THE EXISTENCE OF EIGHT POSITIVE PERIODIC SOLUTIONS FOR A GENERALIZED PREY-PREDATOR SYSTEM WITH DELAY AND STOCKING

By

ZHENGQIU ZHANG (*School of Mathematics and Computing Technology, Central South University,
Changsha, 410083, People's Republic of China*)

AND

XINSHENG XIONG (*Department of Applied Mathematics, Hunan University, Changsha, 410082,
People's Republic of China*)

Abstract. By means of using Mawhin's continuation theorem of coincidence degree theory, we derive criteria for the existence of eight positive periodic solutions of a generalized prey-predator system with time delay and stocking.

1. Introduction. Freedman and Waltman [4] considered a general model of two predators competing for a single prey. They derived criteria for strong persistence in terms of conditions on the system parameters. Farkas and Freedman [2] obtained criteria for global stability of a general model of a two-predator-one-prey system in terms of restrictions on the growth functions. Motivated by the work of Freedman and Ruan [3] for retarded functional differential equations, Mukherjee and Roy [7] investigated a model representing a resource (prey) and two predators with a delay due to gestation and derived conditions for uniform persistence. The existence conditions for a positive equilibrium were discussed. Since global stability is of great interest, they provided sufficient conditions in terms of the parameters of the system to guarantee it. In [8], motivated by the work of Mukherjee and Roy [7] for constructing a generalized model of a two-predator-one-prey system, we constructed a mathematical model, representing a

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E-mail address: z-q-zhang@sina.com.cn

E-mail address: xxs82@163.com

food chain system with delay due to gestation:

$$\begin{cases} \frac{dx_1}{dt} = x_1 g_1(t, x_1) - c_1(t) x_2 p_2(x_1), \\ \frac{dx_2}{dt} = x_2 g_2(t, x_2) + \beta_1(t) x_2 p_2(x_1(t - \tau)) - \beta_2(t) x_3 p_3(x_2), \\ \frac{dx_3}{dt} = x_3 [-d_1(t) + \beta_3(t) p_3(x_2(t - \tau)) - a_{31}(t) x_3]. \end{cases} \quad (1.1)$$

In [8], by using the continuation theorem of coincidence degree theory, we studied the existence of positive periodic solutions of system (1.1). In this paper, we consider the following food chain system with stocking and delay:

$$\begin{cases} \frac{dx_1}{dt} = x_1 g_1(t, x_1) - c_1(t) x_2 p_2(x_1) - h_1(t), \\ \frac{dx_2}{dt} = -x_2 g_2(t, x_2) + \beta_1(t) x_2 p_2(x_1(t - \tau)) - \beta_2(t) x_3 p_3(x_2) - h_2(t), \\ \frac{dx_3}{dt} = x_3 [-d_1(t) + \beta_3(t) p_3(x_2(t - \tau)) - a_{31}(t) x_3(t)] - h_3(t), \end{cases} \quad (1.2)$$

where $\tau > 0$ is a constant time delay. In system (1.2), all parameters are positive continuous ω -periodic functions with period $\omega > 0$; $x_1(t)$ denotes the biomass at time t of resource species x_1 , $x_2(t)$ and $x_3(t)$ represent the densities of the species x_2 and x_3 , respectively, β_1 and β_3 are the conversion of biomass constants, respectively, and d_1 is the death rate of species x_3 , h_1, h_2, h_3 are the stocking of species x_1 , species x_2 and species x_3 , respectively. For biological relevance of system (1.2), see [1].

The following assumptions on the functions $g_i(t, x)$, $i = 1, 2$, $p_i(x)$, $i = 2, 3$, are given below. (H1): $g_i(t, x)$ are continuous and ω -periodic with respect to t , $i = 1, 2$, $g_1(t, 0) > 0$, $g_1(t, x)$ is monotonely decreasing in x for fixed t , $t, x \in R$, $g_2(t, x)$ is monotonely increasing in x for fixed t , $g_2(t, 0) > 0$, $g_2(t, x) > 0$, for $x > 0$, $t, x \in R$. (H2) $p_i(x)$ are monotonely increasing and $p_i(x) > 0$, for $x > 0$, $i = 2, 3$.

On the existence of periodic solutions for system (1.2), so far, few results have been found in the literature. This motivates us to consider the study of the existence for periodic solutions of system (1.2). Since the existence results on multiple periodic solutions of delayed population models obtained by means of coincidence degree theory are very scarce, then in this paper, instead of investigating the existence of a periodic solution, we investigate the existence of multiple positive periodic solutions for system (1.2) by using the continuation theorem of coincidence degree theory. The paper is organized as follows. In Section 2, by applying the continuation theorem of coincidence degree theory, we establish a sufficient condition for the existence of eight positive periodic solutions of system (1.2). In Section 3, we illustrate our result with two examples.

2. Main result. For the reader's convenience, we first summarize a few concepts from the book by Gaines and Mawhin [5, 6].

Let X and Z be normed vector spaces. Let $L: \text{Dom } L \subset X \rightarrow Z$ be a linear mapping and $N: X \rightarrow Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < \infty$ and $\text{Im } L$ is closed in Z . If L is a Fredholm mapping of index zero, then there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$. It follows that $L|_{\text{Dom } L \cap \text{Ker } P}: (I - P)X \rightarrow \text{Im } L$ is invertible and its inverse is denoted by K_p . If Ω is a bounded open subset of X , the mapping N is called L -compact on $\bar{\Omega}$ if $(QN)(\bar{\Omega})$ is

bounded and $K_p(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Because $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

In the proof of our existence result, we need the following continuation theorem.

THEOREM 2.1 (Continuation Theorem, Gaines and Mawhin [5, 6]). Let L be a Fredholm mapping of index zero and let N be L -compact on $\bar{\Omega}$. Suppose

- (a) for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;
- (b) $QNx \neq 0$ for each $x \in \partial\Omega \cap \text{Ker } L$;
- (c) $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution lying in $\text{Dom } L \cap \bar{\Omega}$.

For the sake of convenience and simplicity, we introduce some notation as follows:

$$\max_{t \in [0, \omega]} g_1(t, 0) = a_1^M, \quad f^M = \max_{t \in [0, \omega]} f(t), \quad f^l = \min_{t \in [0, \omega]} f(t);$$

here f is a positive continuous ω -periodic function.

THEOREM 2.2. Assume that the following conditions hold:

- (i) There exists a constant A such that when $x \geq A$,

$$g_1(t, e^x) \leq 0, \quad \forall t, x \in R.$$

- (ii) There exist a constant B such that when $x \geq B$,

$$g_2(t, e^x) \geq \beta_1^M p_2(e^A), \quad \forall t, x \in R.$$

- (iii) There exist two positive constants $l_{\pm}$ with $\frac{h_1^l}{a_1^M} < l_- < l_+ < A$ such that $\forall t, x \in R$,

$$e^x g_1(t, e^x) \geq c_1^M e^B p_2(e^x) + h_1^M, \quad \ln l_- \leq x \leq \ln l_+.$$

- (iv) There exist two positive constants $u_{\pm}$ with $\frac{h_2^l}{\beta_1^M p_2(e^A)} < u_- < u_+ < B$ such that $\forall t, x \in R$;

$$h_2^M + \frac{\beta_2^M \beta_3^M}{a_{31}^l} p_3(e^B) p_3(e^x) + e^x g_2(t, e^x) \leq \beta_1^l e^x p_2\left(\frac{h_1^l}{a_1^M}\right), \ln u_- \leq x \leq \ln u_+.$$

- (v) $\beta_3^l p_3\left[\frac{h_2^l}{\beta_1^M p_2(e^A)}\right] > d_1^M + 2\sqrt{a_{31}^M h_3^M}$.

Then system (1.2) has at least eight positive ω -periodic solutions.

Proof. Since we are concerned with positive periodic solutions of system (1.2), we make the change of variables,

$$x_i(t) = e^{u_i(t)}, \quad i = 1, 2, 3. \quad (2.1)$$

Then system (1.2) is rewritten as

$$\begin{cases} \frac{du_1(t)}{dt} = g_1(t, e^{u_1(t)}) - c_1(t) e^{u_2(t)-u_1(t)} p_2(e^{u_1(t)}) - h_1(t) e^{-u_1(t)}, \\ \frac{du_2(t)}{dt} = -g_2(t, e^{u_2(t)}) + \beta_1(t) p_2(e^{u_1(t-\tau)}) - \beta_2(t) e^{u_3(t)-u_2(t)} p_3(e^{u_2(t)}) - h_2(t) e^{-u_2(t)}, \\ \frac{du_3(t)}{dt} = -d_1(t) + \beta_3(t) p_3(e^{u_2(t-\tau)}) - a_{31}(t) e^{u_3(t)} - h_3(t) e^{-u_3(t)}. \end{cases} \quad (2.2)$$

To prove the existence of positive periodic solutions for system (1.2), because of (2.1), we only need to show the existence of periodic solutions for system (2.2). To apply Theorem 2.1 to system (2.2), we take

$$X = Z = \{u = (u_1, u_2, u_3)^T \in C(R, R^3) : u(t + \omega) = u(t)\}$$

and define

$$\|u\| = \sum_{i=1}^3 \max_{t \in [0, \omega]} |u_i(t)|, \quad u \in X \quad \text{or} \quad Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces.

Let

$$Nu = \begin{bmatrix} g_1(t, e^{u_1(t)}) - c_1(t)e^{u_2(t)-u_1(t)}p_2(e^{u_1(t)}) - h_1(t)e^{-u_1(t)} \\ -g_2(t, e^{u_2(t)}) + \beta_1(t)p_2(e^{u_1(t-\tau)}) - \beta_2(t)e^{u_3(t)-u_2(t)}p_3(e^{u_2(t)}) - h_2(t)e^{-u_2(t)} \\ -d_1(t) + \beta_3(t)p_3(e^{u_2(t-\tau)}) - a_{31}(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)} \end{bmatrix}, u \in X,$$

$$Lu = u' = \frac{du(t)}{dt}, \quad Pu = \frac{1}{\omega} \int_0^\omega u(t)dt, \quad u \in X; \quad Qz = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad z \in Z.$$

Then it follows that

$$Ker L = R^3, Im L = \left\{ z \in Z : \int_0^\omega z(t)dt = 0 \right\} \text{ is closed in } Z, \dim Ker L = 3 = \text{codim } Im L,$$

and P, Q are continuous projectors such that

$$Im P = Ker L, \quad Ker Q = Im L = Im(I - Q).$$

Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : Im L \rightarrow Ker P \cap Dom L$ is given by

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt.$$

Then

$$QNu = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega F_1(s)ds \\ \frac{1}{\omega} \int_0^\omega F_2(s)ds \\ \frac{1}{\omega} \int_0^\omega F_3(s)ds \end{bmatrix}$$

and

$$K_p(I - Q)Nu = \begin{bmatrix} \int_0^t F_1(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_1(s)dsdt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_1(s)ds \\ \int_0^t F_2(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_2(s)dsdt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_2(s)ds \\ \int_0^t F_3(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_3(s)dsdt + \left(\frac{1}{2} - \frac{t}{\omega}\right) \int_0^\omega F_3(s)ds \end{bmatrix},$$

where

$$F_1(s) = g_1\left(s, e^{u_1(s)}\right) - c_1(s)e^{u_2(s)-u_1(s)}p_2(e^{u_1(s)}) - h_1(s)e^{-u_1(s)},$$

$$F_2(s) = -g_2\left(s, e^{u_2(s)}\right) + \beta_1(s)p_2(e^{u_1(s-\tau)}) - \beta_2(s)e^{u_3(s)-u_2(s)}p_3(e^{u_2(s)}) - h_2(s)e^{-u_2(s)},$$

$$F_3(s) = -d_1(s) + \beta_3(s)p_3(e^{u_2(s-\tau)}) - a_{31}(s)e^{u_3(s)} - h_3(s)e^{-u_3(s)}.$$

Obviously, QN and $K_p(I - Q)N$ are continuous. It is not difficult to show that $K_p(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$. we have

$$\begin{cases} \frac{du_1(t)}{dt} = [g_1(t, e^{u_1(t)}) - c_1(t)e^{u_2(t)-u_1(t)}p_2(e^{u_1(t)}) - h_1(t)e^{-u_1(t)}], \\ \frac{du_2(t)}{dt} = [-g_2(t, e^{u_2(t)}) + \beta_1(t)p_2(e^{u_1(t-\tau)}) \\ - \beta_2(t)e^{u_3(t)-u_2(t)}p_3(e^{u_2(t)}) - h_2(t)e^{-u_2(t)}], \\ \frac{du_3(t)}{dt} = [-d_1(t) + \beta_3(t)p_3(e^{u_2(t-\tau)}) - a_{31}(t)e^{u_3(t)} - h_3(t)e^{-u_3(t)}]. \end{cases} \quad (2.3)$$

Assume that $u = u(t) \in X$ is a solution of system (2.3) for a certain $\lambda \in (0, 1)$. Since $(u_1, u_2, u_3)^T \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \max_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \min_{t \in [0, \omega]} u_i(t), \quad i = 1, 2, 3.$$

Then

$$u'_i(\eta_i) = 0, \quad u'_i(\xi_i) = 0, \quad i = 1, 2, 3.$$

From this and system (2.3), we obtain

$$\begin{cases} g_1(\xi_1, e^{u_1(\xi_1)}) - c_1(\xi_1)e^{u_2(\xi_1)-u_1(\xi_1)}p_2(e^{u_1(\xi_1)}) - h_1(\xi_1)e^{-u_1(\xi_1)} = 0, \end{cases} \quad (2.4)$$

$$\begin{cases} -g_2(\xi_2, e^{u_2(\xi_2)}) + \beta_1(\xi_2)p_2(e^{u_1(\xi_2-\tau)}) - \beta_2(\xi_2)e^{u_3(\xi_2)-u_2(\xi_2)}p_3(e^{u_2(\xi_2)}) \\ -h_2(\xi_2)e^{-u_2(\xi_2)} = 0, \end{cases} \quad (2.5)$$

$$\begin{cases} -d_1(\xi_3) + \beta_3(\xi_3)p_3(e^{u_2(\xi_3-\tau)}) - a_{31}(\xi_3)e^{u_3(\xi_3)} - h_3(\xi_3)e^{-u_3(\xi_3)} = 0 \end{cases} \quad (2.6)$$

and

$$\begin{cases} g_1(\eta_1, e^{u_1(\eta_1)}) - c_1(\eta_1)e^{u_2(\eta_1)-u_1(\eta_1)}p_2(e^{u_1(\eta_1)}) - h_1(\eta_1)e^{-u_1(\eta_1)} = 0, \end{cases} \quad (2.7)$$

$$\begin{cases} -g_2(\eta_2, e^{u_2(\eta_2)}) + \beta_1(\eta_2)p_2(e^{u_1(\eta_2-\tau)}) - \beta_2(\eta_2)e^{u_3(\eta_2)-u_2(\eta_2)}p_3(e^{u_2(\eta_2)}) \\ -h_2(\eta_2)e^{-u_2(\eta_2)} = 0, \end{cases} \quad (2.8)$$

$$\begin{cases} -d_1(\eta_3) + \beta_3(\eta_3)p_3(e^{u_2(\eta_3-\tau)}) - a_{31}(\eta_3)e^{u_3(\eta_3)} \\ -h_3(\eta_3)e^{-u_3(\eta_3)} = 0. \end{cases} \quad (2.9)$$

(2.4) implies that

$$g_1(\xi_1, e^{u_1(\xi_1)}) > 0,$$

whereby along with condition (i) in Theorem 2.2, we have

$$u_1(\xi_1) < A. \quad (2.10)$$

(2.7) implies that

$$g_1(\eta_1, 0) > g_1(\eta_1, e^{u_1(\eta_1)}) > h_1^l e^{-u_1(\eta_1)},$$

that is,

$$u_1(\eta_1) > \ln \left[\frac{h_1^l}{g_1(\eta_1, 0)} \right] \geq \ln \frac{h_1^l}{a_1^M}. \quad (2.11)$$

(2.5) implies that

$$g_2(\xi_2, e^{u_2(\xi_2)}) < \beta_1^M p_2(e^A),$$

whereby along with condition (ii) in Theorem 2.2, we have

$$u_2(\xi_2) < B. \quad (2.12)$$

(2.8) implies that

$$h_2^l e^{-u_2(\eta_2)} \leq h_2(\eta_2) e^{-u_2(\eta_2)} < \beta_1^M p_2(e^A),$$

that is,

$$u_2(\eta_2) > \ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right). \quad (2.13)$$

(2.6) implies that

$$a_{31}^l e^{u_3(\xi_3)} \leq a_{31}(\xi_3) e^{u_3(\xi_3)} < \beta_3^M p_3(e^B),$$

that is,

$$u_3(\xi_3) < \ln \frac{\beta_3^M p_3(e^B)}{a_{31}^l}. \quad (2.14)$$

(2.9) implies that

$$\beta_3^M p_3(e^B) \geq \beta_3(\eta_3) p_3(e^{u_2(\eta_3-\tau)}) > h_3^l e^{-u_3(\eta_3)},$$

that is,

$$u_3(\eta_3) > \ln \left[\frac{h_3^l}{\beta_3^M p_3(e^B)} \right]. \quad (2.15)$$

(2.4) implies that

$$e^{u_1(\xi_1)} g_1(\xi_1, e^{u_1(\xi_1)}) < c_1^M e^B p_2(e^{u_1(\xi_1)}) + h_1^M,$$

whereby along with condition (iii) in Theorem 2.2, we have

$$u_1(\xi_1) > \ln l_+ \quad \text{or} \quad u_1(\xi_1) < \ln l_-. \quad (2.16)$$

Similarly, from (2.7), we have

$$u_1(\eta_1) > \ln l_+ \quad \text{or} \quad u_1(\eta_1) < \ln l_-. \quad (2.17)$$

(2.5) implies that

$$-e^{u_2(\xi_2)} g_2(\xi_2, e^{u_2(\xi_2)}) + \beta_1(\xi_2) e^{u_2(\xi_2)} p_2(e^{u_1(\xi_2-\tau)}) - \beta_2(\xi_2) e^{u_3(\xi_2)} p_3(e^{u_2(\xi_2)}) - h_2(\xi_2) = 0,$$

that is,

$$e^{u_2(\xi_2)} g_2(\xi_2, e^{u_2(\xi_2)}) + h_2^M + \frac{\beta_2^M \beta_3^M}{a_{31}^l} p_3(e^B) p_3(e^{u_2(\xi_2)}) > \beta_1^l e^{u_2(\xi_2)} p_2\left(\frac{h_1^l}{a_1^M}\right),$$

whereby along with condition (iv) in Theorem 2.2, we obtain

$$u_2(\xi_2) > \ln u_+ \quad \text{or} \quad u_2(\xi_2) < \ln u_-. \quad (2.18)$$

Similarly, from (2.8), we have

$$u_2(\eta_2) > \ln u_+ \quad \text{or} \quad u_2(\eta_2) < \ln u_-. \quad (2.19)$$

(2.6) implies that

$$a_{31}^M e^{2u_3(\xi_3)} + d_1^M e^{u_3(\xi_3)} + h_3^M > \beta_3^l p_3 \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) e^{u_3(\xi_3)},$$

that is,

$$u_3(\xi_3) > \ln v_+ \quad \text{or} \quad u_3(\xi_3) < \ln v_-, \quad (2.20)$$

where

$$v_{\pm} = \frac{\beta_3^l p_3 \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) - d_1^M \pm \sqrt{\left[\beta_3^l p_3 \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) - d_1^M \right]^2 - 4a_{31}^M h_3^M}}{2a_{31}^M}.$$

Similarly, from (2.9), we obtain

$$u_3(\eta_3) > \ln v_+ \quad \text{or} \quad u_3(\eta_3) < \ln v_-. \quad (2.21)$$

From (2.10), (2.11), (2.16) and (2.17), we have for $t \in [0, \omega]$,

$$\ln \frac{h_1^l}{a_1^M} < u_1(t) < \ln l_- \quad \text{or} \quad \ln l_+ < u_1(t) < A. \quad (2.22)$$

From (2.12), (2.13), (2.18) and (2.19), we obtain for $t \in [0, \omega]$,

$$\ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) < u_2(t) < \ln u_- \quad \text{or} \quad \ln u_+ < u_2(t) < B. \quad (2.23)$$

From (2.14), (2.15), (2.20) and (2.21), we have for $t \in [0, \omega]$,

$$\ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < u_3(t) < \ln v_- \quad \text{or} \quad \ln v_+ < u_3(t) < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right). \quad (2.24)$$

Obviously, $\ln l_{\pm}, \ln u_{\pm}, \ln v_{\pm}, \ln \frac{h_1^l}{a_1^M}, \ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right), \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right), A, B, \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right)$ are independent of λ . Now we let

$$\begin{aligned} \Omega_1 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right) \\ u_2(t) \in \left(\ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right), \ln u_- \right) \\ u_3(t) \in \left(\ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right), \ln v_- \right) \end{array} \right\}, \\ \Omega_2 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right) \\ u_2(t) \in (\ln u_+, B) \\ u_3(t) \in \left(\ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right), \ln v_- \right) \end{array} \right\}, \\ \Omega_3 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in (\ln l_+, A) \\ u_2(t) \in \left(\ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right), \ln u_- \right) \\ u_3(t) \in \left(\ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right), \ln v_- \right) \end{array} \right\}, \\ \Omega_4 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in (\ln l_+, A) \\ u_2(t) \in (\ln u_+, B) \\ u_3(t) \in \left(\ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right), \ln v_- \right) \end{array} \right\}, \\ \Omega_5 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right) \\ u_2(t) \in \left(\ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right), \ln u_- \right) \\ u_3(t) \in \left(\ln v_+, \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right) \right) \end{array} \right\}, \\ \Omega_6 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in \left(\ln \frac{h_1^l}{a_1^M}, \ln l_- \right) \\ u_2(t) \in (\ln u_+, B) \\ u_3(t) \in \left(\ln v_+, \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right) \right) \end{array} \right\}, \\ \Omega_7 &= \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in (\ln l_+, A) \\ u_2(t) \in \left(\ln \left(\frac{h_2^l}{\beta_3^M p_2(e^A)} \right), \ln u_- \right) \\ u_3(t) \in \left(\ln v_+, \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right) \right) \end{array} \right\}, \end{aligned}$$

and

$$\Omega_8 = \left\{ u = (u_1, u_2, u_3)^T \in X \middle/ \begin{array}{l} u_1(t) \in (\ln l_+, A) \\ u_2(t) \in (\ln u_+, B) \\ u_3(t) \in \left(\ln v_+, \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right) \right) \end{array} \right\}.$$

Then $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ are bounded open subsets of X , $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3, 4, 5, 6, 7, 8$, and $\Omega_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ satisfies requirement (a) in Theorem 2.2.

Next, we prove that (b) of Theorem 2.1 holds, i.e., we prove that when $u \in \partial\Omega_i \cap \text{Ker} L = \partial\Omega_i \cap R^3$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, $QNu \neq (0, 0, 0)^T$. There exist three points

$t_i \in [0, \omega], i = 1, 2, 3$, such that when $u \in \partial\Omega_i \cap R^3, i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$QNu = \begin{bmatrix} g_1(t_1, e^{u_1}) - c_1(t)e^{u_2-u_1}p_2(e^{u_1}) - h_1(t_1)e^{-u_1} \\ -g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - \beta_2(t_2)e^{u_3-u_2}p_3(e^{u_2}) - h_2(t_2)e^{-u_2} \\ -d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3} \end{bmatrix}.$$

If when $u \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap R^3, QNu = (0, 0, 0)^T$, then vector u in R^3 with $u \in \partial\Omega_i, i = 1, 2, 3, 4, 5, 6, 7, 8$, satisfies

$$\begin{cases} g_1(t_1, e^{u_1}) = c_1(t)e^{u_2-u_1}p_2(e^{u_1}) + h_1(t_1)e^{-u_1}, & (2.25) \end{cases}$$

$$\begin{cases} g_2(t_2, e^{u_2}) + h_2(t_2)e^{-u_2} + \beta_2(t_2)e^{u_3-u_2}p_3(e^{u_2}) = \beta_1(t_2)p_2(e^{u_1}), & (2.26) \end{cases}$$

$$\begin{cases} d_1(t_3) + h_3(t_3)e^{-u_3} + a_{31}(t_3)e^{u_3} = \beta_3(t_3)p_3(e^{u_2}). & (2.27) \end{cases}$$

From (2.25)-(2.27), following the arguments of (2.22)-(2.24), we obtain

$$\begin{aligned} \ln \frac{h_1^l}{a_1^M} < u_1 < \ln l_- \quad \text{or} \quad \ln l_+ < u_1 < A, \\ \ln \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) < u_2 < \ln u_- \quad \text{or} \quad \ln u_+ < u_2 < B, \\ \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < u_3 < \ln v_- \quad \text{or} \quad \ln v_+ < u_3 < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right). \end{aligned}$$

Thus $u \in \Omega_1 \cap R^3$ or $u \in \Omega_2 \cap R^3$ or $u \in \Omega_3 \cap R^3$ or $u \in \Omega_4 \cap R^3$ or $u \in \Omega_5 \cap R^3$ or $u \in \Omega_6 \cap R^3$, or $u \in \Omega_7 \cap R^3$ or $u \in \Omega_8 \cap R^3$. This leads to a contradiction and proves that (b) in Theorem 2.1 holds.

Finally, we show that (c) in Theorem 2.1 holds; we only prove that when $u \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap R^3, i = 1, 2, 3, 4, 5, 6, 7, 8, \deg \{JQNu, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T\} \neq (0, 0, 0)^T$. Our proof is broken into two steps.

Step 1. We show that for $i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$\begin{aligned} & \deg \{JQNu, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T\} \\ &= \deg \left\{ [g_1(t_1, e^{u_1}) - c_1(t_1)e^{u_2-u_1}p_2(e^{u_1}) - h_1(t_1)e^{-u_1}, -g_2(t_2, e^{u_2}) \right. \\ & \quad \left. + \beta_1(t_2)p_2(e^{u_1}) - \beta_2(t_2)e^{u_3-u_2}p_3(e^{u_2}) - h_2(t_2)e^{-u_2}, -d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) \right. \\ & \quad \left. - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}]^T, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \right\} \\ &= \deg \left\{ [g_1(t_1, e^{u_1}) - h_1(t_1)e^{-u_1}, -g_2(t_2, e^{u_2}) \right. \\ & \quad \left. + \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)e^{-u_2}, -d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) \right. \\ & \quad \left. - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}]^T, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \right\}. \end{aligned}$$

To this end, we define mapping $\phi_1 : \text{Dom}L \times [0, 1] \rightarrow X$ by

$$\phi_1(u_1, u_2, u_3, \mu_1) = \begin{bmatrix} g_1(t_1, e^{u_1}) - \mu_1 c_1(t_1)e^{u_2-u_1}p_2(e^{u_1}) - h_1(t_1)e^{-u_1} \\ -g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - \mu_1 \beta_2(t_2)e^{u_3-u_2}p_3(e^{u_2}) - h_2(t_2)e^{-u_2} \\ -d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3} \end{bmatrix},$$

where $\mu_1 \in [0, 1]$ is a parameter. When $u \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap R^3$, u is a constant vector in R^3 with $u \in \partial\Omega_i$, $i = 1, 2, 3, 4, 5, 6, 7, 8$. We show that when $u \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap \text{Ker}LR^3$, $\phi_1(u_1, u_2, u_3, \mu_1) \neq (0, 0, 0)^T$. If it is not true, then constant vector u in R^3 with $u \in \partial\Omega_i$ ($i = 1, 2, 3, 4, 5, 6, 7, 8$) satisfies

$$\begin{cases} g_1(t_1, e^{u_1}) - \mu_1 c_1(t_1) e^{u_2 - u_1} p_2(e^{u_1}) - h_1(t_1) e^{-u_1} = 0, \end{cases} \quad (2.28)$$

$$\begin{cases} -g_2(t_2, e^{u_2}) + \beta_1(t_2) p_2(e^{u_1}) - \mu_1 \beta_2(t_2) e^{u_3 - u_2} p_3(e^{u_2}) - h_2(t_2) e^{-u_2} = 0, \end{cases} \quad (2.29)$$

$$\begin{cases} -d_1(t_3) + \beta_3(t_3) p_3(e^{u_2}) - a_{31}(t_3) e^{u_3} - h_3(t_3) e^{-u_3}. \end{cases} \quad (2.30)$$

From (2.28)-(2.30), following the arguments of (2.22)-(2.24) and by magnifying μ_1 into 1 or reducing μ_1 into 0, we obtain

$$\begin{aligned} \ln \frac{h_1^l}{a_1^M} &< u_1 < \ln l_- \quad \text{or} \quad \ln l_+ < u_1 < A, \\ \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right] &< u_2 < \ln u_- \quad \text{or} \quad \ln u_+ < u_2 < B, \\ \ln \left[\frac{h_3^l}{\beta_3^M p_3(e^B)} \right] &< u_3 < \ln v_- \quad \text{or} \quad \ln v_+ < u_3 < \ln \left[\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right]. \end{aligned}$$

Thus $u \in \Omega_1 \cap R^3$ or $u \in \Omega_2 \cap R^3$ or $u \in \Omega_3 \cap R^3$ or $u \in \Omega_4 \cap R^3$ or $u \in \Omega_5 \cap R^3$ or $u \in \Omega_6 \cap R^3$ or $u \in \Omega_7 \cap R^3$ or $u \in \Omega_8 \cap R^3$. This is a contradiction. Therefore, when $u \in \partial\Omega_i \cap \text{Ker}L$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, $\phi_1(u_1, u_2, u_3, \mu_1) \neq (0, 0, 0)^T$. Based on topological degree theory, we obtain by taking J as the identity mapping since $\text{Ker}L = \text{Im}Q$, for $i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$\begin{aligned} &\deg \{ JQNu, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \} \\ &= \deg \{ \phi_1(u_1, u_2, u_3, 1), \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \} \\ &= \deg \{ \phi_1(u_1, u_2, u_3, 0), \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \} \\ &= \deg \left\{ [g_1(t_1, e^{u_1}) - h_1(t_1) e^{-u_1}, -g_2(t_2, e^{u_2}) + \beta_1(t_2) p_2(e^{u_1}) - h_2(t_2) e^{-u_2}, \right. \\ &\quad \left. -d_1(t_3) + \beta_3(t_3) p_3(e^{u_2}) - a_{31}(t_3) e^{u_3} - h_3(t_3) e^{-u_3}]^T, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \right\}. \end{aligned}$$

Step 2. We prove that for $i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$\begin{aligned} &\deg \left\{ [g_1(t_1, e^{u_1}) - h_1(t_1) e^{-u_1}, -g_2(t_2, e^{u_2}) + \beta_1(t_2) p_2(e^{u_1}) - h_2(t_2) e^{-u_2}, \right. \\ &\quad \left. -d_1(t_3) + \beta_3(t_3) p_3(e^{u_2}) - a_{31}(t_3) e^{u_3} - h_3(t_3) e^{-u_3}]^T, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \right\} \\ &= \deg \left\{ [a_1 - b_1 e^{u_1} - h_1 e^{-u_1}, -a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2}, \right. \\ &\quad \left. -d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3}]^T, \Omega_i \cap \text{Ker}L, (0, 0, 0)^T \right\}, \end{aligned}$$

where $a_1, b_1, h_1, a_2, b_2, \beta_1, h_2, d_1, \beta_3, a_{31}, h_3$ are all some choosen positive constants such that

$$\begin{aligned} \ln \frac{a_1}{b_1} &< A, \quad b_2 e^B > \beta_1 e^A, \quad a_1 h_1^l < h_1 a_1^M, \beta_1 h_2^l e^A < \beta_1^M h_2 p_2(e^A), \\ a_{31}^l \beta_3 e^B &< a_{31} \beta_3^M p_3(e^B), \quad h_3^l \beta_3 e^B < h_3 \beta_3^M p_3(e^B), \\ \frac{a_1 - \sqrt{a_1^2 - 4b_1 h_1}}{2b_1} &< l_- < l_+ < \frac{a_1 + \sqrt{a_1^2 - 4b_1 h_1}}{2b_1}, \\ \frac{\frac{\beta_1 h_1^l}{a_1^M} - a_2 - \sqrt{\left(\frac{\beta_1 h_1^l}{a_1^M} - a_2\right)^2 - 4b_2 h_2}}{2b_2} &< u_-, \\ u_+ &< \frac{\frac{\beta_1 h_1^l}{a_1^M} - a_2 + \sqrt{\left(\frac{\beta_1 h_1^l}{a_1^M} - a_2\right)^2 - 4b_2 h_2}}{2b_2}, \\ \frac{\frac{\beta_3 h_2^l}{\beta_1^M p_2(e^A)} - d_1 - \sqrt{\left(\frac{\beta_3 h_2^l}{\beta_1^M p_2(e^A)} - d_1\right)^2 - 4a_{31} h_3}}{2a_{31}} &< v_-, \\ v_+ &< \frac{\frac{\beta_3 h_2^l}{\beta_1^M p_2(e^A)} - d_1 + \sqrt{\left(\frac{\beta_3 h_2^l}{\beta_1^M p_2(e^A)} - d_1\right)^2 - 4a_{31} h_3}}{2a_{31}}. \end{aligned}$$

To this end, we define mapping $\phi_2 : Dom L \times [0, 1] \rightarrow X$ by

$$\begin{aligned} \phi_2(u_1, u_2, u_3, \mu_2) &= \mu_2 \begin{bmatrix} a_1 - b_1 e^{u_1} - h_1 e^{-u_1} \\ -a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2} \\ -d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3} \end{bmatrix} \\ &+ (1 - \mu_2) \begin{pmatrix} g_1(t_1, e^{u_1}) - h_1(t_1) e^{-u_1} \\ -g_2(t_2, e^{u_2}) + \beta_1(t_2) p_2(e^{u_1}) - h_2(t_2) e^{-u_2} \\ -d_1(t_3) + \beta_3(t_3) p_3(e^{u_2}) - a_{31}(t_3) e^{u_3} - h_3(t_3) e^{-u_3} \end{pmatrix}, \end{aligned}$$

where $\mu_2 \in [0, 1]$ is a parameter. We show that when $u \in \partial\Omega_i \cap Ker L = \partial\Omega_i \cap R^3$, $\phi_2(u_1, u_2, u_3, \mu_2) \neq (0, 0, 0)^T$. If the conclusion is not true, i.e., constant vector u in R^3 with $u \in \partial\Omega_i$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, satisfies $\phi_2(u_1, u_2, u_3, \mu_2) = (0, 0, 0)^T$, then

$$\begin{cases} \mu_2(a_1 - b_1 e^{u_1} - h_1 e^{-u_1}) + (1 - \mu_2)(g_1(t_1, e^{u_1}) - h_1(t_1) e^{-u_1}) = 0, & (2.31) \end{cases}$$

$$\begin{cases} \mu_2(-a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2}) + (1 - \mu_2)(-g_2(t_2, e^{u_2}) + \beta_1(t_2) p_2(e^{u_1}) - h_2(t_2) e^{-u_2}) = 0, & (2.32) \end{cases}$$

$$\begin{cases} \mu_2(-d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3}) + (1 - \mu_2)(-d_1(t_3) + \beta_3(t_3) p_3(e^{u_2}) - a_{31}(t_3) e^{u_3} - h_3(t_3) e^{-u_3}) = 0. & (2.33) \end{cases}$$

We make the following claims.

Claim 1. $u_1 < A$.

Otherwise, $u_1 \geq A$. From condition (i) in Theorem 2.2, we have

$$\begin{aligned} & (1 - \mu_2) (g_1(t_1, e^{u_1}) - h_1(t_1)e^{-u_1}) + \mu_2(a_1 - b_1e^{u_1} - h_1e^{-u_1}) \\ & \leq (1 - \mu_2) (-h_1(t_1)e^{-u_1}) + \mu_2(a_1 - b_1e^A) \\ & < 0. \end{aligned}$$

This contradicts (2.31). Hence Claim 1 holds.

Claim 2. $u_1 > \ln \frac{h_1^l}{a_1^M}$.

Otherwise, $u_1 \leq \ln \frac{h_1^l}{a_1^M}$. Then

$$\begin{aligned} & (1 - \mu_2) (g_1(t_1, e^{u_1}) - h_1(t_1)e^{-u_1}) + \mu_2(a_1 - b_1e^{u_1} - h_1e^{-u_1}) \\ & < (1 - \mu_2) (g_1(t_1, 0) - h_1^l e^{-u_1}) + \mu_2(a_1 - h_1e^{-u_1}) \\ & < (1 - \mu_2)(a_1^M - h_1^l e^{-u_1}) + \mu_2(a_1 - h_1e^{-u_1}) \\ & < 0. \end{aligned}$$

This contradicts (2.31). Hence Claim 2 holds.

Claim 3. $u_2 < B$.

Otherwise, $u_2 \geq B$. From condition (ii) in Theorem 2.2, we obtain

$$\begin{aligned} & (1 - \mu_2) (-g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)e^{-u_2}) \\ & + \mu_2(-a_2 - b_2e^{u_2} + \beta_1e^{u_1} - h_2e^{-u_2}) \\ & < (1 - \mu_2) (-g_2(t_2, e^{u_2}) + \beta_1^M p_2(e^A)) + \mu(-b_2e^{u_2} + \beta_1e^{u_1}) \\ & < \mu(-b_2e^B + \beta_1e^A) \\ & < 0. \end{aligned}$$

This contradicts (2.32). Hence Claim 3 holds.

Claim 4. $u_2 > \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right]$.

Otherwise, $u_2 \leq \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right]$. Then

$$\begin{aligned} & (1 - \mu_2) (-g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)e^{-u_2}) \\ & + \mu_2(-a_2 - b_2e^{u_2} + \beta_1e^{u_1} - h_2e^{-u_2}) \\ & \leq (1 - \mu_2) (\beta_1^M p_2(e^A)) + \mu_2(\beta_1e^{u_1} - h_2e^{-u_2}) \\ & < (1 - \mu_2) (\beta_1^M p_2(e^A)) + \mu_2(\beta_1e^A - h_2 \frac{\beta_1^M p_2(e^A)}{h_2^l}) \\ & < 0. \end{aligned}$$

This contradicts (2.32). Hence Claim 4 holds.

Claim 5. $u_3 < \ln \left[\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right]$.

Otherwise, $u_3 \geq \ln \left[\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right]$. We have

$$\begin{aligned} & (1 - \mu_2) (-d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}) \\ & + \mu_2(-d_1 + \beta_3 e^{u_2} - a_{31}e^{u_3} - h_3 e^{-u_3}) \\ \leq & (1 - \mu_2) (\beta_3^M p_3(e^B) - a_{31}^l e^{u_3}) + \mu_2(\beta_3 e^B - a_{31} \frac{\beta_3^M p_3(e^B)}{a_{31}^l}) \\ < & 0. \end{aligned}$$

This contradicts (2.33). Hence Claim 5 holds.

Claim 6. $u_3 > \ln \left[\frac{h_3^l}{\beta_3^M p_3(e^B)} \right]$.

Otherwise, $u_3 \leq \ln \left[\frac{h_3^l}{\beta_3^M p_3(e^B)} \right]$. Since

$$\begin{aligned} & (1 - \mu_2) (-d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}) \\ & + \mu_2(-d_1 + \beta_3 e^{u_2} - a_{31}e^{u_3} - h_3 e^{-u_3}) \\ \leq & (1 - \mu_2)(\beta_3^M p_3(e^B) - h_3^l e^{-u_3}) + \mu_2(\beta_3 e^B - h_3 e^{-u_3}) \\ < & \mu_2(\beta_3 e^B - h_3 \frac{\beta_3^M p_3(e^B)}{h_3^l}) \\ < & 0. \end{aligned}$$

This contradicts (2.33). Hence Claim 6 holds.

Claim 7. $u_1 > \ln l_+$ or $u_1 < \ln l_-$.

If it is not true, then $\ln l_- \leq u_1 \leq \ln l_+$. Thus from condition (iii) in Theorem 2.2, we have

$$\begin{aligned} & (1 - \mu_2)(g_1(t_1, e^{u_1}) - h_1(t_1)e^{-u_1}) + \mu_2(a_1 - b_1 e^{u_1} - h_1 e^{-u_1}) \\ = & (1 - \mu_2)e^{-u_1} (e^{u_1} g_1(t_1, e^{u_1}) - h_1(t_1)) + \mu_2 e^{-u_1} (a_1 e^{u_1} - b_1 e^{2u_1} - h_1) \\ \geq & (1 - \mu_2)e^{-u_1} (e^{u_1} g_1(t_1, e^{u_1}) - h_1^M) + \mu_2 e^{-u_1} (a_1 e^{u_1} - b_1 e^{2u_1} - h_1) \\ > & \mu_2 e^{-u_1} (a_1 e^{u_1} - b_1 e^{2u_1} - h_1) \\ > & 0. \end{aligned}$$

This contradicts (2.31). Hence Claim 7 holds.

Claim 8. $u_2 > \ln u_+$ or $u_2 < \ln u_-$.

If the conclusion is not true, then $\ln u_- \leq u_2 \leq \ln u_+$. Thus from condition (iv) in Theorem 2.2, we obtain

$$\begin{aligned} & (1 - \mu_2) (-g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)e^{-u_2}) \\ & + \mu_2(-a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2}) \\ = & (1 - \mu_2)e^{-u_2} (-e^{u_2} g_2(t_2, e^{u_2}) + e^{u_2} \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)) \\ & + \mu_2 e^{-u_2} (-a_2 e^{u_2} - b_2 e^{2u_2} + \beta_1 e^{u_1+u_2} - h_2) \\ > & (1 - \mu_2)e^{-u_2} \left(-e^{u_2} g_2(t_2, e^{u_2}) + \beta_1^l e^{u_2} p_2\left(\frac{h_1^l}{a_1^M}\right) - h_2^M \right) \\ & + \mu_2 e^{-u_2} (-a_2 e^{u_2} - b_2 e^{2u_2} + \frac{\beta_1 h_1^l e^{u_2}}{a_1^M} - h_2) \\ > & 0. \end{aligned}$$

This contradicts (2.32). Hence Claim 8 holds.

Claim 9. $u_3 > \ln v_+$ or $u_3 < \ln v_-$.

If the conclusion is not true, then $\ln v_- \leq u_3 \leq \ln v_+$. Thus from condition (v) in Theorem 2.2, we have

$$\begin{aligned}
& \mu_2(-d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3}) + (1 - \mu_2)(-d_1(t_3) \\
& + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}) \\
> & \mu_2 e^{-u_3}(-d_1 e^{u_3} + \beta_3 e^{u_2+u_3} - a_{31} e^{2u_3} - h_3) \\
& + (1 - \mu_2) \left(-d_1^M e^{u_3} + \beta_3^l p_3 \left(\frac{h_2^l}{\beta_1^M p_2(e^A)} \right) e^{u_3} - a_{31}^M e^{2u_3} - h_3^M \right) \\
> & \mu_2 e^{-u_3} \left(-d_1 e^{u_3} + \frac{\beta_3 h_2^l e^{u_3}}{\beta_1^M p_2(e^A)} - a_{31} e^{2u_3} - h_3 \right) \\
> & 0.
\end{aligned}$$

This contradicts (2.33). Hence Claim 9 holds.

From Claim 1-Claim 9, we have

$$\begin{aligned}
& \ln \frac{h_1^l}{a_1^M} < u_1 < \ln l_- \quad \text{or} \quad \ln l_+ < u_1 < A, \\
& \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right] < u_2 < \ln u_- \quad \text{or} \quad \ln u_+ < u_2 < B, \\
& \ln \left[\frac{h_3^l}{\beta_3^M p_3(e^B)} \right] < u_3 < \ln v_- \quad \text{or} \quad \ln v_+ < u_3 < \ln \left[\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right].
\end{aligned}$$

Thus $u \in \Omega_1 \cap R^3$ or $u \in \Omega_2 \cap R^3$ or $u \in \Omega_3 \cap R^3$ or $u \in \Omega_4 \cap R^3$ or $u \in \Omega_5 \cap R^3$ or $u \in \Omega_6 \cap R^3$ or $u \in \Omega_7 \cap R^3$ or $u \in \Omega_8 \cap R^3$, which is a contradiction. Hence when $u \in \partial\Omega_i \cap \text{Ker} L = \partial\Omega_i \cap R^3$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, then $\phi_2(u_1, u_2, u_3, \mu_2) \neq (0, 0, 0)^T$. Therefore, according to topological degree theory, we obtain for $i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$\begin{aligned}
& \deg \left\{ [g_1(t_1, e^{u_1}) - h_1(t_1)e^{-u_1}, -g_2(t_2, e^{u_2}) + \beta_1(t_2)p_2(e^{u_1}) - h_2(t_2)e^{-u_2}, \right. \\
& \quad \left. -d_1(t_3) + \beta_3(t_3)p_3(e^{u_2}) - a_{31}(t_3)e^{u_3} - h_3(t_3)e^{-u_3}]^T, \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \right\} \\
= & \deg \{ \phi_2(u_1, u_2, u_3, 0), \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \} \\
= & \deg \{ \phi_2(u_1, u_2, u_3, 1), \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \} \\
= & \deg \left\{ [a_1 - b_1 e^{u_1} - h_1 e^{-u_1}, -a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2}, \right. \\
& \quad \left. -d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3}]^T, \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \right\}.
\end{aligned}$$

From Step 1 and Step 2, we have for $i = 1, 2, 3, 4, 5, 6, 7, 8$,

$$\begin{aligned}
& \deg \{ JQNu, \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \} \\
= & \deg \left\{ [a_1 - b_1 e^{u_1} - h_1 e^{-u_1}, -a_2 - b_2 e^{u_2} + \beta_1 e^{u_1} - h_2 e^{-u_2}, \right. \\
& \quad \left. -d_1 + \beta_3 e^{u_2} - a_{31} e^{u_3} - h_3 e^{-u_3}]^T, \Omega_i \cap \text{Ker} L, (0, 0, 0)^T \right\}.
\end{aligned}$$

Note that the system of algebraic equations:

$$\begin{cases} a_1 - b_1 e^x - h_1 e^{-x} = 0, \\ -a_2 - b_2 e^y + \beta_1 e^x - h_2 e^{-y} = 0, \\ -d_1 + \beta_3 e^y - a_{31} e^z - h_3 e^{-z} = 0 \end{cases}$$

has eight distinct solutions:

$$\begin{aligned} (x_1^*, y_1^*, z_1^*) &= \left(\ln x_-, \ln y_+, \ln \left[\frac{\beta_3 y_+^- - d_1 + \sqrt{(\beta_3 y_+^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_2^*, y_2^*, z_2^*) &= \left(\ln x_-, \ln y_+, \ln \left[\frac{\beta_3 y_+^- - d_1 - \sqrt{(\beta_3 y_+^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_3^*, y_3^*, z_3^*) &= \left(\ln x_-, \ln y_-, \ln \left[\frac{\beta_3 y_-^- - d_1 + \sqrt{(\beta_3 y_-^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_4^*, y_4^*, z_4^*) &= \left(\ln x_-, \ln y_-, \ln \left[\frac{\beta_3 y_-^- - d_1 - \sqrt{(\beta_3 y_-^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_5^*, y_5^*, z_5^*) &= \left(\ln x_+, \ln y_+^+, \ln \left[\frac{\beta_3 y_+^+ - d_1 + \sqrt{(\beta_3 y_+^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_6^*, y_6^*, z_6^*) &= \left(\ln x_+, \ln y_+^+, \ln \left[\frac{\beta_3 y_+^+ - d_1 - \sqrt{(\beta_3 y_+^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_7^*, y_7^*, z_7^*) &= \left(\ln x_+, \ln y_-^+, \ln \left[\frac{\beta_3 y_-^+ - d_1 + \sqrt{(\beta_3 y_-^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \\ (x_8^*, y_8^*, z_8^*) &= \left(\ln x_+, \ln y_-^+, \ln \left[\frac{\beta_3 y_-^+ - d_1 - \sqrt{(\beta_3 y_-^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] \right), \end{aligned}$$

where

$$\begin{aligned} x_{\pm} &= \frac{a_1 \pm \sqrt{a_1^2 - 4b_1 h_1}}{2b_1}, \\ y_{\pm}^- &= \frac{\beta_1 x_- - a_2 \pm \sqrt{(\beta_1 x_- - a_2)^2 - 4b_2 h_2}}{2b_2}, \\ y_{\pm}^+ &= \frac{\beta_1 x_+ - a_2 \pm \sqrt{(\beta_1 x_+ - a_2)^2 - 4b_2 h_2}}{2b_2}. \end{aligned}$$

It is not difficult to verify that

$$\begin{aligned}
& \ln l_+ < \ln x_+ < A, \\
& \ln \frac{h_1^l}{a_1^M} < \ln x_- < \ln l_-, \\
& \ln u_+ < \ln y_+^+ < B, \\
& \ln u_+ < \ln y_+^- < B, \\
& \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right] < \ln y_-^+ < \ln u_-, \\
& \ln \left[\frac{h_2^l}{\beta_1^M p_2(e^A)} \right] < \ln y_-^- < \ln u_-, \\
& \ln v_+ < \ln \left[\frac{\beta_3 y_+^+ - d_1 + \sqrt{(\beta_3 y_+^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right), \\
& \ln v_+ < \ln \left[\frac{\beta_3 y_+^- - d_1 + \sqrt{(\beta_3 y_+^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right), \\
& \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < \ln \left[\frac{\beta_3 y_-^- - d_1 + \sqrt{(\beta_3 y_-^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln v_-, \\
& \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < \ln \left[\frac{\beta_3 y_-^+ - d_1 + \sqrt{(\beta_3 y_-^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln v_-, \\
& \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < \ln \left[\frac{\beta_3 y_+^+ - d_1 + \sqrt{(\beta_3 y_+^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln v_-, \\
& \ln \left(\frac{h_3^l}{\beta_3^M p_3(e^B)} \right) < \ln \left[\frac{\beta_3 y_+^- - d_1 + \sqrt{(\beta_3 y_+^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln v_-, \\
& \ln v_+ < \ln \left[\frac{\beta_3 y_-^- - d_1 + \sqrt{(\beta_3 y_-^- - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right), \\
& \ln v_+ < \ln \left[\frac{\beta_3 y_-^+ - d_1 + \sqrt{(\beta_3 y_-^+ - d_1)^2 - 4h_3 a_{31}}}{2a_{31}} \right] < \ln \left(\frac{\beta_3^M p_3(e^B)}{a_{31}^l} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
& (x_1^*, y_1^*, z_1^*) \in \Omega_6, (x_2^*, y_2^*, z_2^*) \in \Omega_2, (x_3^*, y_3^*, z_3^*) \in \Omega_5, (x_4^*, y_4^*, z_4^*) \in \Omega_1, (x_5^*, y_5^*, z_5^*) \in \Omega_8, \\
& (x_6^*, y_6^*, z_6^*) \in \Omega_4, (x_7^*, y_7^*, z_7^*) \in \Omega_7, (x_8^*, y_8^*, z_8^*) \in \Omega_3.
\end{aligned}$$

A direct computation gives for $i = 1, \dots, 8$,

$$\begin{aligned}
& \deg \{JQNu, \Omega_i \cap KerL, (0, 0, 0)^T\} \\
&= \text{sign} \begin{vmatrix} -b_1x^* + \frac{h_1}{x^*} & 0 & 0 \\ \beta_1x^* & -b_2y^* + \frac{h_2}{y^*} & 0 \\ 0 & \beta_3y^* & -a_{31}z^* + \frac{h_3}{z^*} \end{vmatrix} \bigg/ (x^*, y^*, z^*) \\
&= (e^{x_i^*}, e^{y_i^*}, e^{z_i^*}) \\
&= -\text{sign} [(2b_1x^* - a_1)(b_2(y^*)^2 - h_2)(a_{31}(z^*)^2 - h_3)] \bigg/ (x^*, y^*, z^*) \\
&= (e^{x_i^*}, e^{y_i^*}, e^{z_i^*}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \deg \{JQNu, \Omega_1 \cap KerL, (0, 0, 0)^T\} \\
&= \text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_- - a_2 - \sqrt{(\beta_1x_- - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_- - d_1 - \sqrt{((\beta_3y_- - d_1)^2 - 4h_3a_{31})} \right)^2 - 4h_3a_{31} \right] \right\} \\
&= 1, \\
& \deg \{JQNu, \Omega_2 \cap KerL, (0, 0, 0)^T\} \\
&= \text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_- - a_2 + \sqrt{(\beta_1x_- - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_+ - d_1 - \sqrt{((\beta_3y_+ - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= -1, \\
& \deg \{JQNu, \Omega_3 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_+ - a_2 + \sqrt{(\beta_1x_+ - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_- - d_1 - \sqrt{((\beta_3y_- - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= 1, \\
& \deg \{JQNu, \Omega_4 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_+ - a_2 + \sqrt{(\beta_1x_+ - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_+ - d_1 - \sqrt{((\beta_3y_+ - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
& \deg \{JQNu, \Omega_5 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ -\sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_- - a_2 - \sqrt{(\beta_1x_- - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_- - d_1 + \sqrt{((\beta_3y_- - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= -1, \\
& \deg \{JQNu, \Omega_6 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ -\sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_- - a_2 + \sqrt{(\beta_1x_- - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_+ - d_1 + \sqrt{((\beta_3y_+ - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= 1, \\
& \deg \{JQNu, \Omega_7 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_+ - a_2 + \sqrt{(\beta_1x_+ - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_- - d_1 + \sqrt{((\beta_3y_- - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= -1, \\
& \deg \{JQNu, \Omega_8 \cap KerL, (0, 0, 0)^T\} \\
&= -\text{sign} \left\{ \sqrt{a_1^2 - 4b_1h_1} \left[\left(\beta_1x_+ - a_2 + \sqrt{(\beta_1x_+ - a_2)^2 - 4b_2h_2} \right)^2 - 4b_2h_2 \right] \right. \\
&\quad \times \left. \left[\left(\beta_3y_+ - d_1 + \sqrt{((\beta_3y_+ - d_1)^2 - 4h_3a_{31})} \right)^2 - 4a_{31}h_3 \right] \right\} \\
&= -1.
\end{aligned}$$

So far, we have proved that $\Omega_i (i = 1, \dots, 8)$ satisfies all the assumptions in Theorem 2.1. Hence, system (2.2) has at least eight different periodic solutions. By (2.1), system (1.2) has at least eight different positive ω -periodic solutions. This completes the proof. $\square$

3. Examples.

EXAMPLE 1. Consider the following predator-prey system with stocking:

$$\begin{cases} \frac{dx_1}{dt} = x_1[a_1(t) - b_1(t)x_1(t) - c_1(t)x_2(t)] - h_1(t), \\ \frac{dx_2}{dt} = x_2[-a_2(t) - b_2(t)x_2(t) + \beta_1(t)x_1(t - \tau) - \beta_2(t)x_3(t)] - h_2(t), \\ \frac{dx_3}{dt} = x_3[-d_1(t) + \beta_3(t)x_2(t - \tau) - a_{31}(t)x_3(t)] - h_3(t), \end{cases} \quad (3.1)$$

where all coefficients are positive continuous ω -periodic functions and $\tau > 0$ is a constant delay.

In Theorem 2.2, $g_1(t, x_1) = a_1(t) - b_1(t)x_1(t)$, $g_2(t, x_2) = a_2(t) + b_2(t)x_2(t)$, $p_2(x) = p_3(x) = x$. If $x \geq \ln \frac{a_1^M}{b_1^M} \stackrel{\text{def}}{=} A$, then $g_1(t, e^x) \leq 0$, and if $x \geq \ln \frac{\beta_1^M e^A}{b_2^M} \stackrel{\text{def}}{=} B$, then $g_2(t, e^x) \geq \beta_1^M p_2(e^A)$, $t, x \in R$. Take

$$l_{\pm} = \frac{1}{2b_1^M} \left\{ a_1^l - c_1^M e^B \pm \sqrt{(a_1^l - c_1^M e^B)^2 - 4b_1^M h_1^M} \right\},$$

$$u_{\pm} = \frac{1}{2b_2^M} \left\{ \frac{\beta_1^l h_1^l}{a_1^M} - a_2^M - \frac{\beta_2^M \beta_3^M e^B}{a_{31}^l} \pm \sqrt{\left(\frac{\beta_1^l h_1^l}{a_1^M} - a_2^M - \frac{\beta_2^M \beta_3^M e^B}{a_{31}^l} \right)^2 - 4b_2^M h_2^M} \right\},$$

then when $\ln l_- \leq x \leq \ln l_+$,

$$e^x g_1(t, e^x) \geq c_1^M e^B p_2(e^x) + h_1^M,$$

and when $\ln u_- \leq x \leq \ln u_+$,

$$h_2^M + \frac{\beta_2^M \beta_3^M}{a_{31}^l} p_3(e^B) p_3(e^x) + e^x g_2(t, e^x) \leq \beta_1^l e^x p_2\left(\frac{h_1^l}{a_1^M}\right).$$

Hence if the following conditions hold:

- (H1) $a_1^l > c_1^M e^B + 2\sqrt{b_1^M h_1^M}$;
- (H2) $\frac{\beta_1^l h_1^l}{a_1^M} > a_2^M + \frac{\beta_2^M \beta_3^M e^B}{a_{31}^l} + 2\sqrt{b_2^M h_2^M}$;
- (H3) $\frac{\beta_3^l h_3^l}{\beta_1^M e^A} > d_1^M + 2\sqrt{a_{31}^M h_3^M}$,

then all conditions in Theorem 2.2 are satisfied. Therefore, by Theorem 2.2, we have the following theorem.

THEOREM 3.1. Assume that (H1)-(H3) hold. Then system (3.1) has at least eight positive ω -periodic solutions.

EXAMPLE 2. Consider the following predator-prey system with functional response and stocking given by

$$\begin{cases} \frac{dx_1}{dt} = x_1 \left(a_1(t) - b_1(t)x_1(t) - \frac{c_1(t)x_2(t)}{mx_1(t) + \gamma} \right) - h_1(t), \\ \frac{dx_2}{dt} = x_2 \left(-a_2(t) - b_2(t)x_2(t) + \frac{\beta_1(t)x_1(t - \tau)}{mx_1(t - \tau) + \gamma} - \frac{\beta_2(t)x_3(t)}{nx_2(t) + p} \right) - h_2(t), \\ \frac{dx_3}{dt} = x_3 \left(-d_1(t) + \frac{\beta_3(t)x_2(t - \tau)}{nx_2(t - \tau) + p} - a_{31}(t)x_3(t) \right) - h_3(t), \end{cases} \quad (3.2)$$

where all coefficients are positive continuous ω -periodic functions, and $x > 0$ is a constant delay.

In Theorem 2.2, $g_1(t, x) = a_1(t) - b_1(t)x_1(t)$, $g_2(t, x) = a_2(t) + b_2(t)x_2(t)$, $p_2(x) = \frac{x_1(t)}{m(t)x_1(t) + \gamma}$, $p_3(t) = \frac{x_2(t)}{n(t)x_2(t) + p}$. If $x \geq \ln \frac{a_1^M}{b_1^M} \stackrel{\text{def}}{=} A$, then $g_2(t, e^x) \leq 0$, $t, x \in R$ and

if $x \geq \ln \frac{\beta_1^M e^A}{b_2^M} \stackrel{\text{def}}{=} B$, then $g_2(t, e^x) \geq \beta_1^M p_2(e^A)$, $t, x \in R$. Let

$$l_{\pm} = \frac{1}{2b_1^M} \left\{ a_1^l - \frac{c_1^M e^B}{r^l} \pm \sqrt{\left(a_1^l - \frac{c_1^M e^B}{r^l} \right)^2 - 4b_1^M h_1^M} \right\},$$

$$u_{\pm} = \frac{1}{2b_2^M} \left\{ \frac{\beta_1^l h_1^l}{a_1^M r^M + m^M h_1^l} - a_2^M - \frac{\beta_2^M \beta_3^M}{a_{31}^l n^l p^l} \right. \\ \left. \pm \sqrt{\left(\frac{\beta_1^l h_1^l}{a_1^M r^M + m^M h_1^l} - a_2^M - \frac{\beta_2^M \beta_3^M}{a_{31}^l n^l p^l} \right)^2 - 4b_2^M h_2^M} \right\}.$$

Then if $\ln l_- \leq x \leq \ln l_+$, $e^x g_1(t, e^x) \geq c_1^M e^B p_2(e^x) + h_1^M$ and if $\ln u_- \leq x \leq \ln u_+$, then $h_2^M + \frac{\beta_2^M \beta_3^M}{a_{31}^l} p_3(e^B) p_3(e^x) + e^x g_2(t, e^x) \leq \beta_1^l e^x p_2(\frac{h_1^l}{a_1^M})$, $t, x \in R$. Hence if the following conditions hold,

$$(T1) \ a_1^l > \frac{c_1^M e^B}{r^l} + 2\sqrt{b_1^M h_1^M};$$

$$(T2) \ \frac{\beta_1^l h_1^l}{a_1^M r^M + m^M h_1^l} > a_2^M + \frac{\beta_2^M \beta_3^M}{a_{31}^l n^l p^l} + 2\sqrt{b_2^M h_2^M};$$

$$(T3) \ \beta_3^l h_2^l > \left(n^M h_2^l + \frac{p^M e^A \beta_1^M}{m^M e^A + r^M} \right) \left(d_1^M + 2\sqrt{a_{31}^M h_3^M} \right),$$

then all requirements in Theorem 2.2 are satisfied. Therefore, by Theorem 2.2, we obtain the following theorem.

THEOREM 3.2. Assume that (T1)-(T3) hold. Then system (3.2) has at least eight different positive ω -periodic solutions.

EXAMPLE 3. Consider the system

$$\begin{cases} \frac{dx_1}{dt} = x_1[99 + \sin t - (101 + \sin t)x_1(t) - \frac{1}{100 + \sin t}x_2(t)] - 21 - \sin t, \\ \frac{dx_2}{dt} = x_2 \left[-\frac{1}{100 + \cos t} - (101 + \cos t)x_2(t) \right. \\ \quad \left. + (99 + \sin t)x_1(t - \tau) - (9 + \sin t)x_3(t) \right] - \frac{1}{9 + \cos t}, \\ \frac{dx_3}{dt} = x_3 \left[-\frac{1}{11 + \cos t} + (999 + \cos t)x_2(t - \tau) \right. \\ \quad \left. - (10001 + \sin t)x_3(t) \right] - \frac{1}{90001 + \sin t}. \end{cases} \quad (3.3)$$

It is easy to examine that the coefficients of system (3.3) satisfy all assumptions in Theorem 3.1. Thus, by Theorem 3.1, system (3.3) has at least eight positive ω -periodic solutions.

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