ASYMPTOTIC BEHAVIOUR OF PRESSURE AND STRESSES IN A THIN FILM FLOW WITH A ROUGH BOUNDARY

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Abstract. We study the asymptotic behaviour of an incompressible viscous flow in a narrow gap with a thickness of order η and a rough surface. The roughness is defined by a quasi-periodical function with period ϵ . In both cases, when η is smaller or at the same order as ϵ , we obtain different Reynolds equations for the pressure. We have also studied the convergence of the stresses on the rough boundary and we discuss the different cases.

1. Introduction. In this paper, we study the asymptotic behaviour of an incompressible fluid in a narrow gap with a thickness of order η , the boundaries being rough with a roughness period ϵ . Neglecting the inertial terms and the exterior forces, we consider the following Stokes system:

$$(\mathcal{P}^{\eta\epsilon}) \quad \begin{cases} \mu \Delta u^{\eta\epsilon} = \nabla p^{\eta\epsilon} \\ \operatorname{div} u^{\eta\epsilon} = 0 \end{cases}$$

with adhering boundary conditions, where $u^{\eta\epsilon}, p^{\eta\epsilon}$ are the velocity and the pressure and μ is the viscosity of the fluid.

In some papers [1, 4, 17], the roughness of the boundary is taken into account by means of a periodic function, with a small height amplitude of the same order of magnitude as the period, while the flow height is not small. In [14, 13], studies appear related to lubrication from a mechanical point of view. Mathematical asymptotic studies using two parameters η and ϵ , which are supposed to tend to zero, appear in a formal way in [6]

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and rigorously in [5]. The authors proved that the way the two parameters tend to zero is primordial, since the limiting equations are different whether ϵ tends to zero faster, slower, or at the same rate as η . It is a very common situation when problems with several small parameters lead to different limit models [2, 8, 10, 18]. In [5] and [6] the authors considered the case

$$\eta = \lambda\epsilon \quad \text{with } \lambda > 0 \quad (1.1)$$

and they obtained a limit problem $(\mathcal{P}^\lambda)$ under the hypothesis that the gap is uniformly periodic.

In the present study, we consider a quasi-periodic gap. The significant variables for mechanicians are the pressure and the forces exerted by the fluid on the surface walls. We are interested in the asymptotic study of both the pressure and the boundary stresses, so we set:

$$\eta = \lambda\epsilon^\alpha \quad \text{with } \lambda > 0, \quad \alpha \geq 1 \quad (1.2)$$

the case $0 < \alpha < 1$, being meaningless for the lubrication.

The case $\alpha = 1$ corresponds to a narrow gap of the same order of magnitude as the roughness period and generalizes the results in [5] for the pressure. The asymptotic pressure satisfies a homogenized Reynolds equation $(\mathcal{P}^\lambda)$ whose coefficients depend on solutions to local problems being 3D Stokes equations.

The case $\alpha > 1$ corresponds to a narrow gap smaller than the roughness period. The asymptotic pressure satisfies a homogenized Reynolds equation $(\mathcal{P}^*)$ whose coefficients depend on solutions to local problems being 2D elliptic equations. The same limit problem $(\mathcal{P}^*)$ is obtained by passing to the limit $\lambda \rightarrow 0$ in $(\mathcal{P}^\lambda)$. On the other hand, by passing to the limit $\eta \rightarrow 0$ and then $\epsilon \rightarrow 0$ in $(\mathcal{P}^{\eta\epsilon})$, the same limit problem $(\mathcal{P}^*)$ is also obtained (see [5]).

The main contribution of this paper concerns the stress vector on the rough boundary for which the situation is more complex. A priori estimates lead to another critical exponent $\alpha = 3/2$. For $\alpha > 3/2$, the stress limit behaviour is the same as letting first η then ϵ tend to zero in $(\mathcal{P}^{\eta\epsilon})$. For $\alpha = 3/2$, a supplementary term depending on the parameter λ appears in the limit. For $1 < \alpha < 3/2$, the two first components of the stresses are not bounded; nevertheless, we characterise their behaviour by obtaining the convergences after multiplication by $\epsilon^{3-2\alpha}$.

The paper is organized as follows. The problem is presented in Section 2 with two equivalent variational formulations, the first after rescaling and the second after rescaling and flattening the real domain. The main results are given in Section 3 and concern the convergence of the pressure and that of the stresses. The proofs of each previous result are given in Section 4 for the pressure and in Section 5 for the stresses.

2. Statement of the problem. Let us begin with some notation. We write $Z = (z_1, z_2, z_3)$ for a current point in $\mathbb{R}^3$ and $z = (z_1, z_2)$ for its projection in $\mathbb{R}^2$. ω is an open set in $\mathbb{R}^2$ with a Lipschitz boundary and $Y = [0, 1]^2$ is the basic cell in $\mathbb{R}^2$. We consider

a smooth function $h = h(z, y)$, such that:

$$h \in \mathcal{C}^2(w \times \mathbb{R}^2), h \text{ is periodic in } y, \quad \exists a, b > 0 \quad a \leq h(z, y) \leq b \quad \forall z \in w, y \in Y. \quad (2.1)$$

We set

$$h_{min} = \min_{(z,y) \in \omega \times Y} h(z, y), \quad h_{max} = \max_{(z,y) \in \omega \times Y} h(z, y)$$

and

$$\Omega_{min} = \omega \times]0, h_{min}[, \quad \Omega_{max} = \omega \times]0, h_{max}[.$$

We introduce two small parameters, ϵ related to the roughness wavelength and η related to the thickness of the gap between the two surfaces, so that the three-dimensional domain occupied by the fluid is

$$\Omega^{\eta\epsilon} = \{(z, z_3) / z \in \omega \text{ and } 0 < z_3 < \eta h^\epsilon(z)\},$$

where

$$h^\epsilon(z) = h(z, \frac{z}{\epsilon}) \quad \forall z \in \omega. \quad (2.2)$$

In the following, we will need two different scalings.

$\Omega^\epsilon = \omega \times]0, h^\epsilon[$ is the transformation of the domain $\Omega^{\eta\epsilon}$ with rescaled thickness

$$x = z, \text{ and } x_3 = \frac{z_3}{\eta}; \quad (2.3)$$

$\Omega = \omega \times]0, 1[$ is the transformation of the domain $\Omega^{\eta\epsilon}$ by the change of variables

$$t = z, \text{ and } t_3 = \frac{z_3}{\eta h^\epsilon}. \quad (2.4)$$

FIG. 1. Domain $\Omega^{\eta\epsilon}$

We will denote, for any function φ , its average on the basic cell Y by

$$\tilde{\varphi} = \int_Y \varphi dy$$

and for any function φ defined on Ω^ϵ , we denote by

$$\bar{\varphi} = \begin{cases} \varphi & \text{on } \Omega^\epsilon, \\ 0 & \text{on } \Omega_{max} - \Omega^\epsilon \end{cases}$$

its extension by zero to Ω_{max} .

REMARK 2.1. Throughout this paper, for the sake of simplicity, the velocity and pressure will be denoted by u^ϵ and p^ϵ instead of u^{η^ϵ} and p^{η^ϵ} , since η and ϵ are linked by the relation (1.2). Moreover, the same notation will be used for a function defined on the three different domains Ω^{η^ϵ} , Ω^ϵ , and Ω .

Let $s = (s_1, s_2)$ be a given vector in $\mathbb{R}^2$ and let g be a given function in $(H^{\frac{1}{2}}(\partial\Omega_{min}))^3$, which does not depend on η and ϵ , such that

$$g(x, h_{min}) = 0 \quad g(x, 0) = (s_1, s_2, 0) \quad \forall x \in \omega \quad (2.5)$$

and

$$\int_{\partial\Omega_{min}} g \cdot n \, d\sigma = 0. \quad (2.6)$$

From the hypotheses (2.5) and (2.6) on g , $G = (G_1, G_2, G_3)$ in $H^1(\Omega_{min})^3$ exists such that $\operatorname{div} G = 0$ in Ω_{min} and $G = g$ on $\partial\Omega_{min}$ (see [16]). We still denote by G its extension by zero to Ω^ϵ . Let $G^\epsilon = (G_1, G_2, \eta G_3)$, which belongs to $H^1(\Omega^\epsilon)$.

The variational formulation of the Stokes problem $(\mathcal{P}^{\eta^\epsilon})$ in Ω^ϵ follows:

Find u^ϵ in $H^1(\Omega^\epsilon)^3$, p^ϵ in $L_0^2(\Omega^\epsilon)$ such that

$$\mu \int_{\Omega^\epsilon} \nabla_\eta u^\epsilon \cdot \nabla_\eta \phi \, dx \, dx_3 = \int_{\Omega^\epsilon} p^\epsilon \operatorname{div}_\eta \phi \, dx \, dx_3 \quad \forall \phi \in H_0^1(\Omega^\epsilon)^3 \quad (2.7)$$

$$\int_{\Omega^\epsilon} \operatorname{div}_\eta u^\epsilon q \, dx \, dx_3 = 0 \quad \forall q \in L^2(\Omega^\epsilon) \quad (2.8)$$

$$u^\epsilon - G^\epsilon \in (H_0^1)(\Omega^\epsilon)^3 \quad (2.9)$$

where

$$\nabla_\eta = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{1}{\eta} \frac{\partial}{\partial x_3} \right), \quad \operatorname{div}_\eta = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{1}{\eta} \frac{\partial}{\partial x_3}.$$

The variational formulation of the Stokes problem in Ω follows:

Find u^ϵ in $H^1(\Omega)^3$, p^ϵ in $L^2(\Omega)$ with $\int_\Omega h^\epsilon p^\epsilon \, dx \, dt_3 = 0$ such that

$$\begin{aligned} & \mu \int_\Omega \sum_{j=1}^3 \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_j^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_j^\epsilon) \right) \right. \\ & \cdot \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \varphi_j) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \varphi_j) \right) + \frac{\partial u_j^\epsilon}{\partial t_3} \frac{\partial \varphi_j}{\partial t_3} \left. \right] \frac{dt \, dt_3}{\eta h^\epsilon} \\ & = \int_\Omega p^\epsilon \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \varphi_i) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \varphi_i) \right) + \frac{\partial \varphi_3}{\partial t_3} \right] dt \, dt_3 \quad \forall \varphi \in H_0^1(\Omega)^3 \end{aligned} \quad (2.10)$$

$$\operatorname{div}_{tt_3} (\eta h^\epsilon B^\epsilon \cdot u^\epsilon) = 0 \quad \text{in } \Omega \quad (2.11)$$

$$u^\epsilon - G^\epsilon \in (H_0^1)(\Omega)^3$$

where B^ϵ is the Jacobian matrix

$$B^\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-t_3}{h^\epsilon} \frac{\partial h^\epsilon}{\partial t_1} & \frac{-t_3}{h^\epsilon} \frac{\partial h^\epsilon}{\partial t_2} & \frac{1}{\eta h^\epsilon} \end{pmatrix}.$$

3. Main results. In this section, we only state the convergence results; the proofs are shown in Sections 4 and 5.

3.1. *Convergence theorems for the pressure.* It is supposed that $\eta = \lambda \epsilon^\alpha$ and the convergence results are different whether $\alpha = 1$ or $\alpha > 1$. So we consider the two different cases.

3.1.1. *The case $\alpha = 1$.* We first define the sets

$$\begin{aligned} B_x &= \{(y, x_3), \quad y \in Y, \quad 0 < x_3 < h(x, y)\} \\ Q &= \bigcup_{x \in \omega} \{x\} \times B_x = \{(x, x_3, y), \quad x \in \omega, \quad y \in Y, \quad 0 < x_3 < h(x, y)\}, \end{aligned} \quad (3.1)$$

the operators

$$\begin{aligned} \Delta_\lambda &= \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{1}{\lambda^2} \frac{\partial^2}{\partial x_3^2} \\ \nabla_\lambda &= \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{1}{\lambda} \frac{\partial}{\partial x_3} \right) \\ \operatorname{div}_\lambda &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{1}{\lambda} \frac{\partial}{\partial x_3}, \end{aligned}$$

and the spaces

$$\begin{aligned} C_{0\#}^\infty(B_x) &= \{\varphi \in C^\infty(B_x), \quad \varphi \text{ is } y \text{ periodic}, \quad \varphi(y, 0) = \varphi(y, h(x, y)) = 0\} \\ H_{\#}^1(B_x) &= \{\varphi \in H^1(B_x), \quad \varphi \text{ is } y \text{ periodic}\} \\ H_{0\#}^1(B_x) &= \{\phi \in H_{\#}^1(B_x), \quad \varphi(y, 0) = \varphi(y, h(x, y)) = 0\}. \end{aligned}$$

We also introduce the following local problems:

Find $(w^{k\lambda}, \pi^{k\lambda})(k = 1, 2)$ in $\left(H_{0\#}^1(B_x)\right)^3 \times L_0^2(B_x)$ such that

$$\begin{cases} \mu \int_{B_x} \nabla_\lambda w^{k\lambda} \cdot \nabla_\lambda \varphi \, dy \, dx_3 = \int_{B_x} \pi^{k\lambda} \operatorname{div}_\lambda \varphi \, dy \, dx_3 - \int_{B_x} \varphi_k \, dy \, dx_3 & \forall \varphi \in \left(H_{0\#}^1(B_x)\right)^3 \\ \operatorname{div}_\lambda w^{k\lambda} = 0. \end{cases} \quad (3.2)$$

Find $(w^{3\lambda}, \pi^{3\lambda})$ in $H_{\#}^1(B_x) \times L_0^2(B_x)$ such that

$$\left\{ \begin{array}{l} \mu \int_{B_x} \nabla_{\lambda} w^{3\lambda} \cdot \nabla_{\lambda} \varphi \, dy \, dx_3 = \int_{B_x} \pi^{3\lambda} \operatorname{div}_{\lambda} \varphi \, dy \, dx_3 \quad \forall \varphi \in \left(H_{0\#}^1(B_x) \right)^3 \\ \operatorname{div}_{\lambda} w^{3\lambda} = 0 \\ w^{3\lambda} = (s_1, s_2, 0) \text{ on } x_3 = 0 \\ w^{3\lambda} = (0, 0, 0) \text{ on } x_3 = h(x, y). \end{array} \right. \quad (3.3)$$

The convergence of the pressure is given by the following theorem.

THEOREM 3.1. For $\alpha = 1$, $\eta^2 p^{\epsilon}$ two-scale converges in Ω to $p^{\lambda}(x)$, the unique solution in $H^1(\omega)$ of

$$\left\{ \begin{array}{l} \sum_{i,j=1}^2 \int_{\omega} a_{ij}^{\lambda} \frac{\partial p^{\lambda}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, dx = \sum_{i=1}^2 \int_{\omega} b_i^{\lambda} \frac{\partial \varphi}{\partial x_i} - \int_{\partial \omega} f \cdot n \, \varphi \, d\sigma \quad \forall \varphi \in H^1(\omega) \\ \int_{\omega} \tilde{h} p^{\lambda}(t) \, dt = 0 \end{array} \right. \quad (3.4)$$

with

$$\begin{aligned} a_{ij}^{\lambda}(x) &= - \int_{B_x} w_i^{j\lambda} \, dy \, dx_3, \\ b_i^{\lambda} &= \int_{B_x} w_i^{3\lambda} \, dy \, dx_3 \quad i = 1, 2, \\ f_i &= \int_0^{h_{min}} g_i \quad i = 1, 2 \, dx_3, \end{aligned}$$

and $w^{k\lambda}$ the unique solutions of the local problems given in (3.2) and (3.3).

3.1.2. The case $\alpha > 1$. In this case, due to the fact that η tends to zero faster than ϵ , the local problems are two-dimensional problems and independent of λ . Setting

$$H_{per}^1(Y) = \{ \varphi \in H^1(Y), \varphi \text{ is } y\text{-periodic} \},$$

we define the following local problems:

Find π^k ($k = 1, 2$) in $H_{per}^1(Y) \cap L_0^2(Y)$ such that

$$\int_Y h^3 (e_k + \nabla_y \pi^k) \cdot \nabla_y \phi \, dy = 0 \quad \forall \phi \in H_{per}^1(Y) \quad (3.5)$$

where $\{e_1, e_2\}$ is the canonical base in $\mathbb{R}^2$.

Find π^3 in $H_{per}^1(Y) \cap L_0^2(Y)$ such that

$$\int_Y \frac{h^3}{6\mu} \nabla_y \pi^3 \cdot \nabla_y \phi \, dy = \int_Y h s \cdot \nabla_y \phi \, dy \quad \forall \phi \in H_{per}^1(Y). \quad (3.6)$$

The convergence of the pressure is now given by

THEOREM 3.2. For $\alpha > 1$, $\eta^2 p^\epsilon$ two-scale converges in Ω to $p(x)$, the unique solution in $H^1(\omega)$ of

$$\begin{cases} \sum_{i,j=1}^2 \int_{\omega} a_{ij} \frac{\partial p}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = \int_{\omega} \sum_{i=1}^2 b_i(x) \frac{\partial \varphi}{\partial x_i} dx - \int_{\partial \omega} f \varphi dx & \forall \varphi \in H^1(\omega) \\ \int_{\omega} \tilde{h} p(x) dx = 0 \end{cases} \quad (3.7)$$

with f as in Theorem 3.1 and

$$\begin{aligned} a_{ij}(x) &= \int_Y \frac{h^3}{12\mu} (\delta_{ij} + \frac{\partial \pi^i}{\partial y_j}) dy \quad i, j = 1, 2 \\ b_i(x) &= \int_Y \left(-\frac{h^3}{12\mu} \frac{\partial \pi^3}{\partial y_i} + \frac{\tilde{h}}{2} s_i \right) dy \quad i = 1, 2. \end{aligned} \quad (3.8)$$

3.2. *Convergence theorems for the stresses.* The stress tensor $\sigma^\epsilon = (\sigma_{ij}^\epsilon)_{i,j=1,2,3}$ in Ω^{η^ϵ} is defined by

$$\sigma_{ij}^\epsilon = -p^\epsilon \delta_{ij} + \mu \left(\frac{\partial u_i^\epsilon}{\partial z_j} + \frac{\partial u_j^\epsilon}{\partial z_i} \right).$$

We introduce the normalized stress tensor

$$\Sigma^\epsilon = \begin{pmatrix} \eta \sigma_{11}^\epsilon & \eta \sigma_{12}^\epsilon & \eta \sigma_{13}^\epsilon \\ \eta \sigma_{21}^\epsilon & \eta \sigma_{22}^\epsilon & \eta \sigma_{23}^\epsilon \\ \eta^2 \sigma_{31}^\epsilon & \eta^2 \sigma_{32}^\epsilon & \eta^2 \sigma_{33}^\epsilon \end{pmatrix}$$

and the normalized stresses on the boundary $\partial \Omega^{\eta^\epsilon}$, where n^ϵ is the unit normal exterior to $\partial \Omega^{\eta^\epsilon}$:

$$V^\epsilon = \Sigma^\epsilon \cdot n^\epsilon.$$

The only boundary stresses for such a thin domain which have engineering applications are

$$V_+^\epsilon = V_{/\Gamma_+^{\eta^\epsilon}}^\epsilon \quad \text{and} \quad V_-^\epsilon = V_{/\Gamma_-^\epsilon}^\epsilon.$$

Due to the roughness of $\Gamma_+^{\eta^\epsilon}$, the results are different on the upper and lower surfaces.

3.2.1. *Convergence of the stresses on the rough boundary.* The boundary Γ_+^ϵ of the rescaled domain Ω^ϵ still depends on ϵ . So it is necessary to work in the fixed domain Ω . The stress tensor is written in Ω by

$$\begin{aligned} \sigma_{ii}^\epsilon &= -p^\epsilon + 2\mu \left(\frac{\partial u_i^\epsilon}{\partial t_i} - 2 \frac{\partial h^\epsilon}{\partial t_i} \frac{t_3}{h^\epsilon} \frac{\partial u_i^\epsilon}{\partial t_3} \right) \quad i = 1, 2 \\ \sigma_{33}^\epsilon &= -p^\epsilon + \mu \frac{2}{\eta h^\epsilon} \frac{\partial u_3^\epsilon}{\partial t_3} \\ \sigma_{ij}^\epsilon &= \mu \left(\frac{\partial u_i^\epsilon}{\partial t_j} - 2 \frac{\partial h^\epsilon}{\partial t_j} \frac{t_3}{h^\epsilon} \frac{\partial u_i^\epsilon}{\partial t_3} + \frac{\partial u_j^\epsilon}{\partial t_i} - 2 \frac{\partial h^\epsilon}{\partial t_i} \frac{t_3}{h^\epsilon} \frac{\partial u_j^\epsilon}{\partial t_3} \right) \quad i \neq j = 1, 2 \\ \sigma_{i3}^\epsilon &= \sigma_{3j}^\epsilon = \mu \left(\frac{1}{\eta h^\epsilon} \frac{\partial u_i^\epsilon}{\partial t_3} + \frac{\partial u_3^\epsilon}{\partial t_i} - 2 \frac{\partial h^\epsilon}{\partial t_i} \frac{t_3}{h^\epsilon} \frac{\partial u_3^\epsilon}{\partial t_3} \right) \quad i = 1, 2 \end{aligned}$$

and the stress vector V_+^ϵ on the boundary Γ_+ , defined by the equation $t_3 = 1$, is

$$V_+^\epsilon = \begin{pmatrix} -\eta^2 \sigma_{11}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^2 \sigma_{12}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta \sigma_{13}^\epsilon \\ -\eta^2 \sigma_{21}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^2 \sigma_{22}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta \sigma_{23}^\epsilon \\ -\eta^3 \sigma_{31}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^3 \sigma_{32}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta^2 \sigma_{33}^\epsilon \end{pmatrix} \cdot \frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1}\right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2}\right)^2}}. \quad (3.9)$$

As for the pressure, it is necessary to consider different cases. The first convergence result is

THEOREM 3.3. For $\alpha = 1$, $(\epsilon V_{+1}^\epsilon, \epsilon V_{+2}^\epsilon, V_{+3}^{\eta\epsilon})$ converges in $\left(H_{00}^{\frac{1}{2}}(\Gamma_+)\right)'$ to

$$V_+^\lambda = \begin{pmatrix} p^\lambda \left(\int_Y \frac{\lambda^2 \frac{\partial h}{\partial y_1}}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} dy \right) \\ p^\lambda \left(\int_Y \frac{\lambda^2 \frac{\partial h}{\partial y_2}}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} p^\lambda dy \right) \\ p^\lambda \left(\int_Y \frac{-\lambda^2}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} dy \right) \end{pmatrix}.$$

For α greater than 1, a subcritical case $\alpha = \frac{3}{2}$ appears to lead to three supplementary convergence results.

THEOREM 3.4. For $1 < \alpha < \frac{3}{2}$, $(\epsilon^{3-2\alpha} V_{+1}^\epsilon, \epsilon^{3-2\alpha} V_{+2}^\epsilon, V_{+3}^{\eta\epsilon})$ converges in $H^{-1}(\omega) \times H^{-1}(\omega) \times \left(H_{00}^{\frac{1}{2}}(\Gamma_+)\right)'$ to

$$F_+^\lambda = \begin{pmatrix} -\frac{\lambda^2}{2} p \left(\widetilde{\left(\frac{\partial h}{\partial y_1}\right)^3} \right) \\ -\frac{\lambda^2}{2} p \left(\widetilde{\left(\frac{\partial h}{\partial y_2}\right)^3} \right) \\ -p(x) \end{pmatrix}.$$

For $\alpha = \frac{3}{2}$, V_+^ϵ converges in $H^{-1}(\omega) \times H^{-1}(\omega) \times \left(H_{00}^{\frac{1}{2}}(\Gamma_+)\right)'$ to

$$V_+^\lambda = \begin{pmatrix} -\frac{\lambda^2}{2} p \left(\widetilde{\left(\frac{\partial h}{\partial y_1}\right)^3} \right) + \frac{\partial}{\partial t_1}(p\tilde{h}) - \sum_{j=1}^2 \frac{\partial p}{\partial t_j} c_{1j}(t) - d_1(t) \\ -\frac{\lambda^2}{2} p \left(\widetilde{\left(\frac{\partial h}{\partial y_2}\right)^3} \right) + \frac{\partial}{\partial t_2}(p\tilde{h}) - \sum_{j=1}^2 \frac{\partial p}{\partial t_j} c_{2j}(t) - d_2(t) \\ -p(x) \end{pmatrix}.$$

For $\alpha > \frac{3}{2}$, V_+^ϵ converges in $H^{-1}(\omega) \times H^{-1}(\omega) \times \left(H_{00}^{\frac{1}{2}}(\Gamma_+)\right)'$ to

$$V_+ = \begin{pmatrix} \frac{\partial}{\partial t_1}(p\tilde{h}) - \sum_{j=1}^2 \frac{\partial p}{\partial t_j} c_{1j}(x) - d_1(t) \\ \frac{\partial}{\partial t_2}(p\tilde{h}) - \sum_{j=1}^2 \frac{\partial p}{\partial x_j} c_{2j}(t) - d_2(t) \\ -p(x) \end{pmatrix}$$

where

$$\begin{aligned} c_{ij}(t) &= \int_Y \left(\delta_{ij} + \frac{\partial \pi^j}{\partial y_i} \right) \frac{h}{2} dy \quad i, j = 1, 2 \\ d_i(t) &= \int_Y \left(\frac{\partial \pi^3}{\partial y_i} \frac{h}{2} + \mu \frac{s_i}{h} \right) dy \quad i = 1, 2. \end{aligned}$$

3.2.2. Convergence of the stresses on the lower boundary. The lower boundary Γ_- has been supposed to be a flat surface, so it is no longer necessary to straighten the domain. It is sufficient to consider Γ_- as the lower boundary of the fixed subdomain Ω_{min} of Ω^ϵ . We will only give the convergence results, the proofs of which are left to the reader.

THEOREM 3.5. For $\alpha = 1$, the stress vector V_-^ϵ converges in $\left(H_{00}^{\frac{1}{2}}(\Gamma_-)\right)'^3$ to V_-^λ , defined by

$$V_-^\lambda = \begin{pmatrix} \sum_{j=1}^2 c_{1j}^\lambda \frac{\partial p^\lambda}{\partial x_j} + d_1^\lambda(x) \\ \sum_{j=1}^2 c_{2j}^\lambda \frac{\partial p^\lambda}{\partial x_j} + d_2^\lambda(x) \\ \lambda^2 p^\lambda \end{pmatrix},$$

with

$$c_{ij}^\lambda = \left(\frac{\partial \tilde{w}_i^{j\lambda}}{\partial x_3} \right)_{x_3=0}, \quad d_i^\lambda = \left(\frac{\partial \tilde{w}_1^{3\lambda}}{\partial x_3} \right)_{x_3=0} \quad i, j = 1, 2,$$

where $w^{j\lambda}$ ($j = 1, 2, 3$) are the solutions to local problems (3.2) and (3.3).

For $\alpha > 1$, V_-^ϵ converges in $\left(H_{00}^{\frac{1}{2}}(\Gamma_-)\right)'^3$ to V_- , defined by

$$V_- = \begin{pmatrix} \sum_{j=1}^2 c_{1j}(x) \frac{\partial p}{\partial x_j} + d_1(x) \\ \sum_{j=1}^2 c_{2j}(x) \frac{\partial p}{\partial x_j} c_{2j}(x) + d_2(x) \\ p(x) \end{pmatrix}$$

with

$$c_{ij}(x) = \int_Y \left(\delta_{ij} + \frac{\partial \pi^j}{\partial y_i} \right) \frac{h}{2\mu} dy \quad i, j = 1, 2, \quad d_i = \int_Y \left(\frac{\partial \pi^3}{\partial y_i} \frac{h}{2\mu} + \mu \frac{s_i}{h} \right) dy \quad i = 1, 2$$

where π^k ($k = 1, 2, 3$) are the solutions to local problems defined in (3.5) and (3.6).

4. Proofs of the theorems in Section 3.1. In this section, we give the proofs of Theorem 3.1. Also, we will need some results for pressure and velocity. In the first paragraph, we give the estimates of the velocity and pressure in the domain Ω ; they are valid with the hypothesis $\eta \ll \epsilon$. The proofs of the main theorem are in Section 4.2 for the case of $\alpha = 1$ and in Section 4.3 for the case of $\alpha > 1$.

4.1. *Asymptotic behaviour of the velocity and pressure for $\alpha \geq 1$.* First, we give the estimates of the velocity in the rescaled domain Ω^ϵ . On the contrary, the pressure estimates are obtained in the fixed domain Ω and will be deduced in Ω^ϵ .

For any domain $D \subset \mathbb{R}^3$ we introduce the Hilbert space

$$V_{x_3}(D) = \{v \in L^2(D), \frac{\partial v}{\partial x_3} \in L^2(D)\}$$

equipped with the norm

$$\|v\|_{V_{x_3}(D)}^2 = \|v\|_{L^2(D)}^2 + \left\| \frac{\partial v}{\partial x_3} \right\|_{L^2(D)}^2. \quad (4.1)$$

LEMMA 4.1. The following estimates are satisfied in Ω^ϵ (K denotes any constant which does not rely on η and ϵ):

$$\|u_i^\epsilon\|_{V_{x_3}(\Omega^\epsilon)} \leq K, \quad \left\| \eta \frac{\partial u_i^\epsilon}{\partial x_j} \right\|_{L^2(\Omega^\epsilon)^3} \leq K, \quad i = 1, 2, 3, \quad j = 1, 2.$$

Proof. We take $u^\epsilon - G^\epsilon$ as a test function in (2.7). Since $\operatorname{div}_\eta G^\epsilon = 0$,

$$\int_{\Omega^\epsilon} (\nabla_\eta u^\epsilon)^2 dx dx_3 - \int_{\Omega^\epsilon} \nabla_\eta u^\epsilon \nabla_\eta G^\epsilon dx dx_3 = 0.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\|\nabla_\eta u^\epsilon\|_{L^2(\Omega^\epsilon)^9} \leq \|\nabla_\eta G^\epsilon\|_{L^2(\Omega^\epsilon)^9} = \|\nabla_\eta G^\epsilon\|_{L^2(\Omega_{min})^9},$$

so

$$\|\nabla_\eta u^\epsilon\|_{L^2(\Omega^\epsilon)^6} \leq \frac{K}{\eta}.$$

This implies

$$\left\| \eta \frac{\partial u^\epsilon}{\partial x_i} \right\|_{L^2(\Omega^\epsilon)^3} \leq K, \quad i = 1, 2 \quad \text{and} \quad \left\| \frac{\partial u^\epsilon}{\partial x_3} \right\|_{L^2(\Omega^\epsilon)^3} \leq K. \quad (4.2)$$

Using the Friedrich-Poincaré inequality in the x_3 direction for a function in $H^1(\Omega^\epsilon)$ equal to zero on Γ_+^ϵ , we obtain (with b defined in (2.1))

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)^3} \leq b \left\| \frac{\partial u^\epsilon}{\partial x_3} \right\|_{L^2(\Omega^\epsilon)^3} \leq K.$$

□

Now, we can deduce the estimates of velocity and pressure in the domain Ω .

LEMMA 4.2. We have the following estimates for velocity and pressure:

$$\begin{aligned} \|u_i^\epsilon\|_{V_{t_3}(\Omega)} &\leq K & \eta \left\| \frac{\partial u_i^\epsilon}{\partial t_j} \right\|_{L^2(\Omega)^3} &\leq K \quad i = 1, 2, 3, \quad j = 1, 2 \\ & & \|\eta^2 p^\epsilon\|_{L^2(\Omega)} &\leq K. \end{aligned}$$

Proof. Using (2.4) for the third variable,

$$\int_{\Omega} \left(\frac{\partial u^\epsilon}{\partial t_3} \right)^2 dt dt_3 = \int_{\Omega^\epsilon} \left(\frac{\partial u^\epsilon}{\partial x_3} \right)^2 h^\epsilon dx dx_3 \leq K.$$

Using the Poincaré inequality in the t_3 -direction in Ω , as u^ϵ equals zero on the boundary $t_3 = 1$,

$$\int_{\Omega} (u^\epsilon)^2 dt dt_3 \leq K \int_{\Omega} \left(\frac{\partial u^\epsilon}{\partial t_3} \right)^2 dt dt_3 \leq K.$$

Using (2.4) in the domain Ω , we obtain

$$\frac{\partial u^\epsilon}{\partial t_i} = \frac{\partial u^\epsilon}{\partial x_i} + \frac{\partial h^\epsilon}{\partial x_i} \frac{x_3}{h^\epsilon} \frac{\partial u^\epsilon}{\partial x_3}, \quad i = 1, 2,$$

so

$$\int_{\Omega} \left(\frac{\partial u^\epsilon}{\partial t_i} \right)^2 dt dt_3 = \int_{\Omega^\epsilon} \left(\frac{\partial u^\epsilon}{\partial x_i} + \frac{\partial h^\epsilon}{\partial x_i} \frac{x_3}{h^\epsilon} \frac{\partial u^\epsilon}{\partial x_3} \right)^2 \frac{1}{h^\epsilon} dx dx_3 \quad i = 1, 2.$$

Due to the oscillations of the function h^ϵ , we obtain (formula (2.2))

$$\left\| \frac{\partial h^\epsilon}{\partial x_i} \right\|_{L^\infty(w)} \leq \frac{K}{\epsilon},$$

so that estimates from Lemma 4.1 imply

$$\left\| \frac{\partial u^\epsilon}{\partial t_i} \right\|_{L^2(\Omega)} \leq \frac{K}{\eta} \quad i = 1, 2.$$

We now prove the estimates for the pressure. Taking $(0, 0, \varphi)$ with φ in $H_0^1(\Omega)$ in (2.10), we obtain

$$\begin{aligned} \mu \int_{\Omega} \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_3^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_3^\epsilon) \right) \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \varphi) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \varphi) \right) \right. \\ \left. + \frac{\partial u_3^\epsilon}{\partial t_3} \frac{\partial \varphi}{\partial t_3} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_{\Omega} p^\epsilon \frac{\partial \varphi}{\partial t_3} dt dt_3. \end{aligned} \quad (4.3)$$

The estimates of the velocity induce

$$\left| \int_{\Omega} p^\epsilon \frac{\partial \varphi}{\partial t_3} dt dt_3 \right| \leq \frac{K}{\eta} \|\varphi\|_{H_0^1(\Omega)}. \quad (4.4)$$

On the other hand, taking $(\frac{\varphi}{h^\epsilon}, 0, 0)$ in (2.10), we prove that

$$\begin{aligned} I_1^\epsilon = \mu \int_{\Omega} \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_1^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_1^\epsilon) \right) \cdot \left(\frac{\partial}{\partial t_i} (\eta \varphi) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{1}{h^\epsilon} \frac{\partial h^\epsilon}{\partial t_i} \varphi) \right) \right. \\ \left. + \frac{1}{h^\epsilon} \frac{\partial u_1^\epsilon}{\partial t_3} \frac{\partial \varphi}{\partial t_3} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_{\Omega} p^\epsilon \left(\frac{\partial}{\partial t_1} (\eta \varphi) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_1} \frac{\varphi}{h^\epsilon}) \right) dt dt_3. \end{aligned}$$

So we have

$$\int_{\Omega} p^{\epsilon} \frac{\partial \varphi}{\partial t_1} dt dt_3 = \frac{I_1^{\epsilon}}{\eta} + \int_{\Omega} p^{\epsilon} \frac{\partial}{\partial t_3} \left(t_3 \frac{\partial h^{\epsilon}}{\partial t_1} \frac{\varphi}{h^{\epsilon}} \right) dt dt_3.$$

Using the estimates of the velocity, we obtain

$$|I_1^{\epsilon}| \leq \frac{K}{\eta} \|\varphi\|_{H_0^1(\Omega)}.$$

Taking now $(0, 0, t_3 \frac{\partial h^{\epsilon}}{\partial t_1} \frac{\varphi}{h^{\epsilon}})$ in (2.10), we obtain

$$\begin{aligned} & \mu \sum_{i=1}^2 \int_{\Omega} \left(\frac{\partial}{\partial t_i} (\eta h^{\epsilon} u_3^{\epsilon}) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^{\epsilon}}{\partial t_i} u_3^{\epsilon}) \right) \\ & \cdot \left(\frac{\partial}{\partial t_i} (\eta t_3 \frac{\partial h^{\epsilon}}{\partial t_1} \varphi) + \frac{\partial}{\partial t_3} (-\eta \frac{t_3^2}{h^{\epsilon}} \frac{\partial h^{\epsilon}}{\partial t_i} \frac{\partial h^{\epsilon}}{\partial t_1} \varphi) \right) \frac{dt dt_3}{\eta h^{\epsilon}} \\ & + \mu \int_{\Omega} \frac{1}{\eta h^{\epsilon}} \frac{\partial u_3^{\epsilon}}{\partial t_3} \frac{\partial}{\partial t_3} \left(t_3 \frac{\partial h^{\epsilon}}{\partial t_1} \frac{\varphi}{h^{\epsilon}} \right) dt dt_3 \\ & = \int_{\Omega} p^{\epsilon} \frac{\partial}{\partial t_3} \left(t_3 \frac{\partial h^{\epsilon}}{\partial t_1} \frac{\varphi}{h^{\epsilon}} \right) dt dt_3, \end{aligned}$$

which leads to

$$\left| \int_{\Omega} p^{\epsilon} \frac{\partial}{\partial t_3} \left(t_3 \frac{\partial h^{\epsilon}}{\partial t_k} \frac{\varphi}{h^{\epsilon}} \right) \right| \leq \frac{K}{\eta^2} \|\varphi\|_{H_0^1(\Omega)} \quad k = 1, 2. \quad (4.5)$$

So

$$\left| \int_{\Omega} p^{\epsilon} \frac{\partial \varphi}{\partial t_1} dt dt_3 \right| \leq \frac{K}{\eta^2} \|\varphi\|_{H_0^1(\Omega)}. \quad (4.6)$$

In the same way, we prove that

$$\left| \int_{\Omega} p^{\epsilon} \frac{\partial \varphi}{\partial t_2} dt dt_3 \right| \leq \frac{K}{\eta^2} \|\varphi\|_{H_0^1(\Omega)}. \quad (4.7)$$

Using (4.4) and (4.6), this induces

$$\left\| \frac{\partial p^{\epsilon}}{\partial t_i} \right\|_{H^{-1}(\Omega)} \leq \frac{K}{\eta^2} \quad (i = 1, 2) \quad \left\| \frac{\partial p^{\epsilon}}{\partial t_3} \right\|_{H^{-1}(\Omega)} \leq \frac{K}{\eta}. \quad (4.8)$$

p^{ϵ} belongs to $L_0^2(\Omega^{\epsilon})$, so

$$0 = \int_{\Omega^{\epsilon}} p^{\epsilon} dx dx_3 = \int_{\Omega} p^{\epsilon} h^{\epsilon} dt dt_3.$$

From [22], $h^{\epsilon} p^{\epsilon}$ being in $L_0^2(\Omega^{\epsilon})$, ϕ^{ϵ} in $H_0^1(\Omega)^3$ exists with $h^{\epsilon} p^{\epsilon} = \operatorname{div} \phi^{\epsilon}$ and

$$\|\phi^{\epsilon}\|_{H_0^1(\Omega)^3} \leq K \|\operatorname{div} \phi^{\epsilon}\|_{L^2(\Omega)} = K \|h^{\epsilon} p^{\epsilon}\|_{L^2(\Omega)},$$

where K is a constant with respect to ϵ . So we obtain

$$\int_{\Omega} p^{\epsilon} (h^{\epsilon} p^{\epsilon}) dt dt_3 = \int_{\Omega} p^{\epsilon} \left(\frac{\partial \phi_1^{\epsilon}}{\partial t_1} + \frac{\partial \phi_2^{\epsilon}}{\partial t_2} + \frac{\partial \phi_3^{\epsilon}}{\partial t_3} \right) dt dt_3 = - \sum_{i=1}^3 \left\langle \frac{\partial p^{\epsilon}}{\partial t_i}, \phi_i^{\epsilon} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Using (2.1) and estimates (4.8),

$$a \|p^{\epsilon}\|_{L^2(\Omega)}^2 \leq \int_{\Omega} h^{\epsilon} (p^{\epsilon})^2 dt dt_3 \leq \frac{K}{\eta^2} \|\phi^{\epsilon}\|_{H_0^1(\Omega)^3} \leq \frac{K}{\eta^2} \|h^{\epsilon} p^{\epsilon}\|_{L^2(\Omega)} \leq \frac{Kb}{\eta^2} \|p^{\epsilon}\|_{L^2(\Omega)},$$

so we obtain

$$\|p^\epsilon\|_{L^2(\Omega)} \leq \frac{K}{\eta^2},$$

which completes the proof. $\square$

We present a technical lemma in two-scale convergence that will be useful for comparing different limits in the different domains.

LEMMA 4.3. Let f^ϵ be in $L^2(\Omega^\epsilon)$ and $\bar{f}^\epsilon$ its extension by zero to Ω_{max} . Suppose that f^ϵ (respectively $\bar{f}^\epsilon$) two-scale converge to f_1 in $\Omega \times [0, 1]^3$ (respectively to f_2 in $\Omega_{max} \times [0, 1]^3$). If f_1 and f_2 do not rely on y_3 , then

$$f_2(x, x_3, y) = \begin{cases} f_1(x, \frac{x_3}{h(x, y)}, y) & \text{for } x_3 < h(x, y), \\ 0 & \text{elsewhere.} \end{cases}$$

Proof. We choose φ in $\mathcal{D}(\Omega_{max}, \mathcal{C}_\#^\infty([0, 1]^3))$ with $\varphi = 0$ for $x_3 \leq h(x, y)$, and we obtain

$$\begin{aligned} 0 &= \int_{\Omega_{max}} \bar{f}^\epsilon(x, x_3) \varphi(x, x_3, \frac{x}{\epsilon}) dx dx_3 \rightarrow \int_{\Omega_{max} \times [0, 1]^3} f_2(x, x_3, y) \varphi(x, x_3, y) dx dx_3 dy \\ &= \int_{\omega} \int_Y \int_{h(x, y)}^{h_{max}} f_2 \varphi(x, x_3, y) dx dy dx_3, \end{aligned}$$

which shows that $f_2 = 0$ for $x_3 > h(x, y)$.

Now we show that $f_2(x, x_3, y) = f_1(x, \frac{x_3}{h}, y)$ for $x_3 \leq h(x, y)$. Let ψ be in $\mathcal{D}(\Omega, \mathcal{C}_\#^\infty(Y))$. The change of variables (2.4) induces

$$\int_{\Omega} f^\epsilon \psi^\epsilon(t, t_3) dt dt_3 = \int_{\Omega^\epsilon} f^\epsilon \psi^\epsilon(x, \frac{x_3}{h^\epsilon}) \frac{1}{h^\epsilon} dx dx_3.$$

In the second integral, we extended by zero to Ω_{max} and after passing to the two-scale limits in all terms, we obtain

$$\int_{\Omega} \int_Y f_1(t, t_3, y) \psi(t, t_3, y) dt dt_3 dy = \int_{\Omega_{max}} \int_Y f_2(x, x_3, y) \psi(x, \frac{x_3}{h}, y) \frac{1}{h} dx dx_3 dy. \quad (4.9)$$

As $f_2 = 0$ for $x_3 > h(x, y)$, the term on the right-hand side in (4.9) becomes

$$\int_{\omega} \int_Y \int_0^{h(x, y)} f_2(x, x_3, y) \psi(x, \frac{x_3}{h}, y) \frac{1}{h} dx_3 dy dx.$$

Coming back to variables (x, x_3, y) in (4.9),

$$\begin{aligned} \int_{\Omega} \int_Y f_1(t, t_3, y) \psi(t, t_3, y) dt dt_3 dy &= \int_{\omega} \int_Y \int_0^h f_1(x, \frac{x_3}{h}, y) \psi(x, \frac{x_3}{h}, y) \frac{1}{h} dx dy dx_3 \\ &= \int_{\omega} \int_Y \int_0^h f_2(x, x_3, y) \psi(x, \frac{x_3}{h}, y) \frac{1}{h} dy dx_3 dx. \end{aligned}$$

This ends the proof. $\square$

LEMMA 4.4. $p^\lambda(t)$ in $L^2(\omega)$ exists such that in Ω , $\eta^2 p^\epsilon$ two-scale converges to $p^\lambda(t)$ with

$$\int_{\omega} \tilde{h} p^\lambda(t) dt = 0. \quad (4.10)$$

Proof. From the estimates of p^ϵ in Lemma 4.2, and the two-scale convergence results (see [3]), there is $p^\lambda(t, t_3, y, y_3)$ in $L^2(\Omega \times]0, 1[^3)$ and a subsequence of ϵ denoted also by ϵ , such that

$$\eta^2 p^\epsilon \rightharpoonup p^\lambda(t, t_3, y, y_3) \quad \text{in } L^2(\Omega \times]0, 1[^3).$$

First, we will show that p^λ does not rely on y_3 . Let $\phi \in \mathcal{D}(\Omega, \mathcal{C}_\#^\infty([0, 1]^3))$. We set $\phi^\epsilon = \phi(t, t_3, \frac{t}{\epsilon}, \frac{t_3}{\epsilon})$ and take $(0, 0, \phi^\epsilon)$ as a test function in (2.10) such that

$$\begin{aligned} J_1^\epsilon = \mu \int_\Omega \left[\sum_{i=1,2} \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_3^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_3^\epsilon) \right) \cdot \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \phi^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \phi^\epsilon) \right) \right. \\ \left. + \frac{\partial u_3^\epsilon}{\partial t_3} \frac{\partial \phi^\epsilon}{\partial t_3} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_\Omega p^\epsilon \left(\frac{\partial \phi}{\partial t_3} + \frac{1}{\epsilon} \frac{\partial \phi}{\partial y_3} \right) dt dt_3. \end{aligned}$$

Multiplying by $\eta^2 \epsilon$ and using Lemma 4.2, we obtain

$$\left| \int_\Omega \eta^2 p^\epsilon \left(\epsilon \frac{\partial \phi}{\partial t_3} + \frac{\partial \phi}{\partial y_3} \right) dt dt_3 \right| = |\eta^2 \epsilon J_1^\epsilon| \leq K \eta.$$

Passing to the limit, we obtain

$$\int_\Omega \int_{[0,1]^3} p^\lambda \frac{\partial \phi}{\partial y_3} dt dt_3 dy dy_3 = 0,$$

which induces that p^λ does not rely on y_3 . Now we show that p^λ does not rely on t_3 . We choose ϕ in $\mathcal{D}(\Omega, \mathcal{C}_\#^\infty(Y))$ and we take $(0, 0, \phi^\epsilon)$ as a test function in (2.10) with $\phi^\epsilon = \phi(t, t_3, \frac{t}{\epsilon})$:

$$\begin{aligned} \mu \int_\Omega \left[\sum_{i=1,2} \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_3^{\eta^\epsilon}) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_3^\epsilon) \right) \cdot \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \phi^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \phi^\epsilon) \right) \right. \\ \left. + \frac{\partial u_3^{\eta^\epsilon}}{\partial t_3} \frac{\partial \phi^\epsilon}{h^\epsilon \partial t_3} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_\Omega p^\epsilon \frac{\partial \phi^\epsilon}{\partial t_3} dt dt_3. \end{aligned}$$

It is sufficient to multiply by η^2 to obtain

$$\left| \int_\Omega \eta^2 p^\epsilon \frac{\partial \phi^\epsilon}{\partial t_3} dt dt_3 \right| \leq K \eta.$$

Passing to the limit, we obtain

$$\int_\Omega \int_Y p^\lambda \frac{\partial \phi}{\partial t_3} dt dt_3 dy = 0 \quad \forall \phi \in \mathcal{D}(\Omega, \mathcal{C}_\#^\infty(Y)), \quad (4.11)$$

which proves that p^λ does not rely on t_3 .

To show that p^λ is independent of y , we take ϕ^ϵ as above, and $(\frac{\phi^\epsilon}{h^\epsilon}, 0, 0)$ as a test function in (2.10):

$$\begin{aligned} \mu \int_\Omega \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_1^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_1^\epsilon) \right) \cdot \left(\frac{\partial}{\partial t_i} (\eta \phi^\epsilon) + \frac{\partial}{\partial t_3} (-\eta \frac{t_3}{h^\epsilon} \frac{\partial h^\epsilon}{\partial t_i} \phi^\epsilon) \right) \right. \\ \left. + \frac{\partial u_1^\epsilon}{\partial t_3} \frac{\partial \phi^\epsilon}{\partial t_3} \frac{1}{h^\epsilon} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_\Omega p^\epsilon \left(\frac{\partial}{\partial t_1} (\eta \phi) + \frac{1}{\epsilon} \frac{\partial}{\partial y_1} (\eta \phi) + \frac{\partial}{\partial t_3} (-\eta \frac{t_3}{h^\epsilon} \frac{\partial h^\epsilon}{\partial t_1} \phi) \right) dt dt_3. \end{aligned}$$

Multiplying by $\eta\epsilon$ and passing to the limit, we obtain

$$\int_{\Omega} \int_Y p^\lambda \frac{\partial \phi}{\partial y_1} dt dt_3 dy - \int_{\Omega} \int_Y p^\lambda \frac{\partial}{\partial t_3} \left(\frac{t_3}{h} \frac{\partial h}{\partial y_1} \phi \right) dt dt_3 dy = 0.$$

By density we can use (4.11) to obtain

$$\int_{\Omega} \int_Y p^\lambda \frac{\partial}{\partial t_3} \left(\frac{t_3}{h} \frac{\partial h}{\partial y_1} \phi \right) dt dt_3 dy = 0,$$

so that

$$\int_{\Omega} \int_Y p^\lambda \frac{\partial \phi}{\partial y_1} dt dt_3 dy = 0.$$

Proceeding in the same way for y_2 , we deduce that p^λ does not rely on y . $\square$

PROPOSITION 4.5. There is ξ in $L^2(\Omega_{max} \times Y)$ such that $\eta^2 \bar{p}^\epsilon$ two-scale converges to $\xi(x, x_3, y)$ with

$$\xi(x, x_3, y) = \begin{cases} p^\lambda(x) & \forall x_3 < h(x, y) \\ 0 & \forall x_3 \geq h(x, y). \end{cases}$$

Proof. Using the estimate given in Lemma 4.2, we obtain

$$\eta^2 \|\bar{p}^\epsilon\|_{L^2(\Omega_{max})} = \eta^2 \|p^\epsilon\|_{L^2(\Omega^\epsilon)} = \eta^2 \left(\int_{\Omega} (p^\epsilon)^2 h^\epsilon dt dt_3 \right)^{\frac{1}{2}} \leq K.$$

So we can extract a subsequence such that $\eta^2 \bar{p}^\epsilon$ converges weakly to $\xi(t, t_3, y, y_3)$ in $L^2(\Omega_{max} \times [0, 1]^3)$.

Now we show that ξ does not rely on y_3 . For any φ in $\mathcal{D}(\Omega_{max}, \mathcal{C}_\#^\infty([0, 1]^3))$ with $\varphi = 0$ for $x_3 \geq h(x, y)$, taking $\varphi^\epsilon(x, x_3) = \varphi(x, x_3, \frac{x}{\epsilon})$, $(0, 0, \varphi)$ is a test function in (2.7).

$$\begin{aligned} I_1^\epsilon &= \mu \int_{\Omega^\epsilon} \left[\sum_{i=1}^2 \frac{\partial u_3^\epsilon}{\partial x_i} \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_i} \right) + \frac{1}{\eta^2} \frac{\partial u_3^\epsilon}{\partial x_3} \left(\frac{\partial \varphi}{\partial x_3} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_3} \right) \right] dx dx_3 \\ &= \frac{1}{\eta} \int_{\Omega^\epsilon} p^\epsilon \left(\frac{\partial \varphi}{\partial x_3} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_3} \right) dx dx_3. \end{aligned}$$

Extending u^ϵ and p^ϵ by zero to Ω_{max} and passing to the limit after multiplication by $\eta^3 \epsilon$, we obtain

$$\int_{\Omega_{max}} \int_{[0, 1]^3} \xi \frac{\partial \varphi}{\partial y_3} dx dx_3 dy dy_3 = 0.$$

To show that $\xi = 0$ for $x_3 \geq h(x, y)$, we choose φ in $\mathcal{D}(\Omega_{max}, \mathcal{C}_\#^\infty(Y))$ with $\varphi = 0$ for $x_3 \leq h(x, y)$, we obtain

$$\int_{\Omega_{max}} \eta^2 \bar{p}^\epsilon \varphi(x, x_3, \frac{x}{\epsilon}) dx dx_3 = 0.$$

Passing to the two-scale limit, we obtain the result. Lemma 4.3 completes the proof. $\square$

4.2. *Proof of Theorem 3.1.* In this section we prove Theorem 3.1, which specifies the limit pressure in the case $\alpha = 1$, which corresponds to

$$\eta = \lambda \epsilon.$$

To this topic, we first give the two-scale convergence result for the velocity, and then the relation between limit pressure and velocity involving the local problems.

PROPOSITION 4.6. There is u^λ in $(L^2(\Omega_{max}; H_{per}^1(Y)))^3$ with $\frac{\partial u^\lambda}{\partial x_3}$ in $(L^2(\Omega_{max} \times Y))^3$ and there is v^λ in $(L^2(\Omega_{max} \times Y; H_{per}^1([0, 1])))^3$ such that

$$\begin{aligned} \bar{u}^\epsilon &\rightharpoonup u^\lambda, \\ \frac{\partial \bar{u}^\epsilon}{\partial x_3} &\rightharpoonup \frac{\partial u^\lambda}{\partial x_3} + \frac{\partial v^\lambda}{\partial y_3}, \\ \epsilon \frac{\partial \bar{u}^\epsilon}{\partial x_i} &\rightharpoonup \frac{\partial u^\lambda}{\partial y_i} \quad i = 1, 2. \end{aligned}$$

Moreover,

$$\begin{aligned} \operatorname{div}_\lambda u^\lambda &= \frac{\partial u_1^\lambda}{\partial y_1} + \frac{\partial u_2^\lambda}{\partial y_2} + \frac{1}{\lambda} \frac{\partial u_3^\lambda}{\partial x_3} = 0 \quad \text{in } B_x \quad \text{a.e. } x \in \omega, \\ u^\lambda &= 0 \quad \text{for } x_3 \geq h(x, y) \quad \text{and } u^\lambda = (s_1, s_2, 0) \quad \text{for } x_3 = 0. \end{aligned}$$

Proof. From estimates (4.2), we obtain

$$\begin{aligned} \|\bar{u}^\epsilon\|_{L^2(\Omega_{max})^3} &= \|u^\epsilon\|_{L^2(\Omega^\epsilon)^3} \leq K, \\ \epsilon \left\| \frac{\partial \bar{u}^\epsilon}{\partial x_i} \right\|_{L^2(\Omega_{max})^3} &\leq K, \quad i = 1, 2, \\ \left\| \frac{\partial \bar{u}^\epsilon}{\partial x_3} \right\|_{L^2(\Omega_{max})^3} &\leq K. \end{aligned}$$

So applying the two-scale convergence results of Proposition 1.14 in [3], we deduce the existence of u^λ in $L^2(\Omega_{max}; H_{per}^1[0, 1]^3)^3$ and v^λ in $(L^2(\Omega_{max} \times Y; H_{per}^1[0, 1]))^3$ such that

$$\begin{aligned} \bar{u}^\epsilon &\rightharpoonup u^\lambda, \\ \frac{\partial \bar{u}^\epsilon}{\partial x_3} &\rightharpoonup \frac{\partial u^\lambda}{\partial x_3} + \frac{\partial v^\lambda}{\partial y_3}, \\ \epsilon \frac{\partial \bar{u}^\epsilon}{\partial x_i} &\rightharpoonup \frac{\partial u^\lambda}{\partial y_i} \quad i = 1, 2. \end{aligned}$$

Let us prove that u^λ does not rely on y_3 . Exactly as for the pressure, we obtain $u^\lambda = 0$ for $x_3 \geq h(x, y)$. Then, setting $\varphi^\epsilon = \varphi(x, x_3, \frac{x}{\epsilon}, \frac{x_3}{\epsilon})$ with φ in $\mathcal{D}(\Omega_{max} \times [0, 1]^3)$ and $\varphi = 0$ for $x_3 \geq h(x, y)$, we obtain

$$\begin{aligned} \int_{\Omega^\epsilon} \frac{\partial u^\epsilon}{\partial x_3} \cdot \varphi^\epsilon \, dx \, dx_3 &= - \int_{\Omega^\epsilon} u^\epsilon \cdot \left(\frac{\partial \varphi}{\partial x_3} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_3} \right) \, dx \, dx_3 \\ &= - \int_{\Omega_{max}} \bar{u}^\epsilon \cdot \left(\frac{\partial \varphi}{\partial x_3} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_3} \right) \, dx \, dx_3. \end{aligned}$$

Passing to the limit in Ω_{max} after multiplication by ϵ , we obtain

$$0 = \int_{\Omega_{max}} \int_Y u^\lambda \cdot \frac{\partial \varphi}{\partial y_3} dx dx_3 dy,$$

which induces $\frac{\partial u^\lambda}{\partial y_3} = 0$ for $x_3 < h(x, y)$, so u^λ does not rely on y_3 .

Let $\varphi^\epsilon = \varphi(x, x_3, \frac{x}{\epsilon})$ with φ arbitrary in $\mathcal{D}(\Omega_{max}; \mathcal{C}_\#^\infty(Y))$. From (2.8), we obtain

$$\int_{\Omega^\epsilon} \left(\frac{\partial u_1^\epsilon}{\partial x_1} + \frac{\partial u_2^\epsilon}{\partial x_2} + \frac{1}{\eta} \frac{\partial u_3^\epsilon}{\partial x_3} \right) \varphi^\epsilon dx dx_3 = 0.$$

Using the Green formula on each term, we obtain

$$\int_{\Omega^\epsilon} u_1^\epsilon \left(\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_1} \right) dx dx_3 + \int_{\Omega^\epsilon} u_2^\epsilon \left(\frac{\partial \varphi}{\partial x_2} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_2} \right) dx dx_3 + \frac{1}{\eta} \int_{\Omega^\epsilon} u_3^\epsilon \frac{\partial \varphi}{\partial x_3} dx dx_3 = 0.$$

Passing to the two-scale limit in Ω_{max} , after multiplication by ϵ , we obtain

$$\int_Q \left(u_1^\lambda \frac{\partial \varphi}{\partial y_1} + u_2^\lambda \frac{\partial \varphi}{\partial y_2} + \frac{1}{\lambda} u_3^\lambda \frac{\partial \varphi}{\partial x_3} \right) dx dy dx_3 = \int_Q \operatorname{div}_\lambda u^\lambda \varphi dx dy dx_3 = 0,$$

with Q defined in (3.1). Now taking φ in the form

$$\varphi(x, x_3, y) = \theta \psi_x(x_3, y)$$

with θ and ψ_x arbitrary in $\mathcal{D}(\omega)$ (respectively in $\mathcal{D}(B_x)$), we obtain the result. We now prove that the boundary conditions are still valid for the two-scale velocity limit. Passing to the limit in the Green formula

$$\int_{\Omega_{max}} \frac{\partial \bar{u}^\epsilon}{\partial x_3} \cdot \varphi(x, \frac{x}{\epsilon}, x_3) dx dx_3 = - \int_{\Omega_{max}} \bar{u}^\epsilon \cdot \frac{\partial \varphi}{\partial x_3} dx dx_3 + \int_\omega (s_1, s_2, 0) \cdot \varphi(x, \frac{x}{\epsilon}, 0) dx$$

for any φ in $C^\infty(\Omega_{max}; \mathcal{C}_\#^\infty(Y))$, we obtain

$$\begin{aligned} & \int_{\Omega_{max}} \int_Y \frac{\partial \bar{u}^\lambda}{\partial x_3} \cdot \varphi(x, y, x_3) dx dy dx_3 \\ &= - \int_{\Omega_{max}} \int_Y \bar{u}^\lambda \cdot \frac{\partial \varphi}{\partial x_3} dx dy dx_3 + \int_\omega \int_Y (s_1, s_2, 0) \cdot \varphi(x, y, 0) dx dy. \end{aligned}$$

Applying again the Green formula, it is obvious that

$$u^\lambda(x, y, 0) = (s_1, s_2, 0) \quad a.e. \quad (x, y) \in \omega \times Y.$$

□

Now, it is necessary to use the local problems (3.2), (3.3) in order to give a relation between the limit velocity and pressure.

PROPOSITION 4.7. The limit pressure p^λ belongs to $H^1(\omega)$ and the following relation is satisfied:

$$u^\lambda = \sum_{j=1}^2 \frac{\partial p^\lambda}{\partial x_j} w^{j\lambda} + w^{3\lambda}. \quad (4.12)$$

Proof. The proof needs two steps.

Step 1: We will prove the relation

$$\mu \int_w \int_{B_x} \left[\sum_{i=1}^2 \frac{\partial u^\lambda}{\partial y_i} \cdot \frac{\partial \varphi}{\partial y_i} + \frac{1}{\lambda^2} \frac{\partial u^\lambda}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_3} \right] dx \, dx_3 \, dy = \int_w \int_{B_x} p^\lambda(x) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx \, dx_3 \, dy \quad (4.13)$$

for any φ in $\left(\mathcal{D}(\Omega_{max}; C_\#^\infty(Y))\right)^3$ with $\varphi = 0$ for $x_3 \geq h(x, y)$ and $\operatorname{div}_\lambda \varphi = 0$. We set $\varphi^\epsilon(x, x_3) = \varphi(x, \frac{x}{\epsilon}, x_3)$ which can be chosen as a test function in (2.7):

$$\begin{aligned} & \mu \int_{\Omega_{max}} \left[\sum_{i=1}^2 \frac{\partial \bar{u}^\epsilon}{\partial x_i} \cdot \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_i} \right) + \frac{1}{\lambda^2 \epsilon^2} \frac{\partial \bar{u}^\epsilon}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_3} \right] dx \, dx_3 \\ &= \int_{\Omega_{max}} \bar{p}^\epsilon \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{1}{\epsilon} \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{1}{\epsilon} \frac{\partial \varphi_2}{\partial y_2} + \frac{1}{\lambda \epsilon} \frac{\partial \varphi_3}{\partial x_3} \right) dx \, dx_3. \end{aligned} \quad (4.14)$$

As $\operatorname{div}_\lambda \varphi = 0$, we obtain

$$\begin{aligned} & \mu \int_{\Omega_{max}} \left[\sum_{i=1}^2 \frac{\partial \bar{u}^\epsilon}{\partial x_i} \cdot \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{\epsilon} \frac{\partial \varphi}{\partial y_i} \right) + \frac{1}{\lambda^2 \epsilon^2} \frac{\partial \bar{u}^\epsilon}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_3} \right] dx \, dx_3 \\ &= \int_{\Omega_{max}} \bar{p}^\epsilon \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx \, dx_3. \end{aligned}$$

Passing to the limit after multiplication by ϵ^2 , we obtain

$$\begin{aligned} & \mu \int_{\Omega_{max}} \int_Y \sum_{i=1}^2 \frac{\partial u^\lambda}{\partial y_i} \cdot \frac{\partial \varphi}{\partial y_i} dx \, dx_3 \, dy + \int_{\Omega_{max}} \int_Y \frac{1}{\lambda^2} \left(\frac{\partial u^\lambda}{\partial x_3} + \int_Y \frac{\partial v^\lambda}{\partial y_3} dy_3 \right) \cdot \frac{\partial \varphi}{\partial x_3} dx \, dx_3 \, dy \\ &= \int_{\Omega_{max}} \int_Y p^\lambda(x) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx \, dx_3 \, dy. \end{aligned}$$

Using the periodicity of v^λ and the fact that $u^\lambda = 0$ for $x_3 \geq h(x, y)$, we obtain (4.13).

Step 2: We now prove that $p^\lambda \in H^1(\omega)$ and that u^λ satisfies a Stokes equation in B_x , which will lead to (4.25).

Let $\psi = \psi(y_2, x_3)$ be a function that does not rely on y_1 in $C_{0\#}^\infty([0, 1[\times]0, h_{min}[)$. We denote by $\bar{\psi}$ its extension by zero in B_x . We suppose that $\tilde{\psi} = \int_{B_x} \bar{\psi} dx_3 \, dy \neq 0$. Let $\phi = (\frac{\theta \bar{\psi}}{\tilde{\psi}}, 0, 0)$ with θ in $\mathcal{D}(w)$. As $\operatorname{div}_\lambda \phi = 0$, ϕ is a test function in (4.13), so

$$\mu \int_w \int_{B_x} \nabla_\lambda u^\lambda \cdot \nabla_\lambda \phi \, dx \, dx_3 \, dy = \int_w \int_{B_x} p^\lambda(x) \frac{\partial}{\partial x_1} \left(\theta \frac{\bar{\psi}}{\tilde{\psi}} \right) dx \, dx_3 \, dy.$$

As θ and p^λ do not rely on variables y, x_3 , we obtain

$$\frac{\mu}{\tilde{\psi}} \int_w \theta(x) \left(\int_{B_x} \nabla_\lambda u_1^\lambda \nabla_\lambda \bar{\psi} \, dx_3 \, dy \right) dx = \int_w p^\lambda \frac{\partial \theta}{\partial x_1} \left(\int_{B_x} \bar{\psi} \frac{1}{\tilde{\psi}} \, dx_3 \, dy \right) dx = \int_w p^\lambda \frac{\partial \theta}{\partial x_1} dx.$$

As $\nabla_\lambda u_1^\lambda$ belongs to $L^2(Q)^3$, $\left(\int_{B_x} \nabla_\lambda u_1^\lambda \nabla_\lambda \bar{\psi} \, dx_3 \, dy \right)$ lies in $L^2(w)$, which implies that

$\frac{\partial p^\lambda}{\partial x_1}$ is in $L^2(w)$.

In the same way, we prove that $\frac{\partial p^\lambda}{\partial x_2}$ is in $L^2(w)$, so that p^λ belongs to $H^1(w)$. Now we choose in (4.13) $\psi = \theta(x) \cdot \varphi_x(y, x_3)$ with θ in $\mathcal{D}(w)$, φ_x in $(C_{0\#}^\infty(B_x))^3$, and $\operatorname{div}_\lambda \varphi = 0$:

$$\begin{aligned} \mu \int_w \theta \left(\int_{B_x} \nabla_\lambda u^\lambda \cdot \nabla_\lambda \varphi \, dx_3 \, dy \right) dx &= \int_w \int_{B_x} p^\lambda \left(\frac{\partial}{\partial x_1} (\theta \varphi)_1 + \frac{\partial}{\partial x_2} (\theta \varphi)_2 \right) dx \, dx_3 \, dy \\ &= - \int_w \left[\int_{B_x} \left(\frac{\partial p^\lambda}{\partial x_1} \varphi_1 + \frac{\partial p^\lambda}{\partial x_2} \varphi_2 \right) dx_3 \, dy \right] \theta \, dx. \end{aligned}$$

This is true for all θ in $\mathcal{D}(w)$, and by the density argument and Proposition 4.6, we obtain the following Stokes problem:

$$\begin{aligned} \mu \int_{B_x} \nabla_\lambda u^\lambda \cdot \nabla_\lambda \varphi \, dx_3 \, dy &= - \int_{B_x} \left(\varphi_1 \frac{\partial p^\lambda}{\partial x_1} + \varphi_2 \frac{\partial p^\lambda}{\partial x_2} \right) dx_3 \, dy \quad \text{a.e. } x \in w. \\ \forall \varphi &\in \{(H_{0\#}^1(B_x))^3, \operatorname{div}_\lambda \varphi = 0\} \end{aligned} \quad (4.15)$$

and

$$\operatorname{div}_\lambda u^\lambda = 0 \quad \text{a.e. } x \in \omega.$$

Since p^λ depends only on x , we obtain by linearity and using the local problems (3.2), (3.3) the desired relation. $\square$

Proof of Theorem 3.1. Choosing in (2.8) $\varphi \in H^1(\omega)$ and using the Green formula in each term, we obtain

$$- \int_{\Omega^\epsilon} u_1^\epsilon \frac{\partial \varphi}{\partial x_1} dx \, dx_3 + \int_{\partial \Omega^\epsilon} u_1^\epsilon n_1 \varphi \, d\sigma - \int_{\Omega^\epsilon} u_2^\epsilon \frac{\partial \varphi}{\partial x_2} dx \, dx_3 + \int_{\partial \Omega^\epsilon} u_2^\epsilon n_2 \varphi \, d\sigma = 0.$$

Using the boundary conditions on the velocity u^ϵ , we obtain

$$\int_{\Omega^\epsilon} u_1^\epsilon \frac{\partial \varphi}{\partial x_1} dx \, dx_3 + \int_{\Omega^\epsilon} u_2^\epsilon \frac{\partial \varphi}{\partial x_2} dx \, dx_3 = \int_{\Gamma_l} g \cdot n \varphi \, d\sigma.$$

Passing to the limit yields

$$\int_w \int_{B_x} \left(u_1^\lambda \frac{\partial \varphi}{\partial x_1} + u_2^\lambda \frac{\partial \varphi}{\partial x_2} \right) dx \, dx_3 = \int_{\partial w} f \cdot n \varphi \, d\sigma \quad \forall \varphi \in H^1(w) \quad (4.16)$$

with

$$f = \int_0^{h_{min}} g \, dx_3.$$

Now we will use the expression (4.12) of the velocity in (4.16) in order to identify the limit equation for p^λ :

$$\int_w \int_{B_x} \sum_{i=1}^2 \left(\sum_{j=1}^2 \frac{\partial p^\lambda}{\partial x_j} w_i^{j\lambda} + w_i^{3\lambda} \right) \frac{\partial \varphi}{\partial x_i} = \int_{\partial w} f \cdot n \varphi \, d\sigma \quad \forall \varphi \in H^1(w).$$

As p^λ and φ do not rely on variables y and x_3 , this equation gives (3.4).

It remains to prove the uniqueness of the solution of (3.4). For this it is sufficient to prove that the matrix A^λ whose entries $a_{ij}^\lambda(x)$ are defined in Theorem 3.1 is symmetric and positive definite.

Let us choose $w^{1\lambda}$ (resp. $w^{2\lambda}$) as a test function in (3.2) with $k = 2$ (resp. $k = 1$):

$$\mu \int_{B_x} \nabla_\lambda w^{1\lambda} \cdot \nabla_\lambda w^{2\lambda} dy dx_3 = - \int_{B_x} w_1^{2\lambda} = a_{12}^\lambda = a_{21}^\lambda,$$

which proves the symmetry of A^λ . For arbitrary $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, we choose $\xi_i \xi_j w^{j\lambda}$ as a test function in (3.2) with $k = i$:

$$\mu \xi_i \xi_j \int_{B_x} \nabla_\lambda w^{i\lambda} \cdot \nabla_\lambda w^{j\lambda} dx_3 dy = - \int_{B_x} \xi_i \xi_j w_i^{j\lambda} dy dx_3 = \xi_i \xi_j a_{ij}^\lambda.$$

By addition, this implies

$$\sum_{i,j=1}^2 \xi_i \xi_j a_{ij}^\lambda = \mu \int_{B_x} (\xi_1 \nabla_\lambda w^{1\lambda} + \xi_2 \nabla_\lambda w^{2\lambda})^2 dy dx_3 \geq 0.$$

Suppose now that $\sum_{i,j=1}^2 \xi_i \xi_j a_{ij}^\lambda = 0$, that is

$$\int_{B_x} (\nabla_\lambda (\xi_1 w^{1\lambda} + \xi_2 w^{2\lambda}))^2 dy dx_3 = 0.$$

By Poincaré's inequalities, this gives

$$\xi_1 w^{1\lambda} + \xi_2 w^{2\lambda} = 0.$$

From (3.2), taking successively $k = 1$ and 2 , we obtain

$$\int_{B_x} (\xi_1 \varphi_1 + \xi_2 \varphi_2) dy dx_3 = 0$$

for any φ in $(H_{0\#}^1(B_x))^3$ with $\operatorname{div}_\lambda = 0$. Now, taking $\varphi = (\varphi_1(y_2, x_3), 0, 0)$ with $\varphi_1 \in H_{0\#}^1(\omega \times]0, 1[)$ and $\int_{B_x} \varphi dy dx_3 \neq 0$, we deduce $\xi_1 = 0$. In the same way, $\xi_2 = 0$, which ends the proof of Theorem 3.1. $\square$

4.3. *Proof of Theorem 3.2* ($\alpha > 1$). We recall that, in all these sections, the relation $\eta = \lambda \epsilon^\alpha$ with $\alpha > 1$ is satisfied. The limit of the pressure and velocity will be denoted without the λ superscript, since they will not depend on λ . Passing to the limit in the variational formulation in (2.7), only derivatives in x_3 will remain in the limit velocity term. Henceforth the limit u and the test function are no longer in the same functional space and a density lemma is needed.

First, we introduce the following spaces:

$$\begin{aligned} V_0 &= \{\varphi = (\varphi_1, \varphi_2) \in (V_{x_3}(B_x))^2, \varphi(x, 0) = \varphi(x, h) = 0\}, \\ V_{00} &= \{\varphi \in V_0, \int_Y \left(\int_0^h \varphi dx_3 \right) \cdot \nabla_Y \psi dy = 0 \quad \forall \psi \in H_{per}^1(Y)\}, \\ V &= \{(\varphi_1, \varphi_2) \in (C_{0\#}^\infty(B_x))^2 \quad \exists \varphi_3 \in C_{0\#}^\infty(B_x) \text{ with } \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2} + \frac{\partial \varphi_3}{\partial x_3} = 0\}. \end{aligned}$$

It is easy to see that V_0 and V_{00} are Hilbert spaces as they are closed subspaces of $(V_{x_3}(B_x))^2$. We will need the following density lemma.

LEMMA 4.8.

$$\overline{V} = V_{00}$$

where the adherence is in the sense of the norm of $(V_{x_3}(B_x))^2$ (see (4.1)).

Proof. We first prove the inclusion $\overline{V} \subset V_{00}$. Let φ be in $\overline{V}$. There is a sequence $(\varphi_n) = (\varphi_{1n}, \varphi_{2n})$ in V such that $\varphi_n \rightarrow \varphi$ in $(V_{x_3}(B_x))^2$. So for any n there exists φ_{3n} in $C_{0\#}^\infty(B_x)$ such that for any $\psi \in H_{per}^1(Y)$, we obtain

$$\int_{B_x} \left(\frac{\partial \varphi_{1n}}{\partial y_1} + \frac{\partial \varphi_{2n}}{\partial y_2} + \frac{\partial \varphi_{13}}{\partial x_3} \right) \psi \, dy \, dx_3 = 0.$$

Using the Green formula and the fact that ψ is a function of y only, we obtain

$$\int_Y \left(\int_0^h \varphi_n \, dx_3 \right) \nabla_y \psi \, dy = 0,$$

which proves that φ belongs to V_{00} by passing to the limit in n .

To prove that $\overline{V} = V_{00}$, we will show, as in Theorem 1.6 page 18 of [22], that any continuous and linear form L on V_{00} vanishes on $\overline{V}$ is identically equal to zero. Let L be a continuous form on V_{00} that vanishes on $\overline{V}$. Applying the Riesz theorem, there is l in V_{00} such that

$$L(v) = \sum_{i=1}^2 \int_{B_x} \left(l_i v_i + \frac{\partial l_i}{\partial x_3} \frac{\partial v_i}{\partial x_3} \right) \, dy \, dx_3 \quad \forall v \in V_{00}. \quad (4.17)$$

So, we obtain

$$\sum_{i=1}^2 \int_{B_x} \left(l_i v_i + \frac{\partial l_i}{\partial x_3} \frac{\partial v_i}{\partial x_3} \right) \, dy \, dx_3 = 0 \quad \forall v \in V. \quad (4.18)$$

The relation above is in fact true for any (v_1, v_2, v_3) in $C_0^\infty(B_x)$ such that

$$\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{\partial v_3}{\partial x_3} = 0.$$

Applying the De Rham theorem, we deduce by density the existence of q in $L^2(B_x)$ such that

$$\sum_{i=1}^2 \int_{B_x} \left(l_i \phi_i + \frac{\partial l_i}{\partial x_3} \frac{\partial \phi_i}{\partial x_3} \right) \, dy \, dx_3 = \int_{B_x} q \operatorname{div}_{B_x} \phi \, dy_1 \, dy_2 \, dx_3 \quad \forall \phi \in (H_0^1(B_x))^3. \quad (4.19)$$

Choosing successively $\phi = (\varphi, 0, 0)$, $\phi = (0, \varphi, 0)$ and $\phi = (0, 0, \varphi)$ with φ in $H_0^1(B_x)$, we deduce from (4.19)

$$l_i - \frac{\partial^2 l_i}{\partial x_3^2} = -\frac{\partial q}{\partial y_i} \quad i = 1, 2 \quad \text{in } H^{-1}(B_x) \quad (4.20)$$

and

$$\int_{B_x} q \frac{\partial \psi}{\partial x_3} \, dy \, dx_3 = 0 \quad \forall \psi \in H_0^1(B_x),$$

which implies that q does not rely on x_3 . Let us now prove that q lies in $H_{per}^1(Y)$.

Set $\phi = (\frac{x_3}{h^3(x, y)}(x_3 - h)\psi(y), 0, 0)$ with $\psi \in H_{per}^1(Y)$ in (4.19):

$$\int_{B_x} l_1 \frac{x_3}{h^3} (x_3 - h) \psi dy dx_3 + \int_{B_x} \frac{\partial l_1}{\partial x_3} \frac{2x_3 - h}{h^3} \psi dy dx_3 = \int_{B_x} q \frac{\partial}{\partial y_1} \left(\frac{x_3(x_3 - h)}{h^3} \psi \right) dy dx_3.$$

Integrating by parts in the second term on the left-hand side, and taking into account that q and ψ do not rely on x_3 , we obtain by simple calculations

$$\int_{B_x} l_1 \frac{x_3}{h^3} (x_3 - h) \psi dy dx_3 - \int_{B_x} \frac{2}{h^3} l_1 \psi dy dx_3 = -\frac{1}{6} \int_{B_x} q \frac{\partial \psi}{\partial y_1} dy dx_3,$$

which implies

$$\left\langle \frac{\partial q}{\partial y_1}, \psi \right\rangle = 6 \int_Y \left(\int_0^h \frac{l_1}{h^3} [x_3(x_3 - h) - 2] dx_3 \right) \psi dy.$$

As $\int_0^h \frac{l_1}{h^3} [x_3(x_3 - h) - 2] dx_3$ belongs to $L^2(Y)$, we deduce that $\frac{\partial q}{\partial y_1}$ belongs to $L^2(Y)$.

Proceeding in the same manner in y_2 , we prove that $q \in H^1(Y)$. So equality (4.20) is, in fact, in $L^2(B_x)$. Now, for any ϕ in $(H_{0\#}^1(B_x))^3$ with $\operatorname{div}_{B_x} \phi = 0$, multiplying (4.20) by ϕ_i and using the Green formula, we obtain after addition

$$\begin{aligned} & \sum_{i=1}^2 \int_{B_x} \left(l_i \phi_i + \frac{\partial l_i}{\partial x_3} \frac{\partial \phi_i}{\partial x_3} \right) dy_1 dy_2 dx_3 \\ &= \int_{B_x} q \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} \right) dy_1 dy_2 dx_3 - \int_{\partial B_x} q (\phi_1 n_1 + \phi_2 n_2) d\sigma. \end{aligned} \quad (4.21)$$

Since $\int_{B_x} q \frac{\partial \phi_3}{\partial y_3} dy dy_2 dx_3 = 0$, we deduce that

$$\int_{B_x} q \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} \right) dy dx_3 = 0$$

so comparing with (4.18), we obtain

$$\int_{\partial B_x} q (\phi_1 n_1 + \phi_2 n_2) d\sigma = 0 \quad \forall \phi \in (H_{0\#}^1(B_x))^3 \quad \text{with } \operatorname{div}_{B_x} \phi = 0. \quad (4.22)$$

Set

$$B_0 = \{(y_2, x_3) \mid y_2 \in]0, 1[\text{ and } 0 < x_3 < h(x, 0, y_2)\}.$$

For any ψ in $\mathcal{D}(B_0)$, there is $\varphi \in (H_{0\#}^1(B_x))^3$ with $\operatorname{div}_\lambda \varphi = 0$ such that (see [16])

$$\varphi_{/y_1=0} = \varphi_{/y_1=1} = (\psi, 0, 0),$$

$$\varphi_{/y_2=0} = \varphi_{/y_2=1} = \varphi_{/x_3=0} = \varphi_{/x_3=h} = 0.$$

We then deduce from (4.22) the y_1 -periodicity of q , and in a similar way its y_2 -periodicity, so $q \in H_{per}^1(Y)$.

To end the proof of the lemma, we must show that L defined in (4.17) is equal to zero for any v in V_{00} . Multiplying (4.20) by v_i and using the Green formula, we obtain

$$\begin{aligned} \sum_{i=1}^2 \int_{B_x} \left(l_i v_i + \frac{\partial l_i}{\partial x_3} \frac{\partial v_i}{\partial x_3} \right) dy dx_3 &= - \int_{B_x} \left(\frac{\partial q}{\partial y_1} v_1 + \frac{\partial q}{\partial y_2} v_2 \right) dy dx_3 \\ &= - \int_Y \left(\int_0^h v dx_3 \right) \nabla_y q dy, \end{aligned}$$

since q is independent on x_3 . The last term is equal to zero as $v \in V_{00}$ and $q \in H_{per}^1(Y)$, which proves the result. $\square$

PROPOSITION 4.9. There is u in $(L^2(\Omega_{max}; H_{per}^1(Y)))^3$ with $\frac{\partial u}{\partial x_3} \in (L^2(\Omega_{max} \times Y))^3$ and v in $(L^2(\Omega_{max} \times Y; H_{per}^1([0, 1])))^3$ such that

$$\begin{aligned} \bar{u}^\epsilon &\rightharpoonup u, \\ \frac{\partial \bar{u}^\epsilon}{\partial x_3} &\rightharpoonup \frac{\partial u}{\partial x_3} + \frac{\partial v}{\partial y_3}, \\ \eta \frac{\partial \bar{u}_i^\epsilon}{\partial x_3} &\rightharpoonup 0. \end{aligned}$$

Moreover,

$$\int_Y \sum_{i=1}^2 \left(\int_0^h u_i^\lambda dx_3 \right) \frac{\partial \psi}{\partial y_i} dy = 0 \quad \forall \psi \in H^1(Y) \quad (4.23)$$

$$u = 0 \text{ for } x_3 \geq h(x, y) \text{ and } u = (s_1, s_2, 0) \text{ for } x_3 = 0.$$

Proof. The proof of all the results except (4.23) is similar to the ones in Proposition 4.6. Relation (4.23) is immediately deduced from $\text{div}_\eta \bar{u}^\epsilon = 0$ by taking a test function of the form $\theta(x)\psi(y)$ with $\theta \in \mathcal{D}(\omega)$ and $\psi \in H_{per}^1(Y)$. $\square$

Now we have to introduce the relation between the limit velocity and the gradient of the limit pressure. For that purpose, in addition to associated auxiliary problems, we will need a function R in $(V_{x_3}(B_x))^3$, which satisfies

$$R_{/x_3=0} = (s_1, s_2, 0) \text{ and } R_{/x_3=h} = 0.$$

For example we take $R = (r_1, r_2, 0)$ where (for $i = 1, 2$)

$$r_i(x, x_3) = \begin{cases} s_i \left(1 - \frac{x_3}{h_{min}} \right) & \text{if } 0 < x_3 < h_{min}, \\ 0 & x_3 \geq h_{min}. \end{cases} \quad (4.24)$$

PROPOSITION 4.10. The limit pressure p belongs to $H^1(\omega)$ and the following relation is satisfied:

$$u = w^1 \frac{\partial p}{\partial x_1} + w^2 \frac{\partial p}{\partial x_1} + w^3, \quad (4.25)$$

where w^k ($k = 1, 2, 3$) are the unique solutions of the auxiliary problems.

Find w^k ($k = 1, 2$) in V_{00} such that

$$\mu \sum_{i=1}^2 \int_{B_x} \frac{\partial w_i^k}{\partial x_3} \frac{\partial \varphi_i}{\partial x_3} dy dx_3 = - \int_{B_x} \varphi_k dy dx_3 \quad \forall \varphi \in V_{00} \quad (4.26)$$

and

Find w^3 in $R + V_{00}$ such that

$$\mu \sum_{i=1}^2 \int_{B_x} \frac{\partial w_i^3}{\partial x_3} \frac{\partial \varphi_i}{\partial x_3} dy dx_3 = 0 \quad \forall \varphi \in V_{00} \quad (4.27)$$

with R defined in (4.24).

Proof. Let $\psi = (\psi_1, \psi_2, \psi_3) \in \left(\mathcal{D}(\Omega_{max}, C_{\#}^{\infty}(Y)) \right)^3$ with $\operatorname{div}_{B_x} \psi = 0$ and $\psi = 0$ for $x_3 \geq h(x, y)$. Set $\psi^{\epsilon}(x, x_3) = \psi(x, \frac{x}{\epsilon}, x_3)$ and taking $(\psi_1^{\epsilon}, \psi_2^{\epsilon}, \frac{\eta}{\epsilon} \psi_3^{\epsilon})$ as a test function in (2.7), we obtain

$$\begin{aligned} & \mu \int_{\Omega_{max}} \left[\sum_{i,j=1}^2 \frac{\partial \bar{u}_i^{\epsilon}}{\partial x_j} \cdot \left(\frac{\partial \psi_i}{\partial x_j} + \frac{1}{\epsilon} \frac{\partial \psi_i}{\partial y_j} \right) + \frac{\eta}{\epsilon} \frac{\partial \bar{u}_3^{\epsilon}}{\partial x_3} \cdot \left(\frac{\partial \psi_3}{\partial x_j} + \frac{1}{\epsilon} \frac{\partial \psi_3}{\partial y_j} \right) \right. \\ & \left. + \frac{1}{\eta^2} \sum_{i=1}^2 \frac{\partial \bar{u}_i^{\epsilon}}{\partial x_3} \cdot \frac{\partial \psi_i}{\partial x_3} + \frac{1}{\eta \epsilon} \frac{\partial \bar{u}_3^{\epsilon}}{\partial x_3} \cdot \frac{\partial \psi_3}{\partial x_3} \right] dx dx_3 = \int_{\Omega_{max}} p^{\epsilon} \left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} \right) dx dx_3. \end{aligned} \quad (4.28)$$

Passing to the limit after multiplication by η^2 , we obtain

$$\mu \int_{\omega \times B_x} \sum_{i=1}^2 \frac{\partial u_i}{\partial x_3} \cdot \frac{\partial \psi_i}{\partial x_3} dx dy dx_3 = \int_{\omega \times B_x} p \left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} \right) dx dx_3. \quad (4.29)$$

Exactly as in the second step of the proof of Proposition 4.7, we prove that $p \in H^1(\omega)$ and that

$$a.e. \ x \in \omega \quad \mu \sum_{i=1}^2 \int_{B_x} \frac{\partial u_i}{\partial x_3} \cdot \frac{\partial \varphi_i}{\partial x_3} dx dy dx_3 = - \sum_{i=1}^2 \int_{B_x} \frac{\partial p}{\partial x_i} \varphi_i dy dx_3 \quad \forall \varphi \in V. \quad (4.30)$$

By the density Lemma 4.8, the above relation is true for any $\varphi \in V_{00}$.

From Proposition 4.9, $u - R$ belongs to V_{00} . On the other hand, by the Poincaré inequality, the semi-norm

$$\left(\sum_{i=1}^2 \int_{B_x} \left(\frac{\partial \varphi_i}{\partial x_3} \right)^2 dy dx_3 \right)^{\frac{1}{2}}$$

is a norm in V_{00} . So from the Lax-Milgram theorem, (u_1, u_2) is the unique solution in $R + V_{00}$ satisfying (4.30) for all φ in V_{00} . Besides, for the same reasons, the auxiliary problems (4.26) and (4.27) have unique solutions that give (4.25) by linearity and the fact that p depends on x only. $\square$

LEMMA 4.11. The solutions to the auxiliary problems (4.26) and (4.27) satisfy

$$w_i^k = \frac{1}{2\mu} \left(\frac{\partial \pi^k}{\partial y_i} + \delta_{ki} \right) x_3 (x_3 - h) \quad (4.31)$$

and

$$w_i^3 = \frac{1}{2\mu} \frac{\partial \pi^3}{\partial y_i} x_3 (x_3 - h) + s_i \left(1 - \frac{x_3}{h}\right) \quad (4.32)$$

where π^k ($k = 1, 2, 3$) are the solutions of the local problems defined in (3.5) and (3.6).

Proof. As $V \subset V_{00}$, using the De Rham theorem in equation (4.26), there is $\tilde{\pi}^k$ in $L_0^2(B_x)$ such that

$$\begin{aligned} \mu \sum_{i=1}^2 \int_{B_x} \frac{\partial w_i^k}{\partial x_3} \frac{\partial \varphi_i}{\partial x_3} dy_1 dy_2 dx_3 &= - \int_{B_x} \varphi_k dy_1 dy_2 dx_3 \\ &+ \int_{B_x} \tilde{\pi}^k \operatorname{div}_{B_x} \varphi dy_1 dy_2 dx_3 \quad \forall \varphi \in H_0^1(B_x)^3 \end{aligned} \quad (4.33)$$

which implies, setting respectively $(\varphi, 0, 0)$, $(0, \varphi, 0)$ and $(0, 0, \varphi)$ as test functions in (4.33), the following equations:

$$\mu \frac{\partial^2 w_i^k}{\partial x_3^2} = \delta_{ik} + \frac{\partial \tilde{\pi}^k}{\partial y_i} \quad i, k = 1, 2, \quad (4.34)$$

and

$$\frac{\partial \tilde{\pi}^k}{\partial x_3} = 0 \quad k = 1, 2. \quad (4.35)$$

In the same manner, as for q in Lemma 4.10, we prove that $\tilde{\pi}^k$ are in $H_{per}^1(Y)$ so that (4.34) holds in $L^2(B_x)$.

Integrating twice in x_3 , taking into account the boundary conditions and the fact that $\tilde{\pi}^k$ does not rely on x_3 , we obtain

$$w_i^k = \frac{1}{2\mu} \left(\frac{\partial \tilde{\pi}^k}{\partial y_i} + \delta_{ik} \right) x_3 (x_3 - h). \quad (4.36)$$

Finally, w^k belongs to V_{00} , so using (4.36), we obtain that $\tilde{\pi}^k$ satisfy (3.5). By uniqueness $\tilde{\pi}^k = \pi^k$, which proves (4.31). $\square$

Proof of Theorem 3.2. Similarly to the beginning of the proof of Theorem 3.1, we obtain

$$\int_w \int_{B_x} \left(u_1 \frac{\partial \varphi}{\partial x_1} + u_2 \frac{\partial \varphi}{\partial x_2} \right) dx dy dx_3 = \int_{\partial w} \left(\int_0^{h_{min}} g dx_3 \right) . n \varphi d\sigma \quad \forall \varphi \in H^1(\omega). \quad (4.37)$$

Using (4.25), (4.31), and (4.32) in (4.37), we obtain the Reynolds equation (3.7).

We now prove that the matrix A , whose entries are given by (3.8), is symmetric and positive definite to prove the uniqueness of the solution in (3.7).

Set $\phi = \pi^j$ in (3.5):

$$\int_Y \frac{h^3}{12\mu} (e_i + \nabla_y \pi^i) . \nabla_y \pi^j dy = 0$$

so, a_{ij} can be written in the form

$$a_{ij}(x) = \int_Y \frac{h^3}{12\mu} (e_i + \nabla_y \pi^i) . (e_j + \nabla_y \pi^j) dy,$$

which proves that $A(x)$ is symmetric.

For any $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, we have

$$\xi^t A \xi = \int_Y \frac{h^3}{12\mu} \left(\sum_{i=1}^2 \xi_i (e_i + \nabla_y \pi^i) \right)^2 dy.$$

Using (2.1),

$$\begin{aligned} \xi^t A \xi &\geq \frac{a^3}{12\mu} \int_Y \left\| \sum_{i=1}^2 \xi_i (e_i + \nabla_y \pi^i) \right\|_2^2 dy \\ &= \frac{a^3}{12\mu} \int_Y [\xi_1^2 \|e_1 + \nabla_y \pi^1\|_2^2 + \xi_2^2 \|e_2 + \nabla_y \pi^2\|_2^2 + 2\xi_1 \xi_2 (e_1 + \nabla_y \pi^1) \cdot (e_2 + \nabla_y \pi^2)] dy. \end{aligned}$$

On the other hand, for $i = 1, 2$,

$$\int_Y \|(e_i + \nabla_y \pi^i)\|_2^2 dy = \int_Y \left[1 + \|\nabla_y \pi^i\|_2^2 + 2 \frac{\partial \pi^i}{\partial y_i} \right] dy = \int_Y [1 + \|\nabla_y \pi^i\|_2^2] dy$$

since π^i is y -periodic. For the same reason, we have

$$\int_Y (e_1 + \nabla_y \pi^1) \cdot (e_2 + \nabla_y \pi^2) dy = \int_Y \nabla_y \pi^1 \cdot \nabla_y \pi^2 dy.$$

We then obtain

$$\xi^t A \xi \geq \frac{a^3}{12\mu} \int_Y \sum_{i=1}^2 \xi_i^2 dy + \frac{a^3}{12\mu} \int_Y \|\xi_1 \nabla_y \pi^1 + \xi_2 \nabla_y \pi^2\|_2^2 dy \geq \frac{a^3}{12\mu} \|\xi\|_2^2.$$

The uniqueness of the solution of (3.7) is now assured. $\square$

5. Proofs of theorems in Section 3.2.

5.1. *Proof of Theorem 3.3* ($\alpha = 1$). Let us introduce the modified stress matrix

$$\mathcal{A}^\epsilon = \begin{pmatrix} \eta^2 h^\epsilon \sigma_{11}^\epsilon & \eta^2 h^\epsilon \sigma_{12}^\epsilon & t_3 (-\eta^2 \sigma_{11}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^2 \sigma_{12}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta \sigma_{13}^\epsilon) \\ \eta^2 h^\epsilon \sigma_{21}^\epsilon & \eta^2 h^\epsilon \sigma_{22}^\epsilon & t_3 (-\eta^2 \sigma_{21}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^2 \sigma_{22}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta \sigma_{23}^\epsilon) \\ \eta^2 h^\epsilon \sigma_{31}^\epsilon & \eta^2 h^\epsilon \sigma_{32}^\epsilon & t_3 (-\eta^2 \sigma_{31}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - \eta^2 \sigma_{32}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta^2 \sigma_{33}^\epsilon) \end{pmatrix} \cdot \frac{1}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}}. \quad (5.1)$$

Note that the vector V_+^ϵ defined in (3.9) corresponds to the last column of the matrix $\mathcal{A}^\epsilon$ where $t_3 = 1$. So we first study the limit of the matrix $\mathcal{A}^\epsilon$ in order to deduce the limit of the vector V_+^ϵ . For that purpose the following lemma is useful:

LEMMA 5.1. The row vectors $\epsilon \mathcal{A}_i^\epsilon$ ($i = 1, 2$) and $\mathcal{A}_3^\epsilon$ of the matrix $\mathcal{A}^\epsilon$ are bounded in $H(\text{div}; \Omega)$.

Proof. It is easy to prove that for $i = 1, 2, 3$,

$$\frac{\partial}{\partial t_1} (h^\epsilon \sigma_{i1}^\epsilon) + \frac{\partial}{\partial t_2} (h^\epsilon \sigma_{i2}^\epsilon) + \frac{\partial}{\partial t_3} \left(-t_3 \frac{\partial h^\epsilon}{\partial t_1} \sigma_{i1}^\epsilon - t_3 \frac{\partial h^\epsilon}{\partial t_2} \sigma_{i2}^\epsilon + \frac{1}{\eta} \sigma_{i3}^\epsilon \right) = 0 \quad (5.2)$$

so for $i = 1, 2$,

$$\begin{aligned} \operatorname{div}_{tt_3} \mathcal{A}_i^\epsilon &= \eta^2 h^\epsilon \sigma_{i1}^\epsilon \frac{\partial}{\partial t_1} \left(\frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2}} \right) \\ &\quad + \eta^2 h^\epsilon \sigma_{i2}^\epsilon \frac{\partial}{\partial t_2} \left(\frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2}} \right) \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \operatorname{div}_{tt_3} \mathcal{A}_3^\epsilon &= \eta^3 h^\epsilon \sigma_{31}^\epsilon \frac{\partial}{\partial t_1} \left(\frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2}} \right) \\ &\quad + \eta^3 h^\epsilon \sigma_{32}^\epsilon \frac{\partial}{\partial t_2} \left(\frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2}} \right). \end{aligned} \quad (5.4)$$

Using the estimates of Lemma 4.2 on velocity and pressure and

$$\left\| \frac{\partial h^\epsilon}{\partial t_i} \right\|_{L^\infty(w)} = \left\| \frac{\partial h}{\partial t_i} + \frac{1}{\epsilon} \frac{\partial h}{\partial y_i} \right\|_{L^\infty(w)} \leq \frac{K}{\epsilon}, \quad (5.5)$$

we deduce

$$\eta^2 \|\sigma_{ii}^\epsilon\|_{L^2(\Omega)} \leq K \quad \eta \|\sigma_{ij}^\epsilon\|_{L^2(\Omega)} \leq K \quad i \neq j = 1, 2, 3. \quad (5.6)$$

The estimates (5.5) and (5.6) prove that

$$\|\epsilon \mathcal{A}_i^\epsilon\|_{L^2(\Omega)^3} \leq K, \quad \|\epsilon \operatorname{div} \mathcal{A}_i^\epsilon\|_{L^2(\Omega)^3} \leq K \quad i = 1, 2$$

and

$$\|\mathcal{A}_3^\epsilon\|_{L^2(\Omega)^3} \leq K, \quad \|\operatorname{div} \mathcal{A}_3^\epsilon\|_{L^2(\Omega)^3} \leq K.$$

□

Now we prove the convergence of the stresses.

Proof of Theorem 3.3. According to the estimates of the previous lemma, there is a matrix $\mathcal{A}^\lambda$ in $L^2(\Omega)^9$ such that

$$\epsilon \mathcal{A}_i^\epsilon \rightharpoonup \mathcal{A}_i^\lambda \quad i = 1, 2; \quad \text{and} \quad \mathcal{A}_3^\epsilon \rightharpoonup \mathcal{A}_3^\lambda \quad \text{weakly in } H(\operatorname{div}; \Omega).$$

We now identify the limit $\mathcal{A}^\lambda$. It is clear that the first two columns of $\mathcal{A}^\lambda$ are zero. For the third column, the only terms with a nonzero limit are

$$\eta^2 \epsilon t_3 p^\epsilon \frac{\frac{\partial h^\epsilon}{\partial t_i}}{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2} \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2, \quad i = 1, 2$$

and

$$\eta^2 \epsilon p^\epsilon \frac{\frac{\partial h^\epsilon}{\partial t_i}}{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1} \right)^2} \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2} \right)^2, \quad i = 1, 2.$$

We first remark that

$$\frac{\epsilon \frac{\partial h^\epsilon}{\partial t_i}}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1}\right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2}\right)^2}} = \frac{\frac{\partial h}{\partial y_i}}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} + e_i^\epsilon$$

with

$$e_i^\epsilon = \frac{\epsilon \frac{\partial h^\epsilon}{\partial t_i} - \frac{\partial h}{\partial y_i}}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1}\right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2}\right)^2}} + \frac{\partial h}{\partial y_i} \left[\frac{1}{\sqrt{1 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_1}\right)^2 + \eta^2 \left(\frac{\partial h^\epsilon}{\partial t_2}\right)^2}} - \frac{1}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} \right].$$

As e_i^ϵ strongly converges to zero in $L^\infty(\Omega)$, it is sufficient to find the weak limit of

$$\eta^2 p^\epsilon t_3 \frac{\frac{\partial h}{\partial y_i}}{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}.$$

Now from the two-scale convergence of $\eta^2 p^\epsilon$, we obtain

$$\mathcal{A}^\lambda = \begin{pmatrix} 0 & 0 & \lambda^2 t_3 \left(\int_Y \frac{\frac{\partial h}{\partial y_1}}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} dy \right) p^\lambda \\ 0 & 0 & \lambda^2 t_3 \left(\int_Y \frac{\frac{\partial h}{\partial y_2}}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} dy \right) p^\lambda \\ 0 & 0 & \lambda^2 t_3 \left(\int_Y \frac{-1}{\sqrt{1 + \lambda^2 \left(\frac{\partial h}{\partial y_1}\right)^2 + \lambda^2 \left(\frac{\partial h}{\partial y_2}\right)^2}} dy \right) p^\lambda \end{pmatrix},$$

which gives the result. $\square$

5.2. *Proof of Theorem 3.4* ($\alpha > 1$). The divergence of $\mathcal{A}_i^\epsilon$ involves terms like $\frac{\eta^2}{\epsilon^3}$, so

$$\|\operatorname{div} \mathcal{A}_i^\epsilon\|_{L^2(\Omega)^3} \leq K \epsilon^{2\alpha-3} \quad i = 1, 2. \quad (5.7)$$

These estimates cause three different cases to occur: $\alpha > \frac{3}{2}$, $\alpha = \frac{3}{2}$, $\alpha < \frac{3}{2}$. We also remark that

$$\|\mathcal{A}_3^\epsilon\|_{L^2(\Omega)^3} \leq K, \quad \|\operatorname{div} \mathcal{A}_3^\epsilon\|_{L^2(\Omega)^3} \leq K.$$

Proof of Theorem 3.4. We begin by studying the convergence of V_{+3}^ϵ . It is clear that there exists $\mathcal{A}_3^\lambda$ such that

$$\mathcal{A}_3^\epsilon \rightarrow \mathcal{A}_3^\lambda \text{ weakly in } H(\operatorname{div}; \Omega).$$

It is easy to show that $\mathcal{A}_3^\lambda = (0, 0, -p(x))$, which gives the result on V_{+3}^ϵ for $\alpha > 1$.

We now study the convergences of V_{+i}^ϵ , $i = 1, 2$. Since we cannot have a uniform estimate for $\|\mathcal{A}_i^\epsilon\|_{L^2(\Omega)}$, we define the trace on the upper part of the boundary as element of $H^1(\omega)$ in the following manner:

Let $\varphi(t, t_3) = \theta(t)\psi(t_3)$, where $\theta \in H_0^1(\omega)$, $\psi \in H^1([0, 1])$ with $\psi(0) = 0, \psi(1) = 1$ and $\int_0^1 \psi dt_3 = 1$. We define V_{+i}^ϵ by the duality

$$\langle V_{+i}^\epsilon, \theta \rangle = \int_{\Omega} \operatorname{div} \mathcal{A}_i^\epsilon \varphi dt dx_3 + \int_{\Omega} \mathcal{A}_i^\epsilon \nabla \varphi dt dx_3 \quad \text{for } i = 1, 2. \quad (5.8)$$

Exactly as in [22], this definition does not depend on the choice of ψ . We have

$$\begin{aligned} \int_{\Omega} A_i^\epsilon \cdot \nabla \varphi dt dt_3 &= \sum_{j=1,2} \int_{\Omega} \eta^2 \frac{h^\epsilon}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} \frac{\partial \theta}{\partial t_j} \sigma_{ij}^\epsilon \psi dt dt_3 \\ &+ \int_{\Omega} \frac{1}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} \left(-t_3 \eta^2 \sigma_{i1}^\epsilon \frac{\partial h^\epsilon}{\partial t_1} - t_3 \eta^2 \sigma_{i2}^\epsilon \frac{\partial h^\epsilon}{\partial t_2} + \eta \sigma_{i3}^\epsilon \right) \theta \frac{d\psi}{dt_3} dt dt_3. \end{aligned} \quad (5.9)$$

All integrals in the first term in (5.9) converge to zero because of the multiplication by η^2 , except

$$-\eta^2 \int_{\Omega} \frac{h^\epsilon}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} p^\epsilon \frac{\partial \theta}{\partial t_i} \psi dt dt_3.$$

Using the two-scale convergence of $\eta^2 p^\epsilon$ to p , we easily deduce that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(-\eta^2 \int_{\Omega} \frac{h^\epsilon}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} p^\epsilon \frac{\partial \theta}{\partial t_i} \psi dt dt_3 \right) \\ = - \int_{\Omega} \int_Y h p \frac{\partial \theta}{\partial t_i} \psi dt dy dt_3 = - \int_{\omega} \tilde{h} p \frac{\partial \theta}{\partial t_i} dt, \end{aligned} \quad (5.10)$$

the last equality being a consequence of the hypothesis $\int_0^1 \psi dt_3 = 1$.

In the second term in (5.9), all integrals converge to zero, except

$$\int_{\Omega} \eta^2 p^\epsilon t_3 \frac{\frac{\partial h^\epsilon}{\partial t_i}}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} \theta \frac{d\psi}{dt_3} dt dt_3 \quad i = 1, 2 \quad (5.11)$$

and

$$\mu \int_{\Omega} \frac{1}{h^\epsilon} \frac{1}{\sqrt{1 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2 (\frac{\partial h^\epsilon}{\partial t_2})^2}} \frac{\partial u_i^\epsilon}{\partial t_3} \theta \frac{d\psi}{dt_3} dt dt_3 \quad i = 1, 2. \quad (5.12)$$

We now prove that the term in (5.11) tends to zero. We set $\xi(t_3) = \int_0^{t_3} s \frac{d\psi}{ds} ds$
 $= t_3 \psi(t_3) - \int_0^{t_3} \psi(s) ds$, $\xi(0) = \xi(1) = 0$ and $\frac{d\xi}{dt_3} = t_3 \frac{d\psi}{dt_3}$. Choosing $\varphi = (0, 0, \varphi_3(t, t_3))$

with $\varphi_3(t, t_3) = \xi(t_3)\theta(t) \frac{\frac{\partial h^\epsilon}{\partial t_i}}{\sqrt{1 + \eta^2(\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2(\frac{\partial h^\epsilon}{\partial t_2})^2}}$ as a test function in (2.10), we obtain

$$\begin{aligned} \mu \int_{\Omega} \left[\sum_{i=1}^2 \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon u_3^\epsilon) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} u_3^\epsilon) \right) \cdot \left(\frac{\partial}{\partial t_i} (\eta h^\epsilon \varphi_3) + \frac{\partial}{\partial t_3} (-\eta t_3 \frac{\partial h^\epsilon}{\partial t_i} \varphi_3) \right) \right. \\ \left. + \frac{\partial u_3^\epsilon}{\partial t_3} \frac{\partial \varphi_3}{\partial t_3} \right] \frac{dt dt_3}{\eta h^\epsilon} = \int_{\Omega} p^\epsilon \frac{\frac{\partial h^\epsilon}{\partial t_i}}{\sqrt{1 + \eta^2(\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2(\frac{\partial h^\epsilon}{\partial t_2})^2}} \theta(t) t_3 \frac{d\psi}{dt_3} dt dt_3. \end{aligned}$$

Using estimates on the velocity and (5.5), we deduce that the term in (5.11) tends to zero.

In order to pass to the limit in (5.12), we recall that

$$\frac{\partial u_i^\epsilon}{\partial x_3} \rightharpoonup \frac{\partial u_i}{\partial x_3} + \frac{\partial v_i}{\partial y_3} \quad \text{in } \Omega,$$

so passing in Ω_{max} , we obtain

$$\begin{aligned} \mu \int_{\Omega} \frac{1}{h^\epsilon} \frac{1}{\sqrt{1 + \eta^2(\frac{\partial h^\epsilon}{\partial t_1})^2 + \eta^2(\frac{\partial h^\epsilon}{\partial t_2})^2}} \frac{\partial u_i^\epsilon}{\partial t_3} \theta \frac{d\psi}{dt_3} dt dt_3 \\ = \mu \int_{\Omega_{max}} \frac{1}{\sqrt{1 + \eta^2(\frac{\partial h^\epsilon}{\partial x_1})^2 + \eta^2(\frac{\partial h^\epsilon}{\partial x_2})^2}} \frac{\partial u_i^\epsilon}{\partial x_3} \theta \psi' \left(\frac{x_3}{h^\epsilon} \right) \frac{1}{h^\epsilon} dx dx_3. \end{aligned}$$

The limit of this term is

$$\mu \int_{\Omega_{max}} \int_Y \frac{\partial u_i}{\partial x_3} \theta \psi' \left(\frac{x_3}{h} \right) \frac{1}{h} dx dx_3 dy = \mu \int_{\Omega} \int_Y \frac{\partial u_i}{\partial t_3} \theta \psi'(t_3) \frac{1}{h} dt dt_3 dy.$$

Using the Green formula and the fact that $\psi(1) = 1$, we can write the last term as

$$-\mu \int_{\Omega} \int_Y \frac{\partial^2 u_i}{\partial t_3^2} \theta \psi \frac{1}{h} dt dt_3 dy + \mu \int_{\omega} \int_Y \frac{\partial u_i}{\partial t_3}(t, y, 1) \theta \frac{1}{h} dt dy.$$

On the other hand, from (4.25) and Lemma 4.11, we deduce

$$\frac{\partial u_i}{\partial t_3} = \sum_{j=1}^2 \frac{1}{2\mu} \frac{\partial p}{\partial t_j} \left(\frac{\partial \pi^j}{\partial y_i} + \delta_{ij} \right) h^2 (2t_3 - 1) + \frac{1}{2\mu} \frac{\partial \pi^3}{\partial y_i} h^2 (2t_3 - 1) - s_i$$

and

$$\mu \frac{\partial^2 u_i}{\partial t_3^2} = \sum_{k=1}^2 \frac{\partial p}{\partial t_i} h^2 \left(\frac{\partial \pi^k}{\partial y_i} + \delta_{ki} \right) + h^2 \frac{\partial \pi^3}{\partial y_i}.$$

We then deduce that the limit of (5.12) is

$$-\int_{\omega} \sum_{j=1}^2 \frac{\partial p}{\partial t_j} c_{ij}(t) \theta dt - \int_{\omega} d_i \theta dt$$

and with the help of (5.9) (5.10), we deduce

$$\lim_{\epsilon \rightarrow 0} \left(\int_{\Omega} \mathcal{A}_i^\epsilon \cdot \nabla \varphi dt dt_3 \right) = \left\langle \frac{\partial}{\partial t_i} (p\tilde{h}) - \sum_{j=1}^2 \frac{\partial p}{\partial t_j} c_{ij} - d_i, \theta \right\rangle_{H^{-1}(\omega), H_0^1(\omega)}.$$

We now seek the limit of the term with $\operatorname{div} \mathcal{A}_i^\epsilon$ in (5.8). We must consider three different cases on α :

i) For $\alpha > \frac{3}{2}$, (5.7) induces that $\int_{\Omega} \operatorname{div} \mathcal{A}_i^\epsilon \varphi \, dt \, dx_3 \rightarrow 0$, which gives the desired convergence of Theorem 3.4 for $i = 1, 2$.

ii) For $\alpha = \frac{3}{2}$, using (5.3) and the two-scale convergence of $\eta^2 p^\epsilon$, we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \operatorname{div} \mathcal{A}_i^\epsilon \theta \psi \, dt \, dt_3 = \int_{\Omega} \int_Y \lambda^2 p h \frac{\partial h}{\partial y_i} \frac{\partial^2 h}{\partial y_i^2} \theta \psi \, dt \, dy \, dt_3 = -\frac{\lambda^2}{2} \int_{\omega} p \left(\widetilde{\frac{\partial h}{\partial y_i}} \right)^3 \theta \, dt,$$

which gives the result for $i = 1, 2$.

iii) For $1 < \alpha < \frac{3}{2}$, we obtain the result multiplying (5.8) by $\epsilon^{3-2\alpha}$, since

$$\lim_{\epsilon \rightarrow 0} \left(\epsilon^{3-2\alpha} \int_{\Omega} \mathcal{A}_i^\epsilon \cdot \nabla \varphi \, dt \, dt_3 \right) = 0$$

and

$$\lim_{\epsilon \rightarrow 0} \left(\epsilon^{3-2\alpha} \int_{\Omega} \operatorname{div} \mathcal{A}_i^\epsilon \theta \psi \, dt \, dt_3 \right) = - \int_{\omega} \frac{\lambda^2}{2} p \left(\widetilde{\frac{\partial h}{\partial y_i}} \right)^3 \theta \, dt.$$

□

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