ASYMPTOTICS FOR THE TIME-DEPENDENT THERMISTOR PROBLEM

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Abstract. Using the Liapunoff method, the global asymptotic stability of the stationary solution of the time-dependent thermistor problem is proved.

In this paper we study a special case of the time-dependent thermistor problem, for which it is possible to give a fairly complete description of the asymptotic behaviour of the solution. A difference of potential V , in series with an ordinary resistor R , is applied to an indefinite slab of width d and electric conductivity $\sigma(u)$. We model this situation with the following one-dimensional initial-boundary value problem (D) :

$$(\sigma(u)\varphi_x)_x = 0 \quad (1)$$

$$\varphi(0, t) = 0 \quad (2)$$

$$V - R\sigma(0)\varphi_x(d, t) - \varphi(d, t) = 0 \quad (3)$$

$$u_t - u_{xx} = \sigma(u)\varphi_x^2 \quad (4)$$

$$u(0, t) = u(d, t) = 0 \quad (5)$$

$$u(x, 0) = u_0(x) \quad (6)$$

where $\varphi(x, t)$ is the electric potential and $u(x, t)$ the temperature.

In the usual thermistor problem, R is taken equal to 0, but this ignores e.g. the internal resistance of the generator. Thus, the present model is physically more realistic. The method is based on the use of a Liapunoff function. We treat in details the case

$$\sigma(u) = \sigma_0 e^u, \quad (7)$$

which is relevant in the theory of *PTC* thermistor [5].

In recent years, the thermistor problem has been the subject of many mathematical investigations. We quote, among others, the work of W. Allegretto, W. Lin ([1], [2]), and A. Lacey ([8], [9]) for the time-dependent case and, for the steady case, the papers of S. Howison, F. Rodrigues, and M. Shilor ([6], [7]).

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We consider first the stationary problem (S) corresponding to (D), i.e.:

$$(\sigma(u)\varphi')' = 0 \quad (8)$$

$$\varphi(0) = 0 \quad (9)$$

$$V - R\sigma(0)\varphi'(d) - \varphi(d) = 0 \quad (10)$$

$$-u'' = \sigma(u)\varphi'^2 \quad (11)$$

$$u(0) = u(d) = 0. \quad (12)$$

When $R = 0$, problem (S) has one and only one solution if $\ell = \int_0^\infty \frac{dt}{\sigma(t)} > \frac{V^2}{8}$ and no solution when $\frac{V^2}{8} \geq \ell$. We refer for more details to [6]. In the present situation there is always one and only one solution. This can easily be understood on physical grounds, since the electric current cannot exceed a fixed value determined by the presence of R . A formal proof is the following. Let $F : [0, \infty) \rightarrow [0, \ell]$

$$F(u) = \int_0^u \frac{dt}{\sigma(t)} \quad (13)$$

and

$$\theta = \frac{\varphi^2}{2} + F(u), \quad \Gamma = \varphi(d). \quad (14)$$

If $(u(x), \varphi(x))$ is a solution to problem (S) we have

$$(\sigma(u)\theta')' = 0 \quad (15)$$

$$\theta(0) = 0, \quad \theta(d) = \frac{\Gamma^2}{2} \quad (16)$$

$$(\sigma(u)\varphi')' = 0 \quad (17)$$

$$\varphi(0) = 0, \quad \varphi(d) = \Gamma. \quad (18)$$

Whence it follows the functional relation

$$\theta(x) = \frac{\Gamma\varphi(x)}{2}. \quad (19)$$

Let $u = G(w)$, $G : [0, \ell] \rightarrow [0, \infty)$ be the inverse function of $w = F(u)$. From (14) and (19) we have

$$u(x) = G\left(\frac{\Gamma\varphi(x)}{2} - \frac{\varphi^2(x)}{2}\right). \quad (20)$$

Define the bijection $L : [0, \Gamma] \rightarrow [0, L(\Gamma)]$, $\psi = L(\varphi)$ via

$$L(\varphi) = \int_0^\varphi \sigma\left(G\left(\frac{t\Gamma}{2} - \frac{t^2}{2}\right)\right) dt$$

and $\psi(x) = L(\varphi(x))$. By (20) and (17) we have $\psi''(x) = 0$, $\psi(0) = 0$, and $\psi(d) = L(\varphi(d)) = L(\Gamma)$. Hence $\psi(x) = \frac{xL(\Gamma)}{d}$ and $\psi'(d) = \frac{L(\Gamma)}{d}$. Thus condition (10) becomes

$$V - \Gamma = \frac{R}{d} L(\Gamma) \quad (21)$$

where

$$L(\Gamma) = \int_0^\Gamma \sigma\left(G\left(\frac{t\Gamma}{2} - \frac{t^2}{2}\right)\right) dt$$

and, by direct computation, we find

$$L(\Gamma) = \frac{8}{\sqrt{8 - \Gamma^2}} \arctan \left[\frac{\Gamma}{\sqrt{8 - \Gamma^2}} \right].$$

The solutions of problem (S) are in a one-to-one correspondence with those of Eq. (21) and, by elementary means, it can be seen that (21) has a unique solution $\tilde{\Gamma}$. Hence the only solution to problem (S) is given by:

$$\tilde{\varphi}(x) = L^{-1} \left(\frac{xL(\tilde{\Gamma})}{d} \right), \quad \tilde{u}(x) = G \left(\frac{\tilde{\Gamma}\tilde{\varphi}(x)}{2} - \frac{\tilde{\varphi}^2(x)}{2} \right).$$

The norm in $L^2(0, d)$ is denoted by $\| \cdot \|$. Other norms are identified by subscript.

THEOREM 1. Let $T > 0$, $u_0(x) \in H_0^1(0, d)$, and $u_{0m} = \min u_0(x)$. Then, there exists one and only one solution to problem (D) such that

$$u(x, t) \in L^2(0, T; H^2(0, d)) \cap L^\infty(0, T; H_0^1(0, d)), \quad u_t(x, t) \in L^2(0, T; L^2(0, d)). \quad (22)$$

Moreover,

$$\operatorname{ess\,sup}_{[0, T]} \|u(t)\|_{H_0^1(0, d)} + \|u\|_{L^2(0, T; H^2(0, d))} + \|u_t\|_{L^2(0, T; L^2(0, d))} \leq C \|u_0\|_{H_0^1(0, d)} \quad (23)$$

and

$$0 \leq \varphi(x, t) \leq V \quad (24)$$

$$0 \leq u(x, t) \leq U_M, \quad (25)$$

where $U_M = \max U(x, t)$ and $U(x, t)$ is solution of the linear problem:

$$U_t - U_{xx} = \frac{V^2}{R\sigma(u_{0m})} \quad (26)$$

$$U(0, t) = U(d, t) = 0 \quad (27)$$

$$U(x, 0) = u_0(x). \quad (28)$$

Proof. Let (φ, u) be a solution of problem (D). Multiplying (1) by $\varphi(x, t)$ we have, integrating by parts,

$$\varphi(d, t)\sigma(0)\varphi_x(d, t) = \int_0^d \sigma(u(x, t))\varphi_x^2(x, t)dx$$

which implies (24) recalling (3). In view of the positivity of the right hand side of (4), we have $u(x, t) \geq u_{0m}$. Moreover, integrating (1) with respect to x and taking into account of (24), we obtain the “a priori” estimate

$$\sigma(u(x, t))\varphi_x^2(x, t) = \frac{(V - \varphi(d, t))^2}{R^2\sigma(u(d, t))} \leq \frac{V^2}{R^2\sigma(u_{0m})}. \quad (29)$$

Therefore the parabolic maximum principle implies (25). Let $\tilde{\sigma}(\xi) \in C^\infty(\mathbf{R})$ satisfy: (i) $\tilde{\sigma}(\xi) = \sigma_0 e^\xi$, if $-\infty < \xi \leq U_m$; (ii) $\tilde{\sigma}(\xi) = \sigma_0 e^{U_m}$, if $\xi \geq 2U_M$; (iii) $\tilde{\sigma}'(\xi) \geq 0$. The

method of Galerkin can be used in view of the boundedness of $\tilde{\sigma}$ to prove the existence of a solution of the following auxiliary problem (A):

$$\begin{aligned}(\tilde{\sigma}(u)\varphi_x)_x &= 0 \\ \varphi(0, t) = 0, \quad V - R\tilde{\sigma}(0)\varphi_x(d, t) - \varphi(d, t) &= 0 \\ u_t - u_{xx} &= \tilde{\sigma}(u)\varphi_x^2 = 0 \\ u(0, t) &= u(d, t) = 0 \\ u(x, 0) &= u_0(x).\end{aligned}$$

On the other hand, the “a priori” estimate (25) holds for the solutions of (A) and this guarantees the existence of a solution of (D). Uniqueness can be obtained reasoning as in [3] with application of the Gronwall inequality. $\square$

The solution of problem (D) defines a dynamical system

$$S(t) : H_0^1(0, d) \rightarrow H_0^1(0, d), \quad t \geq 0, \quad u(t) = S(t)u_0,$$

since $u \in C([0, \infty); H_0^1(0, d))$. The solution of the stationary problem (S) gives the only fixed point $\tilde{u}$ of $S(t)$. In the next theorem we prove that $\tilde{u}$ is globally asymptotically stable in $H_0^1(0, d)$.

THEOREM 2. Let $(u(x, t), \varphi(x, t))$ and $(\tilde{u}(x), \tilde{\varphi}(x))$ be solutions to problem (D) and (S) respectively. Then

$$\lim_{t \rightarrow \infty} \|u(t) - \tilde{u}\|_{H_0^1(0, d)} = 0 \quad (30)$$

$$\lim_{t \rightarrow \infty} \|\varphi(t) - \tilde{\varphi}\|_{H_0^1(0, d)} = 0. \quad (31)$$

Proof. Let $\Phi(u) : H_0^1(0, d) \rightarrow \mathbf{R}$ be the continuous functional

$$\Phi(u) = \int_0^d \left(\frac{1}{2} u'^2 - \sigma(u)\varphi^2 \right) dx - \frac{1}{R} [\varphi(d) - V]^2,$$

where $\varphi = T(u)$ is solution to problem

$$\begin{aligned}(\sigma(u)\varphi')' &= 0 \\ \varphi(0) &= 0, \quad V - R\sigma(0)\varphi'(d) - \varphi(d) = 0.\end{aligned}$$

By (24) and (29), $\Phi(u)$ is bounded below on the solutions of problem (D). Using the choice $\sigma(u) = \sigma_0 e^u$ and integrating by parts we find, on the solutions of problem (D),

$$\frac{d}{dt} \Phi(u(t)) = - \int_0^d (u_{xx} + \sigma(u)\varphi_x^2)^2 dx \leq 0. \quad (32)$$

Recalling (23) we conclude that every orbit $u(t; u_0)$ is bounded in $H_0^1(0, d)$. We claim that $u(t; u_0)$ is also compact in $H_0^1(0, d)$ if $t \geq t_0 > 0$. Indeed, let $A = \frac{d^2}{dx^2}$ and $D(A) = H_0^1(0, d) \cap H^2(0, d)$. The operator A is positive, selfadjoint, and with a compact resolvent. Thus, we can define A^α and the corresponding Banach spaces $D(A^\alpha)$ equipped with the graph norm $\|u\|_\alpha = \|A^\alpha u\|$. In particular $D(A^{1/2}) = H_0^1(0, d)$. Moreover, the following estimate holds:

$$\|A^\alpha e^{At} u\| \leq C t^{-\alpha} \|u\|. \quad (33)$$

If $0 < \beta < \alpha$, we have the compact inclusion $D(A^\alpha) \subset D(A^\beta)$. Let $u \in H_0^1(0, d)$ and $f(u) = \tilde{\sigma}(u)(T(u)')^2 = \tilde{\sigma}(u)\varphi'^2$. With these notations, problem (D) can be written as an abstract equation of evolution

$$u' = Au + f(u), u(0) = u_0. \quad (34)$$

Using the variation of constant formula we have

$$u(t; u_0) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(u(s))ds. \quad (35)$$

Noting that f maps bounded subsets of $H_0^1(0, d)$ into bounded subsets of $L^2(0, d)$, we have, for all $t \geq 0$,

$$\|f(u(t; u_0))\| \leq M \quad (36)$$

since orbits $u(t; u_0) = S(t)u_0$ are bounded in $H_0^1(0, d)$. Let $\frac{1}{2} < \beta < 1$. From (35) and (36) we have, by (33),

$$\begin{aligned} \|u(t; u_0)\|_\beta &\leq \|e^{At}u_0\|_\beta + \int_0^t \|e^{A(t-s)}f(u(s))\|_\beta ds = \|A^\beta e^{At}u_0\| \\ &\quad + \int_0^t \|A^\beta e^{A(t-s)}f(u(s))\| ds \\ &\leq Ct^{-\beta}\|u_0\| + \int_0^t C(t-s)^{-\beta}M ds = Ct^{-\beta}\|u_0\| + \frac{CMt^{1-\beta}}{1-\beta}. \end{aligned}$$

If $t \geq t_0 > 0$, the orbit $u(t; u_0)$ is bounded in $D(A^\beta)$ and thus compact in $H_0^1(0, d)$. $\Phi(S(t)u_0)$ is nonincreasing; hence $\ell = \lim_{t \rightarrow \infty} \Phi(S(t)u_0)$ exists and, by compactness, a subsequence $t_k \rightarrow \infty$ can be found such that

$$\|u(t_k; u_0) - \tilde{u}\|_{H_0^1(0, d)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore $\Phi(\tilde{u}) = \ell$. On the other hand, $\ell \leq \Phi(S(t)\tilde{u}) \leq \Phi(\tilde{u}) = \ell$ and $\frac{d}{dt}\Phi(S(t)\tilde{u}) = 0$. So, by (32), we conclude that $(\tilde{\varphi} = T(\tilde{u}), \tilde{u})$ is a solution to the stationary problem (S). This solution, however, is unique; thus the entire sequence converges and we obtain (30). By difference from (1) and (8), multiplying the result by $\tilde{\varphi} - \varphi$ and integrating by parts, we have

$$\frac{1}{R}[\tilde{\varphi}(\ell) - \varphi(d, t)]^2 + \int_0^d \sigma(\tilde{u})(\varphi' - \varphi_x)^2 dx = - \int_0^d \varphi_x[\sigma(\tilde{u}) - \sigma(u)](\tilde{\varphi}' - \varphi_x) dx.$$

This implies (31). $\square$

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