PERIODIC SOLUTION OF A NONAUTONOMOUS STAGE-STRUCTURED SINGLE SPECIES MODEL WITH DIFFUSION

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Abstract. A stage-structured single species model with diffusion is considered in which the coefficients are time-dependent. By using the continuation theorem of coincidence degree theory, a sufficient condition is obtained for the existence of a positive periodic solution for this model.

1. Introduction. Recently the so-called single species model was considered and many results were obtained; see [1]–[6]. In [1], the stage-structured single species model, given by

$$\begin{aligned}\dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} \phi(t - \tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} \phi(t - \tau) - \beta x_m^2(t), \quad 0 < t \leq \tau, \\ \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t - \tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t - \tau) - \beta x_m^2(t), \quad t > \tau,\end{aligned}$$

was studied, where $x_i(t)$ and $x_m(t)$ represent the immature and mature population densities, respectively; ϕ is the observed or assumed birth rate of $x_i(t)$ at time t ($-\tau \leq t \leq 0$), and τ represents a constant time length for the immature to become mature. α is the birth rate of mature population. γ is the death rate of the immature population and β is the preventing function caused by loss of limited resource of the mature population. Some sufficient conditions for the stability of the positive equilibrium were also given.

In [4], the nonautonomous single species models with diffusion were considered and the following model was proposed:

$$\begin{cases} \dot{x}_1 = x_1(r_1(t) - a_{11}x_1) + D_1(t)(x_2 - x_1), \\ \dot{x}_2 = x_2(r_2(t) - a_{22}x_2) + D_2(t)(x_1 - x_2). \end{cases}$$

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Species x grows in both patch 1 and patch 2; i.e., species x can diffuse between patch 1 and patch 2, where $x_1(t)$ is the density of the species in patch 1 at time t ($t \geq 0$) and $x_2(t)$ denotes the density in patch 2. $D_1(t)$ and $D_2(t)$ are diffusion coefficients. In [4], criteria for the existence and uniqueness of periodic solutions of these systems were established.

However, for a single species which has both stage structure and the ability of diffusion, there is no proper model studied as yet. For many species, their rate of death and the ability to diffuse are greatly different in each stage of their life stories. For example, the immature red-crowned crane is unable to migrate until it reaches a certain life stage, while in its mature stage, it is able to migrate. As another example, some fish have much higher death rates in their immature stage than in their mature stage. Hence it is not proper to take for granted that a given species has the same death rate in every stage of its life. Therefore the models proposed in [4]-[6], in which stage-structures were ignored, are not necessarily correct.

On the other hand, in papers [1]-[3], the diffusion is ignored and all the coefficients are constants; that is, the environments do not vary with time. Obviously those populations with time-dependence and diffusion cannot be described by these models. Hence it is significant for us to study stage-structured nonautonomous single species models with diffusion.

In this paper, we assume that the single species populations are growing in a two-patch environment, each of which is closed and homogeneous. Let $x_1(t)$ and $x_2(t)$ denote the densities of immature and mature populations in patch 1. Let $y_1(t)$ and $y_2(t)$ denote the densities of the immature and mature populations in patch 2. Now, our model is described by the following system:

$$(1.1) \quad \begin{cases} \dot{x}_1 = b_1(t)x_2 - g_1(t)x_1 - d_1(t)x_1^2, \\ \dot{x}_2 = g_1(t)x_1 - D_1(t)x_2^2 + \lambda_1(t)(y_2 - x_2), \\ \dot{y}_1 = b_2(t)y_2 - g_2(t)y_1 - d_2(t)y_1^2, \\ \dot{y}_2 = g_2(t)y_1 - D_2(t)y_2^2 + \lambda_2(t)(x_2 - y_2), \end{cases}$$

where $b_1(t)$ is the birth rate of the immature population in patch 1 and $b_2(t)$ is the birth rate in patch 2; $g_1(t)$ and $g_2(t)$ denote the transformation rates of the immature populations into the mature in patch 1 and patch 2, respectively. $d_1(t)$ and $d_2(t)$ are the density-dependent coefficients of the immature population in patch 1 and 2, respectively, and $D_1(t)$ and $D_2(t)$ are the same for the mature population in the corresponding patch. $\lambda_1(t)$ and $\lambda_2(t)$ are diffusion coefficients. We assume that all the coefficients in the model are strictly positive continuous ω -periodic functions.

Our purpose in this paper is, by using Mawhin's continuation theorem [7], [8], to study the existence of positive ω -periodic solutions for system (1.1). For work concerning the existence of periodic solutions of delay differential equations, which was done using coincidence degree theory, we refer to [9], [10] and references cited therein.

2. Existence of a positive periodic solution. In this section, by using Mawhin's continuation theorem, we shall show the existence of at least one positive periodic solution of system (1.1). To do so, we need to make some preparations.

Let X and Y be real Banach spaces, let $L : \text{Dom} L \subset X \rightarrow Y$ be a Fredholm mapping of index zero, and let $P : X \rightarrow X$, $Q : Y \rightarrow Y$ be continuous projectors such that $\text{Im} P = \text{Ker} L$, $\text{Ker} Q = \text{Im} L$, and $X = \text{Ker} L \oplus \text{Ker} P$, $Y = \text{Im} L \oplus \text{Im} Q$. Denote by L_p the restriction of L to $\text{Dom} L \cap \text{Ker} P$, $K_p : \text{Im} L \rightarrow \text{Ker} P \cap \text{Dom} L$ the inverse (to L_p), and $J : \text{Im} Q \rightarrow \text{Ker} L$ an isomorphism of $\text{Im} Q$ onto $\text{Ker} L$.

For convenience, we introduce Mawhin's continuation theorem [7, p. 40] as follows.

LEMMA 2.1. Let $\Omega \subset X$ be an open bounded set and let $N : X \rightarrow Y$ be a continuous operator which is L -compact on $\overline{\Omega}$ (i.e., $QN : \overline{\Omega} \rightarrow Y$ and $K_p(I - Q)N : \overline{\Omega} \rightarrow Y$ are compact). Assume

- (i) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom} L$, $Lx \neq \lambda Nx$;
- (ii) for each $x \in \partial\Omega \cap \text{Ker} L$, $QNx \neq 0$;
- (iii) $\deg\{JQN, \Omega \cap \text{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom} L$.

In what follows, we use the notation

$$\overline{f} = \frac{1}{\omega} \int_0^\omega f(t) dt, \quad f^l = \min_{t \in [0, \omega]} |f(t)|, \quad f^u = \max_{t \in [0, \omega]} |f(t)|,$$

where f is a periodic continuous function with period $\omega > 0$.

In system (1.1), we always assume the following.

ASSUMPTION (H1). $b_i(t), g_i(t), d_i(t), D_i(t), \lambda_i(t)$ ($i = 1, 2$) are positive periodic continuous functions with period $\omega > 0$.

Now we state our fundamental theorem about the existence of a positive ω -periodic solution of system (1.1).

THEOREM 2.1. In addition to Assumption (H1), we assume

- (H2) $g_1^l b_1^l > g_1^u \lambda_1^u$;
- (H3) $g_2^l b_2^l > g_2^u \lambda_2^u$.

Then system (1.1) has at least one positive ω -periodic solution.

Proof. Consider the system

$$(2.1) \quad \begin{cases} \frac{du_1(t)}{dt} = b_1(t)e^{u_2(t)-u_1(t)} - g_1(t) - d_1(t)e^{u_1(t)}, \\ \frac{du_2(t)}{dt} = g_1(t)e^{u_1(t)-u_2(t)} - D_1(t)e^{u_2(t)} + \lambda_1(t)(e^{u_4(t)-u_2(t)} - 1), \\ \frac{du_3(t)}{dt} = b_2(t)e^{u_4(t)-u_3(t)} - g_2(t) - d_2(t)e^{u_3(t)}, \\ \frac{du_4(t)}{dt} = g_2(t)e^{u_3(t)-u_4(t)} - D_2(t)e^{u_4(t)} + \lambda_2(t)(e^{u_2(t)-u_4(t)} - 1), \end{cases}$$

where $b_i(t), g_i(t), d_i(t), D_i(t), \lambda_i(t)$ ($i = 1, 2$) are the same as those in Assumption (H1). Let $x_i(t) = e^{u_i(t)}$, $i = 1, 2$, $y_1(t) = e^{u_3(t)}$, $y_2(t) = e^{u_4(t)}$. Then system (1.1) changes into system (2.1). Hence it is easy to see that if system (2.1) has an ω -periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t))^T$, then $(e^{u_1^*(t)}, e^{u_2^*(t)}, e^{u_3^*(t)}, e^{u_4^*(t)})^T$ is a positive ω -periodic solution of system (1.1). Therefore, for (1.1) to have at least one positive ω -periodic solution, it is sufficient that (2.1) have at least one ω -periodic solution. In order to apply

Lemma 2.1 to system (2.1), we take

$$X = Y = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in C(R, R^4) : u_i(t + \omega) = u_i(t), i = 1, 2, 3, 4\}$$

and

$$\|(u_1(t), u_2(t), u_3(t), u_4(t))^T\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)| + \max_{t \in [0, \omega]} |u_3(t)| + \max_{t \in [0, \omega]} |u_4(t)|,$$

where $|\cdot|$ denotes the Euclidean norm in R . With the norm $\|\cdot\|$, X is a Banach space. Set $L : \text{Dom} L \cap X$, $L(u_1(t), u_2(t), u_3(t), u_4(t))^T = (u'_1(t), u'_2(t), u'_3(t), u'_4(t))^T$, where $\text{Dom} L = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in C^1(R, R^4)\}$, $N : X \rightarrow X$, and

$$N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} b_1(t)e^{u_2(t)-u_1(t)} - g_1(t) - d_1(t)e^{u_1(t)} \\ g_1(t)e^{u_1(t)-u_2(t)} - D_1(t)e^{u_2(t)} + \lambda_1(t)(e^{u_4(t)-u_2(t)} - 1) \\ b_2(t)e^{u_4(t)-u_3(t)} - g_2(t) - d_2(t)e^{u_3(t)} \\ g_2(t)e^{u_3(t)-u_4(t)} - D_2(t)e^{u_4(t)} + \lambda_2(t)(e^{u_2(t)-u_4(t)} - 1) \end{bmatrix}.$$

Define two projectors P and Q as follows:

$$P \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = Q \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega u_1(t) dt \\ \frac{1}{\omega} \int_0^\omega u_2(t) dt \\ \frac{1}{\omega} \int_0^\omega u_3(t) dt \\ \frac{1}{\omega} \int_0^\omega u_4(t) dt \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in X.$$

Clearly, $\text{Ker} L = R^4$, $\text{Im} L = \{(u_1, u_2, u_3, u_4)^T \in X : \int_0^\omega u_i(t) dt = 0, i = 1, 2, 3, 4\}$ is closed in X and $\dim \text{Ker} L = \text{codim} \text{Im} L = 4$. Hence L is a Fredholm mapping of index zero. Furthermore, through an easy computation, we can find that the inverse K_p of L_p has the form $K_p : \text{Im} L \rightarrow \text{Dom} L \cap \text{Ker} P$,

$$K_p \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_0^t u_1(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^\eta u_1(s) dt d\eta \\ \int_0^t u_2(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^\eta u_2(s) dt d\eta \\ \int_0^t u_3(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^\eta u_3(s) dt d\eta \\ \int_0^t u_4(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^\eta u_4(s) dt d\eta \end{bmatrix}.$$

Obviously, we can prove that QN and $K_p(I-Q)N$ are continuous by Lebesgue's theorem and that $QN(\overline{\Omega})$, $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$ by the Arzela-Ascoli theorem. Therefore, N is L -compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

(2.2)

$$\begin{cases} \frac{du_1(t)}{dt} = \lambda [b_1(t)e^{u_2(t)-u_1(t)} - g_1(t) - d_1(t)e^{u_1(t)}], \\ \frac{du_2(t)}{dt} = \lambda [g_1(t)e^{u_1(t)-u_2(t)} - D_1(t)e^{u_2(t)} + \lambda_1(t)(e^{u_4(t)-u_2(t)} - \lambda_1(t))], \\ \frac{du_3(t)}{dt} = \lambda [b_2(t)e^{u_4(t)-u_3(t)} - g_2(t) - d_2(t)e^{u_3(t)}], \\ \frac{du_4(t)}{dt} = \lambda [g_2(t)e^{u_3(t)-u_4(t)} - D_2(t)e^{u_4(t)} + \lambda_2(t)(e^{u_2(t)-u_4(t)} - \lambda_2(t))]. \end{cases}$$

Suppose that $(u_1(t), u_2(t), u_3(t), u_4(t))^T \in X$ is a solution of system (2.2) for a certain $\lambda \in (0, 1)$. Integrating (2.2) from 0 to ω gives

$$(2.3) \quad \int_0^\omega b_1(t)e^{u_2(t)-u_1(t)}dt = \int_0^\omega g_1(t)dt + \int_0^\omega d_1(t)e^{u_1(t)}dt,$$

$$(2.4) \quad \int_0^\omega g_1(t)e^{u_1(t)-u_2(t)}dt + \int_0^\omega \lambda_1(t)e^{u_4(t)-u_2(t)}dt = \int_0^\omega D_1(t)e^{u_2(t)}dt + \int_0^\omega \lambda_1(t)dt,$$

$$(2.5) \quad \int_0^\omega b_2(t)e^{u_4(t)-u_3(t)}dt = \int_0^\omega g_2(t)dt + \int_0^\omega d_2(t)e^{u_3(t)}dt$$

and

$$(2.6) \quad \int_0^\omega g_2(t)e^{u_3(t)-u_4(t)}dt + \int_0^\omega \lambda_2(t)e^{u_2(t)-u_4(t)}dt = \int_0^\omega D_2(t)e^{u_4(t)}dt + \int_0^\omega \lambda_2(t)dt.$$

From (2.2)-(2.6), it follows that

$$(2.7) \quad \begin{aligned} \int_0^\omega |u'_1(t)|dt &\leq \int_0^\omega b_1(t)e^{u_2(t)-u_1(t)}dt + \int_0^\omega g_1(t)dt + \int_0^\omega d_1(t)e^{u_1(t)}dt \\ &= 2\bar{g}_1\omega + 2 \int_0^\omega d_1(t)e^{u_1(t)}dt \\ &\leq 2\bar{g}_1\omega + 2d_1^u \int_0^\omega e^{u_1(t)}dt, \end{aligned}$$

$$(2.8) \quad \begin{aligned} \int_0^\omega |u'_2(t)|dt &\leq \int_0^\omega g_1(t)e^{u_1(t)-u_2(t)}dt + \int_0^\omega D_1(t)e^{u_2(t)}dt + \int_0^\omega \lambda_1(t)e^{u_4(t)-u_2(t)}dt + \int_0^\omega \lambda_1(t)dt \\ &= 2\bar{\lambda}_1\omega + 2 \int_0^\omega D_1(t)e^{u_2(t)}dt \\ &\leq 2\bar{\lambda}_1\omega + 2D_1^u \int_0^\omega e^{u_2(t)}dt, \end{aligned}$$

$$(2.9) \quad \begin{aligned} \int_0^\omega |u'_3(t)|dt &\leq \int_0^\omega b_2(t)e^{u_4(t)-u_3(t)}dt + \int_0^\omega g_2(t)dt + \int_0^\omega d_2(t)e^{u_3(t)}dt \\ &= 2\bar{g}_2\omega + 2 \int_0^\omega d_2(t)e^{u_3(t)}dt \\ &\leq 2\bar{g}_2\omega + 2d_2^u \int_0^\omega e^{u_3(t)}dt \end{aligned}$$

and

$$\begin{aligned}
 (2.10) \quad & \int_0^\omega |u_4'(t)| dt \\
 & \leq \int_0^\omega g_2(t) e^{u_3(t)-u_4(t)} dt + \int_0^\omega D_2(t) e^{u_4(t)} dt + \int_0^\omega \lambda_2(t) e^{u_2(t)-u_4(t)} dt + \int_0^\omega \lambda_2(t) dt \\
 & = 2\bar{\lambda}_2\omega + 2 \int_0^\omega D_2(t) e^{u_4(t)} dt \\
 & \leq 2\bar{\lambda}_2\omega + 2D_2^u \int_0^\omega e^{u_4(t)} dt.
 \end{aligned}$$

From (2.2), we get

$$(2.11) \quad \int_0^\omega d_1(t) e^{2u_1(t)} dt + \int_0^\omega g_1(t) e^{u_1(t)} dt = \int_0^\omega b_1(t) e^{u_2(t)} dt,$$

$$(2.12) \quad \int_0^\omega D_1(t) e^{2u_2(t)} dt + \int_0^\omega \lambda_1(t) e^{u_2(t)} dt = \int_0^\omega g_1(t) e^{u_1(t)} dt + \int_0^\omega \lambda_1(t) e^{u_4(t)} dt,$$

$$(2.13) \quad \int_0^\omega d_2(t) e^{2u_3(t)} dt + \int_0^\omega g_2(t) e^{u_3(t)} dt = \int_0^\omega b_2(t) e^{u_4(t)} dt$$

and

$$(2.14) \quad \int_0^\omega D_2(t) e^{2u_4(t)} dt + \int_0^\omega \lambda_2(t) e^{u_4(t)} dt = \int_0^\omega g_2(t) e^{u_3(t)} dt + \int_0^\omega \lambda_2(t) e^{u_2(t)} dt.$$

By using the inequality

$$\int_0^\omega e^{2u_i(t)} dt \geq \frac{1}{\omega} \left(\int_0^\omega e^{u_i(t)} dt \right)^2, \quad \text{for } i = 1, 2, 3, 4,$$

from (2.11)-(2.14), we obtain

$$(2.15) \quad \frac{d_1^l}{\omega} \left(\int_0^\omega e^{u_1(t)} dt \right)^2 + g_1^l \int_0^\omega e^{u_1(t)} dt \leq b_1^u \int_0^\omega e^{u_2(t)} dt,$$

$$(2.16) \quad \frac{D_1^l}{\omega} \left(\int_0^\omega e^{u_2(t)} dt \right)^2 + \lambda_1^l \int_0^\omega e^{u_2(t)} dt \leq g_1^u \int_0^\omega e^{u_1(t)} dt + \lambda_1^u \int_0^\omega e^{u_4(t)} dt,$$

$$(2.17) \quad \frac{d_2^l}{\omega} \left(\int_0^\omega e^{u_3(t)} dt \right)^2 + g_2^l \int_0^\omega e^{u_3(t)} dt \leq b_2^u \int_0^\omega e^{u_4(t)} dt$$

and

$$(2.18) \quad \frac{D_2^l}{\omega} \left(\int_0^\omega e^{u_4(t)} dt \right)^2 + \lambda_2^l \int_0^\omega e^{u_4(t)} dt \leq g_2^u \int_0^\omega e^{u_3(t)} dt + \lambda_2^u \int_0^\omega e^{u_2(t)} dt.$$

Inequality (2.15) gives

$$\frac{d_1^l}{\omega} \left(\int_0^\omega e^{u_1(t)} dt \right)^2 < b_1^u \int_0^\omega e^{u_2(t)} dt,$$

that is,

$$(2.19) \quad \int_0^\omega e^{u_1(t)} dt < \sqrt{\frac{b_1^u \omega}{d_1^l}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}}.$$

Substituting (2.19) into (2.16) gives

$$\frac{D_1^l}{\omega} \left(\int_0^\omega e^{u_2(t)} dt \right)^2 + \lambda_1^l \int_0^\omega e^{u_2(t)} dt < g_1^u \sqrt{\frac{b_1^u \omega}{d_1^l}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}} + \lambda_1^u \int_0^\omega e^{u_4(t)} dt,$$

from which, by using the inequality

$$1 \cdot \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}} \leq \frac{1}{2} + \frac{\int_0^\omega e^{u_2(t)} dt}{2},$$

we get

$$(2.20) \quad \frac{D_1^l}{\omega} \left(\int_0^\omega e^{u_2(t)} dt \right)^2 < \frac{g_1^u}{2} \sqrt{\frac{b_1^u \omega}{d_1^l}} \int_0^\omega e^{u_2(t)} dt + \lambda_1^u \int_0^\omega e^{u_4(t)} dt + \frac{g_1^u}{2} \sqrt{\frac{b_1^u \omega}{d_1^l}}.$$

Then

$$(2.21) \quad \frac{2D_1^l}{\omega} \int_0^\omega e^{u_2(t)} dt < \frac{g_1^u}{2} \sqrt{\frac{b_1^u \omega}{d_1^l}} + \left\{ \frac{(g_1^u)^2 b_1^u \omega}{4d_1^l} + \frac{4D_1^l \lambda_1^u}{\omega} \int_0^\omega e^{u_4(t)} dt + \frac{2D_1^l g_1^u}{\omega} \sqrt{\frac{b_1^u \omega}{d_1^l}} \right\}^{\frac{1}{2}}.$$

By using the inequality

$$(2.22) \quad (a + b)^{\frac{1}{2}} < a^{\frac{1}{2}} + b^{\frac{1}{2}}, \quad \text{for } a \geq 0, b \geq 0,$$

and from (2.21), we obtain

$$\frac{2D_1^l}{\omega} \int_0^\omega e^{u_2(t)} dt < g_1^u \sqrt{\frac{b_1^u \omega}{d_1^l}} + 2 \sqrt{\frac{D_1^l \lambda_1^u}{\omega}} \left(\int_0^\omega e^{u_4(t)} dt \right)^{\frac{1}{2}} + \sqrt{\frac{2D_1^l g_1^u}{\omega}} \sqrt{\frac{b_1^u \omega}{d_1^l}}.$$

Therefore there exists a positive constant d_3 such that

$$(2.23) \quad \int_0^\omega e^{u_2(t)} dt < d_3 + \frac{\omega}{D_1^l} \sqrt{\frac{D_1^l \lambda_1^u}{\omega}} \left(\int_0^\omega e^{u_4(t)} dt \right)^{\frac{1}{2}}.$$

From (2.17) and (2.18), a parallel argument to (2.23) shows that

$$(2.24) \quad \int_0^\omega e^{u_4(t)} dt < d_4 + \frac{\omega}{D_2^l} \sqrt{\frac{D_2^l \lambda_2^u}{\omega}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}},$$

where d_4 is a positive constant.

Substituting (2.24) into (2.23) gives

$$\int_0^\omega e^{u_2(t)} dt < d_3 + \frac{\omega}{D_1^l} \sqrt{\frac{D_1^l \lambda_1^u}{\omega}} \left[d_4 + \frac{\omega}{D_2^l} \sqrt{\frac{D_2^l \lambda_2^u}{\omega}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}} \right]^{\frac{1}{2}},$$

from which, by using inequality (2.22), we obtain

$$\begin{aligned} \int_0^\omega e^{u_2(t)} dt &< d_3 + \frac{\omega}{D_1^l} \sqrt{\frac{D_1^l \lambda_1^u}{\omega}} \left[\sqrt{d_4} + \sqrt{\frac{\omega}{D_2^l}} \sqrt{\frac{D_2^l \lambda_2^u}{\omega}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{4}} \right] \\ &= d_3 + \frac{\omega}{D_1^l} \sqrt{\frac{D_1^l \lambda_1^u d_4}{\omega}} + \frac{\omega}{D_1^l} \sqrt{\frac{D_1^l \lambda_1^u}{D_2^l}} \sqrt{\frac{D_2^l \lambda_2^u}{\omega}} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{4}}. \end{aligned}$$

Thus there exists a positive constant ρ_2 such that

$$(2.25) \quad \int_0^\omega e^{u_2(t)} dt < \rho_2.$$

Substituting (2.25) into (2.24), it follows that there exists a positive constant ρ_4 such that

$$(2.26) \quad \int_0^\omega e^{u_4(t)} dt < \rho_4.$$

From (2.17), (2.19), (2.25) and (2.26), there must exist two positive constants ρ_1 and ρ_3 such that

$$(2.27) \quad \int_0^\omega e^{u_1(t)} dt < \rho_1, \quad \int_0^\omega e^{u_3(t)} dt < \rho_3.$$

Thus from (2.7)-(2.10), (2.25)-(2.27), we obtain

$$(2.28) \quad \int_0^\omega |u_1'(t)| dt \leq 2\bar{g}_1\omega + 2d_1^u \rho_1 \stackrel{def}{=} \delta_1,$$

$$(2.29) \quad \int_0^\omega |u_2'(t)| dt \leq 2\bar{\lambda}_1\omega + 2D_1^u \rho_2 \stackrel{def}{=} \delta_2,$$

$$(2.30) \quad \int_0^\omega |u_3'(t)| dt \leq 2\bar{g}_2\omega + 2d_2^u \rho_3 \stackrel{def}{=} \delta_3,$$

and

$$(2.31) \quad \int_0^\omega |u_4'(t)| dt \leq 2\bar{\lambda}_2\omega + 2D_2^u \rho_4 \stackrel{def}{=} \delta_4.$$

Choose $t_i \in [0, \omega]$, $i = 1, 2, 3, 4$, such that

$$u_i(t_i) = \min_{t \in [0, \omega]} u_i(t), \quad i = 1, 2, 3, 4.$$

Then it is clear that

$$u_i'(t_i) = 0, \quad i = 1, 2, 3.$$

From this and (2.2), we have

$$(2.32) \quad b_1(t_1)e^{u_2(t_1)-u_1(t_1)} - g_1(t_1) - d_1(t_1)e^{u_1(t_1)} = 0,$$

$$(2.33) \quad g_1(t_2)e^{u_1(t_2)-u_2(t_2)} - D_1(t_2)e^{u_2(t_2)} + \lambda_1(t_2)(e^{u_4(t_2)-u_2(t_2)} - 1) = 0,$$

$$(2.34) \quad b_2(t_3)e^{u_4(t_3)-u_3(t_3)} - g_2(t_3) - d_2(t_3)e^{u_3(t_3)} = 0,$$

and

(2.35)

$$g_2(t_4)e^{u_3(t_4)-u_4(t_4)} - D_2(t_4)e^{u_4(t_4)} + \lambda_2(t_4)(e^{u_2(t_4)-u_4(t_4)} - 1) = 0.$$

Thus we obtain

(2.36)

$$d_1(t_1)e^{2u_1(t_1)} + g_1(t_1)e^{u_1(t_1)} = b_1(t_1)e^{u_2(t_1)},$$

(2.37)

$$D_1(t_2)e^{2u_2(t_2)} + \lambda_1(t_2)e^{u_2(t_2)} = g_1(t_2)e^{u_1(t_2)} + \lambda_1(t_2)e^{u_4(t_2)},$$

(2.38)

$$d_2(t_3)e^{2u_3(t_3)} + g_2(t_3)e^{u_3(t_3)} = b_2(t_3)e^{u_4(t_3)}$$

and

(2.39)

$$D_2(t_4)e^{2u_4(t_4)} + \lambda_2(t_4)e^{u_4(t_4)} = g_2(t_4)e^{u_3(t_4)} + \lambda_2(t_4)e^{u_2(t_4)}.$$

Equation (2.37) implies that

(2.40)

$$D_1^u e^{2u_2(t_2)} + \lambda_1^u e^{u_2(t_2)} > g_1^l e^{u_1(t_1)}.$$

From (2.36) and (2.40), we have

$$\begin{aligned} b_1^l e^{u_2(t_2)} &< b_1(t_1)e^{u_2(t_1)} < d_1^u e^{2u_1(t_1)} + g_1^u e^{u_1(t_1)} \\ &< \frac{g_1^u}{g_1^l} (D_1^u e^{2u_2(t_2)} + \lambda_1^u e^{u_2(t_2)}) + \frac{d_1^u}{(g_1^l)^2} (D_1^u e^{2u_2(t_2)} + \lambda_1^u e^{u_2(t_2)})^2, \end{aligned}$$

that is,

$$\begin{aligned} (g_1^l)^2 b_1^l e^{u_2(t_2)} &< g_1^u g_1^l D_1^u e^{2u_2(t_2)} + g_1^u g_1^l \lambda_1^u e^{u_2(t_2)} \\ &+ d_1^u (D_1^u)^2 e^{4u_2(t_2)} + d_1^u (\lambda_1^u)^2 e^{2u_2(t_2)} + 2d_1^u D_1^u \lambda_1^u e^{3u_2(t_2)}. \end{aligned}$$

Thus

$$\begin{aligned} d_1^u (D_1^u)^2 e^{3u_2(t_2)} + 2d_1^u D_1^u \lambda_1^u e^{2u_2(t_2)} + [g_1^u g_1^l D_1^u + d_1^u (\lambda_1^u)^2] e^{u_2(t_2)} \\ > (g_1^l)^2 b_1^l - g_1^u g_1^l \lambda_1^u = g_1^l (g_1^l b_1^l - g_1^u \lambda_1^u). \end{aligned}$$

Let $e^{u_2(t_2)} = z$. Then there exist some positive constants p_i ($i = 0, 1, 2$) such that $z^3 + p_2 z^2 + p_1 z = p_0$. Let $F(z) = z^3 + p_2 z^2 + p_1 z - p_0$. Then $F(0) = -p_0 < 0$, $F(+\infty) = +\infty > 0$. Hence there exists a point $\xi_2 \in [0, \omega]$ such that $F(\xi_2) = 0$. Thus from $z^3 + p_2 z^2 + p_1 z - p_0 > 0$, we have $(z - \xi_2) \left[z^2 + (p_2 + \xi_2)z + \frac{p_0}{\xi_2} \right] > 0$. Therefore $z > \xi_2$, a.e., $e^{u_2(t_2)} > \xi_2$. From this, for $\forall t \in [0, \omega]$, we have

(2.41)

$$e^{u_2(t)} > \xi_2.$$

Using a parallel argument to (2.41), from (2.38) and (2.39), it follows that there exists a positive constant ξ_4 such that

(2.42)

$$e^{u_4(t)} > \xi_4.$$

From (2.36), (2.38), (2.41) and (2.42), we have

(2.43)

$$e^{u_1(t)} > \xi_1, \quad e^{u_3(t)} > \xi_3,$$

where ξ_1 and ξ_3 are two positive constants.

From (2.25)-(2.27), it follows that there exist four points $\eta_i \in [0, \omega]$, $i = 1, 2, 3, 4$, such that

$$(2.44) \quad u_i(\eta_i) < \ln \frac{\rho_i}{\omega}, \quad i = 1, 2, 3, 4.$$

From (2.41)-(2.43), it follows that there exist four positive constants ρ_i^* such that

$$(2.45) \quad u_i(t) > -\rho_i^*.$$

Since for $\forall t \in [0, \omega]$,

$$u_i(t) = u_i(\eta_i) - \int_{\eta_i}^t u_i'(s) ds, \quad i = 1, 2, 3, 4,$$

from (2.28)-(2.31) and (2.44), we have

$$(2.46) \quad u_i(t) < \ln \frac{\rho_i}{\omega} + \int_0^\omega |u_i'(s)| ds < \left| \ln \frac{\rho_i}{\omega} \right| + \delta_i, \quad i = 1, 2, 3, 4.$$

From (2.45) and (2.46), we can obtain

$$|u_i(t)| \leq \max \left\{ \rho_i^*, \left| \ln \frac{\rho_i}{\omega} \right| + \delta_i \right\} \stackrel{\text{def}}{=} R_i, \quad i = 1, 2, 3, 4.$$

Clearly, R_i ($i = 1, 2, 3, 4$) are independent of λ . Denote $M = R_1 + R_2 + R_3 + R_4 + R_0$. Here R_0 is taken sufficiently large such that each solution $(\alpha^*, \beta^*, \gamma^*, \delta^*)^T$ of the system

$$(2.47) \quad \begin{cases} \bar{b}_1 e^{\beta-\alpha} - \bar{g}_1 - \bar{d}_1 e^\alpha = 0, \\ \bar{g}_1 e^{\alpha-\beta} - \bar{D}_1 e^\beta + \bar{\lambda}_1 (e^{\delta-\beta} - 1) = 0, \\ \bar{b}_2 e^{\delta-\gamma} - \bar{g}_2 - \bar{d}_2 e^\gamma = 0, \\ \bar{g}_2 e^{\gamma-\delta} - \bar{D}_2 e^\delta + \bar{\lambda}_2 (e^{\beta-\delta} - 1) = 0 \end{cases}$$

satisfies $\|(\alpha^*, \beta^*, \gamma^*, \delta^*)^T\| = |\alpha^*| + |\beta^*| + |\gamma^*| + |\delta^*| < M$, provided that system (2.47) has a solution or a number of solutions and that

$$\begin{aligned} & \max\{\rho_2^*, |\ln l|\} + \max \left\{ \rho_1^*, \ln \sqrt{\frac{\bar{b}_1 l}{\bar{d}_1}} \right\} \\ & + \max \left\{ \rho_3^*, \left| \frac{1}{2} \ln \frac{\bar{b}_2}{\bar{d}_2} \right| + \left| \ln \left(\frac{\bar{g}_2}{2\bar{D}_2} \sqrt{\frac{\bar{b}_2}{\bar{d}_2}} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}} \right) \right| \right\} \\ & + \max \left\{ \rho_4^*, \left| \ln \left(\frac{\bar{g}_2 \sqrt{\frac{\bar{b}_2}{\bar{d}_2}}}{2\bar{D}_2} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}} \right) \right| \right\} < M. \end{aligned}$$

Here, l is the only real root of equation

$$\bar{D}_1 x = \frac{\bar{g}_1}{2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}} + \sqrt{\bar{D}_1 \bar{\lambda}_1} \left[\sqrt{\frac{\bar{g}_2}{2\bar{D}_2} \sqrt{\frac{\bar{b}_2}{\bar{d}_2}}} + \sqrt[4]{\frac{\bar{\lambda}_2}{\bar{D}_2}} x^{\frac{1}{4}} + \sqrt[4]{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}} \right] + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}}.$$

Now we take $\Omega = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in X : \|(u_1, u_2, u_3, u_4)^T\| < M\}$. This satisfies condition (i) of Lemma 2.1. When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap \text{Ker} L = \partial\Omega \cap R^4$,

$(u_1, u_2, u_3, u_4)^T$ is a constant vector in R^4 with $|u_1| + |u_2| + |u_3| + |u_4| = M$. If system (2.47) has a solution or a number of solutions, then

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 e^{u_2 - u_1} - \bar{g}_1 - \bar{d}_1 e^{u_1} \\ \bar{g}_1 e^{u_1 - u_2} - \bar{D}_1 e^{u_2} + \bar{\lambda}_1 (e^{u_4 - u_2} - 1) \\ \bar{b}_2 e^{u_4 - u_3} - \bar{g}_2 - \bar{d}_2 e^{u_3} \\ \bar{g}_2 e^{u_3 - u_4} - \bar{D}_2 e^{u_4} + \bar{\lambda}_2 (e^{u_2 - u_4} - 1) \end{bmatrix} \neq 0.$$

If system (2.47) does not have a solution, then naturally

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This proves that condition (ii) of Lemma 2.1 is satisfied. Finally we will prove that condition (iii) of Lemma 2.1 is satisfied. To this end, we define $\phi : \text{Dom} L \times [0, 1] \rightarrow X$ by

$$\phi(u_1, u_2, u_3, u_4, \mu) = \begin{bmatrix} \bar{b}_1 e^{u_2 - u_1} - \bar{d}_1 e^{u_1} \\ \bar{g}_1 e^{u_1 - u_2} - \bar{D}_1 e^{u_2} \\ \bar{b}_2 e^{u_4 - u_3} - \bar{d}_2 e^{u_3} \\ \bar{g}_2 e^{u_3 - u_4} - \bar{D}_2 e^{u_4} \end{bmatrix} + \mu \begin{bmatrix} -\bar{g}_1 \\ \bar{\lambda}_1 (e^{u_4 - u_2} - 1) \\ -\bar{g}_2 \\ \bar{\lambda}_2 (e^{u_2 - u_4} - 1) \end{bmatrix},$$

where $\mu \in [0, 1]$ is a parameter. When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap \text{Ker} L = \partial\Omega \cap R^4$, $(u_1, u_2, u_3, u_4)^T$ is a constant vector in R^4 with $|u_1| + |u_2| + |u_3| + |u_4| = M$. We will show that when $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap \text{Ker} L$, $\phi(u_1, u_2, u_3, u_4, \mu) \neq 0$. If the conclusion is not true, a.e., the constant vector $(u_1, u_2, u_3, u_4)^T$ with $|u_1| + |u_2| + |u_3| + |u_4| = M$ satisfies $\phi(u_1, u_2, u_3, u_4, \mu) = 0$, and then from

$$\begin{cases} \bar{b}_1 e^{u_2 - u_1} - \bar{d}_1 e^{u_1} - \mu \bar{g}_1 = 0, \\ \bar{g}_1 e^{u_1 - u_2} - \bar{D}_1 e^{u_2} + \mu \bar{\lambda}_1 (e^{u_4 - u_2} - 1) = 0, \\ \bar{b}_2 e^{u_4 - u_3} - \bar{d}_2 e^{u_3} - \mu \bar{g}_2 = 0, \\ \bar{g}_2 e^{u_3 - u_4} - \bar{D}_2 e^{u_4} + \mu \bar{\lambda}_2 (e^{u_2 - u_4} - 1) = 0, \end{cases}$$

by following the argument of (2.44) and (2.45), we obtain

$$|u_1| < \max \left\{ \rho_1^*, \left| \ln \sqrt{\frac{\bar{b}_1 l}{\bar{d}_1}} \right| \right\},$$

$$|u_2| < \max \{ \rho_2^*, |\ln l| \},$$

$$|u_3| < \max \left\{ \rho_3^*, \left| \frac{1}{2} \ln \frac{\bar{b}_2}{\bar{d}_2} + \left| \ln \left(\frac{\bar{g}_2}{2\bar{D}_2} \sqrt{\frac{\bar{b}_2}{\bar{d}_2}} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}} \right) \right| \right\}$$

and

$$|u_4| < \max \left\{ \rho_4^*, \left| \ln \left(\frac{\bar{g}_2 \sqrt{\frac{\bar{b}_2}{\bar{d}_2}}}{2\bar{D}_2} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}} \right) \right| \right\}.$$

Thus

$$\begin{aligned} \sum_{i=1}^4 |u_i| &< \max\{\rho_2^*, |\ln l|\} + \max\left\{\rho_1^*, \ln \sqrt{\frac{\bar{b}_1 l}{\bar{d}_1}}\right\} \\ &+ \max\left\{\rho_3^*, \left|\frac{1}{2} \ln \frac{\bar{b}_2}{\bar{d}_2}\right| + \left|\ln \left(\frac{\bar{g}_2}{2\bar{D}_2} \sqrt{\frac{\bar{b}_2}{\bar{d}_2}} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}}\right)\right|\right\} \\ &+ \max\left\{\rho_4^*, \left|\ln \left(\frac{\bar{g}_2 \sqrt{\frac{\bar{b}_2}{\bar{d}_2}}}{2\bar{D}_2} + \sqrt{\frac{\bar{\lambda}_2 l}{\bar{D}_2}} + \sqrt{\frac{\bar{D}_1 \bar{g}_1}{2\bar{D}_2} \sqrt{\frac{\bar{b}_1}{\bar{d}_1}}}\right)\right|\right\} < M, \end{aligned}$$

which contradicts the fact that $\sum_{i=1}^4 |u_i| = M$. Therefore when $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap \text{Ker} L$, $\phi(u_1, u_2, u_3, u_4, \mu) \neq 0$. According to topological degree theory and by taking $J = I : \text{Im} Q \rightarrow \text{Ker} L$, $(u_1, u_2, u_3, u_4)^T \rightarrow (u_1, u_2, u_3, u_4)^T$, we have

$$\begin{aligned} &\deg(JQN(u_1, u_2, u_3, u_4)^T, \Omega \cap \text{Ker} L, (0, 0, 0, 0)^T) \\ &= \deg(\phi(u_1, u_2, u_3, u_4, 1), \Omega \cap \text{Ker} L, (0, 0, 0, 0)^T) \\ &= \deg(\phi(u_1, u_2, u_3, u_4, 0), \Omega \cap \text{Ker} L, (0, 0, 0, 0)^T) \\ &= \deg((\bar{b}_1 e^{u_2 - u_1} - \bar{d}_1 e^{u_1}, \bar{g}_1 e^{u_1 - u_2} - \bar{D}_1 e^{u_2}, \bar{b}_2 e^{u_4 - u_3} - \bar{d}_2 e^{u_3}, \\ &\quad \bar{g}_2 e^{u_3 - u_4} - \bar{D}_2 e^{u_4})^T, \Omega \cap \text{Ker} L, (0, 0, 0, 0)^T). \end{aligned}$$

In view of the fact that the system of algebraic equations

$$\begin{cases} \bar{b}_1 \frac{y}{x} - \bar{d}_1 x = 0, \\ \bar{g}_1 \frac{x}{y} - \bar{D}_1 y = 0, \\ \frac{\bar{b}_2 u}{z} - \bar{d}_2 z = 0, \\ \frac{\bar{g}_2 z}{u} - \bar{D}_2 u = 0 \end{cases}$$

has a unique solution $(x^*, y^*, z^*, u^*)^T$ which satisfies $x^* > 0, y^* > 0, z^* > 0$ and $u^* > 0$, it follows that

$$\begin{aligned} &\deg((\bar{b}_1 e^{u_2 - u_1} - \bar{d}_1 e^{u_1}, \bar{g}_1 e^{u_1 - u_2} - \bar{D}_1 e^{u_2}, \bar{b}_2 e^{u_4 - u_3} - \bar{d}_2 e^{u_3}, \\ &\quad \bar{g}_2 e^{u_3 - u_4} - \bar{D}_2 e^{u_4})^T, \Omega \cap \text{Ker} L, (0, 0, 0, 0)^T) \\ &= \text{sign} \begin{vmatrix} -\frac{\bar{b}_1 y^*}{x^*} - \bar{d}_1 x^* & \frac{\bar{b}_1 y^*}{x^*} & 0 & 0 \\ \frac{\bar{g}_1 x^*}{y^*} & -\frac{\bar{g}_1 x^*}{y^*} - \bar{D}_1 y^* & 0 & 0 \\ 0 & 0 & -\frac{\bar{b}_2 u^*}{z^*} - \bar{d}_2 z^* & \frac{\bar{b}_2 u^*}{z^*} \\ 0 & 0 & \frac{\bar{g}_2 z^*}{u^*} & -\frac{\bar{g}_2 z^*}{u^*} - \bar{D}_2 u^* \end{vmatrix} \\ &= \text{sign} \left[\left(\frac{\bar{b}_1 \bar{D}_1 (y^*)^2}{x^*} + \frac{\bar{d}_1 \bar{g}_1 (x^*)^2}{y^*} + \bar{d}_1 \bar{D}_1 x^* y^* \right) \right. \\ &\quad \left. \times \left(\frac{\bar{d}_2 \bar{g}_2 (z^*)^2}{u^*} + \bar{d}_2 \bar{D}_2 u^* z^* + \frac{\bar{b}_2 \bar{D}_2 (u^*)^2}{z^*} \right) \right] \\ &= 1. \end{aligned}$$

This completes the proof of condition (iii) of Lemma 2.1. Now Ω verifies all the requirements of Lemma 2.1 and then system (2.1) has at least one ω -periodic solution. This completes the proof of Theorem 2.1.

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