}{\partial x} dx = \int_{\Lambda} (-2\tilde{u}_1^* + u_1^s + u_1^i) \theta h dx + 2N^2 \int_{\Lambda} \theta \int_0^h \hat{\omega}(2y-h) dy dx.$$

Since θ is any function in $H_0^1(\Lambda)$, (4.5) is proved.

Since h is a smooth strictly positive function and since the right-hand side of (4.5) lies in $L^2(\Lambda)$, we deduce that $\frac{\partial p^*}{\partial x}$ lies in $L^2(\Lambda)$, which ends the proof. $\square$

PROPOSITION 4.4. For any ϕ in $H^1(\Lambda)$, we get

i) If $\delta \leq 0$,

$$\int_{\Lambda} h \tilde{u}_1^* \frac{d\phi}{dx} dx = \int_{\Lambda} (-J_1^s h'(x) + J_2^s - J_2^i) \phi dx + \int_{\partial\Lambda} h \tilde{J}_1 \phi n_1 d\sigma. \quad (4.6)$$

ii) If $\delta > 0$,

$$\int_{\Lambda} h \tilde{u}_1^* \frac{d\phi}{dx} dx = 0. \quad (4.7)$$

Proof. Let us consider the divergence free equation in the weak form:

$$\int_{\Omega} \left(\frac{\partial u_1^\varepsilon}{\partial x} + \frac{1}{\varepsilon} \frac{\partial u_2^\varepsilon}{\partial y} \right) q dx dy = 0. \quad (4.8)$$

Choosing $q = \phi(x)$ for any ϕ in $H^1(\Lambda)$, and applying Green's formula:

$$\int_{\Omega} u_1^\varepsilon \frac{d\phi}{dx} dx dy = - \int_{\Lambda} J_1^s \phi h' dx + \int_{\Lambda} (J_2^s - J_2^i) \phi dx + \int_{\Gamma_L} h \tilde{J}_1 \phi n_1 d\sigma.$$

Passing to the limit when $\delta \leq 0$, we get (4.6). For $\delta > 0$, we multiply both sides of this equation by ε^δ before passing to the limit and get (4.7). $\square$

THEOREM 4.5. Assuming $\gamma < 2$ and (H1)–(H4), then the pressure p^* is the unique solution of the following classical Reynolds equation and the rotation ω^* is a linear function of y . More precisely:

If $\delta < 0$: $p^* \in H^1(\Lambda)$ and for any $\phi \in H^1(\Lambda)$,

$$\int_{\Lambda} \frac{h^3}{12} \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx = \int_{\Lambda} (J_1^s h'(x) - J_2^s + J_2^i) \phi dx + \int_{\Lambda} (J_1^s + J_1^i) \frac{h}{2} \frac{\partial \phi}{\partial x} dx - \int_{\partial\Lambda} h \tilde{J}_1 \phi n_1 d\sigma, \quad (4.9)$$

$$\omega^*(x, y) = 0. \quad (4.10)$$

If $\delta = 0$: $p^* \in H^1(\Lambda)$ and for any $\phi \in H^1(\Lambda)$,

$$\begin{aligned} \int_{\Lambda} \frac{h^3}{12} \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx &= \int_{\Lambda} (J_1^s h'(x) - J_2^s + J_2^i) \phi dx \\ &+ \int_{\Lambda} \frac{h}{2} \left(J_1^s + J_1^i + \frac{hN^2}{3} (F^s - F^i) \right) \frac{\partial \phi}{\partial x} dx - \int_{\partial\Lambda} h \tilde{J}_1 \phi n_1 d\sigma, \end{aligned} \quad (4.11)$$

$$\omega^*(x, y) = \frac{F^s - F^i}{h} y + F^i. \quad (4.12)$$

If $\delta > 0$: $p^* \in H^1(\Lambda)$ and for any $\phi \in H^1(\Lambda)$,

$$\int_{\Lambda} \frac{h^3}{12} \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx = \frac{N^2}{6} \int_{\Lambda} h^2 (F^s - F^i) \frac{\partial \phi}{\partial x} dx, \quad (4.13)$$

$$\omega^*(x, y) = \frac{F^s - F^i}{h} y + F^i, \quad (4.14)$$

with the additional condition for all δ :

$$\int_{\Lambda} h p^* dx = 0. \quad (4.15)$$

Proof. Let us first consider the case $\delta < 0$. From equation (4.2.ii) and Prop. 4.1 (iii), we deduce that $\omega^* = 0$. From (4.2) and Prop. 4.1 (iii), we get

$$u_1^*(x, y) = \frac{\partial p^*}{\partial x} \frac{y(y-h)}{2} + (J_1^s - J_1^i) \frac{y}{h} + J_1^i. \quad (4.16)$$

Averaging (4.16) and taking (4.6) into account, we obtain the required relation (4.9).

Assume now $\delta = 0$. Then (4.2) and Prop. 4.1 (iii) allow us to compute ω^* , by (4.12) and

$$u_1^*(x, y) = \frac{1}{2} \left[\frac{\partial p^*}{\partial x} - 2N^2 \left(\frac{F^s - F^i}{h} \right) \right] y(y-h) + (J_1^s - J_1^i) \frac{y}{h} + J_1^i. \quad (4.17)$$

Using (4.5), (4.6), we obtain (4.11).

At last, assuming $\delta > 0$, we obtain the same value for ω^* . However, since Prop. 4.1 (iii) implies $u_1^s = u_1^i = 0$ and $\omega^* = F$ on $\Gamma^s \cup \Gamma^i$, we obtain

$$u_1^*(x, y) = \frac{1}{2} \left[\frac{\partial p^*}{\partial x} - 2N^2 \left(\frac{F^s - F^i}{h} \right) \right] y(y-h). \quad (4.18)$$

Averaging (4.18) and inserting this value in (4.7), we obtain (4.13). $\square$

THEOREM 4.6. Assuming $\gamma = 2$ and (H1)–(H4), the asymptotic pressure p^* is the unique solution of the following equations:

If $\delta \leq 0$: $p^* \in H^1(\Lambda)$ and for all $\phi \in H^1(\Lambda)$,

$$\int_{\Lambda} \Xi \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx = \int_{\Lambda} (J_1^s h' - J_2^s + J_2^i) \phi dx + \int_{\Lambda} \left((J_1^s + J_1^i) \frac{h}{2} + \Psi \right) \frac{\partial \phi}{\partial x} dx - \int_{\partial\Lambda} h \tilde{J}_1 \phi n_1 d\sigma. \quad (4.19)$$

If $\delta > 0$: $p^* \in H^1(\Lambda)$ and for all $\phi \in H^1(\Lambda)$,

$$\int_{\Lambda} \Xi(x) \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx = \int_{\Lambda} \Psi(x) \frac{\partial \phi}{\partial x} dx \quad (4.20)$$

with the additional condition for all δ :

$$\int_{\Lambda} h p^* dx = 0. \quad (4.21)$$

The functions Ξ and Ψ are defined in the proof.

Proof. For any x , we solve the ordinary differential system (4.3) in which the unknowns are the rotations ω^* and u_1^* , and where $\frac{\partial p^*}{\partial x}$ is considered as a “given” parameter; so we get

$$\begin{aligned} u_1^*(x, y) &= \frac{y^2}{2(1-N)^2} \frac{\partial p^*}{\partial x} + A_1(\beta sh(ky) + 2y) + \beta B_1(ch(ky) - 1) + u_1^i - 2\omega^i y, \\ \omega^*(x, y) &= A_1(ch(ky) - 1) + B_1 sh(ky) - \frac{y}{2(1-N^2)} \frac{\partial p^*}{\partial x} + \omega^i \end{aligned} \quad (4.22)$$

with

$$\begin{aligned} A_1 &= \frac{(u_1^s - u_1^i + \omega^i h) sh(kh)}{2(h sh(kh) + \beta(ch(kh) - 1))} - \frac{h}{4(1-N^2)} \frac{\partial p^*}{\partial x} \\ &\quad + \frac{\beta(\omega^i - \omega^s)(ch(kh) - 1) + \omega^i h sh(kh)}{2(h sh(kh) + \beta(ch(kh) - 1))}, \\ B_1 &= \frac{1}{sh(kh)} \left(\omega^s - \omega^i + \frac{h}{2(1-N^2)} \frac{\partial p^*}{\partial x} - A_1(ch(kh) - 1) \right), \end{aligned} \quad (4.23)$$

where

$$k = 2N \sqrt{\frac{1-N^2}{\lambda}}, \quad \beta = \frac{\lambda k}{2N^2} - \frac{2}{k}. \quad (4.24)$$

This enables us to compute

$$N^2 \int_0^h \omega^*(h-2y) dy = \theta(x) \frac{\partial p^*}{\partial x} - \Psi(x), \quad (4.25)$$

where θ and Ψ are given by

$$\begin{aligned} \theta(x) &= \frac{N^2}{4(1-N^2)} \left[\frac{h^3}{3} + \frac{4h}{k^2} - \frac{2h^2}{k} \coth\left(\frac{kh}{2}\right) \right], \\ \Psi(x) &= \frac{N^2}{4} \left[\frac{2Z}{kD} \left[hS - \frac{2(C-1)}{k} \right] + \frac{1}{kS} \left(\omega^s - \omega^i - \frac{2Z}{D} \right) \left(h(C+1) - \frac{2S}{k} \right) \right], \end{aligned}$$

with $C = ch(kh)$, $S = sh(kh)$, $D = hS + \beta(C-1)$, and $Z = \beta(\omega^s - \omega^i)(C-1) + \omega^i hS$.

From (4.5) and Proposition 4.1 (iii), we get

$$\begin{aligned} h\tilde{u}_1^* &= - \left(\frac{h^3}{12} + \theta(x) \right) \frac{\partial p^*}{\partial x} + (J_1^s + J_1^i) \frac{h}{2} + \Psi(x) \quad \text{if } \delta \leq 0, \\ \tilde{u}_1^* h &= - \left(\frac{h^3}{12} + \theta(x) \right) \frac{\partial p^*}{\partial x} + \Psi(x) \quad \text{if } \delta > 0. \end{aligned}$$

Denoting $\Xi(x) = \frac{h^3}{12} + \theta(x)$ and using (4.6)–(4.7), we get (4.19) and (4.20).

The uniqueness of the solution of equations (4.19)–(4.21) is deduced from the ellipticity of the operator $\int_{\Lambda} \Xi(x) \frac{\partial \varphi}{\partial x} \frac{\partial \phi}{\partial x} dx$, which will come from the positivity of

$$\Xi(x) = \frac{h^3(x)}{1-N^2} \left(\frac{1}{12} + \frac{\lambda}{4h^2(1-N^2)} - \frac{1}{4h} \sqrt{\frac{\lambda N^2}{1-N^2}} \coth \left(Nh \sqrt{\frac{1-N^2}{\lambda}} \right) \right).$$

Let $H = h \sqrt{\frac{1-N^2}{\lambda}}$ and $Y = NH$. The positivity of the function Ξ is equivalent to proving that

$$\frac{Y}{3N^2} + \frac{1}{Y} = m_1(Y, N^2) \geq m_2(Y) = \coth(Y) \quad \forall 0 < Y < +\infty.$$

Since $m_1(Y, N^2)$ is a decreasing function of N^2 , it suffices to prove that $m_1(Y, 1) \geq m_2(Y)$.

The function $m_1(Y, 1)$ achieves a minimum for $Y = Y^* = \sqrt{3}$ and $\frac{\partial m_2}{\partial Y}(Y) < 0$ for all Y . Since $m_1(Y^*, 1) \geq m_2(Y^*)$, we get $m_1(Y, 1) \geq m_2(Y)$ for $Y > Y^*$.

Moreover, $m_1(Y, 1) - m_2(Y)$ is increasing in $[0, Y^*]$ and tends to 0 when Y tends to 0 so that it is positive for all Y . $\square$

THEOREM 4.7. Assuming $\gamma > 2$ and (H1)–(H4), then p^* is the unique solution of a Reynolds-like equation and the rotation ω_* is a linear function of y :

If $\delta \leq 0$: $p^* \in H^1(\Lambda)$ and for all $\phi \in H^1(\Lambda)$,

$$\begin{aligned} \int_{\Lambda} \frac{h^3}{12(1-N^2)} \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx &= \int_{\Lambda} (J_1^s h' - J_2^s + J_2^i) \phi dx \\ &+ \int_{\Lambda} (J_1^s + J_1^i) \frac{h}{2} \frac{\partial \phi}{\partial x} dx - \int_{\partial \Lambda} h \tilde{J}_1 \phi n_1 d\sigma. \end{aligned} \quad (4.26)$$

If $\delta > 0$: $p^* \in H^1(\Lambda)$ and for all $\phi \in H^1(\Lambda)$,

$$\int_{\Lambda} h^3 \frac{\partial p^*}{\partial x} \frac{\partial \phi}{\partial x} dx = 0 \quad \text{which gives } p^* = 0 \quad (4.27)$$

with the additional condition for all δ :

$$\int_{\Lambda} h p^* dx = 0. \quad (4.28)$$

Proof. For any x , we solve the differential system (4.4). We get

$$\begin{aligned} u_1^*(x, y) &= \frac{1}{2(1-N^2)} \frac{\partial p^*}{\partial x} y(y-h) + \frac{J_1^s - J_1^i}{h} + J_1^i \quad \text{if } \delta \leq 0, \\ u_1^*(x, y) &= \frac{1}{2(1-N^2)} \frac{\partial p^*}{\partial x} y(y-h) \quad \text{if } \delta > 0, \end{aligned} \quad (4.29)$$

$$\begin{aligned} \omega_*(x, y) &= \frac{1}{2(1-N^2)} \frac{\partial p^*}{\partial x} \left(\frac{h}{2} - y \right) + \frac{J_1^i - J_1^s}{2h} \quad \text{if } \delta \leq 0, \\ \omega_*(x, y) &= \frac{1}{2(1-N^2)} \frac{\partial p^*}{\partial x} \left(\frac{h}{2} - y \right) \quad \text{if } \delta > 0. \end{aligned} \quad (4.30)$$

Inserting these values in (4.6)–(4.7), we obtain (4.27)–(4.28). $\square$

REMARK 4.8. From the expression of p^* and u_1^* , we deduce that for $\delta > 0$, the asymptotic pressure, velocity and rotation do not rely on the data on Γ_L . For $\delta \leq 0$, they rely only on the average of the velocity on Γ_L and not on the pointwise values. Moreover, for $\gamma > 2$, ω_* does not rely on the values of ω^ε on $\partial\Omega$.

In all cases, choosing $q = \varphi(x) \in \mathcal{D}(\Lambda)$ in (3.24), we get the following mass-flow average conservation property:

$$\int_0^{h(x)} u_1^*(x, y) dy = \int_0^{h(x)} u_1^\varepsilon(x, y) dy = \int_0^{h(x)} J_1(x, y) dy, \quad x \in \partial\Lambda. \quad (4.31)$$

5. Second-order estimates. In this section, we will get second-order estimates for velocity, rotation, and pressure when $\delta = 0$. The result enforces the validity of the thin film approximation since the following terms in the asymptotic expansion will be proved to be of order $O(\varepsilon^2)$ if $\gamma \geq 2$, $O(\varepsilon^{2-\gamma})$ if $1 < \gamma < 2$, and $O(\varepsilon)$ if $0 < \gamma \leq 1$.

Although the convergence of u_1^ε and ω^ε towards u_1^* and $\hat{\omega}$ takes place in $V_y(\Omega)$ or $L^2(\Omega)$ and not in $H^1(\Omega)$, inducing possible boundary layers on Γ_L for the velocity and rotation, and on $\Gamma^s \cup \Gamma^i$ for the rotation if $\gamma > 2$, (4.12), (4.14), (4.16), (4.17), (4.18), (4.22), (4.29), and (4.30) demonstrate that for smooth data, u_1^* and ω^* lie in $H^1(\Omega)$. By choosing specific values related to u_1^* and ω^* for f^ε and g^ε , it will be possible to cancel the occurrence of boundary layers, allowing us to choose particular test functions in the initial problem to obtain the required second-order estimates.

Let us assume in this section:

- (H5) h, J^s, J^i, F^s, F^i lie in $\mathcal{C}^2(\bar{\Lambda})$,

so that p^* belongs to $H^3(\Omega)$ while u_1^* belongs to $H^2(\Omega)$.

We define

$$v^*(x, y) = \int_y^{h(x)} \frac{\partial u_1^*}{\partial x}(x, \xi) d\xi + J_2^s(x)$$

and we choose the following boundary conditions:

- (H6)

$$g_1^\varepsilon = (u_1^* \text{ on } \Gamma_L, J_1^s \text{ on } \Gamma^s, J_1^i \text{ on } \Gamma^i),$$

$$g_2^\varepsilon = (v^* \text{ on } \Gamma_L, J_2^s \text{ on } \Gamma^s, J_2^i \text{ on } \Gamma^i),$$

$$f^\varepsilon = \begin{cases} (\omega^* \text{ on } \Gamma_L, F^s \text{ on } \Gamma^s, F^i \text{ on } \Gamma^i) & \text{if } \gamma \leq 2, \\ (\omega_* \text{ on } \partial\Omega) & \text{if } \gamma > 2. \end{cases}$$

PROPOSITION 5.1. The boundary conditions given by (H6) satisfy the hypotheses (H2)–(H3).

Proof. Let us first prove (H2). This is obvious for $\gamma > 2$ due to the regularity of u_1^* because $\omega_* = -\frac{1}{2} \frac{\partial u_1^*}{\partial y}$. For $\gamma \leq 2$, we know that $\omega^* = F$ on $\Gamma^s \cup \Gamma^i$ due to the V_y -convergence of ω^ε . Since $\hat{\omega}$ is a regular function of y from (4.12), (4.22), (4.30), $\bar{f}$ is continuous on $\partial\Gamma_L$, belongs to $H^{\frac{1}{2}}(\partial\Omega)$ and can be considered as the trace of an ε -independent function of $H^1(\Omega)$.

For the velocity, the divergence-free property is obvious by construction of v^* . We need to prove the continuity of $\bar{g}_2$ on Γ_L by computing:

$$\begin{aligned}\bar{g}_2(x, 0) &= v^*(x, 0) = \int_0^{h(x)} \frac{\partial u_1^*}{\partial x}(x, \xi) d\xi + J_2^s(x) \\ &= \frac{\partial}{\partial x} \int_0^{h(x)} u_1^*(x, \xi) d\xi - h'(x)u_1^*(x, h(x)) + J_2^s(x) \\ &= \frac{\partial}{\partial x}(h\tilde{u}_1^*) - h'(x)u_1^*(x, h(x)) + J_2^s(x) \\ &= J_2^i(x), \quad x \in \partial\Lambda,\end{aligned}$$

from (4.31).

$$\bar{g}_2(x, h(x)) = J_2^s(x), \quad \forall x \in \partial\Lambda. \quad \square$$

REMARK 5.2. The choice of boundary conditions (H6) could appear as restrictive ones. This is not the case, especially for the lubrication problem. Let us consider two physical devices with different given boundary conditions (g, f) and $(\bar{g}, \bar{f})$ on $\partial\Omega$, the asymptotic behaviour being described by the pressure fields p^* and $\bar{p}^*$.

Suppose that $(\bar{g}, \bar{f})$ are such that

$$\bar{g} = g \quad \text{on } \Gamma^i \cup \Gamma^s, \quad \int_{\Gamma_L} \bar{g} = \int_{\Gamma_L} g, \quad \bar{f} = f \quad \text{if } \gamma \leq 2;$$

we deduce from Remark 4.9 that $\bar{p}^* = p^*$.

In other words, it is possible, starting with any boundary conditions satisfying (H2) and (H3) to obtain new ones that give the same asymptotic pressure and for which no boundary layers occur. $\square$

Let us introduce $U^\varepsilon = u^\varepsilon - v^\varepsilon$ where $v^\varepsilon = (u_1^*, \varepsilon v^*)$ and $W^\varepsilon = \omega^\varepsilon - \hat{\omega}$.

PROPOSITION 5.3. Assuming (H1)–(H6), we have

i) If $0 \leq \gamma \leq 1$,

$$\left\| \frac{\partial U_1^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial W^\varepsilon}{\partial x} \right\| \leq K, \quad (5.1)$$

$$\left\| \frac{\partial U_1^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial W^\varepsilon}{\partial y} \right\| \leq K\varepsilon. \quad (5.2)$$

ii) If $1 \leq \gamma < 2$,

$$\left\| \frac{\partial U_1^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial W^\varepsilon}{\partial x} \right\| \leq K\varepsilon^{1-\gamma}, \quad (5.3)$$

$$\left\| \frac{\partial U_1^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial W^\varepsilon}{\partial y} \right\| \leq K\varepsilon^{2-\gamma}. \quad (5.4)$$

iii) If $\gamma \geq 2$,

$$\left\| \frac{\partial U_1^\varepsilon}{\partial x} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial x} \right\| \leq K\varepsilon, \quad (5.5)$$

$$\left\| \frac{\partial U_1^\varepsilon}{\partial y} \right\| + \left\| \frac{\partial U_2^\varepsilon}{\partial y} \right\| + \|W^\varepsilon\| \leq K\varepsilon^2, \quad (5.6)$$

$$\varepsilon^{\frac{\gamma}{2}} \left\| \frac{\partial W^\varepsilon}{\partial x} \right\| + \varepsilon^{\frac{\gamma}{2}-1} \left\| \frac{\partial W^\varepsilon}{\partial y} \right\| \leq K\varepsilon^2. \quad (5.7)$$

Proof. Due to the particular choice of the boundary conditions, U_i^ε and W^ε lie in $H_0^1(\Omega)$ and may be taken as test functions in (3.4). Moreover, $\operatorname{div}_\varepsilon U^\varepsilon = 0$. Denote by

$$\begin{aligned} A &= \left\| \frac{\partial U_1^\varepsilon}{\partial x} \right\|, \quad B = \left\| \frac{\partial U_1^\varepsilon}{\partial y} \right\|, \quad C = \left\| \frac{\partial U_2^\varepsilon}{\partial x} \right\|, \\ D &= \left\| \frac{\partial U_2^\varepsilon}{\partial y} \right\|, \quad E = \left\| \frac{\partial W^\varepsilon}{\partial x} \right\|, \quad G = \left\| \frac{\partial W^\varepsilon}{\partial y} \right\|. \end{aligned}$$

For $\gamma \leq 2$, we get

$$\begin{aligned} &\varepsilon^2(A^2 + C^2) + B^2 + D^2 + \lambda\varepsilon^2 E^2 + \lambda G^2 + 4N^2\varepsilon^{2-\gamma}|W^\varepsilon|^2 \\ &= -\varepsilon^2 \int_\Omega \frac{\partial u_1^*}{\partial x} \frac{\partial U_1^\varepsilon}{\partial x} dx dy - \int_\Omega \frac{\partial u_1^*}{\partial y} \frac{\partial U_1^\varepsilon}{\partial y} dx dy \\ &\quad - \varepsilon^3 \int_\Omega \frac{\partial v^*}{\partial x} \frac{\partial U_2^\varepsilon}{\partial x} dx dy - \varepsilon \int_\Omega \frac{\partial v^*}{\partial y} \frac{\partial U_2^\varepsilon}{\partial y} dx dy \\ &\quad - \lambda\varepsilon^2 \int_\Omega \frac{\partial \omega^*}{\partial x} \frac{\partial W^\varepsilon}{\partial x} dx dy - \lambda \int_\Omega \frac{\partial \omega^*}{\partial y} \frac{\partial W^\varepsilon}{\partial y} dx dy - 4N^2\varepsilon^{2-\gamma} \int_\Omega \omega^* W^\varepsilon dx dy \\ &\quad + 2N^2\varepsilon^{2-\gamma} \int_\Omega \left[\varepsilon^2 \frac{\partial v^*}{\partial x} - \frac{\partial u_1^*}{\partial y} \right] W^\varepsilon dx dy + 2N^2\varepsilon^{2-\gamma} \int_\Omega \left[\varepsilon \frac{\partial U_2^\varepsilon}{\partial x} - \frac{\partial U_1^\varepsilon}{\partial y} \right] W^\varepsilon dx dy \\ &\quad + 2N^2 \int_\Omega \left[\frac{\partial W^\varepsilon}{\partial y} U_1^\varepsilon - \varepsilon \frac{\partial W^\varepsilon}{\partial x} U_2^\varepsilon \right] dx dy + 2N^2 \int_\Omega \left[\frac{\partial \omega^*}{\partial y} U_1^\varepsilon - \varepsilon \frac{\partial \omega^*}{\partial x} U_2^\varepsilon \right] dx dy. \end{aligned} \quad (5.8)$$

For $\gamma < 2$ the limit system (4.2) gives

$$\begin{aligned} -\lambda \int_\Omega \frac{\partial u_1^*}{\partial y} \frac{\partial U_1^\varepsilon}{\partial y} dx dy + 2N^2 \int_\Omega \frac{\partial \omega^*}{\partial y} U_1^\varepsilon dx dy &= \int_\Omega \frac{\partial p^*}{\partial x} U_1^\varepsilon dx dy \\ &= -\frac{1}{\varepsilon} \int_\Omega \frac{\partial p^*}{\partial y} U_2^\varepsilon dx dy = 0, \end{aligned} \quad (5.9)$$

$$-\lambda \int_\Omega \frac{\partial \omega^*}{\partial y} \frac{\partial W^\varepsilon}{\partial y} dx dy = \lambda \int_\Omega \frac{\partial^2 \omega^*}{\partial^2 y} W^\varepsilon dx dy = 0. \quad (5.10)$$

Using the same kind of technical estimates as the one in the proof of Proposition 3.2, we get from (5.8) the required estimates under hypothesis (H4) and using (5.9), (5.10).

For $\gamma = 2$ the limit system (4.3) gives the same first equation (5.9) and

$$\lambda \int_{\Omega} \frac{\partial \omega^*}{\partial y} \frac{\partial W^\varepsilon}{\partial y} dx dy + 4N^2 \int_{\Omega} \omega^* W^\varepsilon dx dy + 2N^2 \int_{\Omega} \frac{\partial u_1^*}{\partial y} W^\varepsilon dx dy = 0. \quad (5.11)$$

For $\gamma > 2$, we use (3.6) in Lemma (3.4) and the limit system (4.4) and we obtain the result. $\square$

PROPOSITION 5.4 (Estimates for the pressure).

Assuming (H1) to (H6), we get

i) If $0 < \gamma \leq 1$,

$$\|p^\varepsilon - p^*\|_{L_0^2(\Omega)} \leq C\varepsilon. \quad (5.12)$$

ii) If $1 < \gamma < 2$,

$$\|p^\varepsilon - p^*\|_{L_0^2(\Omega)} \leq C\varepsilon^{2-\gamma}. \quad (5.13)$$

iii) If $\gamma \geq 2$,

$$\|p^\varepsilon - p^*\|_{L_0^2(\Omega)} \leq C\varepsilon^2. \quad (5.14)$$

Proof. Let us define $q^\varepsilon = p^\varepsilon - p^*$. For any φ in $H_0^1(\Omega)$, we get from (3.2.1)–(3.2.2) and Proposition 4.2:

$$-\int_{\Omega} \frac{\partial q^\varepsilon}{\partial x} \varphi = \varepsilon^2 \int_{\Omega} \frac{\partial U_1^\varepsilon}{\partial x} \frac{\partial \varphi}{\partial x} + \int_{\Omega} \frac{\partial U_1^\varepsilon}{\partial y} \frac{\partial \varphi}{\partial y} - 2N^2 \int_{\Omega} \frac{\partial W^\varepsilon}{\partial y} \varphi + \varepsilon^2 \int_{\Omega} \frac{\partial u_1^*}{\partial x} \frac{\partial \varphi}{\partial x}, \quad (5.15)$$

$$\begin{aligned} -\int_{\Omega} \frac{\partial q^\varepsilon}{\partial y} \varphi dx dy &= \varepsilon^3 \int_{\Omega} \frac{\partial U_2^\varepsilon}{\partial x} \frac{\partial \varphi}{\partial x} dx dy + \varepsilon \int_{\Omega} \frac{\partial U_2^\varepsilon}{\partial y} \frac{\partial \varphi}{\partial y} dx dy + 2N^2 \varepsilon^2 \int_{\Omega} \frac{\partial W^\varepsilon}{\partial x} \varphi dx dy \\ &+ \varepsilon^4 \int_{\Omega} \frac{\partial v^*}{\partial x} \frac{\partial \varphi}{\partial x} dx dy + \varepsilon^2 \int_{\Omega} \frac{\partial v^*}{\partial y} \frac{\partial \varphi}{\partial y} dx dy + 2N^2 \varepsilon^2 \int_{\Omega} \frac{\partial \omega^*}{\partial x} \varphi dx dy. \end{aligned} \quad (5.16)$$

Estimates from Proposition 5.3 allow us to obtain estimates for $\|\frac{\partial q^\varepsilon}{\partial x}\|$ and $\|\frac{\partial q^\varepsilon}{\partial y}\|$ in $H^{-1}(\Omega)$. Since q^ε lies in $L_0^2(\Omega)$, the results are deduced from (3.16). $\square$

6. Second-order limit equations. The estimates of Proposition 5.3 allow us to define the limit of the second-order terms in the suitable corresponding Sobolev spaces.

For $\gamma \leq 1$,

$$U^{**} = \lim_{V_y} \frac{U^\varepsilon}{\varepsilon}, \quad W^{**} = \lim_{V_y} \frac{W^\varepsilon}{\varepsilon}, \quad q^{**} = \lim_{L_0^2} \frac{q^\varepsilon}{\varepsilon}.$$

For $1 < \gamma < 2$,

$$U^{**} = \lim_{V_y} \frac{U^\varepsilon}{\varepsilon^{2-\gamma}}, \quad W^{**} = \lim_{V_y} \frac{W^\varepsilon}{\varepsilon^{2-\gamma}}, \quad q^{**} = \lim_{L_0^2} \frac{q^\varepsilon}{\varepsilon^{2-\gamma}}.$$

For $\gamma \geq 2$,

$$U^{**} = \lim_{V_y} \frac{U^\varepsilon}{\varepsilon^2}, \quad W^{**} = \lim_{V_y} \frac{W^\varepsilon}{\varepsilon^{3-\frac{\gamma}{2}}}, \quad q^{**} = \lim_{L_0^2} \frac{q^\varepsilon}{\varepsilon^2} \quad \text{and} \quad W_{**} = \lim_{L^2} \frac{W^\varepsilon}{\varepsilon^2}.$$

We give first the equations satisfied by the limits for each case.

PROPOSITION 6.1. Under assumptions (H1) to (H6), we get

i) If $0 < \gamma < 2$,

$$\begin{aligned} \frac{\partial^2 U_1^{**}}{\partial y^2} + 2N^2 \frac{\partial W^{**}}{\partial y} &= \frac{\partial q^{**}}{\partial x}, \\ \frac{\partial q^{**}}{\partial y} &= 0, \\ W^{**} &= 0, & \text{if } \gamma < 1, \\ \lambda \frac{\partial^2 W^{**}}{\partial y^2} &= 2N^2 \left(2\omega^* + \frac{\partial u_1^*}{\partial y} \right) & \text{if } 1 \leq \gamma < 2. \end{aligned} \quad (6.1)$$

ii) If $\gamma = 2$,

$$\begin{aligned} \frac{\partial^2 U_1^{**}}{\partial y^2} + 2N^2 \frac{\partial W^{**}}{\partial y} &= \frac{\partial q^{**}}{\partial x} - \frac{\partial^2 u_1^*}{\partial x^2}, \\ \frac{\partial q^{**}}{\partial y} &= \frac{\partial^2 v^*}{\partial y^2} - 2N^2 \frac{\partial \omega^*}{\partial x}, \\ \lambda \frac{\partial^2 W^{**}}{\partial y^2} + 4N^2 W^{**} + 2N^2 \frac{\partial U_1^{**}}{\partial y} &= 2N^2 \frac{\partial v^*}{\partial x} - \lambda \frac{\partial^2 \omega^*}{\partial x^2}. \end{aligned} \quad (6.2)$$

iii) If $\gamma > 2$,

$$\begin{aligned} \frac{\partial^2 U_1^{**}}{\partial y^2} + 2N^2 \frac{\partial W^{**}}{\partial y} &= \frac{\partial q^{**}}{\partial x} - \frac{\partial^2 u_1^*}{\partial x^2}, \\ \frac{\partial q^{**}}{\partial y} &= \frac{\partial^2 v^*}{\partial y^2} - 2N^2 \frac{\partial \omega^*}{\partial x}, \\ 2W^{**} + \frac{\partial U_1^{**}}{\partial y} &= 2N^2 \frac{\partial v^*}{\partial x}. \end{aligned} \quad (6.3)$$

Proof. It is the same kind of proof as the one of Prop. 4.2. □

LEMMA 6.2. For all $\gamma > 0$, we have

$$\int_{\Lambda} h \tilde{U}_1^{**} \frac{\partial \psi}{\partial x} dx = 0, \quad \forall \psi \in H^1(\Lambda). \quad (6.4)$$

Proof. Since $\operatorname{div}_{\varepsilon} U^{\varepsilon} = 0$, we get for all $\psi \in H^1(\Lambda)$,

$$\int_{\Omega} \frac{\partial U_1^{\varepsilon}}{\partial x} \psi dx + \int_{\Omega} \frac{\partial U_2^{\varepsilon}}{\partial y} \psi dx = 0.$$

ψ being independent of y , we obtain, after using Green's formula,

$$\int_{\Omega} U_1^{\varepsilon} \frac{\partial \psi}{\partial x} dx = 0, \quad \forall \psi \in H^1(\Lambda).$$

We divide this equation by ε if $\gamma \leq 1$, by $\varepsilon^{2-\gamma}$ if $1 < \gamma < 2$, and by ε^2 if $\gamma \geq 2$. Passing to the limit in all cases, we have

$$\int_{\Omega} U_1^{**} \frac{\partial \psi}{\partial x} dx = 0, \quad \forall \psi \in H^1(\Lambda),$$

which gives by Fubini:

$$\int_{\Lambda} \frac{\partial \psi}{\partial x} \left(\int_0^h U_1^{**} dy \right) dx = 0,$$

which is (6.4). $\square$

Now going on as in Sec. 4, we obtain the limit of the second-order terms. More precisely:

THEOREM 6.3. Assuming (H1) to (H6),

i) If $0 < \gamma < 1$, we have $U_1^{**} = 0$, $W^{**} = 0$, and $q^{**} = 0$.

ii) If $1 \leq \gamma < 2$, q^{**} is the unique solution of the Reynolds-like problem:

Find $q^{**} \in H^1(\Lambda)$ such that

$$\int_{\Lambda} \frac{h^3}{12} \frac{\partial q^{**}}{\partial x} \frac{\partial \psi}{\partial x} dx = \int_{\Lambda} \left[L(x) \frac{\partial p^*}{\partial x} + N(x) \right] \frac{\partial \psi}{\partial x} dx, \quad \forall \psi \in H^1(\Lambda), \quad (6.5)$$

where L and N are given by (6.13)–(6.14), with the additional condition

$$\int_{\Lambda} h(x) q^{**}(x) dx = 0. \quad (6.6)$$

iii) If $\gamma \geq 2$, we have

$$q^{**} = \alpha(x) + \frac{\partial v^*}{\partial y} + l(x, y), \quad (6.7)$$

where

$$l(x, y) = -2N^2 \int_0^y \frac{\partial \widehat{W}}{\partial x}(x, \xi) d\xi,$$

$\widehat{W} = W^{**}$ if $\gamma = 2$, $\widehat{W} = W_{**}$ if $\gamma > 2$, and α is the unique solution of the Reynolds-like problem:

Find $\alpha(x) \in H^1(\Lambda)$ such that

$$\int_{\Lambda} \mathcal{A}(x) \frac{\partial \alpha}{\partial x} \frac{\partial \psi}{\partial x} dx = \int_{\Lambda} \mathcal{B}(x) \frac{\partial \psi}{\partial x} dx, \quad \forall \psi \in H^1(\Lambda), \quad (6.8)$$

where

$$\mathcal{A} = \begin{cases} \Xi(x) & \text{if } \gamma = 2, \\ \frac{h^3(x)}{12(1 - N^2)} & \text{if } \gamma > 2, \end{cases} \quad (6.9)$$

and $\mathcal{B}(x)$ is given by (6.23), with the additional condition

$$\int_{\Lambda} h(x) \alpha(x) dx = - \int_{\Omega} \left(l + \frac{\partial v^*}{\partial y} \right) dx dy. \quad (6.10)$$

Proof.

i) $0 < \gamma < 1$: The limit equations satisfied by W^{**} , q^{**} , and U_1^{**} are the same as the equations satisfied by ω^* , p^* , and u_1^* (system (4.2)), but they are associated with homogeneous boundary conditions on $\partial\Omega$. This leads to

$$\int_{\Lambda} h^3 \frac{\partial q^{**}}{\partial x} \frac{\partial \psi}{\partial x} dx = 0 \quad \forall \psi \in H^1(\Lambda); \text{ so } q^{**} = 0.$$

Moreover,

$$\frac{\partial^2 U_1^{**}}{\partial y^2} = \frac{\partial q^{**}}{\partial x} = 0 \quad \text{and} \quad U_1^{**} = 0.$$

Similarly $W^{**} = 0$.

ii) $1 \leq \gamma < 2$: The same calculations as in Proposition 4.3 give

$$\frac{h^3}{6} \frac{\partial q^{**}}{\partial x} = -2\tilde{U}_1^{**}h + 2N^2 \int_0^{h(x)} W^{**}(2y-h) dy \quad \text{in } H^{-1}(\Omega) \cap L^2(\Omega), \quad (6.11)$$

which implies $q^{**} \in H^1(\Lambda)$.

We resolve a second-order differential equation (6.14), taking into account the fact that $W^{**} = 0$ on $\Gamma^s \cup \Gamma^i$ and that

$$\begin{aligned} \omega^* &= \frac{F^s - F^i}{h} y + F^i, \\ u_1^* &= \frac{1}{2} \left(\frac{\partial p^*}{\partial x} - 2N^2 \frac{F^s - F^i}{h} \right) y(y-h) + \frac{J^s - J^i}{h} y + J^i. \end{aligned}$$

We obtain

$$N^2 \int_0^{h(x)} W^{**}(2y-h) dy = L(x) \frac{\partial p^*}{\partial x} + N(x), \quad (6.12)$$

where

$$L(x) = \frac{h^5}{18\lambda}, \quad (6.13)$$

$$N(x) = \frac{h^3}{6\lambda} \left[7(J^s - J^i) + h \left(26F^i + (F^s - F^i) \left(\frac{37}{3} - N^2 \right) \right) \right]. \quad (6.14)$$

Equation (6.11) becomes

$$h\tilde{U}_1^{**} = -\frac{h^3}{6} \frac{\partial q^{**}}{\partial x} + L(x) \frac{\partial p^*}{\partial x} + N(x).$$

Inserting this value in (6.4) yields (6.5). Since q^{**} lies in $L_0^2(\Omega)$, we obtain (6.6).

iii) $\gamma \geq 2$: The second-order differential systems (6.2) or (6.3) for $U_1^{**}, \widehat{W}, q^{**}$ are similar to the first-order systems (4.3)–(4.4) for $u_1^*, \hat{\omega}, p^*$, the difference being that we have homogeneous boundary conditions and nonhomogeneous right-hand side. So the resolution is exactly the same but the results are different.

From equations (6.2.2) or (6.3.2), we can write

$$\frac{\partial}{\partial y} \left(q^{**} - \frac{\partial v^*}{\partial y} \right) = -2N^2 \frac{\partial \hat{\omega}}{\partial x} \quad \text{in } L^2(\Omega) \cap H^{-1}(\Omega),$$

which implies, by integration, that there exists $\alpha \in L^2(\Lambda)$ such that

$$q^{**} - \frac{\partial v^*}{\partial y} = \alpha(x) + l(x, y), \quad (6.15)$$

where $l(x, y) = -2N^2 \int_0^y \frac{\partial \hat{\omega}}{\partial x}(x, \xi) d\xi$.

Using Green's formula in equations (6.2.1) or (6.3.1), we obtain for all $\varphi \in H_0^1(\Omega)$:

$$\int_{\Omega} q^{**} \frac{\partial \varphi}{\partial x} dx dy = \int_{\Omega} \frac{\partial U_1^{**}}{\partial y} \frac{\partial \varphi}{\partial y} dx dy - 2N^2 \int_{\Omega} \frac{\partial \widehat{W}}{\partial y} \varphi dx dy + \int_{\Omega} \frac{\partial u_1^*}{\partial x} \frac{\partial \varphi}{\partial x} dx dy. \quad (6.16)$$

Choosing now $\varphi(x, y) = \theta(x)y(y - h(x))$ where θ lies in $H_0^1(\Lambda)$, and taking (6.16) into account, we can write

$$\begin{aligned}\int_{\Omega} q^{**} \frac{\partial \varphi}{\partial x} dx dy &= \int_{\Lambda} \frac{h^3}{6} \frac{\partial \alpha_1}{\partial x} \theta(x) dx - \int_{\Omega} \left(\frac{\partial^2 v^*}{\partial y \partial x} + \frac{\partial l}{\partial x} \right) \varphi dx dy, \\ \int_{\Omega} \frac{\partial U_1^{**}}{\partial y} \frac{\partial \varphi}{\partial y} dx dy &= -2 \int_{\Lambda} h \tilde{U}_1^{**} \theta(x) dx, \\ \int_{\Omega} \frac{\partial u_1^*}{\partial x} \frac{\partial \varphi}{\partial x} dx dy &= - \int_{\Omega} \frac{\partial^2 u_1^*}{\partial x^2} y(y - h) \theta(x) dx dy.\end{aligned}$$

It remains to calculate

$$I = -N^2 \int_{\Omega} \frac{\partial \widehat{W}}{\partial y} \phi dx dy.$$

For any x , we solve the ordinary differential system (6.2) or (6.3) where $\frac{d\alpha}{dx}$ is considered as a parameter; so we get

i) For $\gamma = 2$,

$$\begin{aligned}U_1^{**}(x, y) &= A_2(\beta sh(ky) + 2y) + \beta B_2(ch(ky) - 1) + \frac{2N^2 y^2}{\lambda k^2} \frac{d\alpha_1}{dx} + \zeta_1(x, y), \\ W^{**}(x, y) &= A_2(ch(ky) - 1) + B_2 sh(ky) - \frac{2N^2 y}{\lambda k^2} \frac{d\alpha_1}{dx} + \zeta_2(x, y),\end{aligned}\tag{6.17}$$

where k, β are given by (4.24), and ζ_1 and ζ_2 are such that

$$\begin{aligned}\zeta_1(x, y) &= \int_0^y \left[\frac{\lambda}{2N^2} \left(\frac{\partial^2 \zeta_2}{\partial y^2}(x, \xi) - \frac{\partial^2 \omega^*}{\partial x^2}(x, \xi) \right) - 2\zeta_2(x, \xi) + \frac{\partial v^*}{\partial x}(x, \xi) \right] d\xi, \\ \zeta_2(x, y) &= \frac{1}{k^2} \int_0^y \left[\frac{\partial^3 \omega^*}{\partial x^2 \partial y} + \frac{2N^2}{\lambda} \left(\frac{\partial l}{\partial x} - \frac{\partial^2 u_1^*}{\partial x^2} \right) \right] (1 + ch(k(y - \xi))) d\xi,\end{aligned}$$

and

$$\begin{aligned}A_2 &= \frac{\beta \zeta_2(x, h)(ch(kh) - 1) - \zeta_1(x, h)sh(kh)}{h sh(kh) + \beta(ch(kh) - 1)} - \frac{h}{4(1 - N^2)} \frac{d\alpha}{dx}, \\ B_2 &= \frac{1}{sh(kh)} \left[\frac{h}{2(1 - N^2)} \frac{d\alpha}{dx} - A_2(ch(kh) - 1) - \zeta_2(x, h) \right].\end{aligned}$$

ii) For $\gamma > 2$,

$$\begin{aligned}U_1^{**}(x, y) &= \frac{1}{2(1 - N^2)} \frac{d\alpha}{dx} y(y - h) + \zeta_3(x, y), \\ W_{**}(x, y) &= -\frac{1}{4(1 - N^2)} \frac{d\alpha}{dx} (2y - h) - \frac{1}{2} \frac{\partial \zeta_3}{\partial y}(x, y) + \frac{1}{2} \frac{\partial v}{\partial x},\end{aligned}\tag{6.18}$$

where

$$\zeta_3(x, y) = \frac{1}{1 - N^2} \left[\int_0^y \zeta_4(x, z) dz - \frac{y}{h} \int_0^h \zeta_4(x, z) dz \right]$$

with

$$\zeta_4(x, z) = \int_0^z \left(\frac{\partial l}{\partial x}(x, \tau) - \frac{\partial^2 u_1^*}{\partial x^2}(x, \tau) + (1 - 2N^4) \frac{\partial^2 v^*}{\partial x \partial y}(x, \tau) \right) d\tau.$$

This enables us to compute

$$-N^2 \int_0^{h(x)} \frac{\partial \widehat{W}}{\partial y} \varphi \, dx \, dy = \Phi_1(x) \frac{\partial \alpha_1}{\partial x} + \Phi_2(x), \quad (6.19)$$

where

$$\Phi_1 = \begin{cases} \frac{h^3(x)}{12} - \Xi(x) & \text{if } \gamma = 2, \\ -\frac{N^2 h^3(x)}{12(1 - N^2)} & \text{if } \gamma > 2, \end{cases} \quad (6.20)$$

and

$$\Phi_2 = \begin{cases} \int_0^{h(x)} [t_2(x, h)(ch(ky) - 1)(sh(ky) - 1) - \zeta_2(x, h)sh(ky)](2y - h) \, dy & \text{if } \gamma = 2, \\ \int_0^{h(x)} \left[-\frac{\partial \zeta_3}{\partial y}(x, y) + \frac{\partial v^*}{\partial x} \right] \left(y - \frac{h}{2} \right) \, dy & \text{if } \gamma > 2, \end{cases} \quad (6.21)$$

with

$$t_2(x, h) = \frac{\zeta_1(x, h)sh(ky) - \beta \zeta_2(x, h)(ch(ky) - 1)}{h sh(ky) + \beta(ch(ky) - 1)}.$$

Equation (6.16) may be rewritten so that

$$\begin{aligned} \int_{\Lambda} \left[\frac{h^3}{6} - 2\Phi_1(x) \right] \frac{d\alpha}{dx} \theta(x) \, dx &= -2 \int_{\Lambda} h \widetilde{U}_1^{**} \theta(x) \, dx \\ &+ \int_{\Lambda} [2\Phi_2(x) + M(x)] \theta(x) \, dx, \quad \forall \theta \in H_0^1(\Lambda), \end{aligned}$$

where M is defined by

$$M(x) = \int_0^{h(x)} \left(\frac{\partial^2 v^*}{\partial y \partial x} + \frac{\partial l}{\partial x} - \frac{\partial^2 u_1^*}{\partial x^2} \right) y(y - h) \, dy \quad (6.22)$$

and we get

$$\left[\frac{h^3}{6} - 2\Phi_1(x) \right] \frac{d\alpha}{dx} = 2\Phi_2(x) + M(x) - 2h \widetilde{U}_1^{**} \quad \text{in } H^{-1}(\Lambda) \cap L^2(\Lambda). \quad (6.23)$$

Inserting this value of $\widetilde{U}_1^{**}$ in (6.4) yields (6.8), where

$$\mathcal{B}(x) = \Phi_2(x) + \frac{1}{2} M(x). \quad \square \quad (6.24)$$

Conclusion. The purpose was to study how the various heuristic equations (2.6)–(2.10) are rigorously obtained from the initial Stokes system. We have proved a little more if $\delta = 0$ by the computation of the next term in the asymptotic expansion so that we can write

$$\text{If } 0 < \gamma < 1 : \quad p^\varepsilon \cong p_1^* + \varepsilon q^{**} + \dots, \quad \text{with } q^{**} = 0$$

$$\text{If } 1 \leq \gamma < 2 : \quad p^\varepsilon \cong p_1^* + \varepsilon^{2-\gamma} q^{**} + \dots$$

$$\text{If } \gamma = 2 : \quad p^\varepsilon \cong p_{2,\lambda}^* + \varepsilon^2 q^{**} + \dots$$

$$\text{If } \gamma > 2 : \quad p^\varepsilon \cong p_3^* + \varepsilon^2 q^{**} + \dots$$

FIG. 1. The value of $P_{2,\lambda}^*(0)$ as a function of the parameter λ

An interesting feature which may induce further studies is the behaviour of the asymptotic $p_{2,\lambda}^*$ with respect to λ . In Fig. 1, we draw the variation of $p_{2,\lambda}^*(0)$ when λ goes from 0 to ∞ for three values of N^2 in the following situation where an analytical solution can be computed for the Reynolds micropolar equation (notation of Sec. 3):

$$h \equiv 1, \quad F = 0, \quad g_2^\varepsilon = 0, \quad g_{1/\Gamma^s}^\varepsilon = 0, \quad g_{1/\Gamma^i}^\varepsilon = 1, \quad g_{1/\Gamma^i}^\varepsilon = 1 - y^2,$$

$$p_{2,\lambda}^*(0) = \frac{1}{\Xi} \text{ with } \Xi = \frac{1}{1 - N^2} \left(1 + \frac{3\lambda}{1 - N^2} - 3\sqrt{\frac{\lambda N^2}{1 - N^2}} \coth \left(Nh \sqrt{\frac{1 - N^2}{\lambda}} \right) \right).$$

It appears that the two limit values of $p_{2,\lambda}^*(0)$ when λ goes to 0 and ∞ are numerically equal to

$$I_1(N) = p_3^*(0) = \frac{1}{1 - N^2} \quad \text{and} \quad I_2(N) = p_1^*(0) = 1.$$

This perhaps explains why engineers generally use $p_{2,\lambda}^*$, every physical situation being covered by a single model. It would be interesting to prove rigorously these convergence results with respect to λ .

REMARK. Figure 1 is related to the asymptotic non-dimensional pressure $p_{2,\lambda}^*$. Coming back to the physical asymptotic pressure $\bar{p}_{2,\lambda} = \frac{s\nu p_{2,\lambda}^*}{\eta^2(1 - N^2)R}$ (cf. Remark in Sec. 2)

implies that the limit value of $\bar{p}_{2,\lambda}$ becomes independent from N^2 if λ tends to 0 and is a function of N^2 when λ tends to ∞ .

As mentioned in the introduction, the present study can be extended to a three-dimensional Stokes problem, both for the computation of the Reynolds equation, and for the second-order estimates.

Let us consider for example a micropolar Stokes system with boundary values g^ε for the velocity and f^ε for the rotation, deduced from (1.3)–(1.5) in the thin domain:

$$\Omega^\varepsilon = \Lambda \times \{0, \varepsilon h(x_1, x_3)\},$$

where Λ is a smooth domain in the $\{x_1, x_3\}$ -plane whose characteristic dimension is R . Using the notation from sections 2 and 3, the domain Ω^ε is changed into Ω by introducing

$$X_1 = \frac{x_1}{R}, \quad X_2 = \frac{x_2}{\varepsilon R}, \quad X_3 = \frac{x_3}{R}.$$

The conditions (H1) and (H2) are the same while (H3) is linked to the existence of J_i and F in $H^1(\Omega)$ such that

$$\frac{\partial J_1}{\partial X_1} + \frac{\partial J_2}{\partial X_2} + \frac{\partial J_3}{\partial X_3} = 0 \quad \text{in } \Omega,$$

$$J_i = g_i^\varepsilon, \quad i = \{1, 3\} \text{ in } \partial\Omega,$$

$$J_2 = \frac{g_2^\varepsilon}{\varepsilon} \quad \text{in } \partial\Omega,$$

$$f^\varepsilon = \frac{F}{\varepsilon\delta} \varepsilon \quad \text{in } \partial\Omega.$$

With the same assumptions as for Theorems 4.5–4.6 and 4.7, the asymptotic pressure obeys the following equations:

$$\begin{aligned} \mathcal{L}(p^*, \phi) = k_\delta \sum_{l=\{1,3\}} \left[\int_\Lambda \left(J_l^s \frac{\partial h}{\partial X_l} \phi + h \frac{J_l^s + J_l^i}{2} \frac{\partial \phi}{\partial X_l} \right) dX_1 dX_3 + \int_{\partial\Lambda} h \tilde{J}_l n_l \phi d\sigma \right] \\ + k_\delta \int_\Lambda (J_2^s + J_2^i) \phi dX_1 dX_3 + \sum_{l=\{1,3\}} \int_\Lambda \Psi_{\gamma,\delta} \frac{\partial \phi}{\partial X_l} dX_1 dX_3, \end{aligned}$$

where

$$\mathcal{L}(p^*, \phi) = \begin{cases} \sum_{l=\{1,3\}} \int_\Lambda \frac{h^3(X_1, X_3)}{12} \frac{\partial p^*}{\partial X_l} \frac{\partial \phi}{\partial X_l} dX_1 dX_3 & \text{if } \gamma < 2, \\ \sum_{l=\{1,3\}} \int_\Lambda \Xi(X_1, X_3) \frac{\partial p^*}{\partial X_l} \frac{\partial \phi}{\partial X_l} dX_1 dX_3 & \text{if } \gamma = 2, \\ \sum_{l=\{1,3\}} \int_\Lambda \frac{h^3(X_1, X_3)}{12(1 - N^2)} \frac{\partial p^*}{\partial X_l} \frac{\partial \phi}{\partial X_l} dX_1 dX_3 & \text{if } \gamma > 2, \end{cases}$$

with

$$k_\delta = \begin{cases} 0 & \text{if } \delta > 0, \\ 1 & \text{if } \delta \leq 0, \end{cases}$$

and

$$\Psi_{\gamma,\delta} = \begin{cases} 0 & \text{if } \delta < 0 \text{ and } \gamma < 2, \\ \frac{N^2 h^2(X_1, X_3)}{6} (F^s - F^i) & \text{if } \delta \geq 0 \text{ and } \gamma < 2, \\ \Psi \text{ defined in Remark (4.7)} & \text{if } \gamma = 2, \\ 0 & \text{if } \gamma > 2. \end{cases}$$

REFERENCES

- [1] E. L. Aero, A. N. Bulygin, and E. V. Kuvshinskii, *Asymmetric hydromechanics*, J. Appl. Math. Mech. **2**, 297–308 (1965) (in Russian)
- [2] A. Assemien, G. Bayada, and M. Chambat, *Inertial effects in the asymptotic behaviour of a thin film flow*, Asymptotic Anal. **9**, 177–208 (1994)
- [3] G. Bayada and M. Chambat, *The transition between the Stokes equation and the Reynolds equation: A mathematical proof*, Appl. Math. Optim. **14**, 73–93 (1986)
- [4] G. Bayada and G. Lukaszewicz, *On micropolar fluids in the theory of lubrication. Rigorous derivation of an analogue of the Reynolds equation*, Internat. J. Engrg. Sci. **34**, 1477–1490 (1996)
- [5] N. M. Bessonov, *A new generalization of the Reynolds equation for a micropolar fluid and its application to bearing theory*, J. Tribology, Trans. AMSE **116**, 654–657 (1994)
- [6] A. C. Eringen, *Simple microfluids*, Internat. J. Engrg. Sci. **2**, 205–217 (1964)
- [7] J. Frene, D. Nicolas, B. Degueurce, D. Berthe, and M. Godet, *Lubrification Hydrodynamique*, Collec. DER-EDF, Eyrolles, Paris, 1990
- [8] S. R. Gamouana, *Analyse asymptotique de fluide micropolaire en film mince et milieu poreux*, Thèse-Mathématiques, Univ. Claude Bernard Lyon I, no. 170–99, 1999
- [9] G. Lukaszewicz, *Micropolar fluids—Theory and Applications*, Birkhäuser - Modeling and Simulation in Science, Boston, MA, 1999
- [10] G. Lukaszewicz, *On stationary flows of asymmetric fluids*, Rendiconti, Academia naz. delle scienze detta dei XL, Memorie di matematica **106**, Vol. XII, fasc. 3, 35–44 (1988)
- [11] M. S. Mostefai, *Deduction rigoureuse de l'équation de Reynolds à partir d'un système modelisant l'écoulement à faible épaisseur d'un fluide micropolaire et étude de deux problèmes à frontières libres. Hele-Shaw généralisé et Stefan à deux phases pour un fluide non Newtonien*. Thèse - Mathématiques, Univ. Jean Monnet, Saint-Etienne, 1997
- [12] L. G. Pestrosyan, *Some problems of mechanics of fluids with antisymmetric stress tensor*, Erevan University Press, Erevan, 1984 (in Russian)
- [13] J. Prakash and P. Sinha, *Lubrication theory for micropolar fluid and its application to a journal bearing*, Internat. J. Engrg. Sci. **13**, 217–232 (1975)
- [14] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979