CONVEX POLYHEDRA QUANTUM BILLIARDS IN $\mathbb{R}^n$

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Abstract. “S-quantum billiards” are defined in $\mathbb{R}^n$. These billiards include the regular convex polyhedra as a subset. The “first-excited-state theorem” states that for such quantum billiards: (a) a first excited state (second eigenstate of the Laplacian) exists whose nodal surface is a plane of bisecting symmetry of the billiard; (b) the degeneracy of this state is equal to the dimension in which the billiard exists. It is shown that for such billiards, its bisecting nodal surface is one of minimum energy. The (a) component of the preceding theorem is proved with the latter proof and a conjecture that is based on recent theorems of Lin, Melas, and Alessandrini and a described smoothing procedure. Components of group theory, as well as an ansatz addressing the higher dimensions, come into play in establishing the (b) component of the theorem. An appendix is included describing properties of nodal intersection with a boundary.

1. Introduction. In the quantum-billiard problem, one examines solutions to the Schrödinger equation (i.e., the Helmholtz equation) for a point particle that moves freely in a simply-connected convex domain bounded by a perfectly reflecting surface [1–10]. We introduce the acronyms “*fes*” for the “first excited state” (i.e., second eigenstate of the Laplacian) and “*fest*” for the “first excited-state theorem”. This theorem states that for “S-quantum billiards” (defined below) in $\mathbb{R}^n$, (a) a first excited state exists whose nodal surface is its plane of bisecting symmetry; (b) the degeneracy (or, multiplicity) of this state is equal to the dimension in which the billiard exists. In $\mathbb{R}^n$ it is first shown, for this class of billiards, that the surface of bisecting symmetry is a surface of minimum energy. The possibility of a closed nodal surface (for this eigenstate) is eliminated in an argument based on three recent works [3, 11, 12] and component (a) of *fest* is established in $\mathbb{R}^n$. Elements of group theory, as well as an ansatz addressing S-quantum billiards in higher dimensions, are employed in establishing component (b) of *fest* in all integer $\mathbb{R}^n$.

2. Review of theorems. For purposes of reference we cite theorems proved by Courant (I, II) [4], Alessandrini (III) [3], and Liboff (IV) [7].

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Ia. Nodal partitioning theorem: If eigenfunctions of the Helmholtz equation are ordered according to increasing eigenvalues, then the n th eigenfunction partitions the domain into not more than n subdomains.

Ib. Relative ground-state theorem: The ground-state energy of a convex quantum billiard is larger than that of any convex billiard in which it is inscribed.

II. Cell-energy symmetry theorem: In any nodal partition of the fundamental domain corresponding to a given eigenstate, respective ground-state energies of partitioned cells are equal and equal to the eigenenergy of the given eigenstate.

III. Non-smooth boundary theorem: The nodal of the first excited state of a convex quantum billiard with a non-smooth boundary is a simple curve that connects two distinct points of the boundary.

IV. Circular-sector theorem: The nodal line of the first excited state of a circular-sector quantum billiard is a circular-arc segment for vertex angle θ in the interval $0 < \theta < (0.354\dots)\pi$ and is a bisecting radius of the sector for θ in the interval $(0.354\dots)\pi < \theta \leq \pi$.

3. S-quantum billiards in $\mathbb{R}^2$. An “S-quantum billiard” is convex with the following properties. (a) The ratio of minimum to maximum diameters of the billiard, x , is such that $1 > x > 0$, and $1 - x \ll 1$. (b) The billiard includes a plane of reflection symmetry (“ σ -surface”) that separates the billiard into two congruent sections of maximum volume. (c) The interior angle subtended by the surface of the billiard at the intersection of a σ -surface is $\geq \pi/3$. (d) The symmetry of this billiard in $\mathbb{R}^n$ is that of one of the convex regular polyhedra in $\mathbb{R}^n$. Thus, for example, the regular convex polyhedra as well as their truncations are subsets of the S-billiards.

To establish the *fest* in $\mathbb{R}^2$ for this class of billiards, we recall Theorems (III) and (IV). The first of these asserts that the nodal of the first excited state of a convex quantum billiard is a chord that connects two distinct points of the boundary. As will be shown below, a bisecting nodal line is a nodal of minimum energy. By property (c) of an S-quantum billiard, we see that the interior vertex angle exceeds $(0.354\dots)\pi$, so that by Theorem (IV), the nodal of the first excited state in a vertex domain of the S-quantum billiard is a bisecting line of the vertex. It follows that the bisecting line is a nodal of minimum energy and is asymptotic to first excited states in vertex domains. This establishes the *fest* for the S-quantum billiards. The *fest* for the equilateral triangle billiard was established by Pinsky [2].

4. Minimum energy theorem. We wish to show that a σ -nodal surface of an S-quantum billiard is a surface of minimum energy. Let the two halves of the billiard be labeled L and R, respectively. Let the ground states of the L and R domains be labeled φ_L, φ_R with respective polarities, $+, -$. (By Courant’s nodal partitioning theorem (Ia), each ground state is of one sign.) By the principle of odd reflection [5], the extended state

$$\varphi_{LR} \equiv \varphi_L(+) + \varphi_R(-) \quad (1a)$$

exists. If the ground state of, say, φ_L is labeled E_1 , then with Theorem (II), this is the eigenenergy of φ_{LR} and we may write

$$\hat{H}\varphi_{LR} \equiv E_1\varphi_{LR}, \quad (1b)$$

where $\hat{H}$ is the Hamiltonian and, up to a constant factor,

$$\hat{H} = -\Delta, \quad \mathbf{r} \in \partial\Omega, \quad (1c)$$

where Δ is the Laplacian in $\mathbb{R}^3$ and Ω is the domain of the polyhedron. Dirichlet boundary conditions apply for $\mathbf{r} \in \partial\Omega$, and $\mathbf{r}$ is a three-dimensional radius.

We wish to show that a σ -surface of φ_{LR} is a surface of minimum energy and that this minimum is global with respect to perturbation. To establish this global property, we show first that an infinitesimal variation of the plane surface increases the energy of the eigenstate. It is then shown that an infinitesimal displacement of the surface likewise increases the energy of the eigenstate.

Consider an infinitesimal perturbation that gives the change in wavefunction

$$\psi(x, y, z) = \varphi_{LR} + \varepsilon g(x, y, z), \quad (2a)$$

$$\varepsilon \ll 1. \quad (2b)$$

The nodal plane of φ_{LR} is $x = 0$; so φ_{LR} is odd in x . The function $g(x, y, z)$ has the following properties: it is odd in x and has the same polarity as φ_{LR} . Furthermore, $|g|$ is a rapidly decaying function of x away from maxima at $(x, y, z) = (\pm\varepsilon, 0, 0)$ and vanishes on the boundary of the nodal plane, $x = 0$. In the vicinity of this plane, the function g satisfies the constraint equations

$$\frac{g_{yy}}{g} = -k_1^2(x, y, z) \leq 0, \quad (2c)$$

$$\frac{g_{zz}}{g} = -k_2^2(x, y, z) \leq 0, \quad (2d)$$

where letter subscripts denote differentiation. We wish to show that

$$\delta E \equiv \langle \psi | \hat{H} \psi \rangle - \langle \varphi_{LR} | \hat{H} \varphi_{LR} \rangle > 0. \quad (3a)$$

The form δE may be written

$$\delta E = \langle (\varphi_{LR} + \varepsilon g) | \hat{H} (\varphi_{LR} + \varepsilon g) \rangle - \langle \varphi_{LR} | \hat{H} \varphi_{LR} \rangle$$

so that

$$\delta E = 2\varepsilon \langle g | \hat{H} \varphi_{LR} \rangle + \varepsilon^2 \langle g | \hat{H} g \rangle. \quad (3b)$$

In these expressions, $\langle \rangle$ represents an inner product over the domain of the billiard. With (1b), (3b) reduces to

$$\delta E = 2\varepsilon E_1 \langle g | \varphi_{LR} \rangle + \varepsilon^2 \langle g | \hat{H} g \rangle \equiv \delta E_1 + \delta E_2, \quad (3c)$$

where δE_1 and δE_2 are as implied. Since g is odd in x with the same polarity as φ_{LR} , δE_1 is positive. The second term, with (2c,d) and (1c) gives

$$\delta E_2 = k_1^2(x, y, z) + k_2^2(x, y, z) \geq 0. \quad (3d)$$

Note that if $g(x, y, z)$ is taken as a function that oscillates in sign in the (y, z) -plane, then $k_1^2(x, y, z)$ and $k_2^2(x, y, z)$ increase, thereby indicating that the given perturbation surface

FIG. 1. Schematic of nodals in the Lin-Melas Theorems: Configurations at: (a) $t = 0$, (b) $t = t_i$, (c) $t = 1$

is one of minimum energy. Consider next that an infinitesimal perturbation causes a parallel displacement of the bisecting nodal plane. The corresponding energy increases by theorems (1b), (II). It follows that a σ -nodal surface is a global minimum of energy. This proof extends to $\mathbb{R}^n$.

Having shown that a σ -nodal surface is a surface of minimum energy, to establish that $\varphi_{L,R}$ is a first excited state of the billiard we must show further [with Theorem (Ia)] that the nodal surface of the first excited state is not a closed surface. To do this we recall the following theorems.

Linn-Melas-Alessandrini Theorems. We review three recent theorems for $\mathbb{R}^2$ for the purpose of generalizing these theorems to $\mathbb{R}^n$. Again, let Ω denote a bounded convex domain in the plane, and let $\partial\Omega$ denote its boundary. The outward normal derivative of an eigenfunction, φ on $\partial\Omega$ is labeled $\partial\varphi/\partial\nu$.

Let **P** be the following proposition: the nodal of the *fes* in Ω intersects $\partial\Omega$ at two distinct points. The following theorem then is due to C.-S. Lin [11]: Let Ω_0 be a bounded convex domain with C^∞ boundary such that **P** fails in Ω_0 . Namely, with Theorem (Ia) the nodal of this function is a simple closed curve in Ω_0 . Then there exists a convex bounded domain Ω with C^∞ boundary and a *fes*, φ , in Ω , such that $\partial\varphi/\partial\nu$ has exactly one zero on $\partial\Omega$. Note that if $\partial\varphi/\partial\nu = 0$ at a point on $\partial\Omega$, then the nodal curve at that point intersects $\partial\Omega$, normally. (See Appendix.)

The main element of this theorem is as follows. Let $\Omega(t)$ be a smooth deformation of Ω with $\Omega(0) = \Omega_0$ and $\Omega(1)$ a disk, which is known to have a nodal diameter for the *fes*, and $\Omega(t)$ is a bounded convex domain with a smooth boundary. One defines t_i as the largest value of $t \in (0, 1)$ for which **P** fails. Based on the property that φ is a continuous function of t , one finds that the related function, $\varphi(t_i)$, is a *fes* in $\Omega(t_i)$ and that $\partial\varphi_i/\partial\nu$ has exactly one zero on $\partial\Omega(t_i)$ (Fig. 1). It was then shown [12] that if $\Omega \subseteq \mathbb{R}^2$ is a bounded convex domain with C^∞ boundary and φ is a *fes* in Ω , then $\partial\varphi/\partial\nu$ cannot have exactly one zero on $\partial\Omega$. It follows that the assumption of a closed nodal for the *fes* leads to a contradiction, which establishes Melas' theorem.

Arguing on a parallel basis, G. Alessandrini [3] generalized these results to a bounded convex domain with non-smooth walls. To apply the preceding theorems in the domain of a vertex of the boundary, a conformal transformation is introduced that permits application of the above theorems in a domain of a vertex. It has been further argued by

Alessandrini and Magnanini [14] that the Lin-Melas arguments apply also to a convex C^k boundary, where $k > 1$.

Conjecture for $\mathbb{R}^3$. The following is a generalization of preceding results to the S-quantum billiards in $\mathbb{R}^3$. Since the Alessandrini mapping is not defined in $\mathbb{R}^3$, we first introduce a conjecture for a convex bounded billiard with a smooth boundary in $\mathbb{R}^3$ and then describe a procedure that establishes the *fest* for the S-quantum billiards.

Let Ω denote the domain of a convex bounded billiard with a smooth boundary $\partial\Omega$. In addition, let $Q(\alpha)$ represent a simple closed, smooth curve on $\partial\Omega$ where $0 \leq \alpha \leq 2\pi$. An “open curve” corresponds to a portion of $Q(\alpha)$ with $0 \leq \alpha \leq \alpha_0 < 2\pi$. Furthermore, $\partial\varphi/\partial\mathbf{n}$ denotes the outward normal derivative of φ on $\partial\Omega$.

Lin’s argument [11] may be generalized to $\mathbb{R}^3$ in the following manner. Let $\mathbf{P}'$ be the proposition: the nodal surface of the *fes* in Ω intersects $\partial\Omega$ in a simple closed curve. Let Ω_0 be a smooth bounded domain in $\mathbb{R}^3$ in which $\mathbf{P}'$ fails. Introduce a continuous mapping $\Omega(t)$ with $\Omega(0) = \Omega_0$ and $\Omega(1)$ a sphere which is known to have a bisecting nodal plane of mirror symmetry for the *fes*. One defines t_i as the largest value of $t \in (0, 1)$, for which $\mathbf{P}'$ fails. Again, with Theorem (Ia), it follows that the nodal of the *fes* is a closed surface within $\partial\Omega(t_i)$. If the nodal surface of the *fes* is a bisecting plane in $\Omega(1)$ and $\mathbf{P}'$ fails for t_i , then the nodal surface of $\varphi(t_i)$ connects to $\partial\Omega(t_i)$ such that $\partial\varphi/\partial\mathbf{n} = 0$ on only one open curve on $\partial\Omega(t_i)$. Again, based on the continuity of φ , the related function, $\varphi(t_i)$, is a *fes* in $\Omega(t_i)$. One then has the following:

Conjecture. If φ is a *fes* in Ω , then $\partial\varphi/\partial\mathbf{n}$ cannot be zero only on one open curve on $\partial\Omega$.

(This is the generalization of Melas’ theorem to $\mathbb{R}^3$.) It follows that the preceding assumption is invalid and $\mathbf{P}'$ is correct, which indicates that the nodal surface of the *fes* of a bounded convex billiard with a smooth boundary is a surface that intersects the boundary in a simple closed curve.

Smoothing procedure. To apply these results to the S-quantum billiards, the following smoothing procedure for the vertices of the billiard is introduced. Let the axis from the center of the billiard through a vertex be $\mathbf{r}$. Consider a plane that intersects this axis normally in a vicinity of the vertex, at $\mathbf{r} = \mathbf{r}'$. Remove the vertex section so defined. The “vertex figure” [16] formed by this removal is a flat domain whose largest radius is labeled δ . Now cover this domain with a smooth convex surface of rotation about $\mathbf{r}$. The intersection of this rotational surface and a plane that includes $\mathbf{r}$ is a symmetric curve about $\mathbf{r}$ and may be described by $\rho = \rho(\theta)$, where $\theta = 0$ corresponds to the intersection of the curve and the $\mathbf{r}$ -axis and $\rho = 0$ corresponds to the point on the axis, $\mathbf{r} = \mathbf{r}'$. Let the radius of curvature of $\rho(\theta)$ be labeled $\tau(\theta)$. Then $\tau(0) \equiv \lambda$ and we take $\rho(0) = \varepsilon = K_1\lambda = K_2\delta$, where $K_1, K_2 > 1$. Furthermore, $\tau(\theta)$ increases continuously and monotonically from the value $\tau(0) = \lambda$ to the value $\tau(\pi/4) = \infty$, where $\rho(\theta \geq \pi/4)$ is parallel to adjoining edges of the polyhedron. Consider the finite surface of revolution corresponding to $0 \leq \theta \leq \pi/4$ that lies exterior to the vertex (Fig. 2). Affix this surface to each vertex and let a sheet be pulled tightly over the extended

FIG. 2. Schematic of the covering surface of revolution about a vertex of an S-quantum billiard

S-billiard. The resulting surface is C^∞ and the preceding conjecture is applicable to these smooth convex billiards. In the limit that $\varepsilon \rightarrow 0$, the plane at $\mathbf{r} = \mathbf{r}'$ approaches the related vertex and the smooth billiard approaches the given S-billiard. In this same limit, $\partial\varphi/\partial\mathbf{n} \rightarrow \langle \partial\varphi/\partial\mathbf{n} \rangle$, where the average includes values of $\partial\varphi/\partial\mathbf{n}$ on the adjoining flat surfaces. This process defines $\partial\varphi/\partial\mathbf{n}$ either at an edge or a vertex of the regular polyhedra quantum billiards. For a truncated vertex, smoothing is applied to the edge about the truncation, in a manner similar to that described above. With validity of the given conjecture, this process establishes the *fest* for the S-quantum billiards in $\mathbb{R}^3$.

5. Extension to $\mathbb{R}^n$. We wish to extend the preceding results to $\mathbb{R}^n$, where $n \geq 2$ is an integer. In this case, Ω denotes a bounded convex domain in $\mathbb{R}^n$ and $\partial\Omega$ denotes its boundary, which is a closed hypersurface in $\mathbb{R}^{n-1}$. The outward normal derivative of $\varphi(\mathbf{x}^n)$ on $\partial\Omega$ is labeled $\partial\varphi/\partial\mathbf{n}$, and $\mathbf{x}^n$ are the spatial coordinates in $\mathbb{R}^n$. We first note that the argument for the surface of minimum energy extends readily to $\mathbb{R}^n$. Consider the conjecture described above. Its generalization is:

Conjecture in $\mathbb{R}^n$. If φ is a *fes* in Ω , then $\partial\varphi/\partial\mathbf{n}$ cannot be zero on only one open hypercurve on $\partial\Omega$. (We recall that Courant's theorems (I) and (II) are valid in $\mathbb{R}^n$, where $n \geq 2$ is an integer.)

Again let $\mathbf{P}''$ be the proposition: the nodal surface of the *fes* in Ω intersects $\partial\Omega$ in a simple closed hypercurve in $\mathbb{R}^{n-1}$. Let Ω_0 be a smooth bounded domain in $\mathbb{R}^n$ in which $\mathbf{P}''$ fails. Introduce a continuous mapping $\Omega(t)$ with $\Omega(0) = \Omega_0$ and $\Omega(1)$ a hypersurface, which is known to have a bisecting nodal hyperplane of mirror symmetry for the *fes* [17]. One defines t_i as the largest value of $t \in (0, 1)$, for which $\mathbf{P}''$ fails. In this event, with Theorem (Ia), it follows that the nodal of the *fes* is a closed surface in $\Omega(t_i)$. If the nodal surface of the *fes* is a bisecting plane in $\Omega(1)$, and $\mathbf{P}'$ fails at t_i , then the nodal surface of $\varphi(t_i)$ connects to $\partial\Omega(t_i)$ such that $\partial\varphi/\partial\mathbf{n} = 0$ only on one open hypercurve on $\partial\Omega(t_i)$. This is in violation with the conjecture cited above, and we conclude that the nodal hypersurface of the *fes* of a convex bounded billiard in $\mathbb{R}^n$ with a smooth boundary is a hypersurface that intersects the boundary in a simple closed hypercurve.

Smoothing procedure in $\mathbb{R}^n$. The generalization of the smoothing procedure described above is as follows. Let the axis from the center of the S-quantum billiard in $\mathbb{R}^n$ through a vertex be $\mathbf{r}$ in $\mathbb{R}^n$. Consider a plane that intersects this axis normally in a vicinity of the vertex, at $\mathbf{r} = \mathbf{r}'$. Remove the vertex section so defined. The “vertex figure” formed by this removal is a plane surface in $\mathbb{R}^{n-1}$ whose largest radius is labeled δ . Now cover this domain with a smooth convex hypersurface of rotation about $\mathbf{r}$. The intersection of this rotational surface and a hyperplane that includes $\mathbf{r}$ is a symmetric hypercurve in $\mathbb{R}^{n-1}$ about $\mathbf{r}$ and may be described by

$$\rho = \rho(\theta_1, \theta_2, \dots, \theta_{n-1}) \equiv \rho(\theta^{n-1}), \quad (4)$$

where $\theta^{n-1} = 0^{n-1}$ corresponds to the intersection of the hypercurve and the $\mathbf{r}$ -axis in $\mathbb{R}^{n-1}$ and ρ is measured from the point on the axis, $\mathbf{r} = \mathbf{r}'$. Let the radius of curvature [18] of $\rho(\theta^{n-1})$ be labeled $\tau(\theta^{n-1})$. Then $\tau(0^{n-1}) \equiv \lambda$ and we take $\rho(0^{n-1}) = \varepsilon = K_1\lambda = K_2\delta$, where $K_1, K_2 > 1$. Furthermore, $\tau(\theta^{n-1})$ increases continuously and monotonically from the value $\tau(0^{n-1}) = \lambda$ to the value $\tau[(\pi/4)^{n-1}] = \infty$, where the hypercurve $\rho[\theta^{n-1} \geq (\pi/4)^{n-1}]$ is parallel to adjoining edges of the polyhedron. Consider the finite hypersurface of revolution corresponding to $0^{n-1} \leq \theta^{n-1} \leq (\pi/4)^{n-1}$ that lies exterior to the vertex. Affix this hypersurface to each vertex and let a hypersurface be pulled tightly over the extended polyhedron. The resulting hypersurface is C^∞ and the preceding conjecture applies to these smooth convex billiards. In the limit that $\varepsilon \rightarrow 0$, the smooth hypersurface approaches the given polyhedron. In this same limit, $\partial\varphi/\partial\mathbf{n} \rightarrow \langle\partial\varphi/\partial\mathbf{n}\rangle$, where the average includes values of $\partial\varphi/\partial\mathbf{n}$ on the adjoining hypersurfaces. With validity of the given conjecture, this establishes the *fest* for the regular polyhedra quantum billiard in $\mathbb{R}^n$.

Degeneracy. To discover degeneracy of the S-quantum billiards we recall property (d) (Sec. 3) and the following rule: Degeneracies of quantum states of a system with symmetries of a given group, G , are given by the characters of the identity element of irreducible representations (“irreps”) of G . Thus one need only consult character tables [13] for these geometric forms to discover degeneracies. For regular polyhedra in $\mathbb{R}^2$ one finds that all quantum states are either nondegenerate or 2-fold degenerate. Identifying the ground state as the nondegenerate state, one concludes that the degeneracy of the *fes*’s of the regular polyhedra in $\mathbb{R}^2$ is 2-fold. For regular polyhedra in $\mathbb{R}^3$, degeneracy for the cube and octahedron is known to be three-fold, which corresponds to the maximum degeneracy implied by the character table for the O_h group. Degeneracy of the dodecahedron and the icosahedron quantum billiards are described by the I_h group. The related character table reveals that the 3-dimensional T irrep is spanned by the p -orbitals. The 5-dimensional H irrep of I_h is spanned by d -orbital quadratic forms whose nodal structure separates the domain into more than two subdomains. With Courant’s theorem (Ia) this 5-fold degeneracy may be associated with a higher excited state. Since the 4-dimensional G irrep is orthogonal to the preceding irrep, including the spherically symmetric A irrep, we may conclude that the 4-fold degeneracy of the G irrep likewise

is associated with a higher excited state. We may conclude that the *fes*'s of the dodacahedron and the icosahedron quantum billiards are likewise 3-fold degenerate which, with the said property, applies as well to an S-quantum billiard.

Degeneracy ansatz. To discover degeneracies of the S-quantum billiard in $\mathbb{R}^n$, we recall the following [13] and state an ansatz.

A. The degenerate eigenfunctions corresponding to an n -dimensional irrep of a given symmetry group, G , span an n -dimensional invariant subspace, H_n , of Hilbert space.

B. By the Great Orthogonality Theorem, the space H_n is orthogonal to the space spanned by the eigenfunctions of any other irrep of G .

C. (Ansatz) Consider a convex billiard in $\mathbb{R}^n$ described by a group whose irreps are of dimensions that include the dimension n . Then the *fes* of this billiard exists in a space H_d of dimension $d \geq n$. Rationale: If $d < n$, then functions in H_n exist that are not contained in H_d .

D. With B, one may conclude that irreps of G of dimension greater than n correspond to higher excited states of the given system. (Note that these conclusions are explicitly valid for the I_h group as described above.)

E. It follows that the *fes* of a regular convex billiard in $\mathbb{R}^n$ is n -fold degenerate, provided that an n -dimensional irrep exists for the related symmetry group.

Regular polyhedra in $\mathbb{R}^n$. Schläfli symbols. Properties of all regular convex polyhedra in all positive integer dimensions are listed in Table 1. In each entry in this table, the cell number and Schläfli symbol of a given polyhedron are cited [16]. (We recall that the Schläfli symbol for, say, a 4-dimensional polyhedron, $\{p, q, r\}$, indicates that the Schläfli symbol of cells of which the polyhedron is composed is $\{p, q\}$ and that the Schläfli symbol of vertex figures of the polyhedron is $\{q, r\}$.) In this table, powers denote repeated integers. Following convention, the first three polyhedra (in cell number) are listed as α_n, β_n and γ_n . The isolated four-dimensional polyhedron is here labeled P_4 . This table is arranged symmetrically with respect to dual properties so that entries in the first and sixth columns are self-dual polyhedra, entries in the second and third columns are duals of each other, as are entries in the fourth and fifth columns. It is noted from this table that only the $\alpha_n, \beta_n, \gamma_n$ polyhedra generalize to all positive integer dimensions and that the number of sets of regular polyhedra is maximum in $\mathbb{R}^4$. As may be readily shown, the degeneracy of the *fes* of the cubical quantum billiard, γ_n , in $\mathbb{R}^n$ is the dimension, n . The same is true for its dual, β_n , the octahedron in $\mathbb{R}^n$. The group of symmetries for the tetrahedron in $\mathbb{R}^n$ is the symmetric group on $(n+1)$ objects, S_{n+1} . For all $n \neq 4$, $n-1$ is the smallest dimension [19, 20] of irreps of S_n above dimension 1. (S_4 , relevant to $\mathbb{R}^3$, is equal to the T_d group for which irrep dimensions are 1, 2, 3.) With statement E above, we conclude that the degeneracy of the *fes* for α_n is n . In four dimensions, the $\{3, 3, 5\}$ regular polyhedron and its dual, $\{5, 3, 3\}$ each have 60 hyperplanes of symmetry. The $\{3, 4, 3\}$ regular polyhedron has 24 hyperplanes of symmetry.

The group of symmetries of a regular polyhedron with the Schläfli symbol $\{x, y, z\}$ is written $[\mathbf{x} \ \mathbf{y} \ \mathbf{z}]$. Dimensions of irreps of the groups $[\mathbf{3} \ \mathbf{3} \ \mathbf{5}]$ and $[\mathbf{3} \ \mathbf{4} \ \mathbf{3}]$ together with the number of each such irrep in each group are given in Table 2 [20]. Standard chemical

TABLE 1. Regular polyhedra with dimension n in Schläfli symbols
(Powers denote repeated integers)

n	α_n	β_n	γ_n	Icos.	Dodec.	P_4
3	4, {3, 3}	8, {3, 4}	6, {4, 3}	20, {3, 5}	12, {5, 3}	...
4	5, {3, 3, 3}	16, {3, 3, 4}	8, {4, 3, 3}	600, {3, 3, 5}	120, {5, 3, 3}	24, {3, 4, 3}
$n \geq 5$	$(n+1), \{3^{n-1}\}$	$2^n, \{3^{n-2}, 4\}$	$2n, \{4, 3^{n-2}\}$	...	...	...

TABLE 2. Dimensions of irreps of the point groups [3 3 5] and [3 4 3]

[3 3 5] or [5 3 3]		[3 4 3]	
irrep	dim.	irrep	dim.
A	1	A	1
G	4	E	2
I_2	6	G	4
I_4	8	I_2	6
I_5	9	I_4	8
J_1	10	I_5	9
J_7	16	J_3	12
J_9	18	J_7	16
K_5	24		
K_6	25		
L_1	30		
L_7	36		
M_1	40		
M_9	48		

symbols for these irreps and suggested notation for higher-dimensional irreps are included in this table. In both cases, 4-dimensional irreps exist, and with statement E above, we conclude that the *fes* of these regular polyhedra is four-fold degenerate. We may conclude also that the degeneracy of the first excited state of the regular convex polyhedra quantum billiards in $\mathbb{R}^n$ is n . This property applies also to the S-quantum billiards.

Conclusions. A class of convex “S-quantum billiards” is defined that includes the regular polyhedra as a subset. It is shown that a particular bisecting plane of this billiard is a surface of minimum energy. With this result, and an ansatz regarding degeneracy, together with a conjecture for bounded convex polyhedra, the first-excited-state theorem is established for S-quantum billiards in all $\mathbb{R}^n$. A table is included listing Schläfli symbols for all regular polyhedra in all dimensions. Another table lists dimensions of irreducible representations for the group of symmetries of the “last” two regular polyhedra in four dimensions.

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Appendix. Nodal intersection with a boundary.

THEOREM A. Let Ω be the domain of a regular convex polygon quantum billiard in $\mathbb{R}^2$, with the boundary $\partial\Omega$. Consider a set of nodals that coincide at a point P on $\partial\Omega$. The set of nodals and the boundary make an equal-angled array in the neighborhood of the intersection.

LEMMA. A theorem of Courant [4] states that if a set of nodals intersects at a point in Ω , then they make an equal-angled array.

Proof. *Case (a):* The coincidence point P lies on a straight segment on $\partial\Omega$. Consider an infinitesimal neighborhood about P in which any of the intersecting nodals is well approximated by a straight line. Odd reflect this neighborhood about the straight segment of $\partial\Omega$ in $\mathbb{R}^2$. Since this segment on $\partial\Omega$ is a nodal line in this description, the nodals and the boundary must make an equal angled array about the boundary point to satisfy Courant's theorem.

Case (b): The coincidence point P is a vertex point on $\partial\Omega$. A number of k nodals incident on a wedge of angle θ_0 partitions the wedge into angles θ_0/k . *Proof:* The angular component solution of the Helmholtz equation in this domain is $\sin(n\pi\theta/\theta_0)$, where $\theta = 0$ is one arm of the wedge and n is an integer. Setting $n = k$ establishes the theorem. $\square$

THEOREM B. Let Ω be the domain of a bounded convex quantum billiard with the smooth boundary $\partial\Omega$. Let Ω be such that at any point P on $\partial\Omega$ a tangent line to $\partial\Omega$ may be drawn that well approximates $\partial\Omega$ in an infinitesimal neighborhood about P . Then all the preceding statements carry over to this case.

COROLLARIES. (i) A single nodal intersects $\partial\Omega$ normally. (ii) A nodal cusp intersects $\partial\Omega$ normally. (iii) A nodal that is tangent to $\partial\Omega$ at any point of $\partial\Omega$ is not allowed.

Since Courant's theorem is valid in all $\mathbb{R}^n$, so are the preceding theorems.

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