STEADY STATES FOR A ONE-DIMENSIONAL MODEL OF THE SOLAR WIND

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Abstract. A one (space) dimensional Vlasov equation is used to model the solar wind (a collisionless plasma) as it moves past an applied magnetic field (an obstacle). The goal is to understand physically reasonable steady states for this situation. When the applied magnetic field is sufficiently small, appropriate states are constructed.

1. Introduction. Consider the following simplified model of the solar wind:

$$\begin{cases} \partial_t f + v_1 \partial_x f - (E + v_2 B) \partial_{v_1} f + v_1 B \partial_{v_2} f = 0, \\ \rho(t, x) = \iint (F(v_1, v_2) - f(t, x, v_1, v_2)) dv_2 dv_1, \\ E(t, x) = \frac{1}{2} \int_{-\infty}^x \rho d\tilde{x} - \frac{1}{2} \int_x^{\infty} \rho d\tilde{x}. \end{cases} \quad (1.1)$$

Here f gives the density in phase space of mobile negative ions with mass one and charge minus one. This is assumed to depend only on time, scalar position $x = x_1$, and velocity $v = (v_1, v_2)$. The positive ions have charge positive one, but are taken to have infinite mass so that they form a fixed background density given by $F(v_1, v_2)$ (see [13]). We take F to be smooth and compactly supported with

$$F(v_1, v_2) = 0 \quad \text{if } v_1 \leq 0.$$

$B = B(x)$ is a given magnetic field which we assume is compactly supported. The speed of the solar wind is a few tenths of one percent of the speed of light; so the omission of the self-consistent magnetic field due to the plasma is reasonable. We consider (1.1) with the initial condition

$$f(0, x, v) = F(v) + g(x, v) \quad (1.2)$$

where g has compact support and zero average value.

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The goal is to understand the possible steady behavior of (1.1), (1.2). Thus, we consider the steady state problem

$$\begin{cases} v_1 \partial_x f - (E + v_2 B) \partial_{v_1} f + v_1 B \partial_{v_2} f = 0, \\ \rho(x) = \int (F - f) dv, \\ E(x) = \frac{1}{2} \int_{-\infty}^x \rho d\tilde{x} - \frac{1}{2} \int_x^{\infty} \rho d\tilde{x}. \end{cases} \quad (1.3)$$

We seek physically appropriate conditions at $x = -\infty$. We consider

$$\lim_{x \rightarrow -\infty} E(x) = 0 \quad (1.4)$$

and

$$\lim_{x \rightarrow -\infty} f = F \text{ uniformly on } v_1 \geq 0. \quad (1.5)$$

Note that (1.5) imposes no restriction for $v_1 < 0$. We will show that solutions of the dynamic problem (1.1), (1.2) satisfy (1.4) and (1.5) for any fixed $t \geq 0$. We will also see that conditions (1.4) and (1.5) are appropriate when B is sufficiently small, but it is not clear that they are for B large.

The plasma physics literature on collisionless shocks is extensive, for example, see [3, 16]. Many mathematical works consider the existence of steady states ([2, 6, 15]) and stability of steady states ([5, 7, 8, 14]) in collisionless plasmas. This work differs from those mentioned above in that the applied field, B , is included to model flow past an obstacle. Then we seek appropriate conditions at infinity to capture the steady behavior of the dynamic problem. We mention also two related references [9] and [12].

Section 2 sketches the existence of smooth solutions to the dynamic problem (1.1), (1.2). Section 3 shows some consequences of (1.3), (1.4), and (1.5) and also constructs solutions when B is sufficiently small. Section 4 presents numerical results where the solution of (1.1), (1.2) tends as $t \rightarrow +\infty$ to a solution of (1.3), (1.4), (1.5). Finally, the appendix contains some of the details from Sec. 3.

2. The dynamic problem. Define

$$A(x) = \int_{-\infty}^x B(\tilde{x}) d\tilde{x}.$$

For a smooth solution of (1.1) define $(X(s, t, x, v), V(s, t, x, v))$ by

$$\begin{aligned} \frac{dX}{ds} &= V_1, & X(t, t, x, v) &= x, \\ \frac{dV_1}{ds} &= -(E(s, X) + V_2 B(X)), & V_1(t, t, x, v) &= v_1, \\ \frac{dV_2}{ds} &= V_1 B(X), & V_2(t, t, x, v) &= v_2. \end{aligned} \quad (2.1)$$

Then

$$\frac{d}{dx}(V_2 - A(X)) = 0 \quad \text{and} \quad \frac{d}{ds}(f(x, X, V)) = 0.$$

THEOREM 2.1. Let $B \in C_0^1(\mathbb{R})$, $F \in C_0^1(\mathbb{R}^2)$ with $F \geq 0$, and $g \in C_0^1(\mathbb{R}^3)$ with

$$\iint g \, dv \, dx = 0$$

and

$$F + g \geq 0.$$

Then (1.1), (1.2) has a unique C^1 solution for all $t \geq 0$ with

$$(x, v) \mapsto f(t, x, v) - F(v)$$

compactly supported for every $t \geq 0$.

The proof of existence is not immediate because $f(0, x, v)$ does not decay as $|x| \rightarrow \infty$ and hence the mass and energy are infinite. Uniqueness and local-in-time existence may be handled as in [1], where the iteration

$$\begin{cases} \partial_t f^{n+1} + v_1 \partial_x f^{n+1} - (E^n + v_2 B) \partial_{v_1} f^{n+1} + v_1 B \partial_{v_2} f^{n+1} = 0, \\ \rho^{n+1} = \iint (F - f^{n+1}) dv_2 \, dv_1, \\ E^{n+1} = \frac{1}{2} \int_{-\infty}^x \rho^{n+1} d\tilde{x} - \frac{1}{2} \int_x^\infty \rho^{n+1} d\tilde{x} \end{cases}$$

is used. We omit this here since it has become standard. To continue the local solution in time it is sufficient that

$$\begin{aligned} R(t) = & \sup\{|x|: f(s, x, v) - F(v) \neq 0 \text{ for some } v \in \mathbb{R}^2 \text{ and } s \in [0, t]\} \\ & + \sup\{|x|: B(x) \neq 0\} \end{aligned}$$

remains finite. Thus, we will focus on obtaining an a priori bound on $R(t)$.

Note that

$$\frac{d}{dt} \iint (f - F) dv \, dx = 0$$

and

$$\iint (f(0, x, v) - F(v)) dv \, dx = 0;$$

so

$$\iint (f - F) dv \, dx = - \int \rho \, dx = 0.$$

Hence

$$E = \frac{1}{2} \int_{-\infty}^x \rho \, d\tilde{x} - \frac{1}{2} \int_x^\infty \rho \, d\tilde{x} = \int_{-\infty}^x \rho \, d\tilde{x}$$

and

$$|x| > R(t) \Rightarrow E(t, x) = 0.$$

Also, if we take

$$j = \int (F - f) v_1 \, dv,$$

then

$$\partial_t \rho = - \int \partial_t f \, dv = \int v_1 \partial_x f \, dv = -\partial_x j_1;$$

so

$$\partial_t E = \int_{-\infty}^x \partial_t \rho d\tilde{x} = - \int_{-\infty}^x \partial_x j_1 d\tilde{x} = -j_1.$$

Next

$$\begin{aligned} & \frac{d}{dt} \left(\iint_{-R(t)}^{R(t)} f|v|^2 dx dv \right) \\ &= \int_{-R(t)}^{R(t)} \int |v|^2 \partial_t f dv dx \\ &= \int_{-R(t)}^{R(t)} \int |v|^2 (-v_1 \partial_x f + [E + v_2 B] \partial_{v_1} f - v_1 B \partial_{v_2} f) dv dx \\ &= - \int f |v|^2 v_1 dv \Big|_{-R(t)}^{R(t)} - \int_{-R(t)}^{R(t)} \int 2v_1 E f dv dx \\ &= 2 \int E \left(j_1 - \int F v_1 dv \right) dx \\ &= - \frac{d}{dt} \left(\int E^2 dx \right) - 2 \int F v_1 dv \int E dx; \end{aligned}$$

so

$$\begin{aligned} & \frac{d}{dt} \left(\int_{-R(t)}^{R(t)} \int f |v|^2 dv dx + \int E^2 dx \right) \\ &= -2 \int F v_1 dv \int E dx. \end{aligned}$$

Let

$$Q(t) = \int_{-R(t)}^{R(t)} \int f |v|^2 dv dx + \int E^2 dx.$$

Then

$$\begin{aligned} |Q'(t)| &\leq C \int_{-R(t)}^{R(t)} |E(t, x)| dx \\ &\leq C \sqrt{2R(t) \int_{-R(t)}^{R(t)} E^2(t, x) dx} \\ &\leq CR^{\frac{1}{2}}(t) Q^{\frac{1}{2}}(t). \end{aligned}$$

Hence

$$\left| \frac{d}{dt} Q^{\frac{1}{2}}(t) \right| \leq CR^{\frac{1}{2}}(t)$$

and

$$\begin{aligned}
 Q(t) &\leq \left(Q^{\frac{1}{2}}(0) + \int_0^t C R^{\frac{1}{2}}(\tau) d\tau \right)^2 \\
 &\leq C(Q(0) + t^2 R(t)) \\
 &\leq C(C + t^2 R(t)) \leq C(1 + t^2)R(t).
 \end{aligned} \tag{2.2}$$

Next, we bound the v_2 support of f . Suppose $f(t, x, v) \neq 0$. Then

$$f(t, x, v) = f(0, X(0, t, x, v), V(0, t, x, v)) \neq 0.$$

So

$$|V_2(0, t, x, v)| \leq C.$$

But

$$v_2 - A(x) = V_2(0, t, x, v) - A(X(0, t, x, v)).$$

So

$$|v_2| = |V_2(0, t, x, v) + A(x) - A(X(0, t, x, v))| \leq C. \tag{2.3}$$

We now use an energy estimate from [10, 11]. Let

$$k = \int f|v|^2 dv.$$

Note that for any $r > 0$

$$\begin{aligned}
 \int f dv &= \iint_{-C}^C f dv_2 dv_1 \\
 &\leq \int_{-r}^r \int_{-C}^C C dv_2 dv_1 + r^{-2}k \\
 &\leq Cr + r^{-2}k.
 \end{aligned}$$

Taking $r = k^{1/3}$ yields

$$\int f dv \leq Ck^{\frac{1}{3}}. \tag{2.4}$$

Next using (2.4) we have

$$\begin{aligned}
 |E(t, x)| &\leq \int_{-R(t)}^{R(t)} \int |F - f| dv d\tilde{x} \\
 &\leq CR(t) + \int_{-R(t)}^{R(t)} \int f dv d\tilde{x} \\
 &\leq CR(t) + \int_{-R(t)}^{R(t)} Ck^{\frac{1}{3}} d\tilde{x} \\
 &\leq CR(t) + CR^{\frac{2}{3}}(t) \left(\int_{-R(t)}^{R(t)} k d\tilde{x} \right)^{\frac{1}{3}}.
 \end{aligned}$$

But since

$$\int_{-R(t)}^{R(t)} k \, dx \leq Q(t),$$

using (2.2) yields

$$|E(t, x)| \leq C(1 + t^2)R(t). \quad (2.5)$$

Consider (t, x, v) with $F(v) - f(t, x, v) \neq 0$. If $f(t, x, v) \neq 0$, then

$$f(0, X(0, t, x, v), V(0, t, x, v)) \neq 0.$$

So

$$|V_1(0, t, x, v)| \leq C$$

and hence by (2.5) and (2.3)

$$\begin{aligned} |V_1(s, t, x, v)| &\leq C + \int_0^s |(E + V_2B)|_{(\tau, X(\tau, t, x, v))} \, d\tau \\ &\leq C + Cs + C(1 + s^2) \int_0^s R(\tau) \, d\tau \end{aligned}$$

for $0 \leq s \leq t$. If $f(t, x, v) = 0$, then $F(v) \neq 0$. So $|v| \leq C$ and hence by (2.5) and (2.3)

$$\begin{aligned} |V_1(s, t, x, v)| &= \left| v_1 - \int_s^t (E + V_2B) \Big|_{(\tau, X(\tau, t, x, v))} \, d\tau \right| \\ &\leq C + C(t - s) + C(1 + t^2) \int_s^t R(\tau) \, d\tau. \end{aligned}$$

Thus, in both cases,

$$|V_1(s, t, x, v)| \leq C + Ct + C(1 + t^2) \int_0^t R(\tau) \, d\tau = S(t) \quad (2.6)$$

for $0 \leq s \leq t$. $S(t)$ is defined by (2.6).

Define

$$\bar{R}(t) = R(0) + \int_0^t S(\tau) \, d\tau$$

and claim that $R(t) \leq \bar{R}(t)$. Define

$$H(t) = \sup\{|F(v) - f(t, x, v)| : |x| > \bar{R}(t) \text{ and } v \in \mathbb{R}^2\}$$

and note that

$$\begin{aligned} F(v) - f(t, x, v) &= F(v) - f(0, X(0, t, x, v), V(0, t, x, v)) \\ &= F(V(0, t, x, v)) - f(0, X(0, t, x, v), V(0, t, x, v)) \\ &\quad - \int_0^t \partial_{v_1} F(V)(E(s, X) + V_2B(X)) \, ds \\ &\quad + \int_0^t \partial_{v_2} F(V)V_1B(X) \, ds \end{aligned} \quad (2.7)$$

where we abbreviate $X = X(s, t, x, v)$. Now if $f(t, x, v) - F(v) \neq 0$ and $|x| > \bar{R}(t)$, then by (2.6) we have

$$|X(s, t, x, v)| = \left| x - \int_s^t V_1(\tau, t, x, v) d\tau \right| > \bar{R}(t) - \int_s^t S(\tau) d\tau = \bar{R}(s).$$

Hence, since $\bar{R}(s) \geq R(0)$,

$$B(X(s, t, x, v)) = 0.$$

Also

$$|X(0, t, x, v)| > \bar{R}(0) = R(0);$$

so

$$F(V(0, t, x, v)) - f(0, X(0, t, x, v), V(0, t, x, v)) = 0.$$

Thus, (2.7) yields

$$|F(v) - f(t, x, v)| \leq C \int_0^t |E(s, X(s, t, x, v))| ds. \quad (2.8)$$

If $X > \bar{R}(s)$, then using (2.3) and (2.6) we have

$$\begin{aligned} |E(s, X)| &= \left| \int_{-\infty}^X \rho(s, y) dy \right| \\ &= \left| - \int_X^\infty \int (F(v) - f(s, y, v)) dv dy \right| \\ &\leq R(s) H(s) C S(s). \end{aligned}$$

Similarly, if $X < -\bar{R}(s)$, then

$$|E(s, X)| \leq R(s) H(s) C S(s);$$

so by (2.8)

$$|F(v) - f(t, x, v)| \leq C \int_0^t R(s) H(s) S(s) ds.$$

It follows that

$$H(t) \leq C \int_0^t R(s) H(s) S(s) ds.$$

By Gronwall's inequality, $H(t) = 0$ for all $t \geq 0$. The claim that $R(t) \leq \bar{R}(t)$ now follows. Finally, by (2.6)

$$\begin{aligned} R(t) &\leq \bar{R}(t) = R(0) + \int_0^t S(\tau) d\tau \\ &= R(0) + \int_0^t \left(C + C\tau + C(1 + \tau^2) \int_0^\tau R(s) ds \right) d\tau \\ &\leq C(1 + t + t^2 + t(1 + t^2)) \int_0^t R(s) ds. \end{aligned}$$

So by Gronwall's inequality, a bound on $R(t)$ follows.

3. The static problem.

As before we assume

$$F \in C_0^1(\mathbb{R}^2), \quad F \geq 0, \quad v_1 \leq 0 \Rightarrow F(v) = 0. \quad (3.1)$$

We define G by

$$F(v) = G(\tfrac{1}{2}|v|^2, v_2)I_{v_1 > 0}$$

(where $I_{v_1 > 0} = 1$ if $v_1 > 0$, 0 else) and assume

$$F(v) = 0 \quad \text{if } \tfrac{1}{2}|v|^2 \leq \mathcal{E}_{\min} \quad \text{or} \quad |v_2| \geq \mathcal{S} \quad (3.2)$$

with $\mathcal{E}_{\min} - \frac{1}{2}\mathcal{S}^2 > 0$. We also assume $B \in C_0^1(\mathbb{R})$ with

$$B(x) = 0 \quad \text{if } x \notin (0, L). \quad (3.3)$$

Now we seek solutions of (1.3) with

$$f \geq 0, \quad (3.4)$$

$$\rho \text{ bounded}, \quad (3.5)$$

$$E \text{ integrable on } (-\infty, x) \text{ for all } x \in \mathbb{R}, \quad (3.6)$$

and with (1.4) and (1.5). By (3.5) E is Lipschitz continuous. So we may define $(X(s, x, v), V(s, x, v))$ by

$$\begin{aligned} \frac{dX}{ds} &= V_1, & X(0, x, v) &= x, \\ \frac{dV_1}{ds} &= -(E(X) + V_2 B(X)), & V_1(0, x, v) &= v_1, \\ \frac{dV_2}{ds} &= V_1 B(X), & V_2(0, x, v) &= v_2. \end{aligned} \quad (3.7)$$

We will interpret the Vlasov equation as the requirement that $f(X(s, x, v), V(s, x, v))$ is constant in s and not assume that f is continuous. To motivate this, consider the following example. Let $U(x)$ be a smooth nondecreasing function with

$$\begin{cases} U(x) = 0 & \text{if } x \leq 0, \\ 0 < U(x) < 1 & \text{if } 0 < x < 1, \\ 1 = U(x) & \text{if } 1 \leq x. \end{cases}$$

Now we take $f(x, v_1) = 1$ (drop v_2) on the set

$$\{(x, v_1) : v_1 > 0\} \cup \{(x, v_1) : \tfrac{1}{2}v_1^2 + U(x) \leq 1\}$$

and $f(x, v_1) = 0$ at all other points. Then f is constant on characteristics, i.e., $f(X(s, x, v_1), V(s, x, v_1))$ is constant in s where

$$\begin{aligned} \frac{dX}{ds} &= V_1, \\ \frac{dV_1}{ds} &= -U'(X). \end{aligned}$$

Also

$$F(v_1) := \lim_{x \rightarrow -\infty} f(x, v_1) = 1 \quad \text{for } v_1 \geq 0$$

is smooth for $v_1 \geq 0$; so the “upstream condition” (1.5) is smooth. However, f is discontinuous at each point $(x, 0)$ for $x \geq 1$.

Define

$$U(x) = - \int_{-\infty}^x E(\tilde{x}) d\tilde{x}$$

and

$$A(x) = \int_{-\infty}^x B(\tilde{x}) d\tilde{x}.$$

Then

$$\frac{d}{ds} \left[\frac{1}{2} |V(s, x, v)|^2 - U(X(s, x, v)) \right] = 0 \quad (3.8)$$

and

$$\frac{d}{ds} [V_2(s, x, v) - A(X(s, x, v))] = 0. \quad (3.9)$$

Let us denote

$$\mathcal{E}(x, v) = \frac{1}{2} |V(s, x, v)|^2 - U(X(s, x, v))$$

and

$$\mathcal{L}(x, v) = V_2(s, x, v) - A(X(s, x, v)).$$

If we further define

$$\mathcal{P}(y) = \frac{1}{2} (\mathcal{L} + A(y))^2 - U(y),$$

then for (x, v) fixed we have

$$\begin{aligned} \frac{d^2 X}{ds^2} &= -(E(X) + V_2(B(X))) \\ &= -(-U'(X) + (\mathcal{L} + A(X))A'(X)) \\ &= -\mathcal{P}'(X). \end{aligned} \quad (3.10)$$

Since (3.10) is reversible in s it follows that if $V_1 = 0$ for two values of s , then X is periodic. If

$$\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty,$$

then $V_1(s, x, v) = 0$ for at most one s . Let s_1 be that value of s if it exists and $s_1 = +\infty$ otherwise. In both cases

$$V_1(s, x, v) > 0 \quad \text{for } s < s_1. \quad (3.11)$$

If s_1 is finite, then by reversibility

$$V_1(s, x, v) < 0 \quad \text{for } s > s_1 \quad (3.12)$$

and

$$\lim_{s \rightarrow +\infty} X(s, x, v) = -\infty. \quad (3.13)$$

LEMMA 3.1. If

$$\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$$

for some (x, v) , then

$$\lim_{s \rightarrow -\infty} V_1(s, x, v) \text{ exists and is nonnegative}$$

and

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)).$$

The proof is deferred to the appendix.

Note that $\lim_{x \rightarrow -\infty} U(x) = 0$; so there exists $x_0 < 0$ such that

$$x < x_0 \Rightarrow A(x) = 0 \quad \text{and} \quad |U(x)| < \tfrac{1}{2}(\mathcal{E}_{\min} - \tfrac{1}{2}\mathcal{S}^2). \quad (3.14)$$

LEMMA 3.2. If $X(s_0, x, v) < x_0$, $V_1(s_0, x, v) > 0$,

$$\tfrac{1}{2}|V(s_0, x, v)|^2 - U(X(s_0, x, v)) > \mathcal{E}_{\min},$$

and

$$|V_2(s_0, x, v)| < \mathcal{S},$$

then

$$V_1(s, x, v) > \sqrt{\mathcal{E}_{\min} - \tfrac{1}{2}\mathcal{S}^2}$$

for $s \leq s_0$ and

$$\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty.$$

The proof of Lemma 3.2 is also deferred.

LEMMA 3.3. For $x < x_0$ and $v_1 > 0$,

$$f(x, v) \geq G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)).$$

If $\mathcal{E}(x, v) > \mathcal{E}_{\min}$ and $|v_2| < \mathcal{S}$ also, then

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)).$$

Proof. Consider $x < x_0$ and $v_1 > 0$. If $\mathcal{E}(x, v) > \mathcal{E}_{\min}$ and $|v_2| < \mathcal{S}$, then by Lemma 3.2 we have $\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$. Hence

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x))$$

by Lemma 3.1. If either $\mathcal{E}(x, v) \leq \mathcal{E}_{\min}$ or $|v_2| \geq \mathcal{S}$, then

$$0 = G(\tfrac{1}{2}|v|^2 - V(x), v_2 - A(x)) \leq f(x, v)$$

since $f(x, v) \geq 0$. This completes the proof. $\square$

LEMMA 3.4. If

$$f(x, v) \neq 0 \quad \text{and} \quad \lim_{s \rightarrow -\infty} X(s, x, v) = -\infty,$$

then

$$V_1(s, x, v) > 0 \quad \text{for all } s$$

and

$$\lim_{s \rightarrow +\infty} X(s, x, v) = +\infty.$$

The proof is deferred, but we mention that the condition (1.4) is central to the proof.

Consider the following assumption:

$$\text{if } f(x, v) \neq 0, \text{ then } \lim_{s \rightarrow -\infty} X(s, x, v) = -\infty. \quad (3.15)$$

This would fail to be true if f were nonzero on some bound periodic orbits.

THEOREM 3.5. Assume (1.3), (1.4), (1.5), and (3.1) through (3.6) hold. If (3.15) also holds, then

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x))$$

for $v_1 > 0$ and $f(x, v) = 0$ for $v_1 \leq 0$.

Proof. Consider (x, v) with $f(x, v) \neq 0$. By assumption $\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$ and so by Lemma 3.1

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)).$$

By Lemma 3.4, $v_1 > 0$. It follows that $f = 0$ on $\mathbb{R} \times (-\infty, 0] \times \mathbb{R}$.

Now consider (x, v) with $v_1 > 0$ and

$$G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)) \neq 0.$$

Choose (y, w) with $y < x_0, w_1 > 0$,

$$\tfrac{1}{2}|w|^2 - U(y) = \tfrac{1}{2}|v|^2 - U(x), \quad (3.16)$$

and

$$w_2 - A(y) = v_2 - A(x). \quad (3.17)$$

We may do this since

$$\tfrac{1}{2}|v|^2 - U(x) > \mathcal{E}_{\min}, \quad |v_2 - A(x)| < \mathcal{S},$$

and

$$|U(y)| < \tfrac{1}{2}(\mathcal{E}_{\min} - \tfrac{1}{2}\mathcal{S}^2).$$

Now by Lemma 3.3

$$\begin{aligned} f(y, w) &\geq G(\tfrac{1}{2}|w|^2 - U(y), w_2 - A(y)) \\ &= G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)) > 0. \end{aligned}$$

Since $f(y, w) \neq 0$, from before we have

$$f(y, w) = G(\tfrac{1}{2}|w|^2 - U(y), w_2 - A(y)).$$

By Lemma 3.4, $\lim_{s \rightarrow +\infty} X(s, y, w) = +\infty$. So there exists s_0 such that $x = X(s_0, y, w)$. It follows by (3.16) and (3.17) that

$$v_1 = V_1(s_0, y, w)$$

and hence

$$\begin{aligned} f(x, v) &= f(y, w) \\ &= G(\mathcal{E}(y, w), \mathcal{L}(y, w)) \\ &= G(\mathcal{E}(x, v), \mathcal{L}(x, v)). \end{aligned}$$

This completes the proof. $\square$

Next we seek to construct solutions that have the form stated in Theorem 3.5. Note that

$$\begin{aligned} G(\tfrac{1}{2}|v|^2 - U(x), w_2 - A(x)) &\neq 0 \\ \Rightarrow \tfrac{1}{2}v_1^2 &> \mathcal{E}_{\min} + U(x) - \tfrac{1}{2}v_2^2 \text{ and } |v_2 - A(x)| < S \\ &\Rightarrow \tfrac{1}{2}v_1^2 > \mathcal{E}_{\min} + U(x) - \tfrac{1}{2}(S + |A(x)|)^2. \end{aligned}$$

So if

$$\mathcal{E}_{\min} + U(x) - \tfrac{1}{2}(S + |A(x)|)^2 \geq C > 0, \quad (3.18)$$

then $G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x))$ vanishes for v_1 near zero and hence

$$G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x))I_{v_1 > 0}$$

is smooth.

THEOREM 3.6. Consider $B(x) = \varepsilon B_1(x)$ and assume (3.1) through (3.3). There exists $\varepsilon_0 = \varepsilon_0(B_1)$ such that for $|\varepsilon| < \varepsilon_0$ there is a solution of (1.3) that satisfies (1.4), (1.5), (3.4) through (3.6), (3.15) and is of the form

$$f(x, v) = G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x))I_{v_1 > 0}.$$

Proof. Define

$$\rho_G(u, a) = \int_0^\infty \int G(\tfrac{1}{2}|v|^2 - u, v_2 - a) dv_2 dv_1$$

and note that

$$\rho_G(u, a) = \int_{-u}^\infty \int G(\mathcal{E} + \tfrac{1}{2}v_2^2, v_2 - a) dv_2 \frac{d\mathcal{E}}{\sqrt{2(\mathcal{E} + u)}}.$$

If

$$\mathcal{E}_{\min} + u - \tfrac{1}{2}(S + |a|)^2 > 0,$$

then

$$G(-u + \tfrac{1}{2}v_2^2, v_2 - a) = 0 \quad \text{for all } v_2,$$

and hence

$$\partial_u \rho_G(u, a) = \frac{-1}{2\sqrt{2}} \int_{-u}^\infty \int G(\mathcal{E} + \tfrac{1}{2}v_2^2, v_2 - a) dv_2 (\mathcal{E} + u)^{-\frac{3}{2}} d\mathcal{E} < 0.$$

U must satisfy

$$\begin{aligned}
 U''(x) &= -\rho(x) = \int f \, dv - \int F \, dv \\
 &= \int_0^\infty \int G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)) dv_2 \, dv_1 \\
 &\quad - \int_0^\infty \int G(\tfrac{1}{2}|v|^2, v_2) dv_2 \, dv_1 \\
 &= \rho_G(U(x), A(x)) - \rho_G(0, 0).
 \end{aligned} \tag{3.19}$$

Define

$$\sigma(u, a) = - \int_0^u (\rho_G(\tilde{u}, a) - \rho_G(0, 0)) d\tilde{u}$$

and note that $A(x) = \text{constant}$ on $(-\infty, 0)$ and on (L, ∞) . So

$$\frac{d}{dx} [\tfrac{1}{2}(U'(x))^2 + \sigma(U(x), A(x))] = 0$$

on $(-\infty, 0) \cup (L, \infty)$. Since $U(x) \rightarrow 0$ and $U'(x) \rightarrow 0$ as $x \rightarrow -\infty$

$$\tfrac{1}{2}(U'(x))^2 + \sigma(U(x), 0) = 0$$

on $(-\infty, 0)$. Since $\sigma(0, 0) = 0$, $\partial_u \sigma(0, 0) = 0$, and

$$\partial_u^2 \sigma(0, 0) = -\partial_u \rho_G(0, 0) > 0$$

it follows that $U(x) = 0$ for all $x \leq 0$.

Next choose C_1 and C_2 such that $(u, a) \in N := (-C_1, C_1) \times (-C_2, C_2) \Rightarrow (3.18)$ holds and

$$\partial_u^2 \sigma = -\partial_u \rho_G > -\tfrac{1}{2} \partial_u \rho_G(0, 0) := C_3 > 0.$$

Then for $(u, A(L)) \in N$ we have

$$\begin{aligned}
 \sigma(u, A(L)) &\geq \sigma(0, A(L)) + \partial_u \sigma(0, A(L))u + \tfrac{1}{2} C_3 u^2 \\
 &= -(\rho_G(0, A(L)) - \rho_G(0, 0))u + \tfrac{1}{2} C_3 u^2.
 \end{aligned} \tag{3.20}$$

For ε sufficiently small we have

$$|A(L)| \leq |\varepsilon| \int_0^L |B_1(x)| dx < C_2.$$

By further restricting ε we may force U , the solution of (3.19), to satisfy

$$|U(x)| < C_1 \quad \text{for } x \leq L$$

and hence

$$(U(x), A(x)) \in N \quad \text{for } x \leq L.$$

Define

$$L^* = \sup\{x > L: |U| < C_1 \text{ on } [L, x]\}.$$

Then for $L < x < L^*$ we have by (3.20)

$$\begin{aligned} \frac{1}{2}C_3U^2(x) &< \frac{1}{2}(U'(x))^2 + \sigma(U(x), A(L)) \\ &\quad + (\rho_G(0, A(L)) - \rho_G(0, 0))U(x) \\ &< \frac{1}{2}(U'(L))^2 + \sigma(U(L), A(L)) \\ &\quad + (\rho_G(0, A(L)) - \rho_G(0, 0))C_1. \end{aligned} \quad (3.21)$$

The right-hand side of (3.21) tends to zero as $\varepsilon \rightarrow 0$. So we may force $|U(x)| < \frac{1}{2}C_1$ to hold on (L, L^*) by restricting ε . In this case it follows that $L^* = \infty$, $(U(x), A(x)) \in N$ for all x , and hence (3.18) holds for all x . The proof is now complete. $\square$

We comment that for $x > L$, $U(x)$ is periodic and will not approach zero as $x \rightarrow +\infty$ in general.

4. Numerical results. In this section we describe the results of approximating solutions of (1.1), (1.2) with a particle method. The case considered is

$$\begin{aligned} B(x) &= 9(4(x-1)(x-2))^2 I_{(1,2)}(x), \\ g(x, v) &= 0, \end{aligned}$$

and

$$F(v) = 10\delta(v_1 - 5)\delta(v_2).$$

Note that previously F was assumed to be smooth and because of this Theorem 2.1 does not apply. In fact, the proof of Theorem 3.6 can be adapted to this case and the solutions constructed there may be computed using the equation

$$\begin{aligned} U''(x) &= \rho_G(U(x), A(x)) - \rho_G(0, 0) \\ &= \frac{50}{\sqrt{25 + 2U(x) - A^2(x)}} - \frac{50}{\sqrt{25}}. \end{aligned} \quad (4.1)$$

The solution to (1.1) was computed on the interval $0 \leq x \leq 10$ for t small enough that $f(t, x, v) = F(v)$ for x near 0 and 10. The spatial step $dx = 3.125 \times 10^{-3}$ was used to compute E with 100 particles per spatial step.

Figures 1 and 2 show the graphs of

$$\int f \, dv = \rho + 10$$

and

$$\int f v_1 \, dv = 50 - j_1$$

at the time $t = 0.8$, $t = 1.0$, and $t = 1.2$. Note that all three graphs in each figure agree for $x < 4$. For $x < 4$ steady conditions have been attained by $t = 0.8$. Note also that for

FIG. 1. Time series of rho

a steady state we must have

$$\int f v_1 dv = 50,$$

which is the case in Fig. 2 for $x < 4$.

Note also the sharp spike which occurs in both figures at $t = 1.2$. To understand this we graph

$$\text{velocity} = \frac{\int f v_1 dv}{\int f dv}$$

at the same three times in Fig. 3. Again the curves match for $x < 4$. For $x > 4$ we see faster velocity particles catching up with slower particles downstream, resulting in a steepening curve.

Figure 4 shows $\int f dv$ at time 1.2 computed with the particle code as a dotted curve. The solid curve is the solution of (4.1). Steady conditions hold for $x < 7$ but the sharp spike is a transient feature. Moreover, the solution of (4.1) correctly predicts the steady behavior of the dynamic problem in this case. If the amplitude of B is increased by 20%, then the solution of (4.1) is not globally defined and particles are reflected by B .

FIG. 2. Time series of j_1

5. The appendix.

The Proof of Lemma 3.1. We have

$$A(x) = 0 \quad \text{for } x < 0.$$

So by (3.9)

$$\lim_{s \rightarrow -\infty} V_2(s, x, v) = v_2 - A(x). \quad (5.1)$$

Similarly

$$\lim_{x \rightarrow -\infty} U(x) = 0.$$

So by (3.8) and (5.1)

$$\frac{1}{2} \lim_{s \rightarrow -\infty} V_1^2(s, x, v) = \frac{1}{2}|v|^2 - U(x) - \frac{1}{2}(v_2 - A(x))^2.$$

It follows that $\lim_{s \rightarrow -\infty} V_1(s, x, v)$ exists. Since $\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$ it follows that $\lim_{s \rightarrow -\infty} V_1(s, x, v) \geq 0$ and

$$\lim_{s \rightarrow -\infty} V_1(s, x, v) = \sqrt{|v|^2 - 2U(x) - (v_2 - A(x))^2}.$$

FIG. 3. Time series of velocity

By (3.11), $V_1(s, x, v) > 0$ for s sufficiently negative. So by (1.5)

$$\lim_{s \rightarrow -\infty} f(X(s, x, v), V(s, x, v)) = F\left(\lim_{s \rightarrow -\infty} V(s, x, v)\right)$$

and hence

$$\begin{aligned} f(x, v) &= \lim_{s \rightarrow -\infty} f(X(s, x, v), V(s, x, v)) \\ &= G\left(\frac{1}{2}\left|\lim_{s \rightarrow -\infty} V\right|^2, \lim_{s \rightarrow -\infty} V_2\right) I \lim_{s \rightarrow -\infty} V_1 \geq 0 \\ &= G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)). \end{aligned}$$

Note that if $\lim_{s \rightarrow -\infty} V_1 = 0$, then $G(\frac{1}{2}|v|^2 - U(x), v_2 - A(x)) = 0$. This completes the proof. $\square$

The Proof of Lemma 3.2. Define

$$\bar{s} = \inf\{s < s_0 : V_1(\tau, x, v) > 0 \text{ on } \tau \in (s, s_0)\}.$$

For all s

$$\mathcal{E}_{\min} < \mathcal{E}(x, v) = \frac{1}{2}|V(s, x, v)|^2 - U(X(s, x, v))$$

FIG. 4. Comparison with exact values

and

$$\mathcal{S} > |\mathcal{L}(x, v)| = |V_2(s, x, v) - A(X(s, x, v))|.$$

So for $s \in (\bar{s}, s_0)$ we have $X(s, x, v) < x_0$. So $A(X(s, x, v)) = 0$,

$$|V_2(s, x, v)| < \mathcal{S},$$

and by (3.14)

$$\begin{aligned} \frac{1}{2}V_1^2(s, x, v) &> \mathcal{E}_{\min} + U(X(s, x, v)) - \frac{1}{2}V_2^2(s, x, v) \\ &> \mathcal{E}_{\min} - \frac{1}{2}(\mathcal{E}_{\min} - \frac{1}{2}\mathcal{S}^2) - \frac{1}{2}\mathcal{S}^2 \\ &= \frac{1}{2}(\mathcal{E}_{\min} - \frac{1}{2}\mathcal{S}^2). \end{aligned}$$

Thus

$$V_1(s, x, v) > \sqrt{\mathcal{E}_{\min} - \frac{1}{2}\mathcal{S}^2}$$

for $s \in (\bar{s}, s_0)$. It follows that $\bar{s} = -\infty$ and the lemma follows. □

Before proving Lemma 3.4 we consider the following:

LEMMA 5.1. If

$$f(x, v) \neq 0 \quad \text{and} \quad \lim_{s \rightarrow -\infty} X(s, x, v) = -\infty,$$

then there exists a neighborhood, N , of v such that

$$w \in N \Rightarrow \lim_{s \rightarrow -\infty} X(s, x, w) = -\infty$$

and

$$w \in N \Rightarrow f(x, w) = G(\tfrac{1}{2}|w|^2 - U(x), w_2).$$

Proof. By Lemma 3.1

$$G(\tfrac{1}{2}|v|^2 - U(x), v_2 - A(x)) = f(x, v) \neq 0.$$

So

$$\tfrac{1}{2}|v|^2 - U(x) > \mathcal{E}_{\min}$$

and

$$|v_2 - A(x)| < \mathcal{S}.$$

There exists $s_0 < 0$ such that

$$X(s_0, x, v) < x_0 \quad \text{and} \quad V_1(s_0, x, v) > 0.$$

By continuity with respect to initial conditions there is a neighborhood, N , of v such that for $w \in N$ we have

$$\begin{aligned} \tfrac{1}{2}|w|^2 - U(x) &> \mathcal{E}_{\min}, \\ |w_2 - A(x)| &< \mathcal{S}, \\ X(s_0, x, w) &< x_0, \end{aligned}$$

and

$$V_1(s_0, x, w) > 0.$$

By Lemma 3.2, $w \in N \Rightarrow \lim_{s \rightarrow -\infty} X(s, x, w) = -\infty$. Now using Lemma 3.1 completes the proof. $\square$

The Proof of Lemma 3.4. We proceed by contradiction. First suppose there is (x, v) with $f(x, v) \neq 0$, $v_1 < 0$, and $\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$. We will show this contradicts (1.4).

By (3.12) and (3.13),

$$s \geq 0 \Rightarrow V_1(s, x, v) < 0$$

and

$$\lim_{s \rightarrow +\infty} X(s, x, v) = -\infty.$$

So without loss of generality we take $v_1 < 0$ and $x < x_0$ (by replacing (x, v) with $(X(s, x, v), V(s, x, v))$ with s large). Let N be the neighborhood of v from Lemma 5.1.

Next choose $\underline{\mathcal{E}}, \bar{\mathcal{E}}, \underline{\mathcal{L}}, \bar{\mathcal{L}}$ as follows. Denote

$$\bar{N} = \{w: w_1 < 0, \underline{\mathcal{E}} < \tfrac{1}{2}|w|^2 - U(x) < \bar{\mathcal{E}}, \underline{\mathcal{L}} < w_2 < \bar{\mathcal{L}}\}$$

and choose $\underline{\mathcal{E}}, \bar{\mathcal{E}}, \underline{\mathcal{L}}, \bar{\mathcal{L}}$ so that $\bar{N} \subset N$, $\bar{N} \neq \emptyset$, and $G(\tfrac{1}{2}|w|^2 - U(x), w_2) \geq C > 0$ for $w \in \bar{N}$.

Now consider $(y, z) \in \mathbb{R} \times \mathbb{R}^2$ with

$$\begin{aligned} y < x, \quad z_1 < 0, \\ \underline{\mathcal{E}} < \tfrac{1}{2}|z|^2 - U(y) < \bar{\mathcal{E}}, \end{aligned}$$

and

$$\underline{\mathcal{L}} < z_2 < \bar{\mathcal{L}}.$$

We may choose $w \in \bar{N}$ such that

$$\tfrac{1}{2}|w|^2 - U(x) = \tfrac{1}{2}|z|^2 - U(y)$$

and

$$w_2 = z_2.$$

There is $\mathbf{s}$ such that

$$y = X(\mathbf{s}, x, w) \quad \text{and} \quad z = V(\mathbf{s}, x, w).$$

Then

$$f(y, z) = f(x, w) = G(\tfrac{1}{2}|w|^2 - U(x), w_2) \geq C.$$

So for $y < x$, $z_1 < 0$, $\tfrac{1}{2}|z|^2 - U(y) \in (\underline{\mathcal{E}}, \bar{\mathcal{E}})$, $z_2 \in (\underline{\mathcal{L}}, \bar{\mathcal{L}})$ we have

$$f(y, z) \geq C.$$

Since $U(y) \rightarrow 0$ as $y \rightarrow -\infty$ this implies

$$\int_{-\infty}^0 \int f(y, z) dz_2 dz_1 \geq C$$

for y sufficiently negative.

Finally, using Lemma 3.3,

$$\begin{aligned} U''(y) &= \iint f(y, z) dz dy - \int F(z) dz \\ &\geq \int_0^\infty \int G(\tfrac{1}{2}|z|^2 - U(y), z_2) dz dy + C \\ &\quad - \int_0^\infty \int G(\tfrac{1}{2}|z|^2, z_2) dz dy. \end{aligned}$$

But since $U(y) \rightarrow 0$ as $y \rightarrow -\infty$

$$U''(y) \geq C$$

for y sufficiently negative. This contradicts the assumption that $E = -U' \rightarrow 0$ as $y \rightarrow -\infty$.

Note that if $V_1(\bar{s}, x, v) = 0$ for some $\bar{s}$, then by (3.12), $V_1(s, x, v) < 0$ for $s > \bar{s}$. So the above argument shows that

$$f(x, v) \neq 0 \quad \text{and} \quad \lim_{s \rightarrow -\infty} X(s, x, v) = -\infty \Rightarrow V_1(s, x, v) > 0 \quad \text{for all } s.$$

Next assume $f(x, v) \neq 0$, $\lim_{s \rightarrow -\infty} X(s, x, v) = -\infty$, and $X(s, x, v)$ is bounded from above. Let

$$\bar{x} = \lim_{s \rightarrow +\infty} X(s, x, v). \quad (5.2)$$

Since $\frac{d^2}{ds^2} X(s, x, v)$ is bounded for $s > 0$, it follows that

$$\lim_{s \rightarrow +\infty} V_1(s, x, v) = 0. \quad (5.3)$$

From (3.10)

$$\frac{1}{2} V_1^2(s, x, v) + \mathcal{P}(X(s, x, v))$$

is independent of s . From (5.2) and (5.3) it follows that

$$\mathcal{P}(\bar{x}) = \mathcal{E}(x, v)$$

and

$$\frac{d}{ds} X(s, x, v) = \sqrt{2(\mathcal{P}(\bar{x}) - \mathcal{P}(X(s, x, v)))}. \quad (5.4)$$

Clearly $\mathcal{P}'(\bar{x}) \geq 0$. If $\mathcal{P}'(\bar{x}) > 0$, then the solution of (5.4) would attain $V_1 = 0$ in finite time, contrary to assumption. Thus

$$\mathcal{P}'(\bar{x}) = 0.$$

Let N be the neighborhood of v from Lemma 5.1. Choose $\underline{\mathcal{E}} < \mathcal{E}(x, v)$ such that

$$\begin{aligned} 0 < w_1 \quad \text{and} \quad \underline{\mathcal{E}} < \mathcal{E}(x, w_1, v_2) < \mathcal{E}(x, v) \\ \Rightarrow (w_1, v_2) \in N \quad \text{and} \quad f(x, w_1, v_2) \neq 0. \end{aligned}$$

By Sard's theorem (p. 61 of [4]) the set

$$\{\mathcal{P}(y) : \mathcal{P}'(y) = 0 \quad \text{and} \quad x \leq y \leq \bar{x}\}$$

has measure zero. Hence, we may choose $\mathcal{E}_0 \in (\underline{\mathcal{E}}, \mathcal{E}(x, v))$ with

$$\mathcal{E}_0 \notin \{\mathcal{P}(y) : \mathcal{P}'(y) = 0 \quad \text{and} \quad x \leq y \leq \bar{x}\}. \quad (5.5)$$

Now choose $w_1 > 0$ with $\mathcal{E}(x, w_1, v_2) = \mathcal{E}_0$ and let $w_2 = v_2$. Then $w \in N$ and so $f(x, w) \neq 0$ and $\lim_{s \rightarrow -\infty} X(s, x, w) = -\infty$.

As above $V_1(s, x, w) > 0$ for all s . Also note that $\mathcal{L}(x, w) = \mathcal{L}(x, v)$. So $\mathcal{P}$ in (3.10) is unchanged. So

$$\begin{aligned} \mathcal{E}_0 &= \frac{1}{2} V_1^2(s, x, w) + \mathcal{P}(X(s, x, w)) \\ &= \frac{1}{2} w_1^2 + \mathcal{P}(x) < \frac{1}{2} v_1^2 + \mathcal{P}(x) \\ &= \mathcal{E}(x, v) = \mathcal{P}(\bar{x}). \end{aligned}$$

Hence, $\mathcal{P}(X(s, x, w)) < \mathcal{P}(\bar{x})$ and $X(s, x, w) < \bar{x}$ for all s . As before it follows that $\lim_{s \rightarrow +\infty} V_1(s, x, w) = 0$. But on the other hand,

$$\mathcal{E}_0 = \mathcal{P} \left(\lim_{s \rightarrow +\infty} X(s, x, w) \right);$$

so by (5.5)

$$\mathcal{P}' \left(\lim_{s \rightarrow +\infty} X(s, x, w) \right) \neq 0.$$

Since

$$\frac{d}{ds}X(s, x, w) = \sqrt{2 \left(\mathcal{P} \left(\lim_{s \rightarrow +\infty} X(s, x, w) \right) - \mathcal{P}(X(s, x, w)) \right)},$$

it follows that $V_1(s, x, w) = 0$ for some s . This is a contradiction and completes the proof. $\square$

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