REMARK ON INDIRECT MATCHING OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

BY

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Abstract. We examine four singularly perturbed boundary value problems. We show that it is possible to simplify the standard matching procedure by studying the boundary layer equations with the gauge function η as a new independent variable.

Introduction. The present paper is inspired by Ciarlet's work (see [2]) on plate equations which can be derived from the balance laws of nonlinear elasticity. The derivations themselves are based on the asymptotic expansion of the true displacement field defined over a solid layer whose thickness ε tends to zero. The difficulty which one encounters there is that the outer expansion of the elastic displacements, in general, fails to satisfy the original boundary conditions beyond the first approximation. Thus one is forced to examine the boundary layer corrections which are described by difficult equations of continuum mechanics.

In case of linearized elasticity the problem of finding the boundary layer corrections was examined and solved in [3], [4]. The solution is based on the ingenious blend of techniques combining multiscale expansions with the advanced theory of linear partial differential equations. However, it seems that the matching procedure based on the detailed knowledge of the boundary layer solutions may become very difficult in the case of the nonlinear elasticity which yields von Karman's plate models. Moreover, when studying the linear case one already wonders whether it is possible to recover the formal simplicity of Ciarlet's original arguments when dealing with the boundary layer corrections of the plate or the beam models (see, for example, [8]). Thus a question appears: How to check that the outer and the boundary layer expansions match without solving the boundary layer equations?

We attempt to examine this question in the case of four one-dimensional, standard boundary value problems. In the first two examples we pretend that we cannot solve the boundary layer equations and we search for conditions which would guarantee the matching of the known outer expansion with its unknown boundary layer counterpart.

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The key device which we use is the transformation of the original boundary layer equations in the local variable x into a chain of singular, irregular, differential equations with the gauge function η as the new independent variable. Having done so, we search for solutions of the new equations in a particular analytical form inspired by the structure of the outer expansion. As the result of our computations we are able to recover from the algebraic manipulations the common part of the inner and the outer expansion of the original boundary value problem. Once that common part is known, we show how to change the boundary layer equations into a sequence of the standard boundary value problems posed over a semi-infinite interval. Consequently we reduce the question of matching to the familiar existence problem for the derived sequence of the boundary value problems.

As we venture further into harder equations, we encounter a shock and transition layer problem and a problem whose outer and inner solution contains logarithmic terms in the small parameter ε . In a case of the shock layer we show that the modified matching procedure yields additional information which is lost in the classical solution of this problem. In a case of the logarithmic scales we show that their appearance can be explained by the necessity of lifting the formal calculations from the ring of analytic functions in ε to the ring of expansions generated by $\langle 1, \varepsilon, \varepsilon \ln(\varepsilon) \rangle$. Consequently the familiar expansions which appear in Van Dyke's book [15] have well-defined algebraic properties which have gone unnoticed in a number of standard books on asymptotic expansions.

The first two problems are taken from Holmes's book [7]. The first one illustrates our idea. The second one shows that our idea works for a nonlinear boundary value problem. We supplement its formal solution with an elementary proof of convergence. The third problem is the Cole-Lagerstrom shock and transition layer problem. It is taken from Cole's and Kevorkian's book [9]. Here we test our method against Cole's analysis. We think that we have a better grip on his shock solution. We also supplement Cole's discussion of the transition layer by new details which seem to provide a closure to his original ideas. The last problem, Lagerstrom's model of the Stokes-Oseen equation, comes from Hinch's book [6]. We investigate how our technique works in the case of logarithmic scales. We show that, paradoxically, this problem is, in a sense, a mirror image of the first two problems: We recover the common part of the outer and the inner expansion from the known outer expansion. Then we replace the standard matching procedure by an analytical compatibility condition imposed on the boundary layer expansions. As the result of our modification we are able to write down the uniform, composite expansion which is missing from [6].

Finally, we would like to clarify our terminology: When we speak about "exponentially small terms", we mean that the terms in question are indeed exponentially small. When we speak about "transcendentally small terms", we mean that the precise analytical form of those terms is not known to us except the fact that the terms under consideration are small and are not fractional power series in ε (see [16]).

1. Model Problem. To explain the main idea of the paper we consider a boundary value problem for the unknown function $u(z, \varepsilon)$,

$$\varepsilon \frac{d^2 u}{dz^2} + \frac{du}{dz} + u = 0, \quad u(0, \varepsilon) = 0, \quad u(1, \varepsilon) = 1, \quad (1.1)$$

where ε is a small positive parameter. Our goal is to construct an explicit sequence of approximations which converges uniformly to $u(z, \varepsilon)$ as $\varepsilon \rightarrow 0$.

Using the standard approach (see [7]), we expand u into a series

$$u(z, \varepsilon) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) \dots, \quad (1.2)$$

and we combine expansion (1.2) with Eq. (1.1). Simple manipulations yield a sequence of equations for the consecutive terms in expansion (1.2),

$$\frac{du_0}{dz} + u_0 = 0, \quad \frac{du_j}{dz} + u_j = -\frac{d^2 u_{j-1}}{dz^2}, \quad j = 1, 2, \dots \quad (1.3)$$

It turns out that series (1.2) cannot satisfy the boundary condition at $z = 0$ lest all the terms u_j will vanish. Therefore expansion (1.2), when written in the explicit form

$$u(z, \varepsilon) = \left[1 + \varepsilon[1 - z] + \varepsilon^2 \left[-\frac{3}{2} + 3z - \frac{1}{2}z^2 \right] \right] \exp(1 - z) + \dots, \quad (1.4)$$

cannot be valid for small z since $u(0, \varepsilon) \neq 0$.

To correct approximation (1.4) near $z = 0$, we introduce the scaled variable $x = \varepsilon^{-1}z$ and write Eq. (1.1) in the form

$$\frac{d^2 w}{dx^2} + \frac{dw}{dx} + \varepsilon w = 0, \quad w(x, \varepsilon) = u(\varepsilon x, \varepsilon). \quad (1.5)$$

Again, assuming that

$$w(x, \varepsilon) = w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \dots, \quad (1.6)$$

we obtain a new sequence of equations

$$\frac{d^2 w_0}{dx^2} + \frac{dw_0}{dx} = 0, \quad \frac{d^2 w_j}{dx^2} + \frac{dw_j}{dx} = -w_{j-1}, \quad j = 1, 2, \dots, \quad (1.7)$$

where all $w_j(x)$'s vanish at $x = 0$. In contrast to (1.5), all $w_j(x)$'s are defined now for $x \in [0, \infty)$. Elementary computations show that

$$\begin{aligned} w_0 &= B_0 [1 - \exp(-x)], \\ w_1 &= B_1 - B_0 x - [B_1 + B_0 x] \exp(-x), \\ w_2 &= B_2 - [B_1 + B_0]x + \frac{1}{2}B_0 x^2 - \left[B_2 + [B_1 + B_0]x + \frac{1}{2}B_0 x^2 \right] \exp(-x). \end{aligned} \quad (1.8)$$

To find the constants B_j , we introduce an intermediate variable ξ ,

$$z = \bar{\eta}(\varepsilon)\xi, \quad x = \eta(\varepsilon)^{-1}\xi, \quad \eta\bar{\eta} = \varepsilon, \quad (1.9)$$

where $\bar{\eta}(\varepsilon), \eta(\varepsilon)$ are the gauge functions,

$$\bar{\eta}(\varepsilon) = \varepsilon^\alpha, \quad \eta(\varepsilon) = \varepsilon^{1-\alpha}, \quad 0 < \alpha < 1. \quad (1.10)$$

Then we expand w and u with respect to ε keeping ξ constant. The matching principle says that if u and w represent the same function on a common interval near $z = 0$, then

the expansion for w , modulo exponentially small terms, must agree with the expansion for u for any choice of α . Thereby,

$$w = B_0 + \varepsilon \left[B_1 - B_0 \frac{\xi}{\eta} \right] + \varepsilon^2 \left[B_2 - [B_1 + B_2] \frac{\xi}{\eta} + \frac{1}{2} B_0 \frac{\xi^2}{\eta^2} \right] + \dots \\ + [\text{exponentially small terms}] \quad (1.11)$$

must be identical with u ,

$$u = e \left[1 + \varepsilon + \varepsilon^2 \left[-\frac{3}{2} \right] + \bar{\eta} [-\xi] + \varepsilon \bar{\eta} [-2\xi] + \bar{\eta}^2 \left[\frac{1}{2} \xi^2 \right] \right] \\ + [\text{higher order terms}]. \quad (1.12)$$

It is obvious that for $\bar{\eta} = \eta^{-1}\varepsilon$, both expansions agree providing that

$$B_0 = e, \quad B_1 = e, \quad B_2 = -\frac{3}{2}e. \quad (1.13)$$

Therefore the approximate solution of Eq. (1.1) has the following representation,

$$u(z, \varepsilon) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) \\ + t_0(\varepsilon^{-1}z) + \varepsilon t_1(\varepsilon^{-1}z) + \varepsilon^2 t_2(\varepsilon^{-1}z) + \dots, \quad (1.14)$$

where the exponentially small terms t_j 's are defined through the relations,

$$w_0(x) = e + t_0(x), \\ w_1(x) = e - ex + t_1(x), \\ w_2(x) = -\frac{3}{2}e - 2ex + \frac{1}{2}ex^2 + t_2(x). \quad (1.15)$$

In explicit form

$$t_0(x) = -\exp(1-x), \\ t_1(x) = -[1+x]\exp(1-x), \\ t_2(x) = -\left[-\frac{3}{2} + 2x + \frac{1}{2}x^2\right]\exp(1-x). \quad (1.16)$$

At this point the formal solution of Eq. (1.1) is essentially finished: We can indefinitely improve expansion (1.14) provided we are willing to compute consecutive u_j 's and w_j 's, and provided we are prepared to continue matching using relations (1.9).

We now pose a question: Can we check the matching conditions without solving the boundary layer equations (1.7)? First of all, to answer this question we observe that the matching procedure is independent of any particular form of the functions $t_j(x)$ —it is only the difference $w_j(x) - t_j(x)$ which matters—as long as we know that t_j 's are exponentially small when expressed in terms of $\xi\eta^{-1}$. Secondly, if we are to take advantage of the smallness of t_j 's, then we must abandon x as independent variable in Equations (1.16). A natural candidate for a new variable is the gauge function η . In fact, the very procedure of matching is carried out by fixing ξ and varying η . Thus it is natural to set

$$p_j(\eta) = w_j(\eta^{-1}\xi), \quad \xi = \text{constant}, \quad (1.17)$$

and to write Equations (1.7) in a new form,

$$\begin{aligned}\frac{d}{d\eta} \left[\eta^2 \frac{dp_0}{d\eta} \right] - \xi \frac{dp_0}{d\eta} &= 0, \\ \frac{d}{d\eta} \left[\eta^2 \frac{dp_1}{d\eta} \right] - \xi \frac{dp_1}{d\eta} &= -\xi^2 \eta^{-2} p_0, \\ \frac{d}{d\eta} \left[\eta^2 \frac{dp_2}{d\eta} \right] - \xi \frac{dp_2}{d\eta} &= -\xi^2 \eta^{-2} p_1.\end{aligned}\quad (1.18)$$

We notice that $\varepsilon^j p_j(\eta(\varepsilon))$'s must remain finite as $\varepsilon \rightarrow 0$ for all α 's if p_j 's are to be matched with u . Consequently, inspired by Eq. (1.12), we try to solve Equations (1.18) using the Laurent expansions,

$$\begin{aligned}p_0(\eta) &= \sum_{k \geq 0} A_k^0 \eta^k, \\ p_1(\eta) &= A_{-1}^1 \eta^{-1} + \sum_{k \geq 0} A_k^1 \eta^k, \\ p_2(\eta) &= A_{-2}^2 \eta^{-2} + A_{-1}^2 \eta^{-1} + \sum_{k \geq 0} A_k^2 \eta^k,\end{aligned}\quad (1.19)$$

where all A_k^j 's are functions of ξ alone. It is a matter of simple computations to show that p_j 's are in fact polynomials in $\xi \eta^{-1}$,

$$\begin{aligned}p_0(\eta) &= A_0^0, \\ p_1(\eta) &= [-A_0^0] \xi \eta^{-1} + A_0^1, \\ p_2(\eta) &= \left[\frac{1}{2} A_0^0 \right] \xi^2 \eta^{-2} + [-A_0^0 - A_0^1] \xi \eta^{-1} + A_0^2.\end{aligned}\quad (1.20)$$

Thus, if we set

$$p(\eta) = p_0(\eta) + \varepsilon p_1(\eta) + \varepsilon^2 p_2(\eta) + \dots, \quad \eta = \bar{\eta}^{-1} \varepsilon, \quad (1.21)$$

then $p(\eta)$ becomes identical with the expansion (1.12) providing that

$$A_0^0 = e, \quad A_0^1 = e, \quad A_0^2 = -\frac{3}{2}e. \quad (1.22)$$

Therefore, $p(\eta)$ represents the common part of $u(z, \varepsilon)$ and $w(x, \varepsilon)$ and $p(\eta)$ appears as a rational solution of the boundary layer equations expressed in terms of the gauge function η . We notice that those equations are singular and irregular at $\eta = 0$ (see [1]). Therefore it is natural that their rational solutions have a simple structure.

To finish our computations, we set

$$t_j(x) = w_j(x) - p_j(x), \quad x = \eta^{-1} \xi, \quad j = 0, 1, 2, \dots, \quad (1.23)$$

and we use Equations (1.20), (1.22) to recover Equations (1.16). Since functions $t_j(x)$ must vanish at $x = \infty$ and functions $w_j(x)$ vanish at $x = 0$ we can replace differential equations (1.7) with the missing boundary conditions at $x = \infty$ by a sequence of the

standard boundary value problems for the unknown $t_j(x)$'s,

$$\begin{aligned}\frac{d^2 t_0}{d^2 x} + \frac{dt_0}{dx} &= 0, \quad t_0(0) = -e, \quad t_0(\infty) = 0, \\ \frac{d^2 t_1}{d^2 x} + \frac{dt_1}{dx} &= -t_0, \quad t_1(0) = -e, \quad t_1(\infty) = 0, \\ \frac{d^2 t_2}{d^2 x} + \frac{dt_2}{dx} &= -t_1, \quad t_2(0) = \frac{3}{2}e, \quad t_2(\infty) = 0.\end{aligned}\tag{1.24}$$

We notice that at this point we may or may not wish to solve Equations (1.24), in order to know the precise form of the outer expansion and the precise conditions of matching.

2. Nonlinear Problem. To test our method further, we consider a nonlinear problem studied by Holmes (see [7]),

$$\varepsilon^2 \frac{d^2 u}{dz^2} + \varepsilon^2 \frac{du}{dz} - \exp(u) = -[2 + z], \quad u(0, \varepsilon) = 0, \quad u(1, \varepsilon) = 1.\tag{2.1}$$

The outer expansion,

$$u(z, \varepsilon) = u_0(z) + \varepsilon^2 u_1(z) + \varepsilon^4 u_2(z) + \dots,\tag{2.2}$$

yields the following chain of equations,

$$\begin{aligned}\exp(u_0) &= 2 + z, \\ \frac{d^2 u_0}{dz^2} + \frac{du_0}{dz} - \exp(u_0) u_1 &= 0, \\ \frac{d^2 u_1}{dz^2} + \frac{du_1}{dz} - \exp(u_0) \left[\frac{1}{2} u_1^2 + u_2 \right] &= 0.\end{aligned}\tag{2.3}$$

Therefore

$$\begin{aligned}u &\simeq \ln(2 + z) \\ &+ \varepsilon^2 \left[[z + 2]^{-2} - [z + 2]^{-3} \right] \\ &+ \varepsilon^4 \left[-\frac{3}{2} [z + 2]^{-4} + 10 [z + 2]^{-5} - \frac{25}{2} [z + 2]^{-6} \right] + \dots.\end{aligned}\tag{2.4}$$

The above expansion fails to satisfy either boundary condition, suggesting that the boundary layers occur at $z = 0$ and $z = 1$.

To examine the boundary layer at $z = 0$, we set $z = \varepsilon x$ and we write Eq. (2.1) in terms of x ,

$$\frac{d^2 w}{dx^2} + \varepsilon \frac{dw}{dx} - \exp(w) = -[2 + \varepsilon x], \quad w(0) = 0.\tag{2.5}$$

Next, we assume that w can be represented by a series,

$$w(x, \varepsilon) = w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \dots,\tag{2.6}$$

and using Eq. (2.5) we obtain a sequence of the boundary layer equations,

$$\begin{aligned}\frac{d^2 w_0}{d^2 x} - \exp(w_0) &= -2, \quad w_0(0) = 0, \\ \frac{d^2 w_1}{d^2 x} - \exp(w_0) w_1 &= -\frac{dw_0}{dx} - x, \quad w_1(0) = 0, \\ \frac{d^2 w_2}{d^2 x} - \exp(w_0) w_2 &= \frac{1}{2} \exp(w_0) w_1^2 - \frac{dw_1}{dx}, \quad w_2(0) = 0,\end{aligned}\quad (2.7)$$

We assume that functions $w_j(x)$ are defined over the entire half line $[0, \infty)$. Now, we proceed to check whether expansion (2.6) can be matched with expansion (2.2). We introduce an intermediate variable ξ ,

$$x = \eta^{-1} \xi, \quad z = \bar{\eta} \xi, \quad \bar{\eta} = \varepsilon^\alpha, \quad \eta = \varepsilon^{1-\alpha}, \quad 0 < \alpha < 1, \quad (2.8)$$

and we define p_j 's using Laurent expansions (1.19). Equations (2.7) written in terms of gauge function η have the form

$$\begin{aligned}\eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{dp_0}{d\eta} \right] - \xi^2 \exp(p_0) &= -2\xi^2, \\ \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{dp_1}{d\eta} \right] - \xi^2 \exp(p_0) p_1 &= \xi \eta^2 \frac{dp_0}{d\eta} - \xi^3 \eta^{-1}, \\ \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{dp_2}{d\eta} \right] - \xi^2 \exp(p_0) p_2 &= \xi \eta^2 \frac{dp_1}{d\eta} + \frac{1}{2} \xi^2 \exp(p_0) p_1^2.\end{aligned}\quad (2.9)$$

Elementary, tedious computations show that

$$p_0(\eta) = \ln(2), \quad p_1(\eta) = \frac{1}{2} \xi \eta^{-1}, \quad p_2(\eta) = -\frac{1}{8} \xi^2 \eta^{-2} + \frac{1}{8}. \quad (2.10)$$

According to our hypothesis, the function p ,

$$p(\eta) = \ln(2) + \varepsilon \left[\frac{1}{2} \xi \eta^{-1} \right] + \varepsilon^2 \left[-\frac{1}{8} \xi^2 \eta^{-2} + \frac{1}{8} \right] + \dots, \quad (2.11)$$

represents the common part of u and w when both functions are expanded with respect to ε . To check this we substitute $z = \bar{\eta} \xi$ into Eq. (2.4) and we get

$$u = \ln(2) + \frac{1}{2} \bar{\eta} \xi - \frac{1}{8} \bar{\eta}^2 \xi^2 + \frac{1}{8} \varepsilon^2 + \dots \quad (2.12)$$

Since $\bar{\eta} \eta = \varepsilon$, Equations (2.11), (2.12) are identical and the matching condition does indeed hold true at $z = 0$. Consequently the transcendental parts of w_j 's

$$\begin{aligned}t_j(x) &= w_j(x) - p_j(x), \quad t_j(\infty) = 0, \\ p_0(x) &= \ln(2), \quad p_1(x) = \frac{1}{2} x, \quad p_2(x) = \frac{1}{8} - \frac{1}{8} x^2,\end{aligned}\quad (2.13)$$

satisfy the following chain of the boundary value problems:

$$\begin{aligned}
 \frac{d^2 t_0}{d^2 x} - 2 \exp(t_0) &= -2, \\
 t_0(0) &= -\ln(2), \quad t_0(\infty) = 0, \\
 \frac{d^2 t_1}{d^2 x} - 2 \exp(t_0) t_1 &= -\frac{dt_0}{dx} + x \exp(t_0) - x, \\
 t_1(0) &= 0, \quad t_1(\infty) = 0, \\
 \frac{d^2 t_2}{d^2 x} - 2 \exp(t_0) t_2 &= -\frac{dt_1}{dx} + \exp(t_0) \left[t_1^2 + x t_1 + \frac{1}{4} \right] - \frac{1}{4}, \\
 t_2(0) &= -\frac{1}{8}, \quad t_2(\infty) = 0.
 \end{aligned} \tag{2.14}$$

To check the matching condition at $z = 1$ we repeat the previous computations. We define $\widehat{w}$, $\widehat{x}$,

$$z = 1 + \varepsilon \widehat{x}, \quad \widehat{w}(\widehat{x}, \varepsilon) = u(1 + \varepsilon \widehat{x}, \varepsilon), \quad -\infty < \widehat{x} \leq 0, \tag{2.15}$$

and we introduce the analogue of Eq. (2.5),

$$\frac{d^2 \widehat{w}}{d^2 \widehat{x}} + \varepsilon \frac{d \widehat{w}}{d \widehat{x}} - \exp(\widehat{w}) = -[3 + \varepsilon \widehat{x}], \quad \widehat{w}(0) = 1. \tag{2.16}$$

The expansion for $\widehat{w}$,

$$\widehat{w}(\widehat{x}, \varepsilon) = \widehat{w}_0(\widehat{x}) + \varepsilon \widehat{w}_1(\widehat{x}) + \varepsilon^2 \widehat{w}_2(\widehat{x}) + \dots, \tag{2.17}$$

implies that

$$\begin{aligned}
 \frac{d^2 \widehat{w}_0}{d^2 \widehat{x}} - \exp(\widehat{w}_0) &= -3, \quad \widehat{w}_0(0) = 1, \\
 \frac{d^2 \widehat{w}_1}{d^2 \widehat{x}} - \exp(\widehat{w}_0) \widehat{w}_1 &= -\frac{d \widehat{w}_0}{d \widehat{x}} - \widehat{x}, \quad \widehat{w}_1(0) = 0, \\
 \frac{d^2 \widehat{w}_2}{d^2 \widehat{x}} - \exp(\widehat{w}_0) \widehat{w}_2 &= \frac{1}{2} \exp(\widehat{w}_0) \widehat{w}_1^2 - \frac{d \widehat{w}_1}{d \widehat{x}}, \quad \widehat{w}_2(0) = 0.
 \end{aligned} \tag{2.18}$$

Thus with the intermediate variable $\xi = \eta \widehat{x}$ and with the common part of u and $\widehat{w}$ near $z = 1$ in the form

$$\widehat{p}(\eta, \varepsilon) = \widehat{p}_0(\eta) + \varepsilon \widehat{p}_1(\eta) + \varepsilon^2 \widehat{p}_2(\eta) + \dots, \quad \widehat{p}_j(\eta) = \widehat{w}_j(\xi \eta^{-1}), \tag{2.19}$$

we obtain that

$$\begin{aligned}
 \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{d \widehat{p}_0}{d\eta} \right] - \xi^2 \exp(\widehat{p}_0) &= -3\xi^2, \\
 \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{d \widehat{p}_1}{d\eta} \right] - \xi^2 \exp(\widehat{p}_0) \widehat{p}_1 &= \xi \eta^2 \frac{d \widehat{p}_0}{d\eta} - \xi^3 \eta^{-1}, \\
 \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{d \widehat{p}_2}{d\eta} \right] - \xi^2 \exp(\widehat{p}_0) \widehat{p}_2 &= \xi \eta^2 \frac{d \widehat{p}_1}{d\eta} + \frac{1}{2} \xi^2 \exp(\widehat{p}_0) \widehat{p}_1^2.
 \end{aligned} \tag{2.20}$$

Since the analytical form of all $\widehat{p}_j$'s remains the same (see Equations (1.19)), we obtain

$$\widehat{p}_0(\eta) = \ln(3), \quad \widehat{p}_1(\eta) = \frac{1}{3} \xi \eta^{-1}, \quad \widehat{p}_2(\eta) = \frac{2}{27} - \frac{1}{18} \xi^2 \eta^{-2}. \tag{2.21}$$

Equations (2.21) yield $\hat{p}$ identical with expanded u from Eq. (2.4), providing that

$$z = 1 + \xi\bar{\eta}, \quad \hat{x} = \xi\eta^{-1}, \quad \eta\bar{\eta} = \varepsilon, \quad \bar{\eta} = \varepsilon^\alpha. \quad (2.22)$$

Expressing $\hat{p}_j$'s in terms of $\hat{x}$ and setting $\hat{w}_j(\hat{x}) = \hat{p}_j(\hat{x}) + \hat{t}_j(\hat{x})$, we obtain the analogue of Equations (2.4) for $z = 1$,

$$\begin{aligned} \frac{d^2\hat{t}_0}{d^2\hat{x}} - 3\exp(\hat{t}_0) &= -3, \\ \hat{t}_0(0) &= 1 - \ln(3), \quad \hat{t}_0(-\infty) = 0, \\ \frac{d^2\hat{t}_1}{d^2\hat{x}} - 3\exp(\hat{t}_0)\hat{t}_1 &= \hat{x}\exp(\hat{t}_0) - \frac{d\hat{t}_0}{d\hat{x}} - \hat{x}, \\ \hat{t}_1(0) &= 0, \quad \hat{t}_1(-\infty) = 0, \\ \frac{d^2\hat{t}_2}{d^2\hat{x}} - 3\exp(\hat{t}_0)\hat{t}_2 &= \frac{2}{9}[\exp(\hat{t}_0) - 1] + \exp(\hat{t}_0)[\hat{x}\hat{t}_1 + \frac{3}{2}\hat{t}_1^2] - \frac{d\hat{t}_1}{d\hat{x}}, \\ \hat{t}_2(0) &= -\frac{2}{27}, \quad \hat{t}_2(-\infty) = 0. \end{aligned} \quad (2.23)$$

Consequently the formal, composite, expansion of $u(z, \varepsilon)$ which solves Eq. (2.1) has the form

$$\begin{aligned} u(z, \varepsilon) &= \ln(z + 2) \\ &+ t_0(\varepsilon^{-1}z) + \hat{t}_0(\varepsilon^{-1}[z - 1]) + \varepsilon t_1(\varepsilon^{-1}z) + \varepsilon \hat{t}_1(\varepsilon^{-1}[z - 1]) \\ &+ \varepsilon^2 \left[[z + 2]^{-2} - [z + 2]^{-3} \right] + \dots \end{aligned} \quad (2.24)$$

In order to show that our computational scheme yields a true approximation of $u(z, \varepsilon)$, we consider the reminder R which describes the difference between the exact solution of the boundary value problem (2.1) and its zero order approximation,

$$R(z, \varepsilon) = u(z, \varepsilon) - \ln(z + 2) - t_0(\varepsilon^{-1}z) - \hat{t}_0(\varepsilon^{-1}[z - 1]). \quad (2.25)$$

Equation (2.25) implies that R is a solution of the following boundary value problem,

$$\begin{aligned} \varepsilon^2 \left[\frac{d^2 R}{dz^2} + \frac{dR}{dz} \right] - [z + 2] \exp(t_0 + \hat{t}_0) [\exp(R) - 1] &= G, \\ R(0, \varepsilon) &= -\hat{t}_0(-\varepsilon^{-1}), \quad R(1, \varepsilon) = -t_0(\varepsilon^{-1}), \end{aligned} \quad (2.26)$$

where

$$\begin{aligned} G(z, \varepsilon) &= [z + 2] \exp \left(t_0(\varepsilon^{-1}z) + \hat{t}_0(\varepsilon^{-1}[z - 1]) \right) - \varepsilon^2 \left[[z + 2]^{-2} - [z + 2]^{-1} \right] \\ &- \frac{d^2 t_0}{d^2 x}(\varepsilon^{-1}z) - \frac{d^2 \hat{t}_0}{d^2 \hat{x}}(\varepsilon^{-1}[z - 1]) - \varepsilon \frac{dt_0}{dx}(\varepsilon^{-1}z) - \varepsilon \frac{d\hat{t}_0}{d\hat{x}}(\varepsilon^{-1}[z - 1]). \end{aligned} \quad (2.27)$$

In order to show that G is uniformly small it is convenient to use the identity (see Equations (2.14), (2.23))

$$\begin{aligned}
 & [z+2] \left[\exp \left(t_0 (\varepsilon^{-1} z) + \widehat{t}_0 (\varepsilon^{-1} [z-1]) \right) - 1 \right] \\
 & \quad - \frac{d^2 t_0}{d^2 x} (\varepsilon^{-1} z) - \frac{d^2 \widehat{t}_0}{d^2 \widehat{x}} (\varepsilon^{-1} [z-1]) \\
 & = \frac{1}{2} z \frac{d^2 t_0}{d^2 x} (\varepsilon^{-1} z) \left[1 + \frac{d^2 \widehat{t}_0}{d^2 \widehat{x}} (\varepsilon^{-1} [z-1]) \right] \\
 & \quad + \frac{1}{3} [z-1] \frac{d^2 \widehat{t}_0}{d^2 \widehat{x}} (\varepsilon^{-1} [z-1]) \left[1 - \frac{d^2 t_0}{d^2 x} (\varepsilon^{-1} z) \right]. \tag{2.28}
 \end{aligned}$$

Since $x = \varepsilon^{-1} z$ and $\widehat{x} = \varepsilon^{-1} [z-1]$, identity (2.28) yields G in the following form,

$$G(z, \varepsilon) = \varepsilon \left[T(x, \widehat{x}) + \widehat{T}(x, \widehat{x}) \right] - \varepsilon^2 F(z), \tag{2.29}$$

where the functions $T, \widehat{T}, F$, are independent of ε ,

$$\begin{aligned}
 T(x, \widehat{x}) &= \frac{1}{2} x \frac{d^2 t_0}{d^2 x} (x) \left[1 + \frac{d^2 \widehat{t}_0}{d^2 \widehat{x}} (\widehat{x}) \right] + \frac{dt_0}{dx} (x) \\
 \widehat{T}(x, \widehat{x}) &= \frac{1}{3} \widehat{x} \frac{d^2 \widehat{t}_0}{d^2 \widehat{x}} (\widehat{x}) \left[1 - \frac{d^2 t_0}{d^2 x} (x) \right] + \frac{d\widehat{t}_0}{d\widehat{x}} (\widehat{x}) \\
 F(z) &= [z+2]^{-2} - [z+2]^{-1}. \tag{2.30}
 \end{aligned}$$

Consequently $G(z, \varepsilon)$ is uniformly bounded from below and from above by two constants (see Appendix A),

$$\begin{aligned}
 \underline{G}(\varepsilon) &\leq G(z, \varepsilon) \leq \overline{G}(\varepsilon), \\
 \underline{G}(\varepsilon) &= -\varepsilon m^2 - \frac{1}{4} \varepsilon^2, \quad \overline{G}(\varepsilon) = +\varepsilon M^2 - \frac{2}{9} \varepsilon^2. \tag{2.31}
 \end{aligned}$$

Moreover, the function

$$b(z, \varepsilon) = \exp \left(t_0 (\varepsilon^{-1} z) + \widehat{t}_0 (\varepsilon^{-1} [z-1]) \right), \tag{2.32}$$

is also uniformly bounded by $\underline{b}$ and $\overline{b}$,

$$\underline{b} = \frac{1}{3} e \leq b(z, \varepsilon) \leq 3 = \overline{b}. \tag{2.33}$$

Therefore Eq. (2.26) defines a boundary value problem suitable for the application of the Maximum Principle (see [14], p. 48).

To estimate the reminder R we recast Eq. (2.26) into an abstract form,

$$\begin{aligned}
 N(R) &\equiv \frac{d^2 R}{dz^2} + H \left(\frac{dR}{dz}, R, z, \varepsilon \right) = 0, \\
 R(0, \varepsilon) &= -\widehat{t}_0 (-\varepsilon^{-1}), \quad R(1, \varepsilon) = -t_0 (\varepsilon^{-1}), \\
 H \left(\frac{dR}{dz}, R, z, \varepsilon \right) &= \frac{dR}{dz} - \varepsilon^{-2} b(z, \varepsilon) \left[\exp(R) - 1 \right] - \varepsilon^{-2} G(z, \varepsilon). \tag{2.34}
 \end{aligned}$$

Then we look for the upper and the lower solution of Eq. (2.34), assuming that

$$\overline{R} = \ln(1 + \overline{C}^2(\varepsilon)), \quad \underline{R} = \ln(1 - \underline{C}^2(\varepsilon)), \quad (2.35)$$

where $\overline{C}(\varepsilon)$, $\underline{C}(\varepsilon)$ are constants. It is easy to check (see Appendix A for details) that for

$$\begin{aligned} \overline{C}^2(\varepsilon) &= -\frac{\underline{G}(\varepsilon)}{\underline{b}} = \frac{3}{e} \left[\varepsilon m^2 + \frac{1}{4} \varepsilon^2 \right], \\ N(\overline{R}) &\leq 0 \text{ and } \overline{R}(\varepsilon) \geq R(0, \varepsilon), \quad \overline{R}(\varepsilon) \geq R(1, \varepsilon). \end{aligned} \quad (2.36)$$

Consequently, by the Maximum Principle

$$R(z, \varepsilon) \leq \overline{R}(\varepsilon). \quad (2.37)$$

Similarly, for

$$\begin{aligned} \underline{C}^2(\varepsilon) &= \frac{\overline{G}(\varepsilon)}{\underline{b}} = \frac{3}{e} \left[\varepsilon M^2 - \frac{2}{9} \varepsilon^2 \right], \\ N(\underline{R}) &\geq 0 \text{ and } \underline{R}(\varepsilon) \leq R(0, \varepsilon), \quad \underline{R}(\varepsilon) \leq R(1, \varepsilon). \end{aligned} \quad (2.38)$$

So, again, by the Maximum Principle

$$\underline{R}(\varepsilon) \leq R(z, \varepsilon). \quad (2.39)$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq z \leq 1} |R(z, \varepsilon)| = 0. \quad (2.40)$$

3. Cole-Lagerstrom shock and transition layer. We consider a boundary value problem (see [9], [13]),

$$\varepsilon \frac{d^2 u}{dz^2} + u \frac{du}{dz} - u = 0, \quad u(-1, \varepsilon) = a, \quad u(1, \varepsilon) = b, \quad (3.1)$$

where a, b are prescribed numbers. We search for particular solutions of Eq. (3.1) which possesses an inflection point $z = d(\varepsilon)$ inside the interval $[-1, 1]$,

$$\frac{d^2 u}{dz^2}(d(\varepsilon), \varepsilon) = 0, \quad |d(\varepsilon)| < 1. \quad (3.2)$$

We assume that the function u is analytic in ε ,

$$u(z, \varepsilon) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) + \dots, \quad (3.3)$$

and we use Eq. (3.1) to generate a chain of equations which describes the unknown functions $u_j(z)$,

$$\begin{aligned} u_0 \left[\frac{du_0}{dz} - 1 \right] &= 0, \\ \frac{d^2 u_0}{dz^2} + \frac{d}{dz} [u_0 u_1] - u_1 &= 0, \\ \frac{d^2 u_1}{dz^2} + \frac{d}{dz} [u_0 u_2] - u_2 &= 0 \dots \end{aligned} \quad (3.4)$$

Equations (3.4) describe two alternative expansions for u , none of which can simultaneously satisfy the boundary conditions (3.1),

$$u(z, \varepsilon) = c(\varepsilon) + z, \quad c(\varepsilon) = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 \dots \text{ or } u(z, \varepsilon) \equiv 0. \quad (3.5)$$

In order to augment the outer expansion (3.5), we introduce the boundary layer variable x ,

$$x = \varepsilon^{-1} [z - d(\varepsilon)], \quad (3.6)$$

and we express Eq. (3.1) in terms of x and $w(x, \varepsilon) = u(d(\varepsilon) + \varepsilon x, \varepsilon)$. The boundary layer solution w describes the unknown u in the neighborhood of the inflection point $d(\varepsilon)$,

$$\frac{d^2 w}{dx^2} + w \frac{dw}{dx} - \varepsilon w = 0. \quad (3.7)$$

To find w we search for the solution of Eq. (3.7) in the form

$$w(x, \varepsilon) = w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \dots, \quad (3.8)$$

where the sequence of w_j 's is described by the following equations:

$$\begin{aligned} \frac{d^2 w_0}{dx^2} + w_0 \frac{dw_0}{dx} &= 0, \\ \frac{d^2 w_1}{dx^2} + \frac{d}{dx} [w_0 w_1] &= w_0, \\ \frac{d^2 w_2}{dx^2} + \frac{d}{dx} [w_0 w_2] &= w_1 - w_1 \frac{dw_1}{dx}. \end{aligned} \quad (3.9)$$

We also explicitly assume that all w_j 's are defined on the whole real line.

Since $d(\varepsilon)$ is an inflection point, Equations (3.1), (3.2) imply that

$$u(d(\varepsilon), \varepsilon) = 0 \quad \text{or} \quad \frac{du}{dz}(d(\varepsilon), \varepsilon) = 1. \quad (3.10)$$

The first condition yields a sequence of initial conditions for all w_j 's,

$$w_j(0) = 0, \quad j = 0, 1, 2, \dots \quad (3.11)$$

The second condition yields an alternative sequence,

$$\frac{dw_0}{dx}(0) = 0, \quad \frac{dw_1}{dx}(0) = 1, \quad \frac{dw_2}{dx}(0) = 0, \quad (3.12)$$

whose applicability, however, turns out to be limited: The sequence (3.12) corresponds to already known expansion (3.5) expressed in terms of x and w and therefore can be ignored.

To check whether expansions (3.5), (3.8) match, we define the standard change of variables,

$$x = \eta^{-1} \xi, \quad z = d(\varepsilon) + \bar{\eta} \xi, \quad \eta = \varepsilon^{1-\alpha}, \quad \bar{\eta} \eta = \varepsilon, \quad 0 < \alpha < 1, \quad (3.13)$$

and, as before, we define functions $p_j(\eta)$ using Equations (1.17), (1.19). Equations (3.9) imply that the common part of u and w is described by equations,

$$p(\eta) = p_0(\eta) + \varepsilon p_1(\eta) + \varepsilon^2 p_2(\eta) + \dots, \quad (3.14)$$

$$\begin{aligned} \frac{d}{d\eta} \left[\eta^2 \frac{dp_0}{d\eta} \right] - \xi p_0 \frac{dp_0}{d\eta} &= 0, \\ \frac{d}{d\eta} \left[\eta^2 \frac{dp_1}{d\eta} \right] - \xi \frac{d}{d\eta} [p_0 p_1] &= \frac{\xi^2}{\eta^2} p_0, \\ \frac{d}{d\eta} \left[\eta^2 \frac{dp_2}{d\eta} \right] - \xi \frac{d}{d\eta} [p_0 p_2] &= \frac{\xi^2}{\eta^2} p_1 + \xi p_1 \frac{dp_1}{d\eta}. \end{aligned} \quad (3.15)$$

Equations (3.15) describe two sequences of solutions. The first sequence can be matched with u from Eq. (3.5),

$$p_0(\eta) = a_0(\xi), \quad p_1(\eta) = \frac{\xi}{\eta} + a_1(\xi), \quad p_2(\eta) = a_2(\xi). \quad (3.16)$$

The second sequence, which starts with

$$p_0(\eta) = 2 \frac{\eta}{\xi} \left[1 - a_0 \frac{\xi \eta}{2} \right]^{-1}, \quad (3.17)$$

displays no similarity with the computed u and has to be examined separately, later.

To find the matching conditions we define a pair of outer solutions,

$$\begin{aligned} u^L(z) &= a + 1 + z, \quad u^L(-1) = a \quad \text{if } -1 \leq z < d(\varepsilon), \\ u^R(z) &= b - 1 + z, \quad u^R(+1) = b \quad \text{if } d(\varepsilon) < z \leq +1, \end{aligned} \quad (3.18)$$

we set $z = d(\varepsilon) + \bar{\eta}\xi$ and we try to match $u^L(z), u^R(z)$ with the corresponding pair $p^L(\eta), p^R(\eta)$,

$$\begin{aligned} p^L(\eta) &= a_0^L(\xi) + \varepsilon \left[\frac{\xi}{\eta} + a_0^L(\xi) \right] + \varepsilon^2 a_2^L(\xi) + \dots, \\ p^R(\eta) &= a_0^R(\xi) + \varepsilon \left[\frac{\xi}{\eta} + a_0^R(\xi) \right] + \varepsilon^2 a_2^R(\xi) + \dots. \end{aligned} \quad (3.19)$$

Standard manipulations imply that,

$$\begin{aligned} a_0^L &= a + 1 + z_0, \quad a_1^L = z_1, \quad a_2^L = z_2, \\ a_0^R &= b - 1 + z_0, \quad a_1^R = z_1, \quad a_2^R = z_2, \end{aligned} \quad (3.20)$$

where, necessarily,

$$d(\varepsilon) = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \dots. \quad (3.21)$$

According to our assumptions,

$$\begin{aligned} w_0(x) &= p_0^L(x) + t_0^L(x) = a_0^L + t_0^L(x) \quad \text{if } x \leq 0, \\ w_0(x) &= p_0^R(x) + t_0^R(x) = a_0^R + t_0^R(x) \quad \text{if } x \geq 0. \end{aligned} \quad (3.22)$$

Moreover, since the functions $t_0^L(x), t_0^R(x)$ are exponentially small the following limits must vanish,

$$\lim_{x \rightarrow -\infty} t_0^L(x) = 0, \quad \lim_{x \rightarrow +\infty} t_0^R(x) = 0. \quad (3.23)$$

Thus to determine a_0^L and a_0^R we have to solve the initial value problem,

$$\frac{d^2 w_0}{dx^2} + w_0 \frac{dw_0}{dx} = 0, \quad w_0(0) = 0, \quad (3.24)$$

subject to the side conditions,

$$\lim_{x \rightarrow -\infty} w_0(x) = a_0^L, \quad \lim_{x \rightarrow +\infty} w_0(x) = a_0^R. \quad (3.25)$$

Equation (3.24) implies that w_0 is described by hyperbolic tangent, known up to an arbitrary constant C_0 ,

$$w_0(x) = C_0 \operatorname{th} \left(\frac{C_0}{2} x \right). \quad (3.26)$$

Moreover, since w_0 is an even function of C_0 ,

$$C_0 \operatorname{th} \left(\frac{C_0}{2} x \right) = -C_0 \operatorname{th} \left(\frac{-C_0}{2} x \right), \quad (3.27)$$

we can assume that C_0 is positive. Consequently conditions (3.25) hold true for

$$a_0^L = -C_0, \quad a_0^R = +C_0. \quad (3.28)$$

Comparing Equations (3.28) with Equations (3.20) we obtain,

$$z_0 = -\frac{(a+b)}{2}, \quad C_0 = \frac{(b-a)}{2} - 1 > 0, \quad |z_0| < 1. \quad (3.29)$$

We notice that Equations (3.26), (3.29) correspond precisely to Cole's formulas which can be found in [9].

To compute z_1 , w_1 , we write again

$$\begin{aligned} w_1(x) &= p_1^L(x) + t_1^L(x) = z_1 + x + t_1^L(x) \text{ if } x \geq 0, \\ w_1(x) &= p_1^R(x) + t_1^R(x) = z_1 + x + t_1^R(x) \text{ if } x \leq 0, \\ \lim_{x \rightarrow -\infty} t_1^L(x) &= 0, \quad \lim_{x \rightarrow +\infty} t_1^R(x) = 0. \end{aligned} \quad (3.30)$$

We consider the initial value problem for w_1 ,

$$\frac{d^2 w_1}{dx^2} + \frac{d}{dx} [w_0 w_1] = w_0, \quad w_1(0) = 0. \quad (3.31)$$

Its general solution

$$\begin{aligned} w_1(x) &= \exp(-f_0(x)) \int_0^x ds [C_1 + f_0(s)] \exp(-f_0(s)), \\ f_0(x) &= \int_0^x ds w_0(s), \end{aligned} \quad (3.32)$$

defines an odd function of x and thus conditions (3.30) hold true providing that $z_1 = 0$ and $C_1 = C_0(4 \ln(2) + 2)$, (see Appendix B for details).

Finally we repeat the computations for z_2, w_2 ,

$$\begin{aligned} w_2(x) &= p_2^L(x) + t_2^L(x) = z_2 + t_2^L(x) \text{ if } x \geq 0, \\ w_2(x) &= p_2^R(x) + t_2^R(x) = z_2 + t_2^R(x) \text{ if } x \leq 0, \\ \lim_{x \rightarrow -\infty} t_2^L(x) &= 0, \quad \lim_{x \rightarrow +\infty} t_2^R(x) = 0. \end{aligned} \quad (3.33)$$

Since w_2 is a solution of the initial value problem

$$\frac{d^2 w_2}{dx^2} + \frac{d}{dx}[w_0 w_2] = w_1 - w_1 \frac{dw_1}{dx}, \quad w_2(0) = 0, \quad (3.34)$$

the formula

$$\begin{aligned} w_2(x) &= \exp(-f_0(x)) \int_0^x ds [C_2 + f_1(s)] \exp(-f_0(s)), \\ f_1(x) &= \int_0^x ds w_1(s) \left[1 - \frac{dw_1}{ds}(s) \right], \end{aligned} \quad (3.35)$$

describes an odd function of x which vanishes at ∞ for $C_2 = -f_1(\infty)$. Moreover, z_2 , like z_1 , must be equal to zero (see Appendix B).

In fact it is not difficult to show that all z_j 's must vanish as soon as $j \geq 1$. Consequently the composite approximation of u has the following form,

$$u(z, \varepsilon) \simeq \begin{cases} u^L(z) + \sum_{j=0}^2 \varepsilon^j t_j^L(\varepsilon^{-1}[z - z_0]) & \text{if } -1 \leq z \leq z_0, \\ u^R(z) + \sum_{j=0}^2 \varepsilon^j t_j^R(\varepsilon^{-1}[z - z_0]) & \text{if } z_0 \leq z \leq +1, \end{cases} \quad (3.36)$$

z_0 being the exact location of the inflection point $d(\varepsilon)$.

Now, we return to formula (3.17) which cannot be matched with expansion (3.5). Following Cole, we introduce the transformation

$$z = d(\varepsilon) + \varepsilon^{\frac{1}{2}} y, \quad u = \varepsilon^{\frac{1}{2}} v, \quad |d(\varepsilon)| \leq 1, \quad (3.37)$$

which yields a new form of Eq. (3.1),

$$\frac{d^2 v}{dy^2} + v \frac{dv}{dy} - v = 0, \quad \frac{1}{\sqrt{\varepsilon}} [-1 - d(\varepsilon)] \leq y \leq \frac{1}{\sqrt{\varepsilon}} [+1 - d(\varepsilon)]. \quad (3.38)$$

Since $y = \varepsilon^{\frac{1}{2}} x$ we expect v to match w , providing that v itself can be matched with u . To check those matching conditions we notice that Eq. (3.38) is independent of ε except its interval of definition. Thus if the limit

$$\lim_{\varepsilon \rightarrow 0} v(y, \varepsilon) = v_0(y) \quad (3.39)$$

exists, then $v_0(y)$ should be defined on the interval

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} [-1 - d(\varepsilon)] \leq y \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} [+1 - d(\varepsilon)], \quad (3.40)$$

whose bounds must be identical with those for x . Consequently if x does not vary from $-\infty$ to $+\infty$, then y must vary either from 0 to $+\infty$ or from $-\infty$ to 0. The first option yields $d(\varepsilon) = -1$, the second one $d(\varepsilon) = +1$. Both options are mirror images of each other in the sense that every u that solves Eq. (3.1) defines $\hat{u}(z) = -u(-z)$, $\hat{u}(-1) = \hat{a} = -b$,

$\widehat{u}(+1) = \widehat{b} = -a$, which also is a solution of (3.1). Thereby we set $d(\varepsilon) = -1$ and we define y and v as follows,

$$z = -1 + \varepsilon^{\frac{1}{2}}y, \quad y = \varepsilon^{\frac{1}{2}}x, \quad w = \varepsilon^{\frac{1}{2}}v. \quad (3.41)$$

Equation (3.38) provides no hint as to the asymptotic structure of its solution except for relation (3.39). But, if we assume that u and v can be matched using Van Dyke matching principle then, for small y 's, the following relation should hold:

$$\lim_{\varepsilon \rightarrow 0} \left[\varepsilon^{-\frac{1}{2}} w \left(\varepsilon^{-\frac{1}{2}} y, \varepsilon \right) - v(y, \varepsilon) \right] = 0. \quad (3.42)$$

Consequently, in view of condition (3.39), $v_0(y)$ can be computed using the limit

$$\lim_{\varepsilon \rightarrow 0} \left[\varepsilon^{-\frac{1}{2}} w \left(\varepsilon^{-\frac{1}{2}} y, \varepsilon \right) \right] = v_0(y). \quad (3.43)$$

Since $w(x, \varepsilon)$ has a prescribed dependence on ε (see Eq. (3.8)), the following limit must exist:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} \left[w_0 \left(\varepsilon^{-\frac{1}{2}} y \right) + \varepsilon w_1 \left(\varepsilon^{-\frac{1}{2}} y \right) + \dots \right]. \quad (3.44)$$

The first term in the formula (3.44) is given by Eq. (3.17) written in terms of x ,

$$w_0(x) = \frac{2}{\chi}, \quad \chi = x + a_0, \quad x = \frac{\xi}{\eta}. \quad (3.45)$$

The next terms can be calculated using Equations (3.9),

$$w_1(x) = \frac{b_0}{\chi^2} + b_1\chi + \frac{2}{3}\chi \ln(\chi), \quad w_2(x) = \dots. \quad (3.46)$$

Unfortunately, limit (3.44) blows up as $\varepsilon \rightarrow 0$, namely,

$$\varepsilon^{-\frac{1}{2}} w \left(\varepsilon^{-\frac{1}{2}} y \right) \sim \frac{2}{y} + b_0 + \frac{2}{3} \ln(y) - \frac{2}{3} y \ln(\varepsilon), \quad (3.47)$$

which implies that we are off by a term $\varepsilon \ln(\varepsilon)x$ in expansion (3.8). The missing term cannot be accounted for by a power series expansion in ε . However, we can construct a new ring $\mathfrak{R}_1(\varepsilon)$, the smallest ring generated by $\langle 1, \varepsilon, \varepsilon \ln(\varepsilon) \rangle$, which contains all expansions of the form

$$f(\varepsilon) = \sum_{n \geq 0} P_n(\varepsilon) \varepsilon^n, \quad P_n(\varepsilon) = \sum_{j=0}^n p_{nj} [\ln(\varepsilon)]^j. \quad (3.48)$$

$\mathfrak{R}_1(\varepsilon)$ also contains, as a subring, $\mathfrak{R}_0(\varepsilon)$, the old ring of all formal power series in ε . Moreover, the linear order in $\mathfrak{R}_0(\varepsilon)$, $1 \succ \varepsilon \succ \varepsilon^2 \succ \dots$, extends to $\mathfrak{R}_1(\varepsilon)$ since $1 \succ \varepsilon \ln(\varepsilon) \succ \varepsilon \succ \varepsilon^2 \ln(\varepsilon)^2 \succ \varepsilon^2 \ln(\varepsilon) \succ \varepsilon^2 \succ \dots$. Hence we can replace expansion (3.8) in $\mathfrak{R}_0(\varepsilon)$ by an augmented expansion in $\mathfrak{R}_1(\varepsilon)$, whose terms can still be computed using Eq. (3.7),

$$\begin{aligned} w(x, \varepsilon) = & w_{00}(x) + [w_{11}(x) \ln(\varepsilon) + w_{10}(x)]\varepsilon \\ & + [w_{22}(x) \ln(\varepsilon)^2 + w_{21}(x) \ln(\varepsilon) + w_{20}(x)]\varepsilon^2 + \dots. \end{aligned} \quad (3.49)$$

Tedious, simple computations yield

$$\begin{aligned} w_{00} &= \frac{2}{\chi} (\text{old } w_0), \quad w_{11} = \frac{d_1}{\chi^2} + d_0 \chi, \\ w_{10} &= \frac{b_0}{\chi^2} + b_1 \chi + \frac{2}{3} \chi \ln(\chi) (\text{old } w_1), \quad \chi = x + a_0, \end{aligned} \quad (3.50)$$

with w_{22}, w_{21}, w_{20} , being similar in character but more elaborate. Computing limit (3.44) with the new w_{jk} 's, we get

$$v_0(y) = \frac{2}{y} + \frac{2}{3} y \ln(y) + b_0 y + \left[c_2 \ln(y)^2 + c_1 \ln(y) + c_0 \right] y^2 + \dots, \quad (3.51)$$

provided that $d_0 = 1/3$ and

$$c_0 = -\frac{113}{15^2 10} + \frac{19}{150} b_0 - \frac{1}{10} b_0^2, \quad c_1 = \frac{19}{15^2} - \frac{2}{15} b_0, \quad c_2 = -\frac{2}{45}. \quad (3.52)$$

Careful computations show that Eq. (3.51) holds true for any choice of b_0 . Thereby the matching and the boundary condition, $w(x, \varepsilon) = a$, $a > 0$, do not determine the value of b_0 . In fact we shall see that expansion (3.51) depends on the global structure of the boundary value problem (3.1).

To match v and u we search for expanded form of v valid for all y 's. It is not difficult to show that for small y 's

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} \left[\varepsilon^{-\frac{1}{2}} w \left(\varepsilon^{-\frac{1}{2}} y \right) - v_0(y) \right] = v_1(y), \quad (3.53)$$

where

$$v_1(y) = a_0 \left[-\frac{2}{y^2} + \frac{2}{3} \ln(y) + \frac{2}{3} + b_0 + \dots \right]. \quad (3.54)$$

Consequently we extend relation (3.53) to all y 's and we write,

$$v(y, \varepsilon) = v_0(y) + \varepsilon^{\frac{1}{2}} v_1(y) + \dots \quad (3.55)$$

Expansion (3.55) and Eq. (3.38) generate equations which describe the boundary layer behavior of u ,

$$\begin{aligned} \frac{d^2 v_0}{d^2 y} + v_0 \frac{dv_0}{dy} - v_0 &= 0, \\ \frac{d^2 v_1}{d^2 y} + \frac{d}{dy} [v_0 v_1] - v_1 &= 0. \end{aligned} \quad (3.56)$$

In order to check whether u can indeed be matched with v , we set

$$\begin{aligned} z &= -1 + \bar{\eta} \xi, \quad y = \frac{\xi}{\eta}, \quad \eta = \left(\varepsilon^{\frac{1}{2}} \right)^{1-\alpha}, \quad \eta \bar{\eta} = \varepsilon^{\frac{1}{2}}, \quad 0 < \alpha < 1, \\ p_j(\eta) &= v_j \left(\frac{\xi}{\eta} \right), \end{aligned} \quad (3.57)$$

and we apply the Chain Rule to Equations (3.56),

$$\begin{aligned} \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{dp_0}{d\eta} \right] - \xi \eta^2 p_0 \frac{dp_0}{d\eta} - \xi^2 p_0 &= 0, \\ \eta^2 \frac{d}{d\eta} \left[\eta^2 \frac{dp_1}{d\eta} \right] - \xi \eta^2 \frac{d}{d\eta} [p_0 p_1] - \xi^2 p_1 &= 0. \end{aligned} \quad (3.58)$$

FIG. 3.1

Since $u = \varepsilon^{\frac{1}{2}} v$, the common part of u and $\varepsilon^{\frac{1}{2}} v$, $\varepsilon^{\frac{1}{2}} p = \varepsilon^{\frac{1}{2}} p_0 + \varepsilon p_1$, remains finite for $\varepsilon = 0$ if

$$p_0(\eta) = \frac{A_{-1}}{\eta} + \sum_{j=0}^{\infty} A_j \eta^j, \quad p_1(\eta) = \frac{B_{-2}}{\eta^2} + \frac{B_{-1}}{\eta} + \sum_{j=0}^{\infty} B_j \eta^j. \quad (3.59)$$

Elementary but lengthy computations yield two pairs of p 's,

$$p_0(\eta) = 0, \quad p_1(\eta) = 0 \quad \text{or} \quad p_0(\eta) = \frac{\xi}{\eta} + A_0, \quad p_1(\eta) = B_0. \quad (3.60)$$

The second pair of p 's represents the scaled, exact solution of Eq. (3.1) and should be rejected (see Appendix C). Hence the only solution u which can be matched with v is the zero solution of Eq. (3.1). Since the common part of u and v is zero, v_0 and v_1 are identical with their corresponding, transcendental parts t_0 and t_1 and, thus, both must vanish at ∞ ,

$$v_0(y) = t_0(y), \quad \lim_{y \rightarrow \infty} v_0(y) = 0 \quad \text{and} \quad v_1(y) = t_1(y), \quad \lim_{y \rightarrow \infty} v_1(y) = 0. \quad (3.61)$$

Therefore $v_0(y)$ that connects w and $u=0$ has to be a solution of the following boundary value problem,

$$\begin{aligned} \frac{d^2 v_0}{dy^2} + v_0 \frac{dv_0}{dy} - v_0 &= 0, \quad v_0(\infty) = 0, \\ \lim_{y \rightarrow 0} \left[v_0(y) - \left[\frac{2}{y} + \frac{2}{3} y \ln(y) + b_0 y + \left[c_2 \ln(y)^2 + c_1 \ln(y) + c_0 \right] y^2 \dots \right] \right] &= 0. \end{aligned} \quad (3.62)$$

In order to find v_0 which solves Eq. (3.62), we examine its first integral,

$$q_0 + \ln(|q_0 - 1|) + \frac{v_0^2}{2} = E, \quad q_0 = \frac{dv_0}{dy}. \quad (3.63)$$

Since we search for v_0 which starts at $v_0(0) = \infty$, $q_0(0) = -\infty$ and ends at $v_0(\infty) = 0$, $q_0(\infty) = 0$, the constant E is equal to 0. The corresponding trajectory of Eq. (3.62) is

shown on Fig. (3.1). Since the analytical form of v_0 , for small y 's, is known, we substitute Eq. (3.51) into Eq. (3.63). We set $y = 0$ and we obtain the value of the constant b_0 which we could not determine at the beginning of the computations,

$$b_0 = -\frac{2}{9} - \frac{1}{3} \ln(2). \quad (3.64)$$

With expansion (3.51) complete, we write the composite approximation of u ,

$$\begin{aligned} u(z, \varepsilon) &\simeq w_0(x) + \varepsilon^{\frac{1}{2}} \left[v_0(y) - \left\{ \text{common part of } v_0 \text{ and } \varepsilon^{-\frac{1}{2}} w_0 \right\} \right] \\ &= \frac{2}{x + 2a^{-1}} + \varepsilon^{\frac{1}{2}} \left[v_0(y) - \frac{2}{y} \right], \\ u(-1) &= a, \quad u(1) = 0. \end{aligned} \quad (3.65)$$

Expansions (3.36), (3.65) describe two fragments of Cole's solution of Eq. (3.1) (see [9]). Our analysis, although similar to Cole's, adds new elements to his approach. First of all, we define the location of the shock layer $d(\varepsilon)$ prior to any computations. The formula for $d(\varepsilon)$ itself is a consequence of the indirect matching which yields additional information otherwise lost in the traditional solution of Eq. (3.1). Consequently the higher order approximations of the shock profile are easier to compute within our scheme. Secondly, our technique survives a modification when a, b become analytic functions of ε . Thirdly, computations related to the transition layer have a logical structure that is different from Cole's. Contrary to Cole's claim, we show that, in a sense, solution $v(y, \varepsilon)$ is contained in $w(x, \varepsilon)$ for small y 's. Thus we are able to deduce the global form of ε expansion of v within the asymptotic scheme itself. Next, using the device of indirect matching we compute the boundary condition for v_0 at ∞ without analyzing the phase plane portrait of Eq. (3.62). As the result of all those changes, we are able to write down the composite expansion (3.65) which does not appear in [9].

To close this section we would like to point out that expansion (3.48) is a particular example of a transseries discussed by Ecalle in [5]. It seems that algebraic structure of $\mathfrak{R}_1$, which appears in many textbooks on asymptotic analysis, was not noticed by a number of authors.

4. Model of Stokes-Oseen Problem. We consider a boundary value problem posed over a semi-infinite interval $[1, \infty)$ (see [6]),

$$\begin{aligned} \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} + \varepsilon f \frac{df}{dr} &= 0, \\ f(1, \varepsilon) &= 0, \quad f(\infty, \varepsilon) = 1. \end{aligned} \quad (4.1)$$

The first two terms of the above equation correspond to a radially symmetric Laplacian in $\mathbf{E}^3$. Inspired by its fundamental solution we change the variables, $z = r^{-1}$, $u(z, \varepsilon) = f(z^{-1}, \varepsilon)$, and we transform Eq. (4.1) into a new boundary value problem posed over a unit interval,

$$\begin{aligned} z^2 \frac{d^2 u}{dz^2} - \varepsilon u \frac{du}{dz} &= 0, \\ u(0, \varepsilon) &= 1, \quad u(1, \varepsilon) = 0. \end{aligned} \quad (4.2)$$

To find its outer expansion we assume that $u(z, \varepsilon)$ can be expanded into a power series in ε ,

$$u(z, \varepsilon) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) + \dots \quad (4.3)$$

and we substitute expansion (4.3) into Eq. (4.2),

$$z^2 \frac{d^2 u_0}{dz^2} = 0, \quad z^2 \frac{d^2 u_1}{dz^2} - \varepsilon u_0 \frac{du_0}{dz} = 0, \quad \dots \quad (4.4)$$

Elementary integration together with the boundary condition at $z = 1$ yields consecutive terms in expansion (4.3),

$$u_0 = 1 - z, \quad u_1 = [1 + z] \ln(z) + C [1 + z], \quad \dots \quad (4.5)$$

Evidently, expansion (4.3) fails to satisfy the left boundary condition. Thus we try to augment the outer expansion by its boundary layer correction at $z = 0$. We define the boundary layer variable $x, z = \varepsilon x, w(x, \varepsilon) = u(\varepsilon x, \varepsilon)$ and we write Eq. (4.2) in terms of w and x ,

$$x^2 \frac{d^2 w}{dx^2} - w \frac{dw}{dx} = 0, \quad w(0, \varepsilon) = 0. \quad (4.6)$$

Before we analyze w , we notice that if expansion (4.3) is indeed true and the boundary scaling is indeed correct then the common part of u and w must be a part of the outer expansion for u ,

$$u \simeq u_0(\bar{\eta}\xi) + \varepsilon u_2(\bar{\eta}\xi) + \varepsilon^2 u_2(\bar{\eta}\xi) + \dots, \quad z = \bar{\eta}\xi, \quad x = \eta^{-1}\xi, \quad z = \varepsilon x. \quad (4.7)$$

However if we set $\bar{\eta} = \varepsilon\eta^{-1}$ in Eq. (4.7) and we reorder it with respect to ε then the new expansion does not produce any transcendentally small terms; it still contains all of the original terms modulo their rearrangement,

$$u\left(\varepsilon \frac{\xi}{\eta}, \varepsilon\right) \simeq 1 + \varepsilon \ln(\varepsilon) + \varepsilon \left[C - \frac{\xi}{\eta} + \ln\left(\frac{\xi}{\eta}\right) \right] + [\text{higher order terms}]. \quad (4.8)$$

Consequently none of the terms present in Eq. (4.8) contributes to a term which could be considered transcendentally small with respect to ε . Thus expansion (4.8) must be equal to the common part of u and w . But expansion (4.8) contains logarithmic term $\varepsilon \ln(\varepsilon)$ which is not an element of $\mathfrak{R}_0$, the ring of all formal power series in ε . Therefore assumption (4.3) cannot hold true and in order to avoid inconsistency we must assume that the outer expansion of u belongs at least to $\mathfrak{R}_1$, the smallest ring generated by $\langle 1, \varepsilon, \varepsilon \ln(\varepsilon) \rangle$, that is

$$\begin{aligned} u(z, \varepsilon) &\simeq u_{00}(z) + [u_{11}(z) \ln(\varepsilon) + u_{10}(z)] \varepsilon \\ &+ [u_{22}(z) \ln^2(\varepsilon) + u_{21}(z) \ln(\varepsilon) + u_{20}(z)] \varepsilon^2 + \dots \end{aligned} \quad (4.9)$$

Thus Equations (4.2), (4.9) together with the boundary condition at $z = 1$, yield a new sequence of u_{jk} 's,

$$\begin{aligned} u_{00}(z) &= 1 - z, \\ u_{11}(z) &= A_1(1 - z), \quad u_{10}(z) = [1 + z] \ln(z) + A_2[1 - z], \\ u_{22}(z) &= A_3[1 - z], \quad u_{21}(z) = 2A_1[1 + z] \ln(z) + A_4[1 - z], \\ u_{20}(z) &= -z \ln^2(z) + [2A_2 + 1] z \ln(z) + 2A_2 \ln(z) + \frac{1}{2} \left[\frac{1}{z} - z \right] + A_5[1 - z]. \end{aligned} \quad (4.10)$$

We repeat now the previous argument: If the boundary layer scaling is correct then expansion (4.9) written in terms of $z = \varepsilon \xi \eta^{-1}$ and ordered with respect to ε contains all of its original terms, none of which is transcendentally small. Thereby expansion (4.9) with $z = \varepsilon \xi \eta^{-1}$ must be the common part of u and w expressed in terms of the intermediate variable ξ . Tedious computations show that

$$\begin{aligned} u(z, \varepsilon) &\simeq u_{00} \left(\varepsilon \frac{\xi}{\eta} \right) + \left[u_{11} \left(\varepsilon \frac{\xi}{\eta} \right) \ln(\varepsilon) + u_{10} \left(\varepsilon \frac{\xi}{\eta} \right) \right] \varepsilon + \dots \\ &= p \left(\frac{\xi}{\eta}, \varepsilon \right) \\ &= p_{00} \left(\frac{\xi}{\eta} \right) + \left[p_{11} \left(\frac{\xi}{\eta} \right) \ln(\varepsilon) + p_{10} \left(\frac{\xi}{\eta} \right) \right] \varepsilon \\ &+ \left[p_{22} \left(\frac{\xi}{\eta} \right) \ln^2(\varepsilon) + p_{21} \left(\frac{\xi}{\eta} \right) \ln(\varepsilon) + p_{20} \left(\frac{\xi}{\eta} \right) \right] \varepsilon^2 + \dots, \end{aligned} \quad (4.11)$$

where

$$\begin{aligned} p_{00} \left(\frac{\xi}{\eta} \right) &= 1, \\ p_{11} \left(\frac{\xi}{\eta} \right) &= 1 + A_1, \quad p_{10} \left(\frac{\xi}{\eta} \right) = -\frac{\xi}{\eta} + \ln \left(\frac{\xi}{\eta} \right) + A_2 + \frac{1}{2} \frac{\eta}{\xi}, \\ p_{22} \left(\frac{\xi}{\eta} \right) &= A_3 + 2A_1, \quad p_{21} \left(\frac{\xi}{\eta} \right) = [1 - A_1] \frac{\xi}{\eta} + 2A_1 \ln \left(\frac{\xi}{\eta} \right) + A_4 + 2A_2, \\ p_{20} \left(\frac{\xi}{\eta} \right) &= \frac{\xi}{\eta} \ln \left(\frac{\xi}{\eta} \right) - A_2 \frac{\xi}{\eta} + 2A_2 \ln \left(\frac{\xi}{\eta} \right) + A_5. \end{aligned} \quad (4.12)$$

To find the boundary layer expansion of w we assume that w , like u , belongs to $\mathfrak{R}_1$,

$$\begin{aligned} w(x, \varepsilon) &= w_{00}(x) + [w_{11}(x) \ln(\varepsilon) + w_{10}(x)] \varepsilon \\ &+ [w_{22}(x) \ln^2(\varepsilon) + w_{21}(x) \ln(\varepsilon) + w_{20}(x)] \varepsilon^2 + \dots \end{aligned} \quad (4.13)$$

We substitute Eq. (4.13) into Eq. (4.6) and we obtain a sequence of equations for w_{jk} 's:

$$\begin{aligned}
 x^2 \frac{d^2 w_{00}}{dx^2} - w_{00} \frac{dw_{00}}{dx} &= 0, \quad w_{00}(0) = 1, \\
 x^2 \frac{d^2 w_{11}}{dx^2} - \frac{d}{dx} [w_{00} w_{11}] &= 0, \quad w_{11}(0) = 0, \\
 x^2 \frac{d^2 w_{10}}{dx^2} - \frac{d}{dx} [w_{00} w_{10}] &= 0, \quad w_{10}(0) = 0, \\
 x^2 \frac{d^2 w_{22}}{dx^2} - \frac{d}{dx} [w_{00} w_{22}] &= w_{11} \frac{dw_{11}}{dx}, \quad w_{22}(0) = 0, \\
 x^2 \frac{d^2 w_{21}}{dx^2} - \frac{d}{dx} [w_{00} w_{21}] &= \frac{d}{dx} [w_{11} w_{10}], \quad w_{21}(0) = 0, \\
 x^2 \frac{d^2 w_{20}}{dx^2} - \frac{d}{dx} [w_{00} w_{20}] &= w_{10} \frac{dw_{10}}{dx}, \quad w_{20}(0) = 0.
 \end{aligned} \tag{4.14}$$

Following our original idea we try to match w with u by extracting the common part of both expansions from Equations (4.14) written in terms of the gauge function η ,

$$x = \eta^{-1} \xi, \quad z = \bar{\eta} \xi, \quad \bar{\eta} \eta = \varepsilon, \quad \bar{\eta} = \varepsilon^\alpha, \quad 0 < \alpha < 1. \tag{4.15}$$

Thus we try to examine the sequence of equations,

$$\begin{aligned}
 \frac{d}{d\eta} \left[\eta^2 \frac{dw_{00}}{d\eta} \right] + \frac{\eta^2}{\xi} w_{00} \frac{dw_{00}}{d\eta} &= 0, \quad w_{00}(\infty) = 1, \\
 \frac{d}{d\eta} \left[\eta^2 \frac{dw_{11}}{d\eta} \right] + \frac{\eta^2}{\xi} \frac{d}{d\eta} [w_{00} w_{11}] &= 0, \quad w_{11}(\infty) = 0, \\
 \frac{d}{d\eta} \left[\eta^2 \frac{dw_{10}}{d\eta} \right] + \frac{\eta^2}{\xi} \frac{d}{d\eta} [w_{00} w_{10}] &= 0, \quad w_{10}(\infty) = 0,
 \end{aligned} \tag{4.16}$$

by searching for w_{jk} 's in a particular analytical form. However, unlike before, the common part p in Eq. (4.11) is not a Laurent expansion with respect to η . Consequently, we try to analyze the complementary differences,

$$t_{jk}(\eta) = w_{jk}(\eta) - p_{jk}(\eta), \tag{4.17}$$

which describe those parts of w_{jk} 's which drop out from the matching procedure. The new functions t_{jk} satisfy the following boundary conditions,

$$t_{jk}(0) = 0, \quad \lim_{\eta \rightarrow \infty} [t_{jk}(\eta) + p_{jk}(\eta)] = w_{jk}(\infty), \tag{4.18}$$

the first one being the consequence of the matching hypothesis that w does match u for small η , the second one being the successor of the original boundary condition for w_{jk} 's at $\eta = \infty$. We notice that since expansion (4.11) is our hypothetical p , the outer expansion u must be contained in w when both written in terms of $x = \eta^{-1} \xi$. Consequently we should expect to find unknown A_j 's from Equations (4.16) while searching for unknown t_{jk} 's. The sequence of equations for t_{jk} 's has the following form (see Equations (4.12),

(4.16), (4.17)),

$$\begin{aligned} \frac{d}{d\eta} \left[\eta^2 \frac{dt_{00}}{d\eta} \right] + \frac{\eta^2}{\xi} [1 + t_{00}] \frac{dt_{00}}{d\eta} &= 0, \\ t_{00}(0) &= 0, \quad t_{00}(\infty) = 0, \\ \frac{d}{d\eta} \left[\eta^2 \frac{dt_{11}}{d\eta} \right] + \frac{\eta^2}{\xi} \frac{d}{d\eta} \left[[1 + t_{00}] [1 + A_1 + t_{11}] \right] &= 0, \\ t_{11}(0) &= 0, \quad t_{11}(\infty) = -1 - A_1, \dots, \end{aligned} \quad (4.19)$$

with the successive equations containing other A_j 's which enter the formulas for p_{jk} 's. At first glance, the task of finding A_j 's and t_{jk} 's using Equations (4.19) seems to be daunting. However there exists a simple resolution of all our difficulties if we assume that all t_{jk} 's belong to $\mathfrak{R}_1(\eta)$. More precisely, if each $t_{jk}(\eta)$ can be expanded into a series

$$t(\eta) = \sum_{\alpha=0}^{\infty} P_{\alpha}(\ln(\eta)) \eta^{\alpha}, \quad P_{\alpha}(\ln(\eta)) = \sum_{\beta=0}^{\alpha} P_{\alpha\beta} [\ln(\eta)]^{\beta}, \quad (4.20)$$

then both the formulas for t_{jk} 's and the values of A_j 's can be found using Equations (4.19).

To show that hypothesis (4.20) is indeed true we start with the equation for t_{00} . It is easy to show that if t_{00} has a logarithmic expansion, then it must be equal to zero. Thus $w_{00}(\eta) = 1$ and the chain (4.16) written in terms of $s = \xi^{-1}\eta$ has the following structure:

$$\begin{aligned} \frac{d}{ds} \left[s^2 \frac{dw_{11}}{ds} \right] + s^2 \frac{dw_{11}}{ds} &= 0, \\ \frac{d}{ds} \left[s^2 \frac{dw_{10}}{ds} \right] + s^2 \frac{dw_{10}}{ds} &= 0, \\ \frac{d}{ds} \left[s^2 \frac{dw_{22}}{ds} \right] + s^2 \frac{dw_{22}}{ds} &= -s^2 w_{11} \frac{dw_{11}}{ds}, \\ \frac{d}{ds} \left[s^2 \frac{dw_{21}}{ds} \right] + s^2 \frac{dw_{21}}{ds} &= -s^2 \frac{d}{ds} [w_{11} w_{10}], \\ \frac{d}{ds} \left[s^2 \frac{dw_{20}}{ds} \right] + s^2 \frac{dw_{20}}{ds} &= -s^2 w_{10} \frac{dw_{10}}{ds}. \end{aligned} \quad (4.21)$$

It is obvious that Equations (4.21) can be integrated exactly. It is also clear that the change of variables $\eta = \xi s$ defines a bijection of $\mathfrak{R}_1(\eta)$ onto $\mathfrak{R}_1(s)$. Consequently it is enough to examine Equations (4.21) subject to two boundary conditions,

$$w_{jk}(\infty) = 0, \quad \lim_{s \rightarrow 0} [w_{jk}(s) - p_{jk}(s)] = 0, \quad (4.22)$$

and to show that all $t_{jk}(s) = w_{jk}(s) - p_{jk}(s)$ belong to $\mathfrak{R}_1(s)$ and, also, that they all vanish at $s = 0$.

The first two Equations (4.21) are identical. We integrate them once and we obtain

$$s^2 \frac{dw}{ds} = -B \exp(-s). \quad (4.23)$$

Equation (4.23) yields

$$w(\infty) - w(s) = -B \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau^2}. \quad (4.24)$$

Since $w(\infty) = 0$,

$$w(s) = BF(s), \quad F(s) = \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau^2}. \quad (4.25)$$

Moreover, function F has the standard asymptotic representation,

$$F(s) = \frac{1}{s} + \ln(s) + [\gamma - 1] - \frac{1}{2}s + F_0(s), \quad (4.26)$$

where $F_0(s)$ is an analytic function of s which vanishes at $s = 0$ and where γ is the Euler's constant (see Appendix D for details). To solve the second boundary condition for $w(s)$, we express p_{10}, p_{11} in terms of s (see Equations (4.12)),

$$p_{11}(s) = 1 + A_1, \quad p_{10}(s) = -\frac{1}{s} - \ln(s) + A_2 + \frac{1}{2}s, \quad (4.27)$$

and we search for constants A_1, A_2, B_{11}, B_{10} such that

$$\begin{aligned} \lim_{s \rightarrow 0} [w_{11}(s) - p_{11}(s)] &= \lim_{s \rightarrow 0} [B_{11}F(s) - [1 + A_1]] = 0, \\ \lim_{s \rightarrow 0} [w_{10}(s) - p_{10}(s)] &= \lim_{s \rightarrow 0} \left[B_{10}F(s) - \left[-\frac{1}{s} - \ln(s) + A_2 + \frac{1}{2}s \right] \right] = 0. \end{aligned} \quad (4.28)$$

The first limit vanishes if and only if $B_{11} = 0$ and $A_1 = -1$. The second limit is zero if and only if $B_{10} = -1$ and $A_2 = 1 - \gamma$. Therefore,

$$\begin{aligned} w_{11}(s) &= 0, \quad A_1 = -1, \quad t_{11}(s) = 0, \\ w_{10}(s) &= -F(s), \quad A_2 = 1 - \gamma, \quad t_{10}(s) = -F_0(s). \end{aligned} \quad (4.29)$$

It is also clear that $t_{11}(s), t_{10}(s)$ belong to $\mathfrak{R}_1(s)$ and that both functions vanish at $s = 0$.

To find $w_{22}(s), w_{21}(s), w_{20}(s)$ first we write (see Equations (4.12)),

$$\begin{aligned} p_{22}(s) &= A_3 - 2, \\ p_{21}(s) &= 2 \left[\frac{1}{s} + \ln(s) + \frac{1}{2}A_4 + [1 - \gamma] \right], \\ p_{20}(s) &= -\frac{\ln(s)}{s} - [1 - \gamma] \frac{1}{s} - 2[1 - \gamma] \ln(s) + A_5. \end{aligned} \quad (4.30)$$

Next, we notice that $w_{22}(s), w_{21}(s)$ are described by equations analogous to those for $w_{11}(s), w_{10}(s)$. Thus

$$\begin{aligned} w_{22}(s) &= 0, \quad A_3 = 2, \quad t_{22}(s) = 0, \\ w_{21}(s) &= 2F(s), \quad A_4 = 4[1 - \gamma], \quad t_{21}(s) = s - 2F_0(s), \end{aligned} \quad (4.31)$$

and functions $t_{22}(s), t_{21}(s)$ are, again, in $\mathfrak{R}_1(s)$ and, again, both vanish at zero. Hence we have to solve the last boundary value problem (4.21),

$$\begin{aligned} \frac{d}{ds} \left[s^2 \frac{dw_{20}}{ds} \right] + s^2 \frac{dw_{20}}{ds} &= -s^2 w_{10} \frac{dw_{10}}{ds} = -s^2 F(s) \frac{dF}{ds}(s), \\ w_{20}(\infty) &= 0, \lim_{s \rightarrow 0} [w_{20}(s) - p_{20}(s)] = 0. \end{aligned} \quad (4.32)$$

Since

$$-s^2 F(s) \frac{dF}{ds}(s) = \exp(-s) F(s), \quad (4.33)$$

the first integration of Eq. (4.32) yields

$$s^2 \frac{dw_{20}}{ds} = -B_{20} \exp(-s) - \exp(-s) G(s), \quad G(s) = \int_s^\infty d\tau F(\tau). \quad (4.34)$$

The second integration and the boundary condition at infinity imply that

$$w_{20}(s) = B_{20} F(s) + H(s), \quad H(s) = \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau^2} G(\tau). \quad (4.35)$$

In Appendix D we show that $H(s)$ has the following asymptotic expansion,

$$H(s) = -\frac{\ln(s)}{s} - \frac{[2 + \gamma]}{s} - 3 \ln(s) + [5 - 4 \ln(2) - 3\gamma] + H_0(s), \quad (4.36)$$

where $H_0(s)$ is a true element of $\mathfrak{R}_1(s)$ which vanishes at $s = 0$. Consequently

$$\begin{aligned} w_{20}(s) &= -\frac{\ln(s)}{s} + \frac{B_{20} - [2 + \gamma]}{s} + [B_{20} - 3] \ln(s) \\ &\quad + [5 - 4 \ln(2) - 3\gamma + B_{20} [\gamma - 1]] + t_{20}(s), \end{aligned} \quad (4.37)$$

where

$$t_{20}(s) = H_0(s) + B_{20} \left[-\frac{1}{2}s + F_0(s) \right] \in \mathfrak{R}_1(s), \quad t_{20}(0) = 0. \quad (4.38)$$

Simple calculations show that limit (4.32) is indeed zero, providing that

$$B_{20} = 2\gamma + 1, \quad A_5 = 2[1 - \gamma]^2 + 2 - 4 \ln(2). \quad (4.39)$$

Hence

$$\begin{aligned} w_{20}(s) &= [2\gamma + 1] F(s) + H(s), \\ t_{20}(s) &= H_0(s) + [2\gamma + 1] \left[-\frac{1}{2}s + F_0(s) \right]. \end{aligned} \quad (4.40)$$

As we have just showed, the outer expansion (4.9) is contained in the boundary layer expansion (4.13). Therefore the composite approximation of the boundary value problem

(4.2) has the following, explicit form:

$$\begin{aligned}
 u(z, \varepsilon) \simeq & [1 - z] + \varepsilon \ln(\varepsilon)[z - 1] + \varepsilon \left[[1 + z] \ln(z) + [1 - \gamma][1 - z] \right] \\
 & + \varepsilon^2 \ln^2(\varepsilon) \left[2[1 - z] \right] + \varepsilon^2 \ln(\varepsilon) \left[-2[1 + z] \ln(z) + 4[1 - \gamma][1 - z] \right] \\
 & + \varepsilon^2 \left[-z \ln^2(z) + [3 - 2\gamma]z \ln(z) + 2[1 - \gamma] \ln(z) \right] \\
 & + \varepsilon^2 \left[\frac{1}{2} \left[\frac{1}{z} - z \right] + \left[2[1 - \gamma]^2 + 2 - 4 \ln(2) \right] [1 - z] \right] \\
 & + \varepsilon t_{10} \left(\frac{\varepsilon}{z} \right) + \varepsilon^2 \ln(\varepsilon) t_{21} \left(\frac{\varepsilon}{z} \right) + \varepsilon^2 t_{20} \left(\frac{\varepsilon}{z} \right). \quad (4.41)
 \end{aligned}$$

To close this section we would like to make the following remark: By pursuing our computations further, we may discover that some of the t_{jk} 's are not in $\mathfrak{R}_1(s)$ but in a larger ring $\mathfrak{R}_i(s)$, whose elements are expansions of the form,

$$t(s) = \sum_{\alpha=0}^{\infty} P_{\alpha}^i(\ln(s)) s^{\alpha}, \quad P_{\alpha}^i(\ln(s)) = \sum_{\beta=0}^{\alpha \cdot i} P_{\alpha\beta} [\ln(\eta)]^{\beta}. \quad (4.42)$$

In such a case it is not necessary to repeat all the computations anew, assuming that

$$u(z, \varepsilon) \in \mathfrak{R}_i(\varepsilon), \quad w(x, \varepsilon) \in \mathfrak{R}_i(\varepsilon), \quad t_{jk}(s) \in \mathfrak{R}_i(s). \quad (4.43)$$

Since $\mathfrak{R}_i$'s form an ascending chain of rings,

$$\mathfrak{R}_0 \subset \mathfrak{R}_1 \subset \mathfrak{R}_2 \subset \dots, \quad (4.44)$$

any expansion which belongs to a given $\mathfrak{R}_k$ will also belong to all $\mathfrak{R}_i$'s with $k \leq i$ without any further modification: Thus what was right before will remain so modulo trivial adjustment of the indices.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

Appendix A. In order to get bounds (2.31) we have to examine two equations,

$$\begin{aligned}\frac{d^2 t_0}{d^2 x} &= 2 [\exp(t_0) - 1], \quad t_0(0) = -\ln 2, \quad t_0(\infty) = 0, \\ \frac{d^2 \hat{t}_0}{d^2 \hat{x}} &= 3 [\exp(\hat{t}_0) - 1], \quad \hat{t}_0(0) = 1 - \ln 3, \quad \hat{t}_0(-\infty) = 0,\end{aligned}\quad (\text{A.1})$$

which are equivalent to a single boundary value problem,

$$\frac{d^2 y}{d^2 s} = \exp(y) - 1, \quad y(0) = -\alpha, \quad y(\infty) = 0, \quad \alpha > 0, \quad (\text{A.2})$$

providing that we use two transformations: $x = \sqrt[3]{2}s$ for the first equation and $\hat{x} = -\sqrt[3]{3}s$ for the second. Integrating Eq. (A.2) once, we obtain

$$\frac{1}{2} \left[\frac{dy}{ds} \right]^2 = \exp(y) - y + E_0, \quad (\text{A.3})$$

where $E_0 = -1$ if the trajectory of Eq. (A.3) is to end at the origin. The phase plane portrait of Eq. (A.3) yields the following inequality (see Fig. 1)

$$-\gamma y \leq v(y) \leq -y, \quad v = \frac{dy}{ds}, \quad (\text{A.4})$$

where $\gamma = \alpha^{-1} v(-\alpha) = \alpha^{-1} \sqrt[3]{2} (\exp(-\alpha) + \alpha - 1)^{\frac{1}{2}}$. Consequently

$$-\alpha \exp(-\gamma s) \leq y(s) \leq -\alpha \exp(-s), \quad (\text{A.5})$$

and inequalities (A.4), (A.5) imply that

$$\begin{aligned}-\ln(2) \exp(-\gamma_1 x) &\leq t_0(x) \leq -\ln(2) \exp(-\sqrt[3]{2}x), \\ \gamma_1 \ln(2) \exp(-\sqrt[3]{2}x) &\leq \frac{dt_0}{dx}(x) \leq \sqrt[3]{2} \ln(2) \exp(-\gamma_1 x), \\ \gamma_1 &= [\ln(2)]^{-1} [4 \ln(2) - 2]^{\frac{1}{2}} \simeq 1.268,\end{aligned}\quad (\text{A.6})$$

and

$$\begin{aligned}-[\ln(3) - 1] \exp(\gamma_2 \hat{x}) &\leq \hat{t}_0(\hat{x}) \leq -[\ln(3) - 1] \exp(\sqrt[3]{3} \hat{x}), \\ -\sqrt[3]{3} [\ln(3) - 1] \exp(\gamma_2 \hat{x}) &\leq \frac{d\hat{t}_0}{d\hat{x}}(\hat{x}) \leq -\gamma_2 [\ln(3) - 1] \exp(\sqrt[3]{3} \hat{x}), \\ \gamma_2 &= [\ln(3) - 1]^{-1} [2e + 6 \ln(3) - 12]^{\frac{1}{2}} \simeq 1.704.\end{aligned}\quad (\text{A.7})$$

The graphs of t_0 , $\hat{t}_0$ and their derivatives are shown in Figures 2, 3, and 4.

We notice, now, that both functions t_0 and $\hat{t}_0$ are negative (see Fig. 2), so the inequality (2.38) is trivial while the inequality (2.36) holds true since $t_0(\varepsilon^{-1})$, $\hat{t}_0(-\varepsilon^{-1})$ are exponentially small when compared with the powers of ε . Inequality (2.33) is also obvious, so we are left to show that T , $\hat{T}$ in Eq. (3.9) are bounded for all x and $\hat{x}$. But comparing inequalities (A.6), (A.7) with the expressions for T and $\hat{T}$, we see that

inequality (2.31) is also trivial modulo tedious and elementary details not worth writing. The final estimates for m^2 and M^2 are

$$\begin{aligned} M^2 &\leq \frac{2}{e} \frac{[\ln(3) - 1]^2}{[2e + 6 \ln(3) - 12]^{\frac{1}{2}}} + \sqrt[3]{4 \ln(2) - 2}, \\ m^2 &\leq \frac{1}{e} \frac{[\ln(2)]^2}{[4 \ln(2) - 2]^{\frac{1}{2}}} + \sqrt[3]{2e + 6 \ln(3) - 12}. \end{aligned} \quad (\text{A.8})$$

They are sufficient to show that $G \simeq O(\varepsilon)$.

Appendix B. To compute the limits of $w_1(x)$ and $w_2(x)$ at $x = \infty$, it is convenient to work with the scaled version of Eq. (3.7) which has the form,

$$\begin{aligned} \frac{d^2 \hat{w}}{d^2 \hat{x}} + 2\hat{w} \frac{d\hat{w}}{d\hat{x}} - 4\hat{\varepsilon} \hat{w} &= 0, \quad \hat{w}(0) = 0, \\ \hat{x} = \frac{1}{2} C_0 x, \quad \hat{w}(\hat{x}) &= \frac{1}{C_0} w(x), \quad \hat{\varepsilon} = \frac{\varepsilon}{C_0^2}. \end{aligned} \quad (\text{B.1})$$

The matching conditions for w_j 's can be expressed then in terms of two limits,

$$\lim_{\hat{x} \rightarrow +\infty} [\hat{w}(\hat{x}, \hat{\varepsilon}) - 1 - \hat{\varepsilon} 2\hat{x}] = 0, \quad \lim_{\hat{x} \rightarrow -\infty} [\hat{w}(\hat{x}, \hat{\varepsilon}) + 1 - \hat{\varepsilon} 2\hat{x}] = 0, \quad (\text{B.2})$$

providing that

$$\hat{w}(\hat{x}, \hat{\varepsilon}) = \hat{w}_0(\hat{x}) + \hat{\varepsilon} \hat{w}_1(\hat{x}) + \hat{\varepsilon}^2 \hat{w}_2(\hat{x}) + \dots \quad (\text{B.3})$$

In the new notation Equations (3.9) have the form,

$$\begin{aligned} \frac{d^2 \hat{w}_1}{d^2 \hat{x}} + 2 \frac{d}{d\hat{x}} [\hat{w}_0 \hat{w}_1] &= 4\hat{w}_0, \quad \hat{w}_0(0) = 0, \\ \frac{d^2 \hat{w}_2}{d^2 \hat{x}} + 2 \frac{d}{d\hat{x}} [\hat{w}_0 \hat{w}_2] &= 4\hat{w}_1 - 2\hat{w}_1 \frac{d\hat{w}_1}{d\hat{x}}, \quad \hat{w}_1(0) = 0, \end{aligned} \quad (\text{B.4})$$

where $\hat{w}_0 = th(\hat{x})$. We notice that if Equations (B.1), (B.2) possess a unique solution then necessarily $\hat{w}(\hat{x})$ must be odd since Eq. (B.1) is invariant under the change of variables $W(\hat{x}) = -\hat{w}(-\hat{x})$. Consequently all $\hat{w}_j$'s are also odd.

It is a matter of elementary computations to show that

$$\begin{aligned} \hat{w}_1(\hat{x}) - 2\hat{x} &= ch^{-2}(\hat{x}) \int_0^{\hat{x}} ds \left[\hat{C}_1 - [4 \ln(2) + 2] \right] ch^2(s) + ch^{-2}(\hat{x}) G(\hat{x}), \\ \hat{w}_2(\hat{x}) &= ch^{-2}(\hat{x}) \int_0^{\hat{x}} ds \left[\hat{C}_2 - \hat{f}_1(s) \right] ch^2(s), \end{aligned} \quad (\text{B.5})$$

where

$$\begin{aligned} G(\hat{x}) &= \int_0^{\hat{x}} ds \left[s \exp(-s) ch(s) + \ln(1 + \exp(-2s)) ch^2(s) \right], \\ \hat{f}_1(\hat{x}) &= -4\hat{x} G(\hat{x}) ch^{-2}(\hat{x}) - G^2(\hat{x}) ch^{-4}(\hat{x}) + 4 \int_0^{\hat{x}} ds G(s) ch^{-2}(s). \end{aligned} \quad (\text{B.6})$$

Since for large $\hat{x}$

$$G(\hat{x}) = \hat{x}^2 + \hat{x} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + O(\hat{x} \exp(-2\hat{x})), \quad (\text{B.7})$$

L'Hospital's Rule yields

$$\lim_{\hat{x} \rightarrow \infty} [\hat{w}_1(\hat{x}) - 2\hat{x}] = 0 \text{ and } \lim_{\hat{x} \rightarrow \infty} \hat{w}_2(\hat{x}) = 0, \quad (\text{B.8})$$

providing that

$$\hat{C}_1 = 4 \ln(2) + 2, \quad \hat{C}_2 = -\hat{f}_1(+\infty) = -4 \int_0^\infty ds G(s) ch^{-2}(s). \quad (\text{B.9})$$

Appendix C. We try to match the second pair of p 's from Equations (3.60) with u from Eq. (3.5),

$$p_0(\eta) = A_0 + \frac{\xi}{\eta}, \quad p_1(\eta) = B_0, \quad u(z, \varepsilon) = c(\varepsilon) + z. \quad (\text{C.1})$$

Using Equations (3.57), we get

$$\begin{aligned} u(-1 + \xi\bar{\eta}, \varepsilon) &= c_0 + \varepsilon c_1 + \xi\bar{\eta} + \dots, \\ \varepsilon^{\frac{1}{2}} p(\eta) &= \varepsilon^{\frac{1}{2}} A_0 + \varepsilon B_0 + \xi\bar{\eta} + \dots \end{aligned} \quad (\text{C.2})$$

Comparing both expansions we obtain

$$A_0 = 0, \quad c_0 - 1 = 0, \quad B_0 = c_1. \quad (\text{C.3})$$

If $u(1, \varepsilon) = b$ then $c(\varepsilon) + 1 = b$ or

$$c_0 = b - 1, \quad c_1 = 0. \quad (\text{C.4})$$

Thereby u has the form,

$$u(z, \varepsilon) = z + 1. \quad (\text{C.5})$$

If there is $v(y, \varepsilon) = v_0(y) + \varepsilon^{\frac{1}{2}} v_1(y) + \dots$ that indeed matches u , then $v_0(y)$ is of the form

$$v_0(y) = y + t_0(y), \quad \lim_{y \rightarrow \infty} t_0(y) = 0, \quad (\text{C.6})$$

and $v_0(y)$ must be a solution of the following boundary value problem with the unknown constant b_0 ,

$$\begin{aligned} \frac{d^2 v_0}{dy^2} + v_0 \frac{dv_0}{dy} - v_0 &= 0, \\ \lim_{y \rightarrow 0} \left[v_0(y) - \left[\frac{2}{y} + \frac{2}{3} y \ln(y) + b_0 y + \left[c_2 \ln(y)^2 + c_1 \ln(y) + c_0 \right] y^2 \dots \right] \right] &= 0, \\ \lim_{y \rightarrow \infty} [v_0(y) - y] &= 0. \end{aligned} \quad (\text{C.7})$$

To identify $v_0(y)$ we examine trajectories on the phase plane (see Fig. 3.1) which satisfy the following conditions:

1. When $q_0 = 0$, $v_0 = m_0$, $m_0 > 0$, is the global minimum of $v_0 = v_0(q_0)$.
2. When $q_0 = 1^-$, $v_0 = +\infty$.
3. When $q_0 = -\infty$, $v_0 = +\infty$.

One can show using Eq. (3.63) that each of those trajectories has the property:

$$\begin{aligned} E &= \frac{1}{2}m_0^2, \\ [v_0(y) - 2y^{-1}] &\rightarrow 0 \text{ as } y \rightarrow 0^+ \text{ and } [q_0(y)] \rightarrow -\infty \text{ as } y \rightarrow 0^+, \\ \left[v_0(y) - y - \exp\left(\frac{1}{2}m_0^2 - 1\right) \int_y^\infty ds \exp\left(-\frac{1}{2}s^2\right) \right] &\rightarrow 0 \text{ as } y \rightarrow 0^+. \end{aligned} \quad (\text{C.8})$$

Hence each of the above trajectories corresponds to a well defined solution $v_0(y, m_0)$ which connects w with $u = 1 + z$, providing that

$$b_0 = \frac{1}{6}m_0^2 - \frac{2}{9} - \frac{1}{3} \ln(2). \quad (\text{C.9})$$

Consequently there is an infinite number of asymptotic approximations of the following boundary value problem,

$$\varepsilon \frac{d^2 u}{dz^2} + u \frac{du}{dz} - u = 0, \quad u(-1, \varepsilon) = a > 0, \quad u(1, \varepsilon) = 2. \quad (\text{C.10})$$

In [11] one can find a theorem which states that the boundary value problem (3.1) has a unique solution for any choice of a and b . Thus the pair of p 's in Eq. (C.1) cannot be a valid solution of Equations (3.58).

Appendix D. Asymptotic expansions of $F(s)$ and $H(s)$ are related to the asymptotic expansion of the incomplete gamma function (see [1]),

$$\begin{aligned} E(s) &= \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau} = -\ln(s) - \gamma + \int_0^s d\tau a_0(\tau), \\ a_0(\tau) &= \frac{1 - \exp(-\tau)}{\tau}. \end{aligned} \quad (\text{D.1})$$

Equation (D.1) and integration by parts yields the relation between $F(s)$ and $E(s)$,

$$\begin{aligned} F(s) &= \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau^2} = \frac{\exp(-s)}{s} - E(s) \\ &= \frac{1}{s} + \ln(s) + \gamma - a_0(s) - \int_0^s d\tau a_0(\tau). \end{aligned} \quad (\text{D.2})$$

Since $a_0(s)$ is an analytic function of s , Eq. (D.2) implies that

$$\begin{aligned} F(s) &= \frac{1}{s} + \ln(s) + [\gamma - 1] - \frac{1}{2}s + F_0(s), \\ F_0(s) &= 1 + \frac{1}{2}s - a_0(s) - \int_0^s d\tau a_0(\tau), \end{aligned} \quad (\text{D.3})$$

where $F_0(s)$ is also analytic and $F_0(0) = 0$.

In order to find the asymptotic expansion of $H(s)$, we notice that

$$G(s) = \int_s^\infty d\tau F(\tau) = -sF(s) - \int_s^\infty d\tau \tau \frac{dF}{d\tau}(\tau) = -sF(s) + E(s). \quad (\text{D.4})$$

Consequently

$$\begin{aligned} \frac{\exp(-s)}{s^2} G(s) &= -E(s) \frac{d}{ds} \left[\frac{\exp(-s)}{s} \right] - \frac{\exp(-2s)}{s^2} \\ &= E(s) \frac{d^2 E}{d^2 s}(s) - \frac{\exp(-2s)}{s^2}. \end{aligned} \quad (\text{D.5})$$

Integrating the last identity we obtain the formula for $H(s)$,

$$H(s) = \int_s^\infty d\tau \frac{\exp(-\tau)}{\tau^2} G(\tau) = \frac{\exp(-s)}{s} E(s) - 4F(2s). \quad (\text{D.6})$$

Since the asymptotic expansions for $E(s)$ and $F(s)$ are known, the asymptotic expansion of $H(s)$ has the form

$$\begin{aligned} H(s) &= -\frac{\ln(s)}{s} - \frac{[2+\gamma]}{s} - 3\ln(s) + [5-3\gamma-4\ln(2)] + H_0(s), \\ H_0(s) &= [\gamma + \ln(s)] a_1(s) + 4a_1(2s) + \frac{1}{s} \int_0^s d\tau a_1(\tau) \\ &\quad + 4 \int_0^s d\tau a_0(\tau) - a_0(s) \int_0^s d\tau a_0(\tau), \end{aligned} \quad (\text{D.7})$$

where

$$a_1(s) = \frac{1-s-\exp(-s)}{s}. \quad (\text{D.8})$$

Evidently $H_0(s)$ is not analytic but it does belong to $\mathfrak{R}_1(s)$ and it vanishes at $s=0$.

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