A REMARK ON THE EXISTENCE OF GLOBAL BV SOLUTIONS FOR A NONLINEAR HYPERBOLIC WAVE EQUATION

BY

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Abstract. By means of a suitable change of variables we obtain, by application of a general result by Dafermos and Hsiao, cf. [2], an existence theorem in $L^\infty \cap BV_{\text{loc}}$ of a weak solution of the system corresponding to the quasilinear hyperbolic equation

$$\phi_{tt} - p'(\phi_x) \phi_{xx} + \phi_t + F(\phi) = 0 \quad \text{in } \mathbb{R} \times [0, +\infty[,$$

for small initial data in BV . This theorem is a partial extension of Dafermos's result for the case with $F(\phi) \equiv 0$, proved in [1].

1. The auxiliary system. Let us consider the following Cauchy problem:

$$\bar{\phi}_{tt} - p'(\bar{\phi}_x) \bar{\phi}_{xx} + \bar{\phi}_t + \bar{F}(\bar{\phi}) = 0 , \quad (x, t) \in \mathbb{R} \times [0, +\infty[, \quad (1.1)$$

$$\bar{\phi}(x, 0) = \bar{\phi}_0(x), \quad \bar{\phi}_t(x, 0) = \bar{v}_0(x) , \quad x \in \mathbb{R} , \quad (1.2)$$

where p is a given smooth function such that $p'(\xi) > 0$, $\forall \xi \in \mathbb{R}$, and $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function verifying $\bar{F}(0) = 0$. We assume, to simplify, the condition $p'(0) = 1$. Putting $u = \phi_x$, $v = \phi_t$ we can write (1.1), (1.2) as a Cauchy problem for a hyperbolic system:

$$\begin{cases} \bar{\phi}_t = \bar{v} \\ \bar{u}_t = \bar{v}_x \\ \bar{v}_t - p'(\bar{u}) \bar{u}_x + \bar{v} + \bar{F}(\bar{\phi}) = 0 \end{cases} \quad (x, t) \in \mathbb{R} \times [0, +\infty[, \quad (1.3)$$

$$\bar{\phi}(x, 0) = \bar{\phi}_0(x), \quad \bar{u}(x, 0) = \bar{u}_0(x) = \bar{\phi}_{0x}(x), \quad \bar{v}(x, 0) = \bar{v}_0(x) , \quad x \in \mathbb{R} . \quad (1.4)$$

For technical reasons, if we choose $k > 1$, we can, by putting $u(x, t) = \bar{u}(kx, kt)$, $v(x, t) = \bar{v}(kx, kt)$, $\phi(x, t) = \frac{1}{k} \bar{\phi}(kx, kt)$, replace (1.3), (1.4) by

$$\begin{cases} \phi_t = v \\ u_t = v_x \\ v_t - p'(u) u_x + k v + F(\phi) = 0 \end{cases} \quad (x, t) \in \mathbb{R} \times [0, +\infty[, \quad (1.3')$$

Received March 14, 2000.

2000 *Mathematics Subject Classification.* Primary 35L70.

where $F(\phi) = k \bar{F}(k\phi)$,

$$\begin{aligned}\phi(x, 0) &= \phi_0(x) = \frac{1}{k} \bar{\phi}_0(kx) , \\ u(x, 0) &= u_0(x) = \bar{u}_0(kx) = \phi_{0x}(x) , \\ v(x, 0) &= v_0(x) = \bar{v}_0(kx) , \quad x \in \mathbb{R} .\end{aligned}\tag{1.4'}$$

For $m \in [1, +\infty[$, let us introduce

$$\begin{aligned}a &= a_m = (m^2 - 2m + 2)^{-1/2} , \\ b &= b_m = (m^2 + 2m + 2)^{-1/2} , \\ c_m &= 2 - \left(\frac{a}{b} + \frac{b}{a}\right)/2 , \\ k_m &= 2 c_m^{-1} \left(a(m-1) + b(m+1)\right) , \\ f_m &= c_m^2 \left[2 \left(\frac{1}{a} + \frac{1}{b}\right) \left(a(m-1) + b(m+1)\right)\right]^{-1}\end{aligned}$$

(notice that $c_1 > 0$ and $k_1 > 1$) and assume

$$|\bar{F}'(0)| < f_1 .\tag{1.5}$$

We fix $m > 1$ such that $|\bar{F}'(0)| < f_m$, $c_m > 0$, and $k_m > 1$ and we put $k = k_m$. We obtain

$$\begin{aligned}\left(2 - \left(\frac{a}{b} + \frac{b}{a}\right)/2\right) k - \left(\frac{1}{2a} + \frac{1}{2b}\right) |F'(0)| - a(m-1) - b(m+1) \\ = c_m k_m - \left(\frac{1}{2a} + \frac{1}{2b}\right) k_m^2 |\bar{F}'(0)| - a(m-1) - b(m+1) > 0 .\end{aligned}\tag{1.6}$$

Now, we consider the following nonsingular linear transformation $(\phi, u, v) \rightarrow (\phi, u, w)$, where $w = v + mu + k\phi$, m and k as above. The Cauchy problem (1.3'), (1.4') takes the form

$$\begin{cases} \phi_t = w - mu - k\phi \\ u_t + k\phi_x + mu_x - w_x = 0 \\ w_t + mk\phi_x + (m^2 - p'(u))u_x - mw_x + F(\phi) = 0 \end{cases} \quad (x, t) \in \mathbb{R} \times [0, +\infty[,\tag{1.3''}$$

$$\begin{aligned}\phi(x, 0) &= \phi_0(x) , \\ u(x, 0) &= u_0(x) = \phi_{0x}(x) , \\ w(x, 0) &= w_0(x) = v_0(x) + mu_0(x) + k\phi_0(x) , \quad x \in \mathbb{R} .\end{aligned}\tag{1.4''}$$

Now we introduce the following auxiliary Cauchy problem in (ϕ, u, w) ,

$$\begin{cases} \phi_t + m \phi_x - w + k \phi = 0 \\ u_t + m u_x - w_x + k u = 0 \\ w_t + (m^2 - p'(u)) u_x - m w_x + m k u + F(\phi) = 0 \end{cases} \quad \text{in } \mathbb{R} \times [0, +\infty[, \quad (1.3''')$$

with the initial data (1.4'').

If (ϕ, u, w) is a C^1 solution of (1.3'''), (1.4'') we easily derive, for $\varphi \in C_c^\infty(\mathbb{R} \times [0, +\infty[)$,

$$\begin{aligned} & - \int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi \varphi_{xt} dx dt - \int_{\mathbb{R}} \phi_0 \varphi_x(\cdot, 0) dx - m \int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi \varphi_{xx} dx dt \\ & \quad - \int_{\mathbb{R}_+} \int_{\mathbb{R}} w \varphi_x dx dt + k \int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi \varphi_x dx dt = 0 \end{aligned}$$

and

$$\begin{aligned} & - \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \varphi_t dx dt - \int_{\mathbb{R}} u_0 \varphi(\cdot, 0) dx - m \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \varphi_x dx dt \\ & \quad + \int_{\mathbb{R}_+} \int_{\mathbb{R}} w \varphi_x dx dt + k \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \varphi dx dt = 0 , \end{aligned}$$

and so, by addition, we obtain

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi [\varphi_t + m \varphi_x - k \varphi]_x dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} u [\varphi_t + m \varphi_x - k \varphi] dx dt = 0 .$$

Now, given $\psi \in \mathcal{D}(\mathbb{R} \times]0, +\infty[)$, it is easy to find $\varphi \in C_c^\infty(\mathbb{R} \times [0, +\infty[)$ such that $\varphi_t + m \varphi_x - k \varphi = \psi$: first, with $\psi_1 = e^{-kt} \psi$, $\varphi_1 = e^{-k\varphi} \varphi$, we reduce to $\varphi_{1t} + m \varphi_{1x} = \psi_1$. We put

$$\varphi_1(mt + c, t) = \int_0^t \psi_1(m\tau + c, \tau) d\tau - \tilde{\varphi}_1(c) ,$$

where

$$\tilde{\varphi}_1(c) = \int_0^{+\infty} \psi_1(mt + c, t) dt , \quad \forall c \in \mathbb{R} .$$

Hence,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi \psi_x dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \psi dx dt = 0 , \quad \forall \psi \in \mathcal{D}(\mathbb{R} \times]0, +\infty[) .$$

We derive $u = \phi_x$. It is now easy to prove

PROPOSITION 1.1. For given initial data $(\phi_0, u_0 = \phi_{0x}, w_0)$ in $C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ the systems (1.3'') and (1.3''') have the same (local in time) C^1 solutions.

Given (ϕ_0, u_0, w_0) in $L^\infty(\mathbb{R})$, we say, as usually, that $(\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ is a weak (global) solution of the Cauchy problem (1.3''), (1.4'') if we have

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi \varphi_t dx dt + \int_{\mathbb{R}} \phi_0 \varphi(\cdot, 0) dx + \int_{\mathbb{R}_+} \int_{\mathbb{R}} (w - m u - k \phi) \varphi dx dt \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \psi_t dx dt + \int_{\mathbb{R}} u_0 \psi(\cdot, 0) dx + \int_{\mathbb{R}_+} \int_{\mathbb{R}} (k \phi + m u - w) \psi_x dx dt \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}} w \theta_t dx dt + \int_{\mathbb{R}} w_0 \theta(\cdot, 0) dx \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}} [m^2 u - p(u) - m w] \theta_x dx dt - \int_{\mathbb{R}_+} \int_{\mathbb{R}} F(\phi) \theta dx dt = 0, \\ & \forall \varphi, \psi, \theta \in C_c^\infty(\mathbb{R} \times [0, +\infty[), \end{aligned} \quad (1.7)$$

and a similar definition for the Cauchy problem (1.3'''), (1.4'').

We can repeat the calculations made to prove Proposition 1.1 for a $(\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ weak solution of (1.3'''), (1.4'') and we obtain $\phi_x = u$ in $\mathcal{D}'(\mathbb{R} \times]0, +\infty[)$. We derive $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$ and it is now easy to prove that (ϕ, u, w) is a weak solution of (1.3''), (1.4''). The converse is also true, by similar considerations. Hence, we have

PROPOSITION 1.2. For a given initial data $(\phi_0, u_0 = \phi_{0x}, w_0)$ in $L^\infty(\mathbb{R})$, the systems (1.3'') and (1.3''') have the same weak solutions.

Now let (η, q) be a pair of smooth convex entropy/entropy flux for the system (1.3'') (cf. [4]).

$$\left(\begin{array}{l} \text{Example: } \eta_1(\phi, u, w) = \frac{1}{2} \phi^2 + \int_0^u p(\xi) d\xi + \frac{1}{2} (w - m u - k \phi)^2, \\ q_1(\phi, u, w) = -(w - m u - k \phi) p(u) \end{array} \right).$$

A weak solution (ϕ, u, w) of (1.3''), (1.4'') is called an entropy weak solution if, in $\mathcal{D}'(\mathbb{R} \times]0, +\infty[)$,

$$\eta(\phi, u, w)_t + q(\phi, u, w)_x + \nabla \eta \cdot (-w + m u + k \phi, 0, F(\phi)) \leq 0 \quad (1.8)$$

for all pairs (η, q) , η convex.

The system (1.3''') admits the entropy/entropy flux pair $(\tilde{\eta}_1, \tilde{q}_1)$, $\tilde{\eta}_1$ strictly convex, defined by

$$\begin{aligned} \tilde{\eta}_1(\phi, u, w) &= \frac{1}{2} \phi^2 + \int_0^u p(\xi) d\xi + \frac{1}{2} (w - m u)^2, \\ \tilde{q}_1(\phi, u, w) &= \frac{1}{2} m \phi^2 - (w - m u) p(u). \end{aligned}$$

If $(\phi, u, w) \in (L^\infty_{\text{loc}} \cap BV_{\text{loc}})^3$ is a weak solution of (1.3''), (1.4'') with initial data in $BV(\mathbb{R})$, we can prove (with some tedious computations, taking in mind that $\phi_x =$

u , cf. Proposition 1.2, and applying the theorem in section 13.2 of [5] concerning the differentiation of the composition) that we have, in $\mathcal{D}'(\mathbb{R} \times]0, +\infty[)$,

$$\begin{aligned} & \tilde{\eta}_1(\phi, u, w)_t + \tilde{q}_1(\phi, u, w)_x + \nabla \tilde{\eta}_1 \cdot \left(-w + k\phi, ku, mku + F(\phi) \right) \\ &= \eta_1(\phi, u, w)_t + q_1(\phi, u, w)_x + \nabla \eta_1 \cdot \left(-w + mu + k\phi, 0, F(\phi) \right). \end{aligned}$$

Hence, by Proposition 1.2, we conclude

THEOREM 1.3. Assume that $(\phi_0, u_0 = \phi_{0x}, w_0) \in BV(\mathbb{R})^3$ and let $(\phi, u, w) \in (L_{\text{loc}}^\infty \cap BV_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ be an entropy weak solution of (1.3'''), (1.4''). Then, (ϕ, u, w) is also a weak solution of (1.3''), (1.4'') verifying (1.8) for the pair (η_1, q_1) defined above.

2. Application of Theorem 2 in [2]. Now, in order to apply to the Cauchy problem (1.3'''), (1.4'') Theorem 2 in [2], we give initial data $(\phi_0, u_0 = \phi_{0x}, w_0)$ in $BV(\mathbb{R})$. The system (1.3''') can be written as follows (recall that $p'(0) = 1$ and $m > 1$):

$$\frac{\partial}{\partial t} \begin{pmatrix} \phi \\ u \\ w \end{pmatrix} + A(u) \frac{\partial}{\partial x} \begin{pmatrix} \phi \\ u \\ w \end{pmatrix} + g(\phi, u, w) = 0,$$

where

$$A(u) = \begin{pmatrix} m & 0 & 0 \\ 0 & m & -1 \\ 0 & m^2 - p'(u) & -m \end{pmatrix}, \quad g(\phi, u, w) = \begin{pmatrix} -w + k\phi \\ ku \\ mku + F(\phi) \end{pmatrix}.$$

The eigenvalues of $A(u)$ are $(m, \sqrt{p'(u)}, -\sqrt{p'(u)})$. The matrix of the corresponding (independent) normalized right eigenvectors for $u = 0$ is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & a(m-1) & b(m+1) \end{pmatrix} \quad \text{and} \quad \nabla g(0, 0, 0) = \begin{pmatrix} k & 0 & -1 \\ 0 & k & 0 \\ F'(0) & mk & 0 \end{pmatrix},$$

where $a = (m^2 - 2m + 2)^{-1/2}$ and $b = (m^2 + 2m + 2)^{-1/2}$.

Hence, $R = \{r_{ij}\} = B^{-1} \nabla g(0, 0, 0) B$ is given by

$$R = \begin{pmatrix} k & a(-m+1) & b(-m-1) \\ -\frac{1}{2a} F'(0) & \frac{k}{2} & \frac{b}{a} \frac{k}{2} \\ \frac{1}{2b} F'(0) & \frac{a}{b} \frac{k}{2} & \frac{k}{2} \end{pmatrix}$$

and verifies

$$\begin{aligned} \sum_i r_{ii} - \sum_{i \neq j} |r_{ij}| &= \left(2 - \left(\frac{a}{b} + \frac{b}{a} \right) / 2 \right) k - \left(\frac{1}{2a} + \frac{1}{2b} \right) |F'(0)| \\ &\quad - a(m-1) - b(m+1) > 0 \quad \text{by (1.6)} \end{aligned}$$

and so R is diagonal dominant.

By applying Theorem 2 in [2] we derive

THEOREM 2.1. Let us assume (1.5). Then, there exist two positive constants $a_0, b_0 > 0$ such that, if

$$(\phi_0, u_0 = \phi_{0x}, w_0) \in (BV(\mathbb{R}))^3$$

and

$$\|(\phi_0, u_0, w_0)\|_{L^\infty(\mathbb{R})} \leq a_0, \quad TV_x(\phi_0, u_0, w_0) \leq b_0,$$

then there exists a weak entropy solution $(\phi, u, w) \in (L^\infty \cap BV_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ of (1.3'''), (1.4''). Moreover $(\phi(\cdot, t), u(\cdot, t), w(\cdot, t)) \in (BV(\mathbb{R}))^3$ for each $t \geq 0$, with a uniformly bounded (in t) total variation TV_x .

Hence, by Theorem 1.3, we can derive a similar result for the Cauchy problem (1.3), (1.4) if we replace the general entropy condition (1.8) by the following particular one:

$$\bar{\eta}(\bar{\phi}, \bar{u}, \bar{v})_t + \bar{q}(\bar{\phi}, \bar{u}, \bar{v})_x + \nabla \bar{\eta} \cdot (-\bar{v}, 0, \bar{v} + \bar{F}(\bar{\phi})) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, +\infty[),$$

where

$$(\bar{\eta}, \bar{q}) = \left(\frac{1}{2} \bar{\phi}^2 + \int_0^{\bar{u}} p(\xi) d\xi + \frac{1}{2} \bar{v}^2, -\bar{v} p(\bar{u}) \right).$$

See [1] for the case with $\bar{F} \equiv 0$ and [3] for related results.

Acknowledgments. This work was supported by FCT, PRAXIS XXI, FEDER and Project PRAXIS/2/2.1/MAT/125/94.

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