AN INVERSE PROBLEM FOR THE THREE-DIMENSIONAL MULTI-CONNECTED VIBRATING MEMBRANE WITH ROBIN BOUNDARY CONDITIONS

BY

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Abstract. This paper deals with the very interesting problem concerning the influence of the boundary conditions on the distribution of the eigenvalues of the negative Laplacian in R^3 . The trace of the heat semigroup $\theta(t) = \sum_{v=1}^{\infty} \exp(-t\mu_v)$, where $\{\mu_v\}_{v=1}^{\infty}$ are the eigenvalues of the negative Laplacian $-\nabla^2 = -\sum_{\beta=1}^3 (\frac{\partial}{\partial x^\beta})^2$ in the (x^1, x^2, x^3) -space, is studied for a general multiply-connected bounded domain Ω in R^3 surrounding by simply connected bounded domains Ω_j with smooth bounding surfaces S_j ($j = 1, \dots, n$), where a finite number of piecewise smooth Robin boundary conditions on the piecewise smooth components S_i^* ($i = 1 + k_{j-1}, \dots, k_j$) of the bounding surfaces S_j is considered, such that $S_j = \bigcup_{i=1+k_{j-1}}^{k_j} S_i^*$, where $k_0 = 0$. Some applications of $\theta(t)$ for an ideal gas enclosed in the multiply-connected bounded container Ω with Robin boundary conditions are given. We show that the asymptotic expansion of $\theta(t)$ for short-time t plays an important role in investigating the influence of the finite container Ω on the thermodynamic quantities of an ideal gas.

1. Introduction. The underlying inverse problem is to deduce some geometric quantities associated with a bounded domain in R^3 from complete knowledge of the eigenvalues of the negative Laplacian.

Let Ω be a simply connected bounded domain in R^3 with a smooth bounding surface S . Consider the Robin problem

$$-\nabla^2 \phi = \mu \phi \quad \text{in } \Omega \tag{1.1}$$

and

$$\left(\frac{\partial}{\partial n} + \gamma \right) \phi = 0 \quad \text{on } S \tag{1.2}$$

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where $\frac{\partial}{\partial n}$ denotes differentiation along the inward normal to S with $\phi \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_\nu \leq \cdots \rightarrow \infty \quad \text{as } \nu \rightarrow \infty. \quad (1.3)$$

The Robin problem (1.1) and (1.2) has been discussed by Zayed [13] when γ is a positive constant, and by Zayed [21] when γ is a smooth function which is not strictly positive, and have determined some geometric quantities associated with the bounded domain Ω , by using the asymptotic expansion of the trace of the heat semigroup

$$\theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\mu_\nu) \quad \text{as } t \rightarrow 0^+. \quad (1.4)$$

The above problem has been investigated by many authors (see, for example, Pleijel [10], Hsu [6], McKean and Singer [8], Waechter [12], Gottlieb [4]) in the following special cases:

Case 1.1. $\gamma = 0$ (the Neumann problem):

$$\begin{aligned} \theta(t) = & \frac{V}{(4\pi t)^{\frac{3}{2}}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{\frac{3}{2}}t^{\frac{1}{2}}} \int_S H dS \\ & + \frac{7}{128\pi} \int_S (H^2 - N^*) dS + O(t^{\frac{1}{2}}) \quad \text{as } t \rightarrow 0^+. \end{aligned} \quad (1.5)$$

Case 1.2. $\gamma \rightarrow \infty$ (the Dirichlet problem):

$$\begin{aligned} \theta(t) = & \frac{V}{(4\pi t)^{\frac{3}{2}}} - \frac{|S|}{16\pi t} + \frac{1}{12\pi^{\frac{3}{2}}t^{\frac{1}{2}}} \int_S H dS \\ & + \frac{1}{128\pi} \int_S (H^2 - N^*) dS + O(t^{\frac{1}{2}}) \quad \text{as } t \rightarrow 0^+. \end{aligned} \quad (1.6)$$

In these formulae, V , $|S|$, H , and N^* are respectively the volume, the surface area, the mean curvature, and the Gaussian curvature of the surface S of Ω , where $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$ and $N^* = \frac{1}{R_1 R_2}$, while R_1 and R_2 are the principal radii of curvature.

Case 1.3. (the mixed problem):

Let $|S_1|$, H_1 , and N_1 respectively be the surface area, the mean curvature, and the Gaussian curvature of a component S_1 of the surface S with the Neumann boundary conditions and let $|S_2|$, H_2 , and N_2 respectively be the surface area, the mean curvature, and the Gaussian curvature of the remaining component $S_2 = S \setminus S_1$ of S with the Dirichlet boundary conditions. Then in [4, 14, 17, 18] we obtain

$$\begin{aligned} \theta(t) = & \frac{V}{(4\pi t)^{\frac{3}{2}}} + \frac{|S_1| - |S_2|}{16\pi t} + \frac{1}{12\pi^{\frac{3}{2}}t^{\frac{1}{2}}} \left\{ \int_{S_1} H_1 dS_1 + \int_{S_2} H_2 dS_2 \right\} \\ & + \frac{1}{128\pi} \left\{ 7 \int_{S_1} (H_1^2 - N_1) dS_1 + \int_{S_2} (H_2^2 - N_2) dS_2 \right\} \\ & + \left(\frac{t}{\pi^3} \right)^{\frac{1}{2}} \left\{ \frac{13}{1440} \int_{S_1} H_1^3 dS_1 - \frac{1}{315} \int_{S_2} H_2^3 dS_2 \right\} + O(t) \end{aligned} \quad (1.7)$$

as $t \rightarrow 0^+$.

The object of this paper is to discuss the following more general inverse problem: Let Ω be a general-multiply-connected bounded domain in R^3 surrounded internally by simply connected bounded domains Ω_j with smooth bounding surfaces S_j ($j = 1, \dots, n-1$) and externally by a simply connected bounded domain Ω_n with a smooth bounding surface S_n .

Consider the Robin problem

$$-\nabla^2 \phi = \mu \phi \quad \text{in } \Omega, \quad (1.8)$$

and

$$\left(\frac{\partial}{\partial n_i} + \gamma_i \right) \phi = 0 \quad \text{on } S_i^*, \quad (1.9)$$

where the bounding surfaces S_j ($j = 1, \dots, n$) consists of a finite number of piecewise smooth components S_i^* ($i = 1 + k_{j-1}, \dots, k_j$) such that $S_j = \bigcup_{i=1+k_{j-1}}^{k_j} S_i^*$ where $k_0 = 0$ and γ_i are positive constants.

The basic problem is to extract information on the geometry of the multiply-connected domain Ω associated with the problem (1.8) and (1.9) from complete knowledge of its eigenvalues $\{\mu_\nu\}_{\nu=1}^\infty$ by analyzing the asymptotic expansions of $\theta(t)$ for short-time t .

This problem is important both from the mathematical point of view and from many applications. In physics, this problem has applications in solid state theory (e.g., cluster, quantum dots), nuclear physics (e.g., shell model), acoustics, statistical mechanics (e.g., finite size system, an ideal gas), and electromagnetic theory (e.g., resonant cavities) etc. (see [1], [2]).

So, the present paper contains a section with an example in order to show one among several possible applications. Note that the problem (1.8) and (1.9) has been discussed recently by Zayed [25] in the case where Ω is a two-dimensional multiply-connected bounded domain. Therefore, this problem is an extension of the author's recent work [25] and can be considered as a more general one which does not seem to have been investigated elsewhere.

2. Statement of the results. Suppose that the bounding surfaces S_j ($j = 1, \dots, n$) of the multiply-connected bounded domain Ω are given locally by infinitely differentiable functions $x^\beta = y^\beta(\sigma_j)$, ($\beta = 1, 2, 3$) of the parameters σ_j^α ($\alpha = 1, 2$). If these parameters are chosen so that $\sigma_j^\alpha = \text{constant}$, are lines of curvature, the first and second fundamental forms of S_j can be written respectively, in the following forms:

$$II_{1j}(\sigma_j, \Delta\sigma_j) = \sum_{\alpha=1}^2 a_{\alpha j}(\sigma_j)(\Delta\sigma_j^\alpha)^2,$$

and

$$II_{2j}(\sigma_j, \Delta\sigma_j) = \sum_{\alpha=1}^2 b_{\alpha j}(\sigma_j)(\Delta\sigma_j^\alpha)^2.$$

In terms of the coefficients $a_{\alpha j}$ and $b_{\alpha j}$, the principal radii of the curvatures for S_j ($j = 1, \dots, n$) are given by $R_{1j} = a_{1j}/b_{1j}$ and $R_{2j} = a_{2j}/b_{2j}$. Consequently, let H_j and N_j respectively be the mean curvatures and the Gaussian curvatures of S_j ($j = 1, \dots, n$) where $H_j = \frac{1}{2}(\frac{1}{R_{1j}} + \frac{1}{R_{2j}})$ and $N_j = \frac{1}{R_{1j}R_{2j}}$. Let $|S_i^*|$ ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, n$)

be the surface areas of the components S_i^* , respectively. Let $h_i > 0$ be sufficiently small numbers. Let n_i be the minimum distances from a point $\tilde{x} = (x^1, x^2, x^3)$ of the region Ω to the components S_i^* respectively. Let $n_i(\sigma_j)$ denote the inward unit normals to the components S_i^* respectively. We note that the coordinates in the neighborhoods of the components S_i^* ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, N$) are in the same forms as in Sec. 5.2 of Zayed [17] with the interchanges $\sigma_1^1 \leftrightarrow \sigma_j^1, \sigma_1^2 \leftrightarrow \sigma_j^2, n_1 \leftrightarrow n_i, h_1 \leftrightarrow h_i, I_1 \leftrightarrow I_i, \mathfrak{D}(I_1) \leftrightarrow \mathfrak{D}(I_i)$, and $\delta_1 \leftrightarrow \delta_i$. Thus, we have the same formulae (5.2.1)–(5.2.5) of Sec. 5.2 in [17] with the interchanges $n_1 \leftrightarrow n_i, \tilde{n}_1(\sigma_1) \leftrightarrow \tilde{n}_i(\sigma_j), II_1^* \leftrightarrow II_{1j}, II_2^* \leftrightarrow II_{2j}, H_1^* \leftrightarrow H_j$, and $N_1^* \leftrightarrow N_j$, ($j = 1, \dots, N$). Similarly, the coordinates in the neighborhoods of the components S_i^* ($i = 1 + k_{j-1}, \dots, k_j$), ($j = N + 1, \dots, n$) are in the same forms as in Sec. 5.1 of Zayed [17] with the interchanges $\sigma_2^1 \leftrightarrow \sigma_j^1, \sigma_2^2 \leftrightarrow \sigma_j^2, n_2 \leftrightarrow n_i, h_2 \leftrightarrow h_i, I_2 \leftrightarrow I_i, \mathfrak{D}(I_2) \leftrightarrow \mathfrak{D}(I_i)$, and $\delta_2 \leftrightarrow \delta_i$. Thus, we have the same formulae (5.1.1)–(5.1.6) of Sec. 5.1 in [17] with the interchanges $n_2 \leftrightarrow n_i, \tilde{n}_2(\sigma_2) \leftrightarrow \tilde{n}_i(\sigma_j), II_1 \leftrightarrow II_{1j}, II_2 \leftrightarrow II_{2j}, H_1 \leftrightarrow H_j$, and $N_1 \leftrightarrow N_j$, ($j = N + 1, \dots, n$).

THEOREM 2.1. With the assumptions stated above, the asymptotic expansion of the trace of the heat semigroup $\theta(t)$ of the problem (1.8) and (1.9) can be written in the following form:

$$\theta(t) = \frac{a_1}{t^{\frac{3}{2}}} + \frac{a_2}{t} + \frac{a_3}{t^{\frac{1}{2}}} + a_4 + a_5 t^{\frac{1}{2}} + O(t) \quad \text{as } t \rightarrow 0^+, \quad (2.1)$$

where $0 < \gamma_i \ll 1$ ($i = 1 + k_{j-1}, \dots, m_j$) and $\gamma_i \gg 1$ ($i = m_j + 1, \dots, k_j$), ($j = 1, \dots, N, N + 1, \dots, n$).

Here the coefficients a_ν ($\nu = 1 - 5$) have the forms:

$$\begin{aligned} a_1 &= V/8\pi^{\frac{3}{2}}, \\ a_2 &= \frac{1}{16\pi} \sum_{j=1}^n \left\{ \sum_{i=1+k_{j-1}}^{m_j} |S_i^*| - \sum_{i=m_j+1}^{k_j} \left[|S_i^*| - 2\gamma_i^{-1} \int_{S_i^*} H_j dS_i^* \right] \right\}, \\ a_3 &= \frac{1}{12\pi^{\frac{3}{2}}} \sum_{j=1}^n \left\{ \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} (H_j - 3\gamma_i) dS_i^* + \sum_{i=m_j+1}^{k_j} \int_{S_i^*} H_j dS_i^* \right\}, \\ a_4 &= \frac{1}{128\pi} \sum_{j=1}^n \left\{ 7 \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left[(H_j - 3\gamma_i)^2 - \left(N_j - \frac{26}{7}\gamma_i H_j + \frac{47}{7}\gamma_i^2 \right) \right] dS_i^* \right. \\ &\quad \left. + \sum_{i=m_j+1}^{k_j} \int_{S_i^*} [H_j^2 - (N_j - 16H_j\gamma_i^{-1})] dS_i^* \right\}, \\ a_5 &= \frac{1}{\pi^{\frac{3}{2}}} \sum_{j=1}^n \left\{ \frac{13}{1440} \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} (H_j - 3\gamma_i)^3 dS_i^* - \frac{1}{315} \sum_{i=m_j+1}^{k_j} \int_{S_i^*} H_j^3 dS_i^* \right\}. \end{aligned}$$

From this theorem we deduce the following corollaries and remarks:

COROLLARY 2.1. If we consider the problem (1.8) and (1.9) with $0 < \gamma_i \ll 1$ ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, n$), then the asymptotic expansion of $\theta(t)$ follows directly from (2.1) by setting $m_j = k_j$ with $\sum_{i=1+k_j}^{k_j}$ as zero.

COROLLARY 2.2. If we consider the problem (1.8) and (1.9) with $\gamma_i \gg 1$ ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, n$), then the asymptotic expansion of $\theta(t)$ follows directly from (2.1) by setting $m_j = k_{j-1}$ with $\sum_{i=1+k_{j-1}}^{k_{j-1}}$ as zero.

REMARK 2.1. In the case $n = 1$ (i.e., Ω is simply connected), then we have the results of Zayed [13, 15, 16].

REMARK 2.2. In the case $n = 2$ (i.e., Ω is doubly-connected), then we have the results of Zayed [14, 17, 19, 20].

Our main result (2.1) will be used in Sec. 4 to derive some asymptotic expansions of the thermodynamic quantities of an ideal gas enclosed in its container Ω together with the conditions (1.9).

Finally, let us add that the question raised by Kac [7], namely, “Can one hear the shape of a drum?” was answered negatively by Gordon *et al.* [3], who showed explicitly two domains that, although they have different shapes, have the same eigenvalues (i.e., isospectral domains). This theoretical result was experimentally verified recently by employing thin microwave cavities shaped in the form of two different isospectral domains (see [11]). Milnor [9] had already showed an example of two different domains with the same eigenvalues, but they were two 16-dimensional tori. Therefore, we conclude that the boundary conditions cannot define the spectrum uniquely.

3. Proof of Theorem 2.1. In analogy with the two-dimensional problems [7, 24, 25] and with reference to the articles [12–22], it is easy to show that the trace of the heat semigroup $\theta(t)$ associated with the problem (1.8) and (1.9) is given by

$$\theta(t) = \iiint_{\Omega} G(\underline{x}, \underline{x}; t) d\underline{x}, \quad (3.1)$$

where $G(\underline{x}_1, \underline{x}_2; t)$ is the Green's function for the heat equation

$$\left(\nabla^2 - \frac{\partial}{\partial t} \right) \phi = 0, \quad (3.2)$$

subject to the conditions (1.9) and the initial condition

$$\lim_{t \rightarrow 0^+} G(\underline{x}_1, \underline{x}_2; t) = \delta(\underline{x}_1 - \underline{x}_2), \quad (3.3)$$

where $\delta(\underline{x}_1 - \underline{x}_2)$ is the Dirac delta function located at the source point $\underline{x}_1 = \underline{x}_2$. Let us write

$$G(\underline{x}_1, \underline{x}_2; t) = G_0(\underline{x}_1, \underline{x}_2; t) + \kappa(\underline{x}_1, \underline{x}_2; t), \quad (3.4)$$

where

$$G_0(\tilde{x}_1, \tilde{x}_2; t) = (4\pi t)^{-\frac{3}{2}} \exp \left\{ -\frac{\left| \tilde{x}_1 - \tilde{x}_2 \right|^2}{4t} \right\}, \quad (3.5)$$

is the fundamental solution of the heat equation (3.2) while $\varkappa(\tilde{x}_1, \tilde{x}_2; t)$ is the regular solution chosen in such a way that $G(\tilde{x}_1, \tilde{x}_2; t)$ satisfies the conditions (1.9). On setting $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, we find that

$$\theta(t) = \frac{V}{(4\pi t)^{\frac{3}{2}}} + R(t), \quad (3.6)$$

where

$$R(t) = \iiint_{\Omega} \varkappa(\tilde{x}, \tilde{x}; t) d\tilde{x}. \quad (3.7)$$

The problem now is to determine the asymptotic expansions of $R(t)$ for $t \rightarrow 0^+$. In what follows, we shall use Laplace transforms with respect to t , and s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\tilde{x}_1, \tilde{x}_2; s^2) = \int_0^{+\infty} e^{-s^2 t} G(\tilde{x}_1, \tilde{x}_2; t) dt. \quad (3.8)$$

An application of the Laplace transforms to the heat equation (3.2) shows that $\overline{G}(\tilde{x}_1, \tilde{x}_2; s^2)$ satisfies the membrane equation

$$(\nabla^2 - s^2)\overline{G}(\tilde{x}_1, \tilde{x}_2; s^2) = -\delta(\tilde{x}_1 - \tilde{x}_2) \text{ in } \Omega, \quad (3.9)$$

together with the conditions (1.9).

By virtue of a classical Tauberian argument, the asymptotic expansion of $R(t)$ for small positive t may then be deduced directly from the asymptotic expansion of $\overline{R}(s^2)$ for large s , where

$$\overline{R}(s^2) = \iiint_{\Omega} \overline{\varkappa}(\tilde{x}, \tilde{x}; s^2) d\tilde{x}. \quad (3.10)$$

It is well known (see, for example, [13–20]) that the membrane equation (3.9) has the fundamental solution $\overline{G}_0(\tilde{x}_1, \tilde{x}_2; s^2) = \exp(-sr_{\tilde{x}_1\tilde{x}_2})/4\pi r_{\tilde{x}_1\tilde{x}_2}$, where $r_{\tilde{x}_1\tilde{x}_2} = \left| \tilde{x}_1 - \tilde{x}_2 \right|$ is the distance between the points $\tilde{x}_1 = (x_1^1, x_1^2, x_1^3)$ and $\tilde{x}_2 = (x_2^1, x_2^2, x_2^3)$ of the domain Ω . The existence of this solution enables us to construct integral equations for $\overline{G}(\tilde{x}_1, \tilde{x}_2; s^2)$ satisfying the conditions (1.9) for small/large impedances γ_i . Therefore, if we consider the problem (1.8) and (1.9) with $0 < \gamma_i \ll 1$ ($i = 1 + k_{j-1}, \dots, m_j$) and $\gamma_i \gg 1$ ($i = m_j + 1, \dots, k_j$) where ($j = 1, \dots, N, N + 1, \dots, n$), then the Green's theorem gives

the following integral equation:

$$\begin{aligned}
 \overline{G}(\tilde{x}_1, \tilde{x}_2; s^2) = & \exp(-sr_{\tilde{x}_1 \tilde{x}_2})/4\pi r_{\tilde{x}_1 \tilde{x}_2} \\
 & + \frac{1}{2\pi} \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \overline{G}(\tilde{x}_1, \tilde{y}; s^2) \left[\frac{\partial}{\partial n_{i\tilde{y}}} + \gamma_i \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y} \\
 & - \frac{1}{2\pi} \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left[\frac{\partial}{\partial n_{i\tilde{y}}} \overline{G}(\tilde{x}_1, \tilde{y}; s^2) \right] \\
 & \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}}} \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y} \\
 & - \frac{1}{2\pi} \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \overline{G}(\tilde{x}_1, \tilde{y}; s^2) \left[\frac{\partial}{\partial n_{i\tilde{y}}} + \gamma_i \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y} \\
 & + \frac{1}{2\pi} \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left[\frac{\partial}{\partial n_{i\tilde{y}}} \overline{G}(\tilde{x}_1, \tilde{y}; s^2) \right] \\
 & \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}}} \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y}.
 \end{aligned} \tag{3.11}$$

By applying the iteration methods (see, for example, [13, 16, 17]) to the integral equation (3.11), we obtain the Green's function $\overline{G}(\tilde{x}_1, \tilde{x}_2; s^2)$ which has a regular part in the following form:

$$\overline{\mathcal{K}}(\tilde{x}_1, \tilde{x}_2; s^2) = \left[\sum_{k=1}^{20} A_k \right] / 8\pi^2, \tag{3.12}$$

where

$$\begin{aligned}
 A_1 = & \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] \left[\frac{\partial}{\partial n_{i\tilde{y}}} + \gamma_i \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y}, \\
 A_2 = & - \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}}} \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y}, \\
 A_3 = & - \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] \left[\frac{\partial}{\partial n_{i\tilde{y}}} + \gamma_i \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y}, \\
 A_4 = & \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}}} \right] [\exp(-sr_{\tilde{y}\tilde{x}_2})/r_{\tilde{y}\tilde{x}_2}] d\tilde{y},
 \end{aligned}$$

$$\begin{aligned}
A_5 &= \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i}(\tilde{y}, \tilde{y}') \\
&\quad \times \left[\frac{\partial}{\partial n_{i \tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y} d\tilde{y}', \\
A_6 &= \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \int_{S_i^*} \frac{\partial}{\partial n_{i \tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i^{-1}}(\tilde{y}, \tilde{y}') \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i \tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y} d\tilde{y}', \\
A_7 &= \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \int_{S_i^*} \frac{\partial}{\partial n_{i \tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}(\tilde{y}, \tilde{y}') \\
&\quad \times \left[\frac{\partial}{\partial n_{i \tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y} d\tilde{y}', \\
A_8 &= \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \int_{S_i^*} \frac{\partial}{\partial n_{i \tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i^{-1}}(\tilde{y}, \tilde{y}') \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i \tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y} d\tilde{y}', \\
A_9 &= - \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i \tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i^{-1}}^*(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[\frac{\partial}{\partial n_{i \tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{10} &= - \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i}^*(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i \tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{11} &= - \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}^*(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[\frac{\partial}{\partial n_{i \tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}',
\end{aligned}$$

$$\begin{aligned}
A_{12} &= - \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[\frac{\partial}{\partial n_{i\tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{13} &= \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i}^{*-1}(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[\frac{\partial}{\partial n_{i\tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{14} &= \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}^*(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{15} &= \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i}^*(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{16} &= \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}^{*-1}(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[\frac{\partial}{\partial n_{i\tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{17} &= - \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] M_{\gamma_i}^{-1}(\tilde{y}, \tilde{y}') d\tilde{y} \right\} \\
&\quad \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}' \tilde{x}_2})/r_{\tilde{y}' \tilde{x}_2}] d\tilde{y}', \\
A_{18} &= - \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1 \tilde{y}})/r_{\tilde{x}_1 \tilde{y}}] L_{\gamma_i}^{-1}(\tilde{y}, \tilde{y}') d\tilde{y} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}'\tilde{x}_2})/r_{\tilde{y}'\tilde{x}_2}] d\tilde{y}', \\
A_{19} = & - \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \frac{\partial}{\partial n_{i\tilde{y}}} [\exp(-sr_{\tilde{x}_1\tilde{y}})/r_{\tilde{x}_1\tilde{y}}] L_{\gamma_i^{-1}}^*(y, y') d\tilde{y} \right\} \\
& \times \left[\frac{\partial}{\partial n_{i\tilde{y}'}} + \gamma_i \right] [\exp(-sr_{\tilde{y}'\tilde{x}_2})/r_{\tilde{y}'\tilde{x}_2}] d\tilde{y}', \\
A_{20} = & - \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} \int_{S_i^*} \left\{ \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} \int_{S_i^*} [\exp(-sr_{\tilde{x}_1\tilde{y}})/r_{\tilde{x}_1\tilde{y}}] L_{\gamma_i}^*(y, y') d\tilde{y} \right\} \\
& \times \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}'\tilde{x}_2})/r_{\tilde{y}'\tilde{x}_2}] d\tilde{y}',
\end{aligned}$$

where, we deduce also that

$$\begin{aligned}
M_{\gamma_i}(y, y') &= \sum_{\nu=0}^{\infty} K_{\gamma_i}^{(\nu)}(y, y'), \\
K_{\gamma_i}^{(0)}(y, y') &= \left[\frac{\partial}{\partial n_{i\tilde{y}}} + \gamma_i \right] [\exp(-sr_{\tilde{y}'\tilde{y}})/r_{\tilde{y}'\tilde{y}}]/2\pi, \\
M_{\gamma_i^{-1}}(y, y') &= \sum_{\nu=0}^{\infty} (-1)^{\nu} K_{\gamma_i^{-1}}^{(\nu)}(y, y'), \\
K_{\gamma_i^{-1}}^{(0)}(y, y') &= \left[\frac{\partial}{\partial n_{i\tilde{y}'}} + \gamma_i^{-1} \frac{\partial^2}{\partial n_{i\tilde{y}} \partial n_{i\tilde{y}'}} \right] [\exp(-sr_{\tilde{y}'\tilde{y}})/r_{\tilde{y}'\tilde{y}}]/2\pi, \\
L_{\gamma_i}(y, y') &= \sum_{\nu=0}^{\infty} (-1)^{\nu} K_{\gamma_i}^{(\nu)}(y, y'), \\
L_{\gamma_i^{-1}}(y, y') &= \sum_{\nu=0}^{\infty} K_{\gamma_i^{-1}}^{(\nu)}(y, y'), \\
M_{\gamma_i}^*(y, y') &= \sum_{\nu=0}^{\infty} (-1)^{\nu} {}^*K_{\gamma_i}^{(\nu)}(y, y'), \\
{}^*K_{\gamma_i}^{(0)}(y, y') &= \left[\frac{\partial^2}{\partial n_{i\tilde{y}} \partial n_{i\tilde{y}'}} + \gamma_i \frac{\partial}{\partial n_{i\tilde{y}}} \right] [\exp(-sr_{\tilde{y}'\tilde{y}})/r_{\tilde{y}'\tilde{y}}]/2\pi, \\
M_{\gamma_i^{-1}}^*(y, y') &= \sum_{\nu=0}^{\infty} {}^*K_{\gamma_i^{-1}}^{(\nu)}(y, y'),
\end{aligned}$$

$$\begin{aligned}
{}^*K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}') &= \left[1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\underline{y}}} \right] [\exp(-sr_{\underline{y}'\underline{y}})/r_{\underline{y}'\underline{y}}]/2\pi, \\
L_{\gamma_i}^*(\underline{y}, \underline{y}') &= \sum_{\nu=0}^{\infty} {}^*K_{\gamma_i}^{(\nu)}(\underline{y}, \underline{y}'), \\
L_{\gamma_i}^*(\underline{y}, \underline{y}') &= \sum_{\nu=0}^{\infty} (-1)^{\nu} {}^*K_{\gamma_i}^{(\nu)}(\underline{y}, \underline{y}').
\end{aligned}$$

In these formulae, we note, for example, that $K_{\gamma_i}^{(\nu)}(\underline{y}, \underline{y}')$ are the iterates of $K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}')$.

On the basis of (3.12), the function $\overline{\kappa}(\underline{x}_1, \underline{x}_2; s^2)$ will be estimated for $s \rightarrow \infty$ together with small/large impedances γ_i ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, N, N+1, \dots, n$). The case when $\underline{x}_1$ and $\underline{x}_2$ lie in the neighborhood of S_i^* is particularly interesting. For this case we use the local expansions of the functions:

$$[\exp(-sr_{\underline{x}\underline{y}})/r_{\underline{x}\underline{y}}], \frac{\partial}{\partial n_{i\underline{y}}} [\exp(-sr_{\underline{x}\underline{y}})/r_{\underline{x}\underline{y}}], \quad (3.13)$$

when the distance between $\underline{x}$ and $\underline{y}$ is small, which are very similar to those obtained in Secs. 4 and 5 of [13]. Consequently, for small/large impedances γ_i the local behaviour of the kernels:

$$K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}'); {}^*K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}'), \quad (3.14)$$

$$K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}'); {}^*K_{\gamma_i}^{(0)}(\underline{y}, \underline{y}'), \quad (3.15)$$

when the distance between $\underline{y}$ and $\underline{y}'$ is small, follows directly from knowledge of the local expansions of (3.13).

DEFINITION 3.1. If $\underline{\xi}_1$ and $\underline{\xi}_2$ are points in the upper half-plane $\xi^3 > 0$ of the (ξ^1, ξ^2, ξ^3) -space, then we define $\rho_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2 + (\xi_1^3 + \xi_2^3)^2}$.

An $e^\lambda(\underline{\xi}_1, \underline{\xi}_2; s)$ -function is defined for points $\underline{\xi}_1$ and $\underline{\xi}_2$ belonging to sufficiently small domains $\mathcal{D}(I_i)$ ($i = 1 + k_{j-1}, \dots, k_j$), ($j = 1, \dots, N, N+1, \dots, n$) except when $\underline{\xi}_1 = \underline{\xi}_2 \in I_i$, where λ is called the degree of this function. For every positive integer Λ , it has the following local expansions (see [10, 13, 17]):

$$\begin{aligned}
e^\lambda(\underline{\xi}_1, \underline{\xi}_2; s) &= \sum^* f(\xi_1^1, \xi_1^2)(\xi_1^3)^{P_1}(\xi_2^3)^{P_2} \left(\frac{\partial}{\partial \xi_1^1} \right)^{l_1} \left(\frac{\partial}{\partial \xi_1^2} \right)^{l_2} \left(\frac{\partial}{\partial \xi_1^3} \right)^{l_3} \frac{\exp(-s\rho_{12})}{\rho_{12}} \\
&\quad + R^\Lambda(\underline{\xi}_1, \underline{\xi}_2; s),
\end{aligned} \quad (3.16)$$

where $\sum^*$ denotes a sum of a finite number of terms in which $f(\xi_1^1, \xi_1^2)$ is an infinitely differentiable function. In this expansion P_1, P_2, l_1, l_2 , and l_3 are integers, where $P_1 \geq 0, P_2 \geq 0, l_1 \geq 0, l_2 \geq 0, \lambda = \min(P_1 + P_2 - q), q = l_1 + l_2 + l_3$ and the minimum is

taken over all terms which occur in the summation $\sum^*$. The remainder $R^\wedge(\xi_{\tilde{1}}, \xi_{\tilde{2}}; s)$ has continuous derivatives of all orders $d \leq \wedge$ satisfying

$$D^d R^\wedge \left(\xi_{\tilde{1}}, \xi_{\tilde{2}}; s \right) = O[s^{-\wedge} \exp(-As\rho_{12})] \text{ as } s \rightarrow \infty, \quad (3.17)$$

where A is a positive constant.

Thus, using methods similar to those obtained in Secs. 6-10 of [13], we can show that the functions (3.13) are e^λ -functions with degrees $\lambda = -1, -2$ respectively. Consequently, for small impedances γ_i , the functions (3.14) are e^λ -functions with degrees $\lambda = 0, -1$, while for large impedances γ_i , the functions (3.15) are e^λ -functions with degrees $\lambda = 0, 1$, respectively.

DEFINITION 3.2. If $\tilde{x}_1$ and $\tilde{x}_2$ are points in large domains $\Omega + S_i^*$, then we define

$$\begin{aligned} r_i &= \min_y (r_{\tilde{x}_1 \tilde{y}} + r_{\tilde{x}_2 \tilde{y}}) \quad \text{if } \tilde{y} \in S_i^* \quad (i = 1 + k_{j-1}, \dots, m_j, \quad j = 1, \dots, N), \\ R_i &= \min_y (r_{\tilde{x}_1 \tilde{y}} + r_{\tilde{x}_2 \tilde{y}}) \quad \text{if } \tilde{y} \in S_i^* \quad (i = m_j + 1, \dots, k_j, \quad j = 1, \dots, N), \\ r_i^* &= \min_y (r_{\tilde{x}_1 \tilde{y}} + r_{\tilde{x}_2 \tilde{y}}) \quad \text{if } \tilde{y} \in S_i^* \quad (i = 1 + k_{j-1}, \dots, m_j, \quad j = N + 1, \dots, n), \\ R_i^* &= \min_y (r_{\tilde{x}_1 \tilde{y}} + r_{\tilde{x}_2 \tilde{y}}) \quad \text{if } \tilde{y} \in S_i^* \quad (i = m_j + 1, \dots, k_j, \quad j = N + 1, \dots, n). \end{aligned}$$

An $E^\lambda(\tilde{x}_1, \tilde{x}_2; s^2)$ -function is defined and infinitely differentiable with respect to $\tilde{x}_1$ and $\tilde{x}_2$ when these points belong to large domains $\Omega + S_i^*$ except when $\tilde{x}_1 = \tilde{x}_2 \in S_i^*$. Thus, the E^λ -function has a similar local expansion of the e^λ -function (see [10, 13, 17]).

With the help of Secs. 8 and 9 in [13], it is easily seen that (3.12) is an $E^{-2}(\tilde{x}_1, \tilde{x}_2; s^2)$ -function and consequently we get

$$\begin{aligned} \bar{G}(\tilde{x}_1, \tilde{x}_2; s^2) &= \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{m_j} O\{r_i^{-2} \exp(-A_i s r_i)\} \\ &+ \sum_{j=1}^N \sum_{i=m_j+1}^{k_j} O\{R_i^{-2} \exp(-A_i s R_i)\} \\ &+ \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{m_j} O\{r_i^{*-2} \exp(-A_i s r_i^*)\} \\ &+ \sum_{j=N+1}^n \sum_{i=m_j+1}^{k_j} O\{R_i^{*-2} \exp(-A_i s R_i^*)\}, \end{aligned} \quad (3.18)$$

which is valid for $s \rightarrow \infty$ and for small/large impedances γ_i , where A_i are positive constants. The estimate (3.18) shows that $\bar{G}(\tilde{x}_1, \tilde{x}_2; s^2)$ is exponentially small for s -large. This proves that the integral (3.8) converges for $s \rightarrow \infty$.

Following Sec. 10 in [13], if the ϵ^λ -expansions of the functions (3.13)–(3.15) are introduced into (3.12) and if we use formulae similar to (6.4) and (6.9) of Sec. 6 in [13], we obtain the following local expansion of $\overline{\mathfrak{x}}(\tilde{x}_1, \tilde{x}_2; s^2)$ when r_i, R_i, r_i^* and R_i^* are small, which is valid for $s \rightarrow \infty$ and for small/large impedances γ_i :

$$\overline{\mathfrak{x}}(\tilde{x}_1, \tilde{x}_2; s^2) = \sum_{j=1}^n \sum_{i=1+k_{j-1}}^{k_j} \overline{\mathfrak{x}}_i(\tilde{x}_1, \tilde{x}_2; s^2), \quad (3.19)$$

where

(a) if $\tilde{x}_1$ and $\tilde{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_i)$, ($i = 1 + k_{j-1}, \dots, m_j$, $j = 1, \dots, N$), then

$$\overline{\mathfrak{x}}_i(\tilde{x}_1, \tilde{x}_2; s^2) = -\frac{1}{8\pi} \left\{ 1 - \gamma_i \left(\frac{\partial}{\partial \xi_1^3} \right)^{-1} \right\} \frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s \rho_{12})\}. \quad (3.20)$$

(b) if $\tilde{x}_1$ and $\tilde{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_i)$, ($i = m_j + 1, \dots, k_j$, $j = 1, \dots, N$), then

$$\overline{\mathfrak{x}}_i(\tilde{x}_1, \tilde{x}_2; s^2) = \frac{1}{8\pi} \left\{ 1 - \gamma_i^{-1} \left(\frac{\partial}{\partial \xi_1^3} \right) \right\} \frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s \rho_{12})\}. \quad (3.21)$$

(c) if $\tilde{x}_1$ and $\tilde{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_i)$, ($i = 1 + k_{j-1}, \dots, m_j$, $j = N + 1, \dots, n$), then

$$\overline{\mathfrak{x}}_i(\tilde{x}_1, \tilde{x}_2; s^2) = \frac{1}{8\pi} \left\{ 1 - \gamma_i \left(\frac{\partial}{\partial \xi_1^3} \right)^{-1} \right\} \frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s \rho_{12})\}. \quad (3.22)$$

(d) if $\tilde{x}_1$ and $\tilde{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_i)$, ($i = m_j + 1, \dots, k_j$, $j = N + 1, \dots, n$), then

$$\overline{\mathfrak{x}}_i(\tilde{x}_1, \tilde{x}_2; s^2) = -\frac{1}{8\pi} \left\{ 1 - \gamma_i^{-1} \left(\frac{\partial}{\partial \xi_1^3} \right) \right\} \frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s \rho_{12})\}. \quad (3.23)$$

When $r_i \geq \delta_i > 0$ ($i = 1 + k_{j-1}, \dots, m_j$, $j = 1, \dots, N$), $R_i \geq \delta_i > 0$ ($i = m_j + 1, \dots, k_j$, $j = 1, \dots, N$), $r_i^* \geq \delta_i > 0$ ($i = 1 + k_{j-1}, \dots, m_j$, $j = N + 1, \dots, n$), $R_i^* \geq \delta_i > 0$ ($i = m_j + 1, \dots, k_j$, $j = N + 1, \dots, n$), the function $\overline{\mathfrak{x}}(\tilde{x}_1, \tilde{x}_2; s^2)$ is of order $O\{\exp(-Bs)\}$ as $s \rightarrow \infty$ where B is a positive constant. Thus, since

$$\lim_{r_i \rightarrow 0} \frac{r_i}{\rho_{12}} = \lim_{R_i \rightarrow 0} \frac{R_i}{\rho_{12}} = \lim_{r_i^* \rightarrow 0} \frac{r_i^*}{\rho_{12}} = \lim_{R_i^* \rightarrow 0} \frac{R_i^*}{\rho_{12}} = 1, \quad (3.24)$$

(see [10, 13, 17]), then we have the asymptotic formulae (3.20)–(3.23) with ρ_{12} in the small domains $\mathfrak{D}(I_i)$ are replaced by r_i, R_i, r_i^* and R_i^* in the large domains $\Omega + S_i^*$ respectively.

Since for $\xi^3 \geq h_i > 0$, the functions $\overline{\mathfrak{x}}_i(\tilde{x}, \tilde{x}; s^2)$ are of orders $O\{\exp(-2sA_i h_i)\}$, the integral over the region Ω of the function $\overline{\mathfrak{x}}(\tilde{x}, \tilde{x}; s^2)$ can be approximated in the following

way (see (3.10)):

$$\begin{aligned} \bar{R}(s^2) = & \sum_{j=N+1}^n \sum_{i=1+k_{j-1}}^{k_j} \int_{S_i^*} \int_{\xi^3=0}^{h_i} \bar{\pi}_i(\underline{x}, \underline{x}; s^2) \{1 - 2\xi^3 H_j + (\xi^3)^2 N_j\} d\xi^3 dS_i^* \\ & - \sum_{j=1}^N \sum_{i=1+k_{j-1}}^{k_j} \int_{S_i^*} \int_{\xi^3=0}^{h_i} \bar{\pi}_i(\underline{x}, \underline{x}; s^2) \{1 + 2\xi^3 H_j + (\xi^3)^2 N_j\} d\xi^3 dS_i^* \quad (3.25) \\ & + \sum_{j=1}^n \sum_{i=1+k_{j-1}}^{k_j} O\{\exp(-2sA_i h_i)\} \text{ as } s \rightarrow \infty. \end{aligned}$$

If the e^λ -expansions of $\bar{\pi}_i(\underline{x}, \underline{x}; s^2)$ are introduced into (3.25), and with the help of formula (10.2) of Sec. 10 in Zayed [17] (see also [13]), we deduce after inverting Laplace transforms and using (3.6) that our result (2.1) has been constructed and the proof of Theorem 2.1 is completed. $\square$

4. Applications of $\theta(t)$ to an ideal gas. In statistical mechanics, the classical ideal gas is, by definition, a set of non-interacting particles enclosed in a box, where the wave function ϕ is exactly zero outside the box (Dirichlet boundary condition), see [5]. Thus, in the case of Neumann boundary condition we have a very special ideal gas, which is a set of non-interacting particles enclosed in a box, where the rate of change of the wave function ϕ in the direction of the inner normal of the box is exactly zero outside the box. Similarly, in the case of the Robin boundary conditions, we have a very special ideal gas, which is a set of non-interacting particles enclosed in a box, where the rate of change of the wave function ϕ in the direction of the inner normal of the box is proportional to the jump in the wave function ϕ from the exterior to the interior of the box (see [23]).

Following [5], we are interested in examining how the thermodynamic properties of an ideal gas are influenced by the geometry of the multiply-connected bounded domain with the Robin boundary condition (1.9). Thermodynamic properties of an ideal gas can be extracted from the partition function

$$Z(\beta) = \frac{z^r(\beta)}{r!}, \quad (4.1)$$

where “ r ” is the number of particles and $z(\beta)$ is given by

$$z(\beta) = \sum_n \exp(-\beta E_n), \quad (4.2)$$

with $\beta = (K_B T)^{-1}$, K_B being the Boltzmann constant, and T being the absolute temperature. The energy level (eigenvalues) E_n of one particle are obtained from the stationary states $\Psi = \phi \exp(-iEt/\hbar)$ of the time-dependent Schrödinger equation

$$-\frac{\hbar^2}{2M} \nabla^2 \Psi + V(\underline{x}) \Psi = i\hbar \frac{\partial \Psi}{\partial t}, \quad (4.3)$$

with $V(\underline{x}) = 0$, where M is the mass and $\hbar$ is the Planck constant. Thus ϕ satisfies the Helmholtz equation (1.8) with $\mu = 2ME/\hbar^2$. Therefore, we deduce that the asymptotic

expansion of the sum (1.4) of $\theta(t)$ as $t \rightarrow 0$ which formally is the same as the one-particle partition function (4.2) of $z(\beta)$ as $\beta \rightarrow 0$.

The purpose of this section is to investigate the influence of the multiply-connected container Ω on the thermodynamic quantities of an ideal gas with the Robin boundary condition. This is the problem of a set of non-interacting particles which obeys the Schrödinger equation (4.3) with the Robin boundary conditions (1.9) and where the thermodynamic quantities can be extracted from (4.1). By extension, we called this system a very special ideal gas. We note that such physical system could be similar to the Wigner-Seitz method used in electronic structure calculation in solid state theory (see [1], [2]). The intersecting point is that one could obtain the properties of an ideal gas with Dirichlet boundary condition taking suitably the limit $\gamma \rightarrow \infty$ (see [5]).

Following the discussions of Secs. 1 and 2, we can obtain information about the shape of the multiply-connected bounded container Ω by studying the asymptotic expansions of the sum (4.2) when $\beta \rightarrow 0$ (i.e. $T \rightarrow \infty$, the ideal gas case). Noting that the eigenvalue problem of the Schrödinger equation is the same as the eigenvalue problem of the wave equation, we can use the asymptotic expansion (2.1) of $\theta(t)$ of Sec. 2, replacing t by $(\frac{\hbar^2}{2M})\beta$. Let us now consider the general partition function (4.1). Using the result (2.1) with $\beta \rightarrow 0$, equation (4.2) gives

$$z(\beta) = \left(\frac{2M}{\hbar^2}\right)^{\frac{3}{2}} \frac{a_1}{\beta^{\frac{3}{2}}} + \left(\frac{2M}{\hbar^2}\right) \frac{a_2}{\beta} + \left(\frac{2M}{\hbar^2}\right)^{\frac{1}{2}} \frac{a_3}{\beta^{\frac{1}{2}}} + a_4 + \left(\frac{\hbar^2}{2M}\right)^{\frac{1}{2}} a_5 \beta^{\frac{1}{2}} + O(\beta) \text{ as } \beta \rightarrow 0. \quad (4.4)$$

We apply this formula to thermodynamic quantities such as the internal energy $\mathcal{U} = -[\frac{\partial}{\partial \beta} \ln Z(\beta)]_{V,r}$, the pressure $P = \beta^{-1}[\frac{\partial}{\partial V} \ln Z(\beta)]_{T,r}$, and the specific heat $C = (\frac{\partial \mathcal{U}}{\partial T})_{V,r}$, among others (see [5]). In the case of the internal energy, we obtain

$$\mathcal{U} = -r \frac{\partial}{\partial \beta} \ln \left\{ \left(\frac{2M}{\hbar^2}\right)^{\frac{3}{2}} \frac{a_1}{\beta^{\frac{3}{2}}} + \left(\frac{2M}{\hbar^2}\right) \frac{a_2}{\beta} + \left(\frac{2M}{\hbar^2}\right)^{\frac{1}{2}} \frac{a_3}{\beta^{\frac{1}{2}}} + a_4 + \left(\frac{\hbar^2}{2M}\right)^{\frac{1}{2}} a_5 \beta^{\frac{1}{2}} + O(\beta) \right\}. \quad (4.5)$$

Now differentiating, expanding in powers of $\beta = (K_B T)^{-1}$, and using the definition of the thermal wave length $\Lambda(T) = (\frac{2\pi\hbar^2}{MK_B T})^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \mathcal{U}(T) = & \frac{3}{2} r K_B T \left\{ 1 - \frac{a_2}{6a_1\sqrt{\pi}} \Lambda(T) \right. \\ & + \frac{1}{12\pi} \left[\left(\frac{a_2}{a_1}\right)^2 - \frac{2a_3}{a_1} \right] \Lambda^2(T) \\ & - \frac{1}{24\pi^{\frac{3}{2}}} \left[\left(\frac{a_2}{a_1}\right)^3 - \frac{3a_2a_3}{a_1^2} + \frac{3a_4}{a_1} \right] \Lambda^3(T) \\ & \left. + \frac{1}{48\pi^2} \left[\left(\frac{a_2}{a_1}\right)^4 + \frac{4a_2a_4}{a_1^2} - \frac{4a_2^2a_3}{a_1^3} + 2\left(\frac{a_3}{a_1}\right)^2 - \frac{4a_5}{a_1} \right] \Lambda^4(T) + O(\Lambda^5(T)) \right\} \\ & \text{as } T \rightarrow \infty. \end{aligned} \quad (4.6)$$

Similar expansions hold for the pressure and the specific heat.

Gutierrez and Yanez [5] have recently constructed a formula similar to our formula (4.6) but for a simply connected bounded domain with Dirichlet boundary conditions using the well-known formula (1.6).

An examination of the result (4.6) shows that the internal energy $\mathcal{U}(T)$ is obtained as an expansion in powers of the thermal wave length, whose coefficients depend on some geometric properties of the multiply-connected bounded domain $\Omega \subseteq R^3$ with the Robin boundary conditions (1.9). Thus, in principle, an ideal gas could feel some aspects of the shape of its container Ω because its thermodynamic quantities depend on some geometric properties of Ω such as the volume, the surface area, the mean curvature, and the Gaussian curvature. But according to Gordon *et al.* [3], we note that the ideal gas, even with all terms in the expansion of the partition function completely known, is not able to discriminate between two different shapes. So, with reference to articles [3] and [11], there are domains where, although different in shape, the thermodynamic quantities of an ideal gas will be exactly the same independent of the order of the approximation in (4.6). In this sense, an ideal gas cannot feel the shape of its container, although it can feel some geometrical properties of it. From these discussions, we deduce that the answer to either Kac's question [7] or Gutierrez and Yanez's question in [5] is negative.

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REFERENCES

- [1] R. Balian and C. Bloch, *Distribution of eigenfrequencies for the wave equation in a finite domain. I. Three-dimensional problem with smooth boundary surface*, Annals. Phys. **60**, 401–447 (1970)
- [2] H. P. Baltes and E. R. Hilf, *Spectra of finite system*, B. I., Wissenschafts verlag, Mannheim, 1976
- [3] C. Gordon, D. L. Webb and S. Wolpert, *One cannot hear the shape of a drum*, Bull. Amer. Math. Soc. **27**, 134–138 (1992)
- [4] H. P. W. Gottlieb, *Eigenvalues of the Laplacian with Neumann boundary conditions*, J. Aust. Math. Soc. Ser. B **26**, 293–309 (1985)
- [5] G. Gutierrez and J. M. Yanez, *Can an ideal gas feel the shape of its container?* Amer. J. Phys. **65**, 739–743 (1997)
- [6] P. Hsu, *On the θ -function of a Riemannian manifold with boundary*, Trans. Amer. Math. Soc. **333**, 643–671 (1992)
- [7] M. Kac, *Can one hear the shape of a drum?* Amer. Math. Month. **73**, 1–23 (1966)
- [8] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. **1**, 43–69 (1967)
- [9] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci., U.S.A. **51**, 542 (1964)
- [10] A. Pleijel, *On Green's functions and the eigenvalue distribution of the three-dimensional membrane equation*, Skandinav. Mat. Konger **12**, 222–240 (1954)
- [11] S. Sridhar and A. Kudrolli, *Experiments on not hearing the shape of drums*, Phys. Rev. Lett. **72**, 2175–2178 (1994)
- [12] R. T. Waechter, *On hearing the shape of a drum: An extension to higher dimensions*, Proc. Camb. Philos. Soc. **72**, 439–447 (1972)
- [13] E. M. E. Zayed, *An inverse eigenvalue problem for a general convex domain: An extension to higher dimensions*, J. Math. Anal. Appl. **112**, 455–470 (1985)
- [14] E. M. E. Zayed, *Hearing the shape of a general doubly-connected domain in R^3 with impedance boundary conditions*, J. Math. Phys. **31**, 2361–2365 (1990)

- [15] E. M. E. Zayed, *Heat equation for a general convex domain in R^3 with a finite number of piecewise impedance boundary conditions*, Appl. Anal. **42**, 209–220 (1991)
- [16] E. M. E. Zayed, *Hearing the shape of a general convex domain: An extension to a higher dimensions*, Portugal. Math. **48**, 259–280 (1991)
- [17] E. M. E. Zayed, *Hearing the shape of a general doubly-connected domain in R^3 with mixed boundary conditions*, J. Appl. Math. Phys (ZAMP) **42**, 547–564 (1991)
- [18] E. M. E. Zayed, *An inverse eigenvalue problem for an arbitrary multiply-connected bounded domain in R^3 with impedance boundary conditions*, SIAM J. Appl. Math. **52**, 725–729 (1992)
- [19] E. M. E. Zayed, *An inverse problem for a general doubly-connected bounded domain: An extension to higher dimensions*, Tamkang. J. Math. **28**, 277–295 (1997)
- [20] E. M. E. Zayed, *An inverse problem for a general doubly-connected bounded domain in R^3 with a finite number of piecewise impedance boundary conditions*, Appl. Anal. **64**, 69–98 (1997)
- [21] E. M. E. Zayed, *Short-time asymptotics of the heat kernel of the Laplacian of a bounded domain with Robin boundary conditions*, Houston. J. Math. **24**, 377–385 (1998)
- [22] E. M. E. Zayed, *An inverse problem for a general multiply-connected bounded domain: An extension to higher dimensions*, Appl. Anal. **72**, 27–41 (1999)
- [23] E. M. E. Zayed, *On hearing the shape of a bounded domain with Robin boundary conditions*, IMAJ. Appl. Math. **64**, 95–108 (2000)
- [24] E. M. E. Zayed, *The asymptotics of the heat semigroup for a general bounded domain with mixed boundary conditions*, Acta. Math. Sinica (English Series) 16, No. 4, 627–636 (2000).
- [25] E. M. E. Zayed, *Short-time asymptotics of the heat kernel of the Laplacian for a multiply-connected domain in R^2 with Robin boundary conditions*, Appl. Anal. **77**, 177–194 (2001).