A PHASE ANALYSIS OF TRANSONIC SOLUTIONS FOR THE HYDRODYNAMIC SEMICONDUCTOR MODEL

BY

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Abstract. In the present paper we present a phase plane analysis of transonic solutions of the steady state one-dimensional unipolar hydrodynamic model for semiconductors, taking also into consider shocks.

1. Introduction. In this paper we present a phase analysis of transonic solutions of the steady state one-dimensional unipolar hydrodynamic model for semiconductors. We take also into account the occurrence of shocks.

U. M. Asher, P. A. Markowich, P. Pietra and C. Schmeiser [3] have studied the phase plane portrait for the limit case of an isentropic flow with an adiabatic exponent γ equal to one, i.e. the isothermal hydrodynamic model. P. A. Markowich [10] has discussed the more realistic case $\gamma = 5/3$, but under the restrictive assumption of an infinite current relaxation time (electron plasma). In this paper we work simply by fixing $\gamma > 1$ and taking also into account scattering events of the electrons. However, we achieve analogous results to that found in [3] and [10]. Then our only assumption is that the doping concentration is equal to one.

Let us now give a list of the physical variables used to describe a semiconductor together with their main properties:

- $\rho = \rho(t, x) > 0$ the **charge density** in $x \in \mathbf{R}$ at time $t > 0$,
- $u = u(t, x) \in \mathbf{R}$ the **electron velocity** in $x \in \mathbf{R}$ at time $t > 0$,
- $J = \rho u$ the (negative) **electron current density**,
- $E = E(t, x)$ the (negative) **electric field** in $x \in \mathbf{R}$ at time $t > 0$,
- $d = d(x)$ the **doping profile** in $x \in \mathbf{R}$,
- $\varepsilon = \varepsilon(t, x) > 0$ the **specific internal energy** in $x \in \mathbf{R}$ at time $t > 0$,
- $e = \varepsilon + J^2/(2\rho^2) > 0$ the **specific total energy**,

Received April 1, 2003.

2000 *Mathematics Subject Classification.* Primary 82D37; Secondary 35B40, 35L67.

Key words and phrases. Transonic shock waves, stability, hydrodynamic models, semiconductors.

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- $\mathcal{E} = \rho e$ the **total energy**,
- $C_v > 0$ the **specific heat at constant volume**,
- $C_p > 0$ the **specific heat at constant pressure**,
- $\gamma = C_p/C_v > 0$ the **adiabatic exponent**,
- $p = (\gamma - 1)\rho e$ the **pressure**,
- $s = \log(p\rho^{-\gamma})$ the **entropy**,
- $T_l = e/C_v > 0$ the **lattice temperature**,
- $\tau_p > 0$ the **momentum relaxation time**,
- $\tau_w > 0$ the **energy relaxation time**.

In section 2 we analyze in detail the isentropic case. The approach is based on the construction of the orbits of the system in the electron density-electric field phase plane.

We begin the analysis by considering the vacuum case in section 2.1. This case corresponds by taking an infinite momentum relaxation time τ_p . Clearly this does not model a semiconductor, but only a gas of negatively charged particles in a vacuum. The reason for this hypothesis is that the analysis is greatly simplified by it. For this case a first integral can be obtained, and thus the phase plane portrait is given explicitly. We show that the asymptotic behaviour of the trajectories depends on the value of γ . Indeed, there are three main different situations corresponding to the cases $1 < \gamma < 2$, $\gamma = 2$ and $\gamma > 2$. Furthermore, the monotonicity properties of the trajectories depend on the value of the sonic density ρ_s . We prove that necessary conditions to have smooth transonic solutions or transonic shocks are:

- (α) the sonic density ρ_s is strictly greater than one,
- (β) the couple charge density-electric field (ρ, E) belongs to the subsonic branch of the sonic trajectory characterized by a negative electric field less than zero for $x \rightarrow -\infty$.

The discontinuous solutions are represented by a union of trajectory pieces. These pieces are related by the Rankine-Hugoniot jump conditions and by the Lax entropy conditions. Finally, we prove that it is impossible to have a shock with more than one jump.

In section 2.2 we take into account scattering events of the electrons. We study this more realistic case fixing a positive and finite momentum relaxation time τ_p . For this case the phase portrait cannot be obtained explicitly. Thus we have to construct it qualitatively. Its qualitative properties can be obtained by using the first integral of section 2.1 as a Lyapunov function. The results achieved are analogous to those of section 2.1. Again, to have a shock, the conditions (α), (β) have to be satisfied. Furthermore, there are no shocks with more than one jump. Unlike the vacuum case, for the general case there are no smooth transonic solutions.

In section 3 we analyze the adiabatic case. There are three independent variables used to study this case (against the two independent variables of the isentropic case). This forces us to represent the trajectories in a phase volume (in the isentropic case we use a simple phase plane).

As in section 2 we start by studying the vacuum case in section 3.1. This case corresponds by letting both the momentum relaxation time τ_p and the energy relaxation time τ_w go to infinity. We find that for this case the entropy s is constant on each interval

of regularity. Thus the phase portrait follows from that obtained in section 2.1. Indeed, it is sufficient to paste the trajectories found in section 2.1 on particular parallel planes, taking into account if the corresponding sonic density ρ_s is greater, equal or less than one. As for the isentropic case, if the condition (α) is satisfied, then there exist smooth transonic solutions and shocks, which have to satisfy the condition (β) also. Unlike the isentropic vacuum case, in the adiabatic vacuum case there are also shocks with more than one jump under some particular hypotheses.

In section 3.2 we take into account scattering events of the electrons. We study this more realistic case with fixed positive and finite momentum relaxation time τ_p and energy relaxation time τ_w . We do not know if in this case there are shocks with more than one jump (but we think that the answer is positive at least under some hypotheses). Furthermore, we can give the phase portrait only for a very special case. In fact, we have to assume that $\tau_p = 2\tau_w$ and that $C_v = 1/(\gamma - 1)$. This technical assumptions assure us the flow is isentropic on each interval of regularity. Therefore the phase portrait can be derived from that found in section 2.2. For this case there are no smooth transonic solutions, but only shocks. Finally, we show that the shocks must satisfy the conditions (α) , (β) given before.

Unfortunately, in the real case a semiconductor is characterized by a momentum relaxation time strictly less than twice the energy relaxation time, $\tau_p < 2\tau_w$. This is a consequence of the entropy principle (see [2] for the proof). Our knowledge of the behaviour of the trajectories for this general case is far from complete. However, their monotonicity properties seem to imply that there exist transonic trajectories. Furthermore, their behaviours should be similar to those of section 2.2.

2. The isentropic case. In this section we consider the one-dimensional hydrodynamic model for semiconductors in the isentropic case, i.e.,

$$\varepsilon = \varepsilon_0 \rho^{\gamma-1}, \quad (2.1)$$

for a constant $\varepsilon_0 > 0$. For this case the Euler-Poisson equations read:

$$\begin{cases} \rho_t + J_x = 0 & (a), \\ J_t + \left(\frac{J^2}{\rho} + (\gamma - 1)\varepsilon_0 \rho^\gamma \right)_x = \rho E - \frac{J}{\tau_p} & (b), \\ E_x = \rho - 1 & (c). \end{cases} \quad (2.2)$$

The first two equations of (2.2) give the hydrodynamic part of the system and represent, respectively, charge conservation and momentum balance. The last equation (2.2c) is the Poisson equation and the electric field E has to be determined self-consistently from it. The nonconservative form of the system (2.2a,b) is

$$\vec{U}_t + A(\vec{U})\vec{U}_x = \vec{F}(\vec{U}), \quad (2.3)$$

where

$$\vec{U} = \begin{pmatrix} \rho \\ J \end{pmatrix}, A(\vec{U}) = \begin{pmatrix} 0 & 1 \\ \gamma(\gamma - 1)\varepsilon_0 \rho^{\gamma-1} - \frac{J^2}{\rho^2} & \frac{2J}{\rho} \end{pmatrix}, \vec{F}(\vec{U}) = \begin{pmatrix} 0 \\ \rho E - \frac{J}{\tau_p} \end{pmatrix}.$$

In order to guarantee the hyperbolicity of system (2.3), if and only if $\gamma > 1$, we assume that

$$\gamma > 1.$$

REMARK 2.1. If $\gamma > 1$, then the system (2.3) is strictly hyperbolic and symmetrizable hyperbolic.

Our aim is to study a current driven (one-dimensional) steady state, i.e., $\rho_t = J_t = E_t = 0$. Therefore we consider the system

$$\begin{cases} \left(\frac{J^2}{\rho} + (\gamma - 1)\varepsilon_0 \rho^\gamma \right)' = \rho E - \frac{J}{\tau_p}, \\ E' = \rho - 1, \end{cases} \quad (2.4)$$

where J is a constant. In the following, we consider only the current-controlled case, i.e., we assume that J is prescribed. If, for a given J , the couple $(\rho, E)(x)$ is a solution of (2.4), then the couple $(\tilde{\rho}, \tilde{E})(x) = (\rho, -E)(-x)$ is a solution for $-J$. Thus, it suffices to consider the case $J \geq 0$.

The **sound speed** of the flow is defined by

$$c = c(\rho) = \sqrt{\gamma(\gamma - 1)\varepsilon_0 \rho^{\gamma-1}}.$$

Then, the flow is subsonic, iff

$$|u| < c \iff \rho > \rho_s,$$

and it is supersonic iff

$$|u| > c \iff \rho < \rho_s,$$

where $\rho_s = \left(\frac{J^2}{\gamma(\gamma-1)\varepsilon_0} \right)^{\frac{1}{\gamma+1}}$ is the **sonic density**. The practically relevant situation is to assume subsonic flow at infinity.

In transonic flows the occurrence of shocks has to be taken into account. Therefore we have to supplement (2.4) by the Rankine-Hugoniot jump condition and the Lax entropy condition. The Rankine-Hugoniot jump condition for a shock at $x = x_0$ is (see [15])

$$J^2 \left[\frac{1}{\rho} \right]_{x_0} + (\gamma - 1)\varepsilon_0 [\rho^\gamma]_{x_0} = 0, \quad (2.5)$$

where $[\cdot]_{x_0}$ denotes as usual the jump of any quantity q across the point x_0 , i.e., $[q] = q_R - q_L$ with $q_{R,L}(x_0) = \lim_{\varepsilon \rightarrow 0^+} q(x_0 \pm \varepsilon)$. Obviously, the electric field E is continuous. To select a physically relevant weak solution, we have to impose the so-called Lax entropy condition (see [15]). Under our hypotheses, a way of stating it is to require that

$$c_L \rho_L < J < c_R \rho_R, \quad (2.6)$$

i.e., that a shock occurs from a supersonic to a subsonic state in the direction of the flow, which is the x direction since $J > 0$. In the following, two states, (ρ_L, E) and (ρ_R, E) , will be called conjugate points when (2.5), (2.6) hold.

In [16] is proved the existence of subsonic solutions of the hydrodynamic model for semiconductors in the isentropic case. These results have been extended to multidimensional potential flows in [4]. The existence of one-dimensional transonic solutions was

obtained in [5], [6] by means of viscosity method and in [3] by means of a phase plane analysis. In this paper, the problem (2.4), (2.5), (2.6) is solved, generalizing the techniques used in [3] to a general γ -law (2.1) with $\gamma > 1$ arbitrarily fixed. The analysis is greatly simplified by the assumption of an infinite momentum relaxation time. For this case a first integral of the system (2.4) can be obtained, and thus the phase portrait is given explicitly in section 2.1. Section 2.2 deals with the more realistic case $0 < \tau_p < +\infty$. Although an explicit representation of the phase portrait does not exist for this case, its qualitative properties can be obtained by using the first integral of section 2.1 as a Lyapunov function.

2.1. *The vacuum case.* As a somewhat simpler limit case and for further reference we first consider the case of an infinite momentum relaxation time. Obviously this does not model a semiconductor, but only a gas of negatively charged particles in a vacuum. For $\tau_p = +\infty$, the equations (2.4) read

$$\begin{cases} \left(\frac{J^2}{\rho} + (\gamma - 1)\varepsilon_0 \rho^\gamma \right)' = \rho E, & (a) \\ E' = \rho - 1. & (b) \end{cases} \quad (2.7)$$

The following analysis will be based on the phase portrait of (2.7) in the (ρ, E) -plane. Discontinuous solutions of (2.7), (2.5), (2.6) will be represented as the union of “pieces” of trajectories of (2.7). From the equations (2.7) we immediately conclude the following monotonicity properties of the trajectories:

$$\begin{aligned} \rho \uparrow \text{ in } x \text{ if and only if } & \begin{cases} \rho > \rho_s, E > 0, \\ \text{or} \\ 0 < \rho < \rho_s, E < 0, \end{cases} \\ \rho \downarrow \text{ in } x \text{ if and only if } & \begin{cases} \rho > \rho_s, E < 0, \\ \text{or} \\ 0 < \rho < \rho_s, E > 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} E \uparrow \text{ in } x \text{ if and only if } & \rho > 1, \\ E \downarrow \text{ in } x \text{ if and only if } & 0 < \rho < 1. \end{aligned}$$

By integrating

$$\frac{\rho - 1}{\rho^3} (\gamma(\gamma - 1)\varepsilon_0 \rho^{\gamma+1} - J^2) d\rho = E dE,$$

we obtain the first integral

$$H_\gamma(\rho, E, J) = \frac{E^2}{2} + \frac{(1 - \rho)^2 J^2}{2\rho^2} + \varepsilon_0 (\rho^{\gamma-1} (\gamma - (\gamma - 1)\rho) - 1). \quad (2.8)$$

Thus the trajectory passing through (ρ_0, E_0) is

$$H_\gamma(\rho, E, J) = H_\gamma(\rho_0, E_0, J). \quad (2.9)$$

We observe that H_γ vanishes at the stationary point $(\rho, E) = (1, 0)$. Since, furthermore, $H_\gamma(\rho, E, J) = H_\gamma(\rho, -E, J)$, the trajectories are symmetric with respect to the axis

$\{E = 0\}$. Note that pieces of trajectories not passing through $\{E = 0\}$ can be written in the form $|E| = E(\rho; \rho_0, E_0, J)$, where

$$E(\rho; \rho_0, E_0, J) = \sqrt{2(H_\gamma(\rho_0, E_0, J) - H_\gamma(\rho, 0, J))}.$$

Simple computations show that for any fixed point (ρ_0, E_0) , the asymptotic behaviour of $E(\rho; \rho_0, E_0, J)$ and of the first derivative for $\rho \rightarrow +\infty$ are the following:

$$E \sim \sqrt{2(\gamma - 1)\varepsilon_0\rho^\gamma} \xrightarrow{\rho \rightarrow +\infty} +\infty,$$

$$E_\rho \sim \gamma \sqrt{\frac{(\gamma - 1)\varepsilon_0\rho^{\gamma-2}}{2}} \xrightarrow{\rho \rightarrow +\infty} \begin{cases} 0 & \text{if } \gamma \in (1, 2), \\ \sqrt{2\varepsilon_0} & \text{if } \gamma = 2, \\ +\infty & \text{if } \gamma > 2. \end{cases}$$

Thus the phase portrait depends on the value of γ . Furthermore, it depends also on where ρ_s is with respect to 1.

Clearly the sonic line $\{\rho = \rho_s\}$ is a line of singularities of (2.7) because it holds that

$$\partial_\rho \left(\frac{J^2}{\rho} + (\gamma - 1)\varepsilon_0\rho^\gamma \right) = 0.$$

It is only possible to cross this line at the sonic point $(\rho, E) = (\rho_s, 0)$. In fact, the sonic point is a point of nonuniqueness of (2.7) if $\rho_s \neq 1$, in the sense that the initial value problem for (2.7) with initial data $(\rho_s, 0)$ has two solutions.

The phase portrait of (2.7) given by (2.9) is depicted in the next three figures, which correspond to the three main different situations: $\rho_s > 1$, $\rho_s < 1$ and $\rho_s = 1$. Using the monotonicity properties of the trajectories, we can see that there exists a trajectory which crosses the sonic line if and only if $\rho_s > 1$. Furthermore, in this case such a trajectory is unique. For this reason we assume by now that

$$\rho_s > 1.$$

The thickly drawn trajectory, which we shall call sonic trajectory and denote by T_s , passes through the sonic point twice, once on its way from the subsonic into the supersonic region and once on its way back. The sonic trajectory is given by $E = \pm E(\rho; \rho_s, 0, J)$. For further reference the minimal ρ -value along T_s is denoted by ρ_* . Note that $\rho_* \in (1/2, 1)$. The set of points conjugate to those on the supersonic part of T_s is denoted by T^s (drawn by dashed lines). In the following we will use that T^s lies between the two subsonic branches of T_s . In particular, $(\rho^*, 0) \in T^s$ denotes the conjugate point of $(\rho_*, 0) \in T_s$.

Because of (2.7.a), $|\rho'| = \infty$ holds at all points (ρ_s, E) with $E \neq 0$. At any point (ρ_s, E) with $E > 0$ two trajectories start (one going into the supersonic regime and one going into the subsonic regime) and at every point (ρ_s, E) with $E < 0$ two trajectories end (one coming from the supersonic regime and one coming from the subsonic regime). All points at the sonic line $\{\rho = \rho_s\}$ are attained at finite values of x .

Every supersonic point inside T_s lies on exactly one periodic orbit. The stationary point $(1, 0)$ is a center.

In conclusion, since jumps are only allowed from the supersonic to the subsonic region, the transition from a subsonic state to a supersonic state must be smooth. Therefore

we can have a transonic solution $\vec{U}$ if and only if $\rho_s > 1$. Furthermore, $\vec{U}$ have to come to the sonic point following the subsonic branch of T_s characterized by an electric field less than zero. Then $\vec{U}$ must follow T_s through $(\rho_s, 0)$ into the supersonic regime. After some orbits along the supersonic branch of T_s , $\vec{U}$ can “decide” to return into a subsonic regime by following T_s back into the subsonic region, or “decide” to be a shock and jump from a certain supersonic point $(\rho_L, E_D) \in T_s$ to the conjugate subsonic point $(\rho_R, E_D) \in T^s$, where $\rho_R \in (\rho_s, \rho^*)$ because $\rho_L \in (\rho_*, \rho_s)$. In any case, $\vec{U}$ cannot never return to the supersonic region. Thus any transonic solution $\vec{U}(x) = (\rho, E)(x)$ has to satisfy the limiting condition

$$\lim_{x \rightarrow \pm\infty} E(x) = \pm\infty,$$

and thus any shock has exactly one jump.

2.2. The realistic case. We now consider the problem (2.4) with $\tau_p > 0$ finite, i.e., we take into account scattering events of the electrons.

Note that for $\tau_p > 0$ the phase portrait cannot be obtained explicitly. Therefore we shall now construct it qualitatively.

As for $\tau_p = +\infty$, the line $\{\rho = \rho_s\}$ separates the subsonic regime $\{\rho > \rho_s\}$ from the supersonic regime $\{0 < \rho < \rho_s\}$. For this reason we continue to call this line the sonic line. The point $(\rho, E) = (\rho_s, J/(\tau_p \rho_s))$ is the only point on the sonic line through which solutions with $|\rho'| < \infty$ can pass. We shall refer to it as the sonic point. As in the vacuum case, studied in section 2.1, the sonic point is a point of nonuniqueness of (2.4), in the sense that the initial value problem for (2.4) with initial data $(\rho_s, J/(\tau_p \rho_s))$ has two solutions.

The only stationary point is $(\rho, E) = (1, J/\tau_p)$. We observe that the sonic point and the stationary point both lie on the hyperbola branch

$$\left\{ E = \frac{J}{\tau_p \rho}, \rho > 0 \right\}. \quad (2.10)$$

From the equations (2.4) we immediately conclude the following monotonicity properties of the trajectories:

$$\begin{aligned} \rho \uparrow \text{ in } x \text{ if and only if } & \begin{cases} \rho > \rho_s, E > \frac{J}{\tau_p \rho}, \\ \text{or} \\ 0 < \rho < \rho_s, E < \frac{J}{\tau_p \rho}, \end{cases} \\ \rho \downarrow \text{ in } x \text{ if and only if } & \begin{cases} \rho > \rho_s, E < \frac{J}{\tau_p \rho}, \\ \text{or} \\ 0 < \rho < \rho_s, E > \frac{J}{\tau_p \rho}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} E \uparrow \text{ in } x \text{ if and only if } & \rho > 1, \\ E \downarrow \text{ in } x \text{ if and only if } & 0 < \rho < 1. \end{aligned}$$

Directly from these monotonicity properties of the trajectories, it follows that, just as in the vacuum case, to have a transonic solution we have to require that

$$\rho_s > 1.$$

A simple local analysis shows that then, as in the case $\tau_p = +\infty$, there are exactly two trajectories passing through the sonic point $(\rho_s, J/(\tau_p \rho_s))$. One of them passes from the supersonic regime to the subsonic regime and the other one, which we shall call sonic trajectory and denote by T_s , passes from the subsonic regime into the supersonic regime. Also, by computing ρ' and E' at the sonic point $(\rho_s, J/(\tau_p \rho_s))$ along T_s , we see that, locally at $(\rho_s, J/(\tau_p \rho_s))$, T_s lies above the hyperbola branch (2.10) in the supersonic regime and below (2.10) in the subsonic regime.

PROPOSITION 2.2. The stationary point is stable.

Proof. The essential ingredient of the proof is the construction of a Lyapunov function $L = L(\rho, E; J)$ suggested by the solution of the integrable problem with $\tau_p = +\infty$. We set

$$L(\rho, E; J) = H_\gamma \left(\rho, E - \frac{J}{\tau_p}, J \right),$$

where H_γ is defined in (2.8), and compute that

$$\frac{d}{dx} L(\rho(x), E(x); J) = - \frac{J(\rho(x) - 1)^2}{\tau_p \rho(x)} \quad (\spadesuit)$$

along trajectories. Since the stationary point $(\rho, E) = (1, J/\tau_p)$ is the only trajectory along which ρ is constant and equal to one, $(\spadesuit)$ implies that L is strictly monotonically decreasing along trajectories. Thus, the nonexistence of periodic orbits follows. The supersonic part of the level set

$$\{L(\rho, E; J) = H_\gamma(\rho_s, 0, J)\} \quad (\clubsuit)$$

is a closed curve and its interior B is an invariant region, which is attracted to the stationary point. Therefore it remains to prove that T_s enters B . As mentioned above, T_s enters the curved supersonic “triangle” B_T bounded by the level set $(\clubsuit)$, the hyperbola (2.10) and the sonic line. The behaviour of the flow at the boundary of B_T implies that $B \cup B_T$ is also an invariant region. Thus, T_s reaches B after a finite length of travel through B_T . $\square$

A simple computation shows that the stationary point is a stable nonoscillatory point if

$$0 < \tau_p < \frac{1}{2} \text{ or } \tau_p \geq \frac{1}{2}, \quad 1 < \rho_s \leq \left(\frac{4\tau_p^2}{4\tau_p^2 - 1} \right)^{\frac{1}{\gamma+1}},$$

and that it is a stable oscillatory point if

$$\tau_p \geq \frac{1}{2}, \rho_s > \left(\frac{4\tau_p^2}{4\tau_p^2 - 1} \right)^{\frac{1}{\gamma+1}}.$$

The set of points conjugate to those on the supersonic part of T_s is denoted by T^s . To study T^s it is convenient to make the transformation

$$U = E - \frac{J}{\tau_p \rho}.$$

We obtain from (2.4) for smooth trajectories,

$$\begin{cases} \rho' = \frac{U \rho^3}{\gamma(\gamma-1)\varepsilon_0(\rho^{\gamma+1} - \rho_s^{\gamma+1})}, \\ U' = \rho - 1 + \frac{JU\rho}{\gamma(\gamma-1)\varepsilon_0\tau_p(\rho^{\gamma+1} - \rho_s^{\gamma+1})}. \end{cases} \quad (2.11)$$

Therefore, the smooth trajectory passing through the point (ρ_0, U_0) with $U_0 = E_0 - \frac{J}{\tau_p \rho_0}$ satisfies

$$H_\gamma(\rho, U, J) = H_\gamma(\rho_0, U_0, J) + \frac{J}{\tau_p} \int_{\rho_0}^{\rho} \frac{U(\tilde{\rho})}{\tilde{\rho}^2} d\tilde{\rho}, \quad (2.12)$$

wherever U can be represented locally as a function of ρ .

We denote by T_s^{sp} the supersonic part of T_s and T_s^{sb} the subsonic part, i.e.,

$$\begin{aligned} T_s^{sp} &= T_s \cap \{(\rho, E) | 0 < \rho < \rho_s\}, \\ T_s^{sb} &= T_s \cap \{(\rho, E) | \rho_s < \rho\}. \end{aligned}$$

Now let $(\rho_L, E_D) \in T_s^{sp}$ and let (ρ_R, E_D) be the corresponding conjugate point.

LEMMA 2.3. (ρ_R, E_D) lies above T_s^{sb} .

Proof. We denote $U_L = E_D - \frac{J}{\tau_p \rho_L}$, $U_R = E_D - \frac{J}{\tau_p \rho_R}$. It is easy to show that the subsonic part of T_s satisfies $U_s = U < 0$. Let $\tilde{U} < 0$ be the U -coordinate of the point $(\tilde{U}, \rho_R)$ on T_s^{sb} . We have

$$H_\gamma(\rho_R, \tilde{U}, J) \stackrel{(2.12)}{=} H_\gamma(\rho_s, 0, J) + \frac{J}{\tau_p} \int_{\rho_s}^{\rho_R} \frac{U_s(\xi)}{\xi^2} d\xi. \quad (2.13)$$

Note that $\rho_R > \rho_L$ implies $U_R > U_L$. Thus, if (ρ_L, U_L) lies on a segment of T_s^{sp} with $U \geq 0$ we have $U_R \geq 0$ and thus $U_R > U_s$. We therefore assume $U_L < 0$ in the sequel and denote by $\rho_0 < \rho_L$ the ρ -coordinate of the point $(\rho_0, U = 0) \in T_s^{sp}$ such that $(\rho_0, U = 0)$ and $(\rho_L, U = U_L)$ are connected by a segment of T_s^{sp} along which $U_- = U \leq 0$ holds. Then we can write

$$H_\gamma(\rho_L, U_L, J) \stackrel{(2.12)}{=} H_\gamma(\rho_0, 0, J) + \frac{J}{\tau_p} \int_{\rho_0}^{\rho_L} \frac{U_-(\xi)}{\xi^2} d\xi. \quad (2.14)$$

We also have

$$U_R = U_L + \frac{J(\rho_R - \rho_L)}{\tau_p \rho_R \rho_L}. \quad (2.15)$$

Differentiating (2.13) and U_R with respect to ρ_L (ρ_R is regarded as a function of ρ_L determined by the jump condition (2.5)), and using (2.14) gives (after a lengthy but simple computation)

$$\frac{d(U_R - \tilde{U})}{d\rho_L} = \frac{\gamma(\gamma-1)\varepsilon_0 \left(\rho_s^{\gamma+1} - \rho_L^{\gamma+1} \right) \left(\rho_L(\rho_R - 1)U_L - \rho_R(\rho_L - 1)\tilde{U} \right)}{\rho_L^3 \rho_R U_L \tilde{U}}. \quad (2.16)$$

We use (2.15) to calculate

$$\begin{aligned} \rho_L(\rho_R - 1)U_L - \rho_R(\rho_L - 1)\tilde{U} &= -\rho_L(\rho_R - 1)(\tilde{U} - U_R) \\ &+ \frac{J(\rho_L - \rho_R)(\rho_R - 1)}{\tau_p \rho_R} + (\rho_R - \rho_L)\tilde{U}, \end{aligned} \quad (2.17)$$

and rewrite (2.16)

$$\frac{d(U_R - \tilde{U})}{d\rho_L} = a(U_R - \tilde{U}) + b, \quad (2.18)$$

where we set

$$\begin{aligned} a(\rho_L) &:= \frac{\gamma(\gamma - 1)\varepsilon_0(\rho_R - 1)(\rho_s^{\gamma+1} - \rho_L^{\gamma+1})}{\rho_L^2 \rho_R U_L \tilde{U}}, \\ b(\rho_L) &:= \frac{\gamma(\gamma - 1)\varepsilon_0(\rho_s^{\gamma+1} - \rho_L^{\gamma+1})(\rho_L - \rho_R)}{\rho_L^3 \rho_R U_L \tilde{U}} \left(\frac{J(\rho_R - 1)}{\tau_p \rho_R} - \tilde{U} \right) < 0. \end{aligned}$$

Let $\rho_1 > \rho_0$ be the ρ -coordinate of the point at which the segment of T_s^{sp} , which connects $(\rho_0, U = 0)$ and $(\rho_L, U = U_L)$, hits $U = 0$ again. Then $\rho_1 = \rho_L$ gives $U_L = 0$, $\tilde{U} < 0$ and $U_R = J(\rho_R - \rho_L)/(\tau_p \rho_R \rho_L) > 0$. Therefore, there is $\delta > 0$ such that $U_R - \tilde{U} > 0$ for $\rho_L \in [\rho_1 - \delta, \rho_1]$. Solving (2.18) for $\rho_L \in (\rho_0, \rho_1 - \delta]$ gives $U_R - \tilde{U} > 0$ on $[\rho_0, \rho_1]$. $\square$

The phase portrait of (2.4) is depicted in the next figure, which corresponds to the unique interesting case: $\rho_s > 1$.

The thickly drawn trajectory is the sonic trajectory T_s . It passes through the sonic line from the subsonic into the supersonic region. As in the vacuum case, the minimal ρ -value along T_s is denoted by ρ_* and is less than one. The set T^s of points conjugate to those on T_s^{sp} is drawn by dashed lines and lies above T_s^{sb} . In particular, $(\rho^*, E^*) \in T^s$ denotes the conjugate point of $(\rho_*, E_*) \in T_s^{sp}$.

Because of (2.4a), $|\rho'| = \infty$ holds at all points (ρ_s, E) with $E \neq J/(\tau_p \rho_s)$. At any point (ρ_s, E) with $E > 0$ two trajectories start (one going into the supersonic regime and one into the subsonic regime) and at every point (ρ_s, E) with $E < 0$ two trajectories end (one coming from the supersonic regime and one coming from the supersonic regime). All points at the sonic line $\{\rho = \rho_s\}$ are attained at finite values of x .

There are no periodic orbits as proved in Proposition 2.2, but only oscillatory trajectories around the stationary point $(1, J/\tau_p)$, when $(1, J/\tau_p)$ is a stable oscillatory point.

In conclusion, since the transition from a subsonic state to a supersonic state must be smooth, just as in the vacuum case, to have a transonic solution it must be

$$\rho_s > 1.$$

Note that there are no smooth transonic solutions of (2.4). In fact, any transonic solution $\vec{U}(x) = (\rho, E)(x)$ must have a jump, i.e., it has to be a shock. Furthermore, any shock has the following properties:

- (1) $\lim_{x \rightarrow -\infty} E(x) = -\infty$,
- (2) it has to follow the sonic trajectory T_s reaching the supersonic regime crossing the sonic line at the sonic point,
- (3) after maybe some oscillation around the stationary point $(1, J/\tau_p)$ along T_s^{sp} , it jumps from a supersonic point $(\rho_L, E_D) \in T_s^{sp}$ to a subsonic point (ρ_R, E_D) which lies above T_s^{sb} , with $\rho_R \in (\rho_s, \rho^*)$ because $\rho_L \in (\rho_*, \rho_s)$,
- (4) it follows a trajectory which belongs to the subsonic region and finally it has to satisfy the limiting condition $\lim_{x \rightarrow +\infty} E(x) = +\infty$.

Clearly from Lemma 2.3 it follows that there is no shock with more than one jump.

3. The adiabatic case. The equations for the adiabatic case are [7]

$$\begin{cases} \rho_t + J_x = 0, & (a) \\ J_t + \left((\gamma - 1)\mathcal{E} + \frac{3-\gamma}{2} \frac{J^2}{\rho} \right)_x = \rho E - \frac{J}{\tau_p}, & (b) \\ \mathcal{E}_t + \left(\frac{J}{\rho} \left(\gamma \mathcal{E} + \frac{1-\gamma}{2} \frac{J^2}{\rho} \right) \right)_x = JE - \frac{1}{\tau_w} \left(\mathcal{E} - \frac{\rho T_l}{\gamma - 1} \right), & (c) \\ E_x = \rho - 1. & (d) \end{cases} \quad (3.1)$$

Let us recall that the first three equations of (3.1) are the hydrodynamic part of the system and represent respectively charge conservation, momentum balance and energy balance. The last equation is the Poisson equation and the electric field has to be determined self-consistently from it. The nonconservative form of the system (3.1.a,b,c) is

$$\vec{U}_t + A(\vec{U})\vec{U}_x = \vec{F}(\vec{U}), \quad (3.2)$$

where

$$\vec{U} = \begin{pmatrix} \rho \\ J \\ \mathcal{E} \end{pmatrix}, \quad A(\vec{U}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2} \frac{J^2}{\rho^2} & (3-\gamma) \frac{J}{\rho} & \gamma-1 \\ \frac{J}{\rho^2} \left((\gamma-1) \frac{J^2}{\rho} - \gamma \mathcal{E} \right) & \frac{1}{\rho} \left(\gamma \mathcal{E} - \frac{3}{2} (\gamma-1) \frac{J^2}{\rho} \right) & \gamma \frac{J}{\rho} \end{pmatrix},$$

$$\vec{F}(\vec{U}) = \begin{pmatrix} 0 \\ \rho E - \frac{J}{\tau_p} \\ JE - \frac{1}{\tau_w} \left(\left(1 - \frac{1}{(\gamma-1)C_v} \right) \mathcal{E} + \frac{1}{2(\gamma-1)C_v} \frac{J^2}{\rho} \right) \end{pmatrix}.$$

Since the system (3.2) is hyperbolic if and only if $\gamma > 1$, we assume that

$$\gamma > 1.$$

REMARK 3.1. If $\gamma > 1$, then the system (3.2) is strictly hyperbolic and symmetrizable hyperbolic.

Here we only consider the stationary case, i.e., $\rho_t = J_t = \mathcal{E}_t = E_t = 0$. Then the problem can be written as

$$\begin{cases} \left((\gamma - 1)\mathcal{E} + \frac{3-\gamma}{2} \frac{J^2}{\rho} \right)' = \rho E - \frac{J}{\tau_p}, & (b) \\ \left(\frac{J}{\rho} \left(\gamma \mathcal{E} + \frac{1-\gamma}{2} \frac{J^2}{\rho} \right) \right)' = J E - \frac{1}{\tau_w} \left(\mathcal{E} - \frac{\rho T_L}{\gamma - 1} \right), & (c) \\ E' = \rho - 1, & (d) \end{cases} \quad (3.3)$$

where J is a constant. In the following, we consider the current-controlled case, i.e., we assume that J is prescribed. If, for a given J , the triple $(\rho, E, \mathcal{E})(x)$ is a solution of (3.3), then the triple $(\tilde{\rho}, \tilde{E}, \tilde{\mathcal{E}})(x) = (\rho, -E, \mathcal{E})(-x)$ is a solution for $-J$. Thus, it suffices to consider $J \geq 0$.

The **sound speed** of the flow is defined by

$$c = c(\rho) = \sqrt{\gamma(\gamma - 1)\varepsilon}.$$

Then, the flow is subsonic, if

$$|u| < c \iff c\rho > J,$$

and it is supersonic if

$$|u| > c \iff c\rho < J.$$

The practically relevant situation is to assume subsonic flow at infinity.

Since in general quasi-linear systems of conservation laws do not have global smooth solutions, for the presence of jumps of discontinuities in the solutions, the occurrence of shocks has to be taken into account. Thus we have to supplement (3.3) by Rankine-Hugoniot jump conditions, that are

$$\begin{cases} (\gamma - 1)[\mathcal{E}] + \frac{(3-\gamma)J^2}{2} \left[\frac{1}{\rho} \right] = 0, \\ \gamma \left[\frac{\mathcal{E}}{\rho} \right] + \frac{(1-\gamma)J^2}{2} \left[\frac{1}{\rho^2} \right] = 0. \end{cases} \quad (3.4)$$

Obviously, the electric field E is continuous. Furthermore,

$$\mathcal{E}_{R,L} = \frac{((\gamma + 1)\rho_{R,L} + (\gamma - 1)^2 \rho_{L,R})J^2}{2\gamma(\gamma - 1)\rho_R \rho_L}, \quad (3.5)$$

and $\mathcal{E}$ is continuous if and only if $\gamma = 3$. To select a physically relevant weak solution, we have to impose the so-called Lax entropy condition. Under our hypotheses, a way of stating it is to require that

$$c_L \rho_L < J < c_R \rho_R, \quad (3.6)$$

i.e., that a shock occurs from a supersonic to a subsonic state in the direction of the flow, which is the x direction since $J > 0$. In the following, two states (ρ_L, E) and (ρ_R, E) will be called conjugate points when (3.4), (3.6) hold. Finally, a further admissibility condition is needed to select stable solutions. For this reason we introduce the concept of entropy-entropy flux pair (η, q) for (3.2) as a couple of smooth functions of $\vec{U}$ such that

$$\nabla q = A^T \nabla \eta,$$

where ∇ denotes the gradient respect to $\vec{U}$. It can be shown that the unique nontrivial entropy pair of the system (3.2) up to addition of conserved quantities and constants (see [14]) is given by the “mechanical energy”:

$$\eta(\vec{U}) = -\rho \left(\log \left(\mathcal{E} - \frac{J^2}{2\rho} \right) - \gamma \log \rho \right), \quad q(\vec{U}) = -J \left(\log \left(\mathcal{E} - \frac{J^2}{2\rho} \right) - \gamma \log \rho \right),$$

which is a strictly convex entropy pair. Simple computations show that the mathematical entropy condition (see [7]) and also the more realistic physical entropy condition are both equivalent to require that $\rho_L < \rho_R$, and thus they are a simple consequence of the Lax entropy condition (3.6). Let us recall that the difference between the mathematical entropy condition and the physical entropy condition is that the former uses the identity matrix as the diffusion matrix, and the latter uses as a diffusion matrix the following:

$$D_\mu(\vec{U}) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{J}{\rho^2} & \frac{1}{\rho} & 0 \\ \frac{K(T_l)}{\mu C_v} \left(\frac{J^2}{\rho^3} - \frac{\mathcal{E}}{\rho^2} \right) - \frac{J^2}{\rho^3} & \frac{J}{\rho^2} \left(1 - \frac{K(T_l)}{\mu C_v} \right) & \frac{K(T_l)}{\mu C_v \rho} \end{pmatrix},$$

where $\mu > 0$ is the diffusion coefficient and $K(T_l)$ is a continuous positive function.

In [10] is analyzed the adiabatic case with an adiabatic exponent $\gamma = 5/3$ and letting the momentum relaxation time, τ_p , and the energy relaxation time, τ_w , go to infinity. In our paper, the adiabatic problem (3.3), (3.4), (3.6) is studied for any fixed adiabatic exponent γ strictly greater than one.

Proceeding as in section 2, we start analyzing in section 3.1 the limit case with $\tau_p = \tau_w = +\infty$, which model an adiabatic gas of negatively charged particles in a vacuum. Under this assumption we will find a special first integral of the system (3.3), which will imply that the flow is isentropic on the intervals of regularity. Thus the phase volume portrait will follow directly from the phase plane portrait obtained in section 2.1.

In section 3.2 we pass to the more realistic case with $0 < \tau_p, \tau_w < \infty$. As for the isentropic case, also for this case there is no explicit representation of the phase portrait. What we will do is to assume particular hypotheses about the link between the constants τ_p, τ_w and about the link between the constants C_p, C_v , in such a way to obtain an isentropic flow at least on each interval of regularity. Under these assumptions it will be very simple to depict the phase portrait knowing that of section 2.2.

3.1. The vacuum case. In this section we study the adiabatic problem (3.3), (3.4), (3.6) under the assumption that the momentum relaxation time and the energy relaxation time are both equal to $+\infty$. With these hypotheses the system (3.3) becomes

$$\begin{cases} \left(\frac{J^2}{\rho} + (\gamma - 1)\rho\mathcal{E} \right)' = \rho E, \\ \left(\frac{J^2}{2\rho^2} + \gamma\mathcal{E} \right)' = E, \\ E' = \rho - 1. \end{cases} \quad (3.7)$$

The following analysis will be based on the phase portrait of (3.7) in the $(\rho, \varepsilon/\rho^{\gamma-1}, E)$ -volume. By integrating

$$\frac{\gamma-1}{\rho} d\rho = \frac{1}{\varepsilon} d\varepsilon,$$

we obtain the first integral

$$G_\gamma(\rho, \varepsilon) = \log \left(\frac{\varepsilon_0 \rho^{\gamma-1}}{\varepsilon} \right),$$

which implies that the entropy s is constant on every interval of regularity. Thus the trajectories in the (ρ, ε_0, E) -volume are obtained by depicting on each parallel plane to the (ρ, E) -plane the corresponding trajectories found in section 2.1. We denote by $\varepsilon_0^- > 0$ the constant relative to the flow for $x = -\infty$, and introduce $\rho_s(\varepsilon_0^-) = (J^2/(\gamma(\gamma-1)\varepsilon_0^-))^{\frac{1}{\gamma+1}}$ the corresponding sonic density. We will call the hyperplane $\{\rho = \rho_s(\varepsilon_0^-)\}$ the sonic wall. Clearly it is a set of points of singularities of (3.7). Furthermore, if $\rho_s(\varepsilon_0^-)$ is less than or equal to one, we know that there are no transonic trajectories. So let us assume that $\rho_s(\varepsilon_0^-)$ is strictly greater than one. Then we know that on the plane $\{\varepsilon_0 = \varepsilon_0^-\}$ there is a unique trajectory, $T_s(\varepsilon_0^-)$, which crosses the sonic wall, and furthermore there exist smooth transonic solutions and shocks. Let us underline that discontinuous solutions of (3.7), (3.4), (3.6) will be represented as union of “pieces” of isentropic trajectories of (2.7). The set of points conjugate to those on the supersonic part of $T_s(\varepsilon_0^-)$ is denoted by $T^s(\varepsilon_0^-)$. From the Rankine-Hugoniot jump conditions (3.4) it follows that

$$T^s(\varepsilon_0^-) : \begin{cases} \rho_R = \rho_R(\rho_L; \varepsilon_0^-) := \frac{(\gamma+1)J^2\rho_L}{(\gamma-1)(J^2+2\gamma\varepsilon_0^-\rho_L^{\gamma+1})}, \\ \varepsilon_0^+ = \varepsilon_0^+(\rho_L; \varepsilon_0^-) := \frac{(\gamma-1)^{\gamma-1}(2J^2-(\gamma-1)^2\varepsilon_0^-\rho_L^{\gamma+1})(J^2+2\gamma\varepsilon_0^-\rho_L^{\gamma+1})^\gamma}{(\gamma+1)^{\gamma+1}\rho_L^{\gamma+1}J^{2\gamma}}, \\ E_R = E_R(\rho_L; \varepsilon_0^-) := \sqrt{2(H_\gamma(\rho_s(\varepsilon_0^-), 0, J) - H_\gamma(\rho_L, 0, J))}, \end{cases} \quad (3.8)$$

where $\varepsilon_0^+ > 0$ denotes the constant corresponding to the flow after the jump. Note that $\rho_R > \rho_s(\varepsilon_0^+)$ if and only if $\rho_L > \rho_s(\varepsilon_0^-)$. The phase portrait of (3.3) is depicted in the next figure, where we have represented three trajectories: the first two from the left are characterized by a sonic density strictly greater than one, and the third by a sonic density strictly less than one.

Let us suppose that our transonic solution $\vec{U} = (\rho, \varepsilon_0^-(0), E)$ belongs to the subsonic part of the thickly drawn trajectory $T_s(0)$ relative to the constant $\varepsilon_0^-(0) > 0$, such that the relative sonic density $\rho_s(0)$ is strictly greater than one. For further reference the minimal ρ -value along $T_s(0)$ is denoted by $\rho_*(0)$. The set of conjugate points to those on the supersonic part of $T_s(0)$ is denoted by $T^s(0)$ and is drawn by dashed lines. If $E(x) \xrightarrow{x \rightarrow -\infty} -\infty$, then we know that $\vec{U}$ crosses the sonic wall through the sonic point $(\rho, \varepsilon_0, E) = (\rho_s(0), \varepsilon_0^-(0), 0)$ and enters in the supersonic region. After some loops along the supersonic branch of $T_s(0)$, $\vec{U}$ can return into a subsonic region by continuing to follow $T_s(0)$ back, or jump from a certain supersonic point $(\rho_L(0), \varepsilon_0^-(0), E_D(0)) \in T_s(0)$ to the conjugate subsonic point $(\rho_R(0), \varepsilon_0^+(0), E_D(0)) \in T^s(0)$ given by (3.8). Since jumps are only allowed from the supersonic to the subsonic region, in the first case $\vec{U}$ will no longer cross the sonic wall. In the second case, to satisfy the condition of a subsonic regime at infinity, the conjugate point does not have to belong to a trajectory different from the

sonic trajectory which hits the sonic wall. The reason is that any point on the sonic wall is reached for a finite value of x . A particular case is when it happens that the conjugate point $(\rho_R(0), \varepsilon_0^+(0), E_D(0))$ belongs to the sonic trajectory $T_s(1)$ relative to the constant $\varepsilon_0^+(0) = \varepsilon_0^-(1)$. In this case, if the sonic density $\rho_s(1)$ relative to the constant $\varepsilon_0^-(1)$ is strictly greater than one and $E_D(0)$ is strictly negative, then $\vec{U}$ can cross again the sonic wall through the sonic point $(\rho, \varepsilon_0, E) = (\rho_s(1), \varepsilon_0^-(1), 0)$ and enter again in the supersonic region. Now it is clear that a shock can have more than one jump. Long but not difficult computations analogous to that made in [10] show that necessary and sufficient conditions to have a shock with more than one jump are

$$\rho_s(0) > \frac{\gamma}{\gamma-1} \text{ and } \left(f_\gamma \left(\left(\frac{\rho_s(0)}{\rho_*(0)} \right)^{\gamma+1} \right) \right)^{\frac{1}{\gamma+1}} > \frac{(\gamma+1)(2(\gamma-1)\rho_s - \gamma)}{\gamma}, \quad (3.9)$$

where $f_\gamma(y) = \left(-1 + \frac{2}{y} + \gamma\right)^\gamma (1 + (-1 + 2y)\gamma)$. Furthermore, under the assumption (3.9), there are exactly two points denoted by $P_-(0)$ and $P_+(0)$ on the sonic trajectory $T_s(0)$, whose conjugate points belong to the sonic trajectory $T_s(1)$. These points are characterized by the same density and opposite electric field. Thus, if $P_-(0)$ is the point with an electric field less than zero, a shock with more than one jump of discontinuity has to jump from $P_-(0)$.

Now it is clear how to proceed in the construction of shocks with more than two jumps, and so on: it is sufficient to verify the corresponding conditions (3.9) and to find the corresponding point P_- .

3.2. The realistic case. In this last section we consider the problem (3.3) with $\tau_p, \tau_w > 0$ finite, i.e., we take into account scattering events of the electrons.

Note that for this case the phase portrait cannot be obtained explicitly. Thus we shall construct it qualitatively.

The analysis is greatly simplified by the assumption

$$\tau_p = 2\tau_w, \quad C_v = \frac{1}{\gamma - 1}. \quad (3.10)$$

In fact, under this hypothesis, the adiabatic system (3.3) implies that the flow is isentropic on each interval of regularity, and thus the adiabatic system (3.3) is equivalent to the isentropic system (2.4). In this case the sonic wall is $\{\rho = \rho_s(\varepsilon_0)\}$, where

$$\rho_s(\varepsilon_0) = \left(\frac{J^2}{\gamma(\gamma - 1)\varepsilon_0} \right)^{\frac{1}{\gamma+1}}.$$

The sonic wall can be crossed only through the line $\left\{ \rho = \rho_s(\varepsilon_0), E = \frac{J}{\tau_p \rho} \right\}$, which we will call the sonic line. Using the results of section 2.2 we can easily depict now the volume phase portrait of the system (3.3). In the next figure we depict only one transonic trajectory, which passes through the sonic line from the subsonic region to the supersonic region.

Clearly there are no smooth transonic solutions, but only shocks. Unfortunately, at the present time we have not yet found the conditions to have shocks with more than one jump, but we are quite sure that they exist.

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