STABILITY ESTIMATES OF THE BOLTZMANN EQUATION IN A HALF SPACE

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This paper is dedicated to Tai-Ping Liu on the occasion of his sixtieth birthday.

Abstract. In this paper, we study the large-time behavior and the stability of continuous mild solutions to the Boltzmann equation in a half space. For this, we introduce two nonlinear functionals measuring future binary collisions and L^1 -distance. Through the time-decay estimates of these functionals and the pointwise estimate of the gain part of the collision operator, we show that continuous mild solutions approach to collision free flows time-asymptotically in L^1 , and L^1 -distance at time t is uniformly bounded by that of corresponding initial data, when initial datum is a small perturbation of the vacuum.

1. Introduction. This paper is devoted to the study of large-time behavior and L^1 -stability for the motion of dilute gas undergoing binary collisions in the half space $\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\}$ and reflecting specularly on $\partial\Omega$. The issue can be understood via the Boltzmann equation describing the spatial-temporal evolution of a phase space density function $f = f(x, \xi, t)$:

$$\partial_t f + \xi \cdot \nabla_x f = Q(f, f) \quad \text{in } \Omega \times \mathbb{R}^3 \times \mathbb{R}_+, \quad (1.1)$$

subject to initial and boundary conditions:

$$\begin{aligned} f \Big|_{t=0} &= f_0(x, \xi) && \text{on } \Omega \times \mathbb{R}^3, \\ f(x, \xi, t) &= f(x, R(\xi), t), && \text{on } \partial\Omega, \quad \xi_1 \geq 0, \quad t \geq 0, \end{aligned} \quad (1.2)$$

where $R(\cdot)$ is the specular reflection boundary operator:

$$R(\xi) := (-\xi_1, \xi_2, \xi_3), \quad \xi = (\xi_1, \xi_2, \xi_3).$$

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The collision operator $Q(f, f)$ acts on the velocity variable ξ and reads as

$$Q(f, f)(\xi) := \frac{1}{\kappa} \iint_{\mathbb{R}^3 \times \mathbf{S}^2} B(\xi - \xi_*, \omega) \left(f(\xi') f(\xi'_*) - f(\xi) f(\xi_*) \right) d\omega d\xi_*. \quad (1.3)$$

Here κ denotes the Knudsen number which is the ratio between the mean free and the characteristic length of the flow, and scattered velocities (ξ', ξ'_*) are given by the incident velocities (ξ, ξ_*) and $\omega \in \mathbf{S}_+^2$:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega] \omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega] \omega, \quad (1.4)$$

and we used abbreviated notation $f(\xi) := f(x, \xi, t)$.

Let $\{T(t)\}$ denote the one-parameter semigroup generated by the free molecular motion:

$$\partial_t f + \xi \cdot \nabla_x f = 0,$$

i.e., $T(t)(x, \xi)$ is the phase space position of particles issued from (x, ξ) at $t = 0$ (see Section 2.1 for a more detailed description). For given $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$, we set

$$f^\sharp(x, \xi, t) := f(T(t)(x, \xi), t) \quad \text{and} \quad Q^\sharp(f, f)(x, \xi, t) := Q(f, f)(T(t)(x, \xi), t).$$

We integrate (1.1) along $T(t)(x, \xi)$ to find a mild form of (1.1):

$$f^\sharp(x, \xi, t) = f_0(x, \xi) + \int_0^t Q^\sharp(f, f)(x, \xi, s) ds, \quad (x, \xi, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}_+. \quad (1.5)$$

The definition of a global mild solution can be stated as follows.

DEFINITION 1.1. A nonnegative function $f \in C(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$ is a global mild solution of (1.1) with a nonnegative initial datum f_0 if and only if for all $t \in \mathbb{R}_+$ and a.e. $(x, \xi) \in \Omega \times \mathbb{R}^3$,

- (1) $Q^\sharp(f, f)(x, \xi, \cdot) \in L_{loc}^1(\mathbb{R}_+)$;
- (2) f satisfies the integral equation (1.5) pointwise.

In the sequel, we restrict our discussion mainly to a kinetic regime which has a large Knudsen number. For the detailed review near a fluid regime, we refer to [5, 9].

For the Cauchy problem to (1.1), there are many results available, for example the local and global existence of solutions [10, 11, 19], uniqueness and qualitative properties of solutions such as an H -theorem, scattering type results [7, 16, 18, 21, 22] and uniform L^1 -stability [13, 14]. In the presence of a boundary, there are some results available on the global existence of mild solutions [15, 23, 29] (see [31, 32] near a global Maxwellian). The L^1 -stability and the large-time behavior of classical solutions with a cut-off inverse power potential in the whole space have been studied in [14] using the nonlinear functional approach. Unlike the Cauchy problem in the whole space, classical solutions are not available for (1.1)–(1.2), and in fact, the existence of classical solutions is not generic for the initial-boundary value problem [25]. However, the nonlinear functional approach can be applied to the mild solution of (1.1)–(1.2) via the mollification and the usual limiting procedure.

Below we describe our main assumptions. Introduce bounding functions decaying algebraically: For $\alpha, \beta > 0$,

$$h_\alpha(x) := \frac{1}{(1 + |x|^2)^{\alpha/2}} \quad \text{and} \quad m_\beta(\xi) := \frac{1}{(1 + |\xi|^2)^{\beta/2}}.$$

Define a set $\mathcal{S}(\delta, \alpha, \beta)$ equipped with a space-time norm $||| \cdot |||_{\alpha, \beta}$:

$$\begin{aligned} |||f|||_{\alpha, \beta} &:= \sup_{x, \xi, t} h_\alpha^{-1}(x) m_\beta^{-1}(\xi) f^\sharp(x, \xi, t), \\ \mathcal{S}(\delta, \alpha, \beta) &:= \{f \in C_b(\Omega \times \mathbb{R}^3 \times \mathbb{R}_+) : |||f|||_{\alpha, \beta} < \delta\}, \end{aligned}$$

where $C_b(X)$ denotes the space of continuous bounded functions defined on X .

We next summarize assumptions $\mathcal{A}$ used in this paper:

- $\mathcal{A1}$. Collision kernel B satisfies an angular cut-off assumption:

$$\begin{aligned} B(\xi - \xi_*, \omega) &= |\xi - \xi_*|^\gamma b(\theta), \quad -2 < \gamma \leq 1, \\ \frac{b(\theta)}{\cos \theta} &\leq B_* < \infty, \quad \theta := \cos^{-1} \left(\frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|} \right). \end{aligned}$$

- $\mathcal{A2}$. Parameters in $\mathcal{S}(\delta, \alpha, \beta)$ satisfy

$$\delta \ll \kappa, \quad \alpha > 6 + 2\gamma \quad \text{and} \quad \beta > 6 + \gamma.$$

We next briefly discuss the main results of this paper. Let f and $\bar{f}$ be continuous mild solutions corresponding to initial data f_0 and $\bar{f}_0$, respectively, and let f_ε and $\bar{f}_\varepsilon$ be the corresponding mollifications of f and $\bar{f}$, respectively. Then f_ε is smooth in x, t and satisfies an approximate Boltzmann equation:

$$\partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon),$$

where the error term $P(f, f_\varepsilon)$ is given by

$$P(f, f_\varepsilon) := Q_\varepsilon(f, f) - Q(f_\varepsilon, f_\varepsilon),$$

where $Q_\varepsilon(f, f)$ is the mollified collision operator.

The main idea is to construct nonlinear functionals in [14] based on the above approximate Boltzmann equations and to study their time-evolution estimate, and finally to pass the mollification parameter $\varepsilon \rightarrow 0$ to recover the time-decay estimate for nonlinear functionals evaluated along continuous mild solutions.

More precisely, we first construct a “collision potential” $\mathcal{D}(f_\varepsilon)$ measuring all possible future binary collisions between particles and yielding approximate Lyapunov estimates:

$$\frac{d}{dt} \mathcal{D}(f_\varepsilon(t)) \leq \left(-1 + \mathcal{O}(1) \frac{\delta}{\kappa} \right) \Lambda(f_\varepsilon(t)) + \mathcal{O}(1) \delta \iint_{\Omega \times \mathbb{R}^3} |P^\sharp(f, f_\varepsilon)| d\xi dx, \quad (1.6)$$

where $\Lambda(f_\varepsilon(t))$ is a nonnegative functional. Then as $\varepsilon \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega \times \mathbb{R}^3} |P^\sharp(f, f_\varepsilon)| d\xi dx = 0.$$

This yields a Lyapunov estimate for $\mathcal{D}(f(t))$ (see Section 3).

Secondly, we construct a nonlinear functional $\mathcal{H}^\varepsilon(t)$ satisfying the following two properties:

- $\mathcal{H}^\varepsilon(t)$ is equivalent to L^1 -distance:

$$||f_\varepsilon(t) - \bar{f}_\varepsilon(t)||_{L^1} \leq \mathcal{H}^\varepsilon(t) \leq C_0 ||f_\varepsilon(t) - \bar{f}_\varepsilon(t)||_{L^1},$$

where C_0 is a positive constant independent of ε and time t .

- $\mathcal{H}^\varepsilon(t)$ satisfies a stability estimate:

$$\mathcal{H}^\varepsilon(t) + C_1 \int_0^t \Lambda_d^\varepsilon(s) ds \leq C_2 \mathcal{H}^\varepsilon(0) + \mathcal{O}(1) \int_0^t \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\# d\xi dx, \quad t \geq 0,$$

where C_1 and C_2 are positive constants independent of t and ε , and we have used the simplified notation:

$$\|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} := \|f_\varepsilon(\cdot, \cdot, t) - \bar{f}_\varepsilon(\cdot, \cdot, t)\|_{L^1(\Omega \times \mathbb{R}^3)}.$$

The above two key properties yield an approximate L^1 -stability estimate:

$$\begin{aligned} & \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} + C_0 C_1 \int_0^t \Lambda_d^\varepsilon(s) ds \\ & \leq C_0^2 C_2 \|f_{0,\varepsilon} - \bar{f}_{0,\varepsilon}\|_{L^1} + \mathcal{O}(1) \int_0^t \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\# d\xi dx ds. \end{aligned}$$

Finally we let $\varepsilon \rightarrow 0$ to get the following stability result.

THEOREM 1.1. Suppose the main assumptions $\mathcal{A}$ hold, and let f and $\bar{f}$ be two mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$ corresponding to initial data f_0 and $\bar{f}_0$, respectively. Then we have

$$\|f(t) - \bar{f}(t)\|_{L^1} + G_0 \int_0^t \Lambda_d(s) ds \leq G_1 \|f_0 - \bar{f}_0\|_{L^1}, \quad t \geq 0,$$

where G_0 and G_1 are positive constants independent of t and Λ_d is the collision production rate whose explicit form will be given in Section 4.

- REMARK 1.1.**
- (1) The existence of global mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$ was established in Shinbrot's work [23] in a more general setting.
 - (2) For the restricted range of $\gamma \in (-2, 0]$, the L^1 -stability result in Theorem 1.1 can be obtained without the production term Λ_d employing the estimate in [20]. Our result extends to the more interesting hard potential case of $\gamma \in (-2, 1]$.
 - (3) We need the stronger decay rate $\mathcal{A}_2$ for the solution compared to [14] to obtain the uniform estimate of the mollified solutions (see Lemma 4.1).

The rest of this paper is organized in the following manner. In Section 2, we present an explicit representation for curved particle trajectories, the mollification procedure of continuous mild solutions and several basic estimates. In Section 3, we construct explicitly the collision potential $\mathcal{D}(f)$ along smooth mollifications and as mollification parameter $\varepsilon \rightarrow 0$, we show that the resulting functional satisfies (1.6). Finally, in Section 4, we present the nonlinear functional $\mathcal{H}$ which yields the uniform L^1 -stability.

2. Preliminaries. In this section, we present an explicit representation of particle trajectories, the mollification procedure and some basic estimates to be used in later sections.

FIG. 1

2.1. *Curved particle trajectories.* Here we define the semigroup $T(t)$ related to the free molecular motion in the half space:

$$\partial_t f + \xi \cdot \nabla_x f = 0. \quad (2.1)$$

A particle trajectory issuing from (x, ξ) at time 0 will be straight lines when $\xi_1 \geq 0$; in contrast, when $\xi_1 < 0$, the trajectory will hit the wall and then be reflected from the wall (see Figure 1) at time $-\frac{x_1}{\xi_1}$:

$$T(t)(x, \xi) = \begin{cases} (x + t\xi, \xi) & \text{if } x_1 + t\xi_1 \geq 0, \\ (x - (x_1/\xi_1)\xi + (t + x_1/\xi_1)R(\xi), R(\xi)) & \text{if } x_1 + t\xi_1 < 0. \end{cases}$$

For simplicity of presentation, we use the following notation:

$$x^\sharp(t; s, x, \xi) := \mathbb{P}_x T(t - s)(x, \xi), \quad \xi^\sharp(t; s, x, \xi) := \mathbb{P}_\xi T(t - s)(x, \xi), \quad t \geq s, \quad (2.2)$$

$$x^\flat(s; t, x, \xi) := \mathbb{P}_x T^{-1}(t - s)(x, \xi), \quad \xi^\flat(s; t, x, \xi) := \mathbb{P}_\xi T^{-1}(t - s)(x, \xi), \quad t > s, \quad (2.3)$$

$$x^\sharp(t) := x^\sharp(t; 0, x, \xi), \quad \xi^\sharp(t) := \xi^\sharp(t; 0, x, \xi), \quad (2.4)$$

where $\mathbb{P}_x$ and $\mathbb{P}_\xi$ are projections from $\Omega \times \mathbb{R}_\xi^3$ onto Ω and $\mathbb{R}_\xi^3$, respectively.

2.2. *Mollifications of mild solutions and approximate Boltzmann equations.* Here we study the mollification procedure to derive approximate Boltzmann equations satisfied by a mild solution $f \in \mathcal{S}(\delta, \alpha, \beta)$ and its mollification f_ε in x . Since the nonlinear functional approach [14] uses the equation (1.1) directly, the analysis is restricted to the class of classical solutions; however in our framework, the classical solutions do not exist generically. Hence we need to mollify (1.1) in x to use the nonlinear functional approach for the mollified equation. The mollification procedure serves to extend the mild solution to the whole space, then to mollify the extended function and restrict the mollified extended function to the original domain. Below we present an alternative way of performing the mollification procedure in [6] without an extension operator, which may be more easily adapted to general exterior domain problems.

Let f be a mild solution in $\mathcal{S}(\delta, \alpha, \beta)$ and let $\mathcal{C}$ be the open cone with height $h > 0$, the cone's angle $0 < \theta < \pi$:

$$\mathcal{C} := \{x = (x_1, \bar{x}) \in \Omega \mid 0 < x_1 < h, \quad |x'| < x_1 \tan(\theta/2)\}, \quad \bar{x} = (x_2, x_3),$$

FIG. 2. $x - \varepsilon\nu - B(0, \eta(\varepsilon)) \subset \Omega$

i.e., the vertex of the cone is at the origin and the outward normal axis vector ν is $(-1, 0, 0)$. Notice that for all $x \in \bar{\Omega}$, $x + \mathcal{C}$ is in $\bar{\Omega}$. Below we fix $h = 1$, $\theta = \frac{\pi}{3}$. For all $\varepsilon > 0$, we now define

$$\eta(\varepsilon) = \varepsilon \sin(\theta/2) = \frac{\varepsilon}{2} \quad \text{and} \quad \phi_\varepsilon(y) = \eta(\varepsilon)^{-3} \phi(y/\eta(\varepsilon)),$$

where ϕ is a standard mollifier, i.e.,

$$0 \leq \phi \in C_c^\infty(\mathbb{R}^3), \quad \text{supp } \phi \subset B(0, 1) \quad \text{and} \quad \int_{\mathbb{R}^3} \phi(x) dx = 1,$$

where $B(x_0, r)$ is the open ball with a center x_0 and a radius r . Let us set

$$\varepsilon_{\mathcal{C}} = \frac{1}{1 + \sin(\pi/6)}.$$

For all $0 < \varepsilon < \varepsilon_{\mathcal{C}}$ and all $x \in \bar{\Omega}$, we set

$$\begin{aligned} f_\varepsilon(x, \xi, t) &:= f *_{\varepsilon, \nu} \phi_\varepsilon \\ &= \int_{B(0, \eta(\varepsilon))} f(x - \varepsilon\nu - y, \xi, t) \phi_\varepsilon(y) dy. \end{aligned}$$

Note that

$$x - \varepsilon\nu - B(0, \eta(\varepsilon)) \subset x + \mathcal{C} \subset \Omega;$$

therefore, $f_\varepsilon(x)$ is well defined up to the boundary (see Figure 2). Moreover, since ϕ_ε has support in $\bar{B}(0, \eta(\varepsilon))$, we have

$$f_\varepsilon = \int_{\mathbb{R}^3} \tilde{f}(x - \varepsilon\nu - y) \phi_\varepsilon(y) dy$$

for any extension $\tilde{f}$ defined on the whole space. It follows that $f_\varepsilon(\cdot, \xi, t) \in C^\infty(\bar{\Omega})$. The modified mollification f_ε satisfies the same property as the standard one, for instance

$$\begin{aligned} f_\varepsilon &\rightarrow f \text{ a.e.} \quad \text{as } \varepsilon \rightarrow 0, \\ f_\varepsilon(\cdot, \xi, t) &\rightarrow f(\cdot, \xi, t) \text{ in } L^p(\Omega) \quad \text{if } 1 \leq p < \infty \text{ and } f(\cdot, \xi, t) \in L^p(\Omega). \end{aligned}$$

By mollifying the integral equation (1.5), we find f_ε satisfy

$$\begin{cases} \partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon), & (x, \xi, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}_+, \\ f_\varepsilon(x, \xi, 0) = f_0 *_{\varepsilon, \nu} \phi_\varepsilon, \end{cases} \quad (2.5)$$

where

$$P(f, f_\varepsilon) := Q_\varepsilon(f, f) - Q(f_\varepsilon, f_\varepsilon).$$

The following lemma shows that mollification preserves decay rates, but possibly amplifies its magnitude.

LEMMA 2.1. Let f be a function in $\mathcal{S}(\delta, \alpha, \beta)$, and let f_ε be the corresponding mollification. Then for $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned} (1) \quad & f_\varepsilon \in \mathcal{S}(3 \cdot 2^\alpha \delta, \alpha, \beta), \quad \nabla_x f_\varepsilon \in \mathcal{S}(3 \cdot 2^\alpha \varepsilon^{-1} \delta, \alpha, \beta), \\ (2) \quad & \|f_\varepsilon(t)\|_{L^1} \leq \|f(t)\|_{L^1}. \end{aligned}$$

Proof. (1) By definition of a mollifier, we have

$$\begin{aligned} & h_\alpha^{-1}(x) m_\beta^{-1}(\xi) f_\varepsilon^\sharp(x, \xi, t) \\ &= h_\alpha^{-1}(x) m_\beta^{-1}(\xi) \int_{B(0, \eta(\varepsilon))} f^\sharp(x^\sharp(t) - \varepsilon\nu - y, \xi^\sharp(t), t) \phi_\varepsilon(y) dy, \end{aligned} \quad (2.6)$$

where we denote $(x^\sharp(t; 0, x, \xi), \xi^\sharp(t; 0, x, \xi)) = (x^\sharp(t), \xi^\sharp(t))$. We set

$$(X, \Xi) = T^{-1}(t)(x^\sharp(t) - \varepsilon\nu - y, \xi^\sharp(t)).$$

Then the L.H.S of (2.6) is

$$\begin{aligned} & \int_{B(0, \eta(\varepsilon))} \underbrace{h_\alpha^{-1}(X) m_\beta^{-1}(\Xi) f^\sharp(X, \Xi, t)}_{\leq \delta} h_\alpha(X) h_\alpha^{-1}(x) m_\beta(\Xi) m_\beta^{-1}(\xi) \phi_\varepsilon(y) dy \\ & \leq \delta \int_{B(0, \eta(\varepsilon))} \frac{(1 + |x|^2)^{\frac{\alpha}{2}}}{(1 + |X|^2)^{\frac{\alpha}{2}}} \phi_\varepsilon(y) dy \end{aligned} \quad (2.7)$$

for $\Xi = \xi$ or $R(\xi)$.

Now we locate X ; along backward characteristics issuing

$$(x^\sharp(t) - \varepsilon\nu - y, \xi^\sharp(t)) = \begin{cases} (x + t\xi - \varepsilon\nu - y, \xi), & \xi_1 \geq -\frac{x_1}{t}, \\ (x + t_0\xi + (t - t_0)R(\xi) - \varepsilon\nu - y, R(\xi)), & \xi_1 < -\frac{x_1}{t}. \end{cases}$$

Then we have three cases.

Case 1 ($\xi_1 \geq -\frac{x_1}{t}$):

$$X = x - \varepsilon\nu - y.$$

Case 2 ($\xi_1 < -\frac{x_1}{t}, x_1 + \varepsilon\nu_1 + y_1 \leq 0$):

$$\begin{aligned} X &= x - \varepsilon\nu - y + t_0\xi - t_0R(\xi) \\ &= (-x_1 + \varepsilon - y_1, x_2 - y_2, x_3 - y_3). \end{aligned}$$

Case 3 ($\xi_1 < -\frac{x_1}{t}, x_1 + \varepsilon\nu_1 + y_1 > 0$):

$$\begin{aligned} X &= x - \varepsilon\nu - y + t_0\xi + (t - t_0)R(\xi) - \tau_0R(\xi) - (t - \tau_0)\xi \\ &= (x_1 - \varepsilon + y_1, x_2 - y_2, x_3 - y_3), \end{aligned}$$

where $t_0 = -\frac{x_1}{\xi_1}$, $\tau_0 = \frac{t\xi_1 + x_1 + \varepsilon\nu_1 + y_1}{\xi_1}$. Note that in Cases 2 and 3

$$\frac{1 + |x - \varepsilon\nu - y|^2}{1 + |X|^2} \leq \frac{1 + (x_1 + \varepsilon - y_1)^2}{1 + (x_1 - \varepsilon + y_1)^2} \leq 3$$

holds for small ε with $|y| \leq \frac{\varepsilon}{2}$. Then we estimate (2.7) by

$$\begin{aligned} (2.7) &\leq 3\delta \int_{B(0, \eta(\varepsilon))} \frac{(1 + |x|^2)^{\alpha/2} \phi_\varepsilon(y) dy}{(1 + |x - \varepsilon\nu - y|^2)^{\alpha/2}} \\ &= 3 \cdot 2^{\alpha-1} \delta \int_{B(0, \eta(\varepsilon))} \frac{(1 + |x - \varepsilon\nu - y|^2)^{\alpha/2} + (\varepsilon + |y|)^\alpha}{(1 + |x - \varepsilon\nu - y|^2)^{\alpha/2}} \phi_\varepsilon(y) dy \\ &= 3 \cdot 2^{\alpha-1} \delta \left(\int_{B(0, \eta(\varepsilon))} \phi_\varepsilon(y) dy + \int_{B(0, \eta(\varepsilon))} \frac{(\varepsilon + |y|)^\alpha \phi_\varepsilon(y) dy}{(1 + |x - \varepsilon\nu - y|^2)^2} \right) \\ &\leq 3 \cdot 2^\alpha \delta, \end{aligned}$$

where we used

$$|x|^2 \leq 2(|x - \varepsilon\nu - y|^2 + |\varepsilon\nu - y|^2)$$

and

$$(a + b)^\mu \leq 2^{\mu-1}(a^\mu + b^\mu), \quad a, b \geq 0, \mu \geq 1.$$

Similarly we have

$$\begin{aligned} &h_\alpha^{-1}(x) m_\beta^{-1}(\xi) |\nabla_x f_\varepsilon^\sharp(x, \xi, t)| \\ &\leq 3 \cdot 2^{\alpha-1} \delta \int_{B(0, \eta(\varepsilon))} \frac{(1 + |x - \varepsilon\nu - y|^2)^{\frac{\alpha}{2}} + (\varepsilon + |y|)^\alpha}{(1 + |x - \varepsilon\nu - y|^2)^{\frac{\alpha}{2}}} |\nabla_y \phi_\varepsilon(y)| dy \\ &\leq 3 \cdot 2^{\alpha-1} \delta \left(\int_{B(0, \eta(\varepsilon))} |\nabla_y \phi_\varepsilon(y)| dy + \int_{B(0, \eta(\varepsilon))} (\varepsilon + |y|)^\alpha |\nabla_y \phi_\varepsilon(y)| dy \right) \\ &\leq 3 \cdot 2^{\alpha-1} \delta (\varepsilon^{-1} + \varepsilon^{\alpha-1}) \leq 2^\alpha \varepsilon^{-1} \delta. \end{aligned}$$

In the above, $(\nu_1, \nu_2, \nu_3) = (-1, 0, 0)$, as mentioned earlier.

(2) The estimate $\|f_\varepsilon(t)\|_{L^1} \leq \|f(t)\|_{L^1}$ is due to the elementary property of the convolution. $\square$

REMARK 2.1. Unlike the whole space problem, $(f_\varepsilon)^\sharp \neq (f^\sharp)_\varepsilon$ due to curved characteristics. However, both functions behave almost the same with respect to phase space and time decay; if $(1 + t)^\gamma f(t) \in \mathcal{S}(\delta, \alpha, \beta)$, then as seen in the previous lemma, $(f_\varepsilon)^\sharp(x, \xi, t) \leq 3 \cdot 2^\alpha \delta (1 + t)^{-\gamma} h_\alpha(x) m_\beta(\xi)$, while $(f^\sharp)_\varepsilon(x, \xi, t) \leq 2^\alpha \delta (1 + t)^{-\gamma} h_\alpha(x) m_\beta(\xi)$ is easily seen. Throughout this paper, we abuse the notation $(f_\varepsilon)^\sharp = f_\varepsilon^\sharp$.

2.3. *Basic estimates.* Here we present some basic estimates to be used in Section 3 and 4.

LEMMA 2.2 ([14]). Let $\alpha > 1$ and $w \neq 0$. Then we have the following estimate:

$$\sup_{x \in \mathbb{R}^3} \int_0^\infty h_\alpha(x + \tau w) d\tau \leq \frac{2\alpha}{|w|(\alpha - 1)}.$$

Next we study the pointwise estimate of the gain part of the collision operator.

LEMMA 2.3. Let f be a function in $\mathcal{S}(\delta, \alpha, \beta)$. Then we have

$$Q_+^\sharp(f, f)(x, \xi, t) \leq \mathcal{O}(1) \frac{\delta^2}{\kappa} \left[\frac{h_{0.5\alpha-\gamma}(x) m_{\beta-(\gamma+3)}(\xi)}{(1+t)^{\gamma+3}} \right],$$

where $\mathcal{O}(1)$ is a bounded function depending only on α and β .

Proof. Recall that

$$\begin{aligned} Q_+^\sharp(f, f)(x, \xi, t) &= \frac{1}{\kappa} \iint_{\mathbb{R}^3 \times \mathbf{S}^2} B(\xi^\sharp(t) - \xi_*, \omega) f(\mathbb{P}_x T(t)(x, \xi), \xi', t) f(\mathbb{P}_x T(t)(x, \xi), \xi'_*, t) d\omega d\xi_*. \end{aligned}$$

We first locate $T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi')$ by the following routine computation to use the pointwise ansatz of $f^\sharp$:

$$f^\sharp(x, \xi, t) \leq \delta h_\alpha(x) m_\beta(\xi).$$

We consider eight cases depending on the geometry of particle trajectories (**either** a part of a straight line **or** a part of a wedge with one joint).

Case 1 ($\xi_1 \geq -\frac{x_1}{t}$): trajectory $T(t)(x, \xi)$ is a straight line up to t .

(i) $\xi'_1 \leq \frac{x_1}{t} + \xi_1$, $\xi'_{*1} \leq \frac{x_1}{t} + \xi_1$ (straight, straight):

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') = (x + t(\xi - \xi'), \xi'),$$

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) = (x + t(\xi - \xi'_*), \xi'_*).$$

(ii) $\xi'_1 \leq \frac{x_1}{t} + \xi_1$, $\xi'_{*1} > \frac{x_1}{t} + \xi_1$ (straight, bent):

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') = (x + t(\xi - \xi'), \xi'),$$

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) = (R(x) + t(R(\xi) - R(\xi'_*)), R(\xi'_*)).$$

(iii) $\xi'_1 > \frac{x_1}{t} + \xi_1$, $\xi'_{*1} \leq \frac{x_1}{t} + \xi_1$ (bent, straight):

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') = (R(x) + t(R(\xi) - R(\xi')), R(\xi')),$$

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) = (x + t(\xi - \xi'_*), \xi'_*).$$

(iv) $\xi'_1 > \frac{x_1}{t} + \xi_1$, $\xi'_{*1} > \frac{x_1}{t} + \xi_1$ (bent, bent):

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') = (x + t\xi - \tau_0\xi' - (t - \tau_0)R(\xi'), R(\xi'))$$

$$= (R(x) + t(R(\xi) - R(\xi')), R(\xi')),$$

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) = (x + t\xi - \bar{\tau}_0\xi'_* - (t - \bar{\tau}_0)R(\xi'_*), R(\xi'_*))$$

$$= (R(x) + t(R(\xi) - R(\xi'_*)), R(\xi'_*)),$$

where $\tau_0 = \frac{x_1 + t\xi_1}{\xi'_1}$, $\bar{\tau}_0 = \frac{x_1 + t\xi_1}{\xi'_{*1}}$.

Case 2 ($\xi_1 < -\frac{x_1}{t}$): (trajectory $T(t)(x, \xi)$ bent at $t_0 = -\frac{x_1}{\xi_1} < t$).

(i) $\xi'_1 \leq -\frac{x_1}{t} - \xi_1$, $\xi'_{*1} \leq -\frac{x_1}{t} - \xi_1$ (straight, straight):

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') = (x + t_0\xi + (t - t_0)R(\xi) - t\xi', \xi')$$

$$= (R(x) - t(\xi - R(\xi)) + t(\xi - \xi'), \xi'),$$

$$T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) = (x + t_0\xi + (t - t_0)R(\xi) - t\xi'_*, \xi'_*)$$

$$= (R(x) - t(\xi - R(\xi)) + t(\xi - \xi'_*), \xi'_*).$$

(ii) $\xi'_1 \leq -\frac{x_1}{t} - \xi_1$, $\xi'_{*1} > -\frac{x_1}{t} - \xi_1$ (straight, bent):

$$\begin{aligned} T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') &= (R(x) - t(\xi - R(\xi)) + t(\xi - \xi'), \xi'), \\ T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) &= (x + t(\xi - R(\xi)) + t(R(\xi) - R(\xi'_*)), R(\xi'_*)). \end{aligned}$$

(iii) $\xi'_1 > -\frac{x_1}{t} - \xi_1$, $\xi'_{*1} \leq -\frac{x_1}{t} - \xi_1$ (bent, straight):

$$\begin{aligned} T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') &= (x + t(\xi - R(\xi)) + t(R(\xi) - R(\xi')), R(\xi')), \\ T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) &= (R(x) - t(\xi - R(\xi)) + t(\xi - \xi'_*), \xi'_*). \end{aligned}$$

(iv) $\xi'_1 > -\frac{x_1}{t} - \xi_1$, $\xi'_{*1} > -\frac{x_1}{t} - \xi_1$ (bent, bent):

$$\begin{aligned} T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi') &= (x + t_0 \xi + (t - t_0)R(\xi) - \tau_0 \xi' - (t - \tau_0)R(\xi'), R(\xi')) \\ &= (x + t(\xi - R(\xi)) + t(R(\xi) - R(\xi')), R(\xi')), \\ T^{-1}(t)(\mathbb{P}_x T(t)(x, \xi), \xi'_*) &= (x + t_0 \xi + (t - t_0)R(\xi) - \bar{\tau}_0 \xi'_* - (t - \bar{\tau}_0)R(\xi'_*), R(\xi'_*)) \\ &= (x + t(\xi - R(\xi)) + t(R(\xi) - R(\xi'_*)), R(\xi'_*)), \end{aligned}$$

where $\tau_0 = -\frac{x_1 + t\xi_1}{\xi'_1}$, $\bar{\tau}_0 = -\frac{x_1 + t\xi_1}{\xi'_{*1}}$.

The following inequality holds true due to the orthogonality of $\xi - \xi'$ and $\xi - \xi'_*$ (see Lemma 2.4 in [14] for the proof):

$$\begin{aligned} &h_\alpha(x + t(\xi - \xi'))h_\alpha(x + t(\xi - \xi'_*)) \\ &= (1 + |x + t(\xi - \xi')|^2)^{-\frac{\alpha}{2}}(1 + |x + t(\xi - \xi'_*)|^2)^{-\frac{\alpha}{2}} \\ &\leq (1 + |x|^2)^{-\frac{\alpha}{2}}((1 + |x + t(\xi - \xi')|^2)^{-\frac{\alpha}{2}} + (1 + |x + t(\xi - \xi'_*)|^2)^{-\frac{\alpha}{2}} \\ &\quad \times (1 + |x + t(\xi - \xi'_*)|^2)^{-\frac{\alpha}{2}}). \end{aligned}$$

We apply the above inequality to Cases 1 and 2 using the fact that $R(\xi) - R(\xi')$ and $R(\xi) - R(\xi'_*)$ are also orthogonal. Since the rest of the proof is identical to the proof of Lemma 2.4 in [14], we omit it here. For Case 2, we introduce a new variable $x^* := R(x) - t(\xi - R(\xi))$ and use $|x^*| \geq |x|$. $\square$

3. Collision potential and L^1 -scattering. In this section, we construct the collision potential and study its time-decay estimate.

3.1. *Construction of collision potential.* Let f and $\bar{f}$ be two mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$ of (1.1) corresponding to initial data f_0 and $\bar{f}_0$, respectively. Recall that f_ε and $\bar{f}_\varepsilon$ satisfy

$$\partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon),$$

where the error term $P(f, f_\varepsilon)$ is given by

$$P(f, f_\varepsilon) := Q_\varepsilon(f, f) - Q(f_\varepsilon, f_\varepsilon).$$

Below we construct the collision potential $\mathcal{D}(f_\varepsilon(t))$ measuring the future collisions after time t as follows. Let (x, ξ) be a given point in phase space $\Omega \times \mathbb{R}^3$. Then the total

collisions between test particles issued from (x, ξ) and their neighboring field particles can be measured by $\mathcal{D}_{f_\varepsilon}$:

$$\mathcal{D}_{f_\varepsilon}(x, \xi, t) = f_\varepsilon(T(t)(x, \xi), t) \underbrace{\int_{\mathbb{R}^3} \int_0^\infty |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_*}_{:= I(f_\varepsilon)(x, \xi, t)}.$$

We now define the collision potential by

$$D(f_\varepsilon(t)) = \iint_{\Omega \times \mathbb{R}^3} D_{f_\varepsilon}(x, \xi, t) dx d\xi.$$

LEMMA 3.1. Suppose the main assumption $\mathcal{A}$ holds and let f be in $\mathcal{S}(\delta, \alpha, \beta)$. Then $I(f_\varepsilon)$ satisfies a uniform estimate:

$$\sup_{x, \xi, t} I(f_\varepsilon)(x, \xi, t) = \mathcal{O}(1)\delta.$$

Proof. Let $f \in \mathcal{S}(\delta, \alpha, \beta)$. Then Lemma 2.1 yields

$$\begin{aligned} f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) \\ \leq 2^\alpha \delta h_\alpha(x^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*)) m_\beta(\xi^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*)) \\ = 2^\alpha \delta h_\alpha(x^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*)) m_\beta(\xi_*), \end{aligned}$$

where we used

$$\xi^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*) \in \{\xi_*, R(\xi_*)\} \quad \text{and} \quad m_\beta(\xi_*) = m_\beta(R(\xi_*)).$$

Following the particle trajectory $T(t)$, we have

$$x^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*) = \begin{cases} x^\sharp(t + \tau) - (t + \tau)\xi_*, & \text{either } \xi_{*1} \leq 0 \text{ or } (\xi_{*1} > 0 \text{ and } \tau \leq t_1 - t); \\ x^\sharp(t + \tau) - t_1\xi_* + (t + \tau - t_1)R(\xi_*), & \xi_{*1} > 0 \text{ and } 0 \leq t_1 - t < \tau. \end{cases}$$

where $t_0 = -x_1/\xi_1$, $t_1 = x_1^\sharp(t + \tau)/\xi_{*1}$,

$$x^\sharp(t + \tau) = \begin{cases} x + (t + \tau)\xi, & \text{either } \xi_1 \geq 0 \text{ or } (\xi_1 < 0 \text{ and } \tau \leq t_0 - t); \\ x + t_0\xi + (t + \tau - t_0)R(\xi), & \xi_1 < 0 \text{ and } \tau > t_0 - t. \end{cases}$$

We use Lemma 2.2 to obtain

$$\int_0^\infty \frac{d\tau}{(1 + |x^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*)|^2)^{\alpha/2}} \leq \frac{4\alpha}{(\alpha - 1)} \left(\frac{1}{|\xi - \xi_*|} + \frac{1}{|\xi - R(\xi_*)|} \right).$$

Hence,

$$\begin{aligned}
 I(f_\varepsilon)(x, \xi, t) &\leq C\delta \int_{\mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma m_\beta(\xi_*) \left(\int_0^\infty \frac{d\tau}{(1 + |x^\flat(0; t + \tau, x^\sharp(t + \tau), \xi_*)|^2)^{\frac{\alpha}{2}}} \right) d\xi_* \\
 &\leq C\delta \int_{\mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma \left(\frac{1}{|\xi - \xi_*|} + \frac{1}{|\xi - R(\xi_*)|} \right) m_\beta(\xi_*) d\xi_* \\
 &\leq C\delta \int_{\mathbb{R}^3} (|\xi - \xi_*|^\gamma + |R(\xi) - \xi_*|^\gamma) \left(\frac{1}{|\xi - \xi_*|} + \frac{1}{|\xi - R(\xi_*)|} \right) m_\beta(\xi_*) d\xi_* \\
 &\leq C\delta \int_{\mathbb{R}^3} |\xi - \xi_*|^{\gamma-1} m_\beta(\xi_*) + \frac{|\xi - R(\xi_*)|^\gamma}{|\xi - \xi_*|} m_\beta(\xi_*) d\xi_*.
 \end{aligned}$$

For the first integral, we change a variable $\xi - \xi_* = \xi^*$ to show

$$\begin{aligned}
 \int_{\mathbb{R}^3} |\xi - \xi_*|^{\gamma-1} m_\beta(\xi_*) d\xi_* &= \int_{|\xi^*| \leq 1} \frac{|\xi^*|^{\gamma-1}}{(1 + |\xi - \xi^*|^2)^{\frac{\beta}{2}}} + \int_{|\xi^*| > 1} \frac{|\xi^*|^{\gamma-1}}{(1 + |\xi - \xi^*|^2)^{\frac{\beta}{2}}} \\
 &\leq \int_{|\xi^*| \leq 1} |\xi^*|^{\gamma-1} d\xi^* + \int_{\mathbb{R}^3} \frac{d\xi_*}{(1 + |\xi - \xi^*|^2)^{\frac{\beta}{2}}} \\
 &\leq C \quad \text{if } \beta > 3, \quad -2 < \gamma \leq 1.
 \end{aligned} \tag{3.1}$$

For the second integral we note that

$$\begin{aligned}
 |\xi - \xi_*| &\leq |\xi - R(\xi_*)|, & \text{if } \xi_1 \cdot \xi_{*1} \geq 0, \\
 |\xi - \xi_*| &> |\xi - R(\xi_*)|, & \text{if } \xi_1 \cdot \xi_{*1} < 0.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 \int_{\mathbb{R}^3} \frac{|\xi - R(\xi_*)|^\gamma}{|\xi - \xi_*|} m_\beta(\xi_*) d\xi_* \\
 \leq \int_{\xi_1 \cdot \xi_{*1} \geq 0} |\xi - \xi_*|^{\gamma-1} m_\beta(\xi_*) d\xi_* + \int_{\xi_1 \cdot \xi_{*1} < 0} \frac{|\xi - R(\xi_*)|^\gamma}{|\xi - \xi_*|} m_\beta(\xi_*) d\xi_*.
 \end{aligned}$$

The integral over $\{\xi_* \in \mathbb{R}^3 | \xi_1 \cdot \xi_{*1} < 0\}$ is bounded by

$$\int_{\mathbb{R}^3} |\xi - R(\xi_*)|^{\gamma-1} m_\beta(\xi_*) d\xi_*, \quad \text{if } \gamma < 0$$

or

$$\int_{\mathbb{R}^3} |\xi - R(\xi_*)|^{\gamma-1} m_\beta(\xi_*) + |\xi - R(\xi_*)|^{-1} m_{\beta-\gamma}(\xi_*) d\xi_*, \quad \text{if } \gamma > 0,$$

where the inequality $|\xi - \xi_*| < |\xi - R(\xi_*)| + 2|\xi_*|$ is used for $\gamma > 0$. Both cases can be estimated by (3.1). $\square$

REMARK 3.1. Note that $\mathcal{D}(f_\varepsilon(t))$ is well-defined due to Lemma 3.1:

$$\mathcal{D}(f_\varepsilon(t)) \leq \left(\sup_{x, \xi, t} I(f_\varepsilon)(x, \xi, t) \right) \iint_{\Omega \times \mathbb{R}^3} f_\varepsilon(T(t)(x, \xi), t) dx d\xi = \mathcal{O}(1)\delta^2 < \infty.$$

3.2. *Time-decay estimates.* Here we present the time-decay estimates of the collision potential $\mathcal{D}(f_\varepsilon(t))$.

LEMMA 3.2. Let f be a solution of (1.1) in $\mathcal{S}(\delta, \alpha, \beta)$, and let f_ε be the corresponding mollifier. Then we have

- (1) $f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)$ is differentiable in t and τ unless (x, ξ, t, τ) satisfies the relations $x_1 + (t+\tau)\xi_1 = 0$ or $x_1^\sharp(t+\tau) - \tau\xi_{*1}$. In this case, we have

$$\begin{aligned} & \partial_t[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)] - \partial_\tau[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)] \\ &= [Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon)](T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t). \end{aligned}$$

- (2) Under the same condition as (1), $\partial_\tau[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)]$ is integrable in τ for $\xi \neq \xi_*, R(\xi_*)$.

Proof. (1) Under the given conditions, both of $x^\sharp(t+\tau)$ and $x^b(\tau; t+\tau, x^\sharp(t+\tau), \xi_*)$ are located in Ω . Thus differentiation of f_ε is permissible.

Case 1 ($\xi_1 < 0$ and $t \leq t_0 := -\frac{x_1}{\xi_1}$): In this case we consider two subcases.

Case 1.1 ($\tau < t_0 - t$ and $\tau < x_1^\sharp(t+\tau)/\xi_{*1}$): We have

$$T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*) = (x^\sharp(t+\tau) - \tau\xi, \xi_*) = (x + t\xi + \tau(\xi - \xi_*), \xi_*).$$

Hence

$$\begin{aligned} & \partial_t[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)] \\ &= \partial_t[f_\varepsilon(x^\sharp(t+\tau) - \tau\xi_*, \xi_*, t)] \\ &= \partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon \quad \text{using } x^\sharp(t+\tau) = x + (t+\tau)\xi \\ &= \partial_t f_\varepsilon + \xi_* \cdot \nabla_x f_\varepsilon + (\xi - \xi_*) \cdot \nabla_x f_\varepsilon \\ &= [Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon)](T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t) + \partial_\tau f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t), \end{aligned}$$

where all partial derivatives are evaluated at $(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)$.

Case 1.2 ($\tau < t_0 - t$ and $\tau > x_1^\sharp(t+\tau)/\xi_{*1}$): Since

$$T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*) = x^\sharp(t+\tau) - \tau_0\xi_* - (\tau - \tau_0)R(\xi_*), R(\xi_*) \quad \text{where } \tau_0 = x_1^\sharp(t+\tau)/\xi_{*1},$$

we have

$$\begin{aligned} & \partial_t[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)] \\ &= \partial_t[f_\varepsilon(x^\sharp(t+\tau) - \tau_0\xi_* - (\tau - \tau_0)R(\xi_*), R(\xi_*), t)] \\ &= \partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon \\ &= \partial_t f_\varepsilon + R(\xi_*) \cdot \nabla_x f_\varepsilon + (\xi - R(\xi_*)) \cdot \nabla_x f_\varepsilon \\ &= [Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon)](T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t) + \partial_\tau f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t). \end{aligned}$$

Case 2 ($\xi_1 < 0, t > t_0 = -\frac{x_1}{\xi_1}$):

Case 2.1 ($\tau < x_1^\sharp(t+\tau)/\xi_{*1}$): Since

$$T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*) = (x^\sharp(t+\tau) - \tau\xi_*, \xi_*) = (x + t_0\xi + (t+\tau-t_0)R(\xi) - \tau\xi_*, \xi_*),$$

by direct calculation we have

$$\begin{aligned} & \partial_t[f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)] \\ &= \partial_t[f_\varepsilon(x^\sharp(t+\tau) - \tau\xi_*, \xi_*, t)] \\ &= \partial_t f_\varepsilon + R(\xi) \cdot \nabla_x f_\varepsilon \end{aligned}$$

$$\begin{aligned}
&= \partial_t f_\varepsilon + \xi_* \cdot \nabla_x f_\varepsilon + (R(\xi) - \xi_*) \cdot \nabla_x f_\varepsilon \\
&= [Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon)](T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) + \partial_\tau f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t).
\end{aligned}$$

The other cases $\xi_1 \geq 0$ can be treated similarly.

(2) Under the same conditions as (1) we have

$$\begin{aligned}
&|\partial_\tau f_\varepsilon(t, T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*))| \\
&= |\partial_\tau(x^\sharp(t + \tau) - \tau\xi_*) \cdot \nabla_x f_\varepsilon(t, x^\sharp(t + \tau) - \tau\xi_*, \xi_*)| \\
&\leq \frac{\varepsilon^{-1}|\partial_\tau(x^\sharp(t + \tau) - \tau\xi_*)|}{(1 + |\mathbb{P}_x T^{-1}(t)(x^\sharp(t + \tau) - \tau\xi_*, \xi_*)|^2)^{\frac{\alpha}{2}}(1 + |\xi_*|^2)^{\frac{\beta}{2}}} \quad \text{by Lemma 2.1}
\end{aligned} \tag{3.2}$$

which is integrable in τ by Lemma 2.2. □

We now return to the time-evolution estimate of $\mathcal{D}(f_\varepsilon)$ as follows.

LEMMA 3.3. $\mathcal{D}(f_\varepsilon)$ is differentiable in t ,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{D}(f_\varepsilon(t)) \\
&= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+}}_{\substack{\times (x, \xi, t) f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* d\xi dx}} |\xi^\sharp(t) - \xi_*|^\gamma (Q^\sharp(f_\varepsilon, f_\varepsilon) + P^\sharp(f, f_\varepsilon)) \\
&+ \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+}}_{\substack{\times (T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* d\xi dx}} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) (Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon)) \\
&- \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3}}_{\substack{\times (T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* d\xi dx}} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) f_\varepsilon(x^\sharp(t; 0, x, \xi), \xi_*, t) d\xi_* d\xi dx \\
&= \mathcal{J}_\varepsilon^1(t) + \mathcal{J}_\varepsilon^2(t) - \Lambda_\varepsilon(t).
\end{aligned}$$

Proof. The proof is rather tedious and straightforward. We divide the domain of the integral into

$$(1) \quad \xi_1 < 0 \quad (\xi_1 \geq 0),$$

$$(2) \quad -x_1/\xi_1 < t \quad (-x_1/\xi_1 \geq t),$$

$$(3) \quad 0 < \tau \leq -x_1/\xi_1 - t \quad (-x_1/\xi_1 - t < \tau < \infty) \text{ in case } -x_1/\xi_1 > t \text{ holds in (2),}$$

$$(4) \quad \xi_{*1} > x_1^\sharp(t + \tau)/\tau \quad (\xi_{*1} \leq x_1^\sharp(t + \tau)/\tau).$$

Below we abuse the notation $(x, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ instead of (x_1, ξ_1) to simplify the presentation. We write

$$\begin{aligned}
 D(f_\varepsilon(t)) = & \left[\int_{-\infty}^0 \int_0^{-t\xi} \int_0^\infty \int_{x^\sharp(t+\tau)/\tau}^\infty + \int_{-\infty}^0 \int_0^{-t\xi} \int_0^\infty \int_{-\infty}^{x^\sharp(t+\tau)/\tau} \right. \\
 & + \int_{-\infty}^0 \int_{-t\xi}^\infty \int_0^{-x/\xi-t} \int_{x^\sharp(t+\tau)/\tau}^\infty + \int_{-\infty}^0 \int_{-t\xi}^\infty \int_0^{-x/\xi-t} \int_{-\infty}^{x^\sharp(t+\tau)/\tau} \\
 & + \int_{-\infty}^0 \int_{-t\xi}^\infty \int_{-x/\xi-t}^\infty \int_{x^\sharp(t+\tau)/\tau}^\infty + \int_{-\infty}^0 \int_{-t\xi}^\infty \int_{-x/\xi-t}^\infty \int_{-\infty}^{x^\sharp(t+\tau)/\tau} \\
 & \left. + \int_0^\infty \int_0^\infty \int_0^\infty \int_{x^\sharp(t+\tau)/\tau}^\infty + \int_0^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^{x^\sharp(t+\tau)/\tau} \right] \\
 & |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t) d\xi_* d\tau dx d\xi \\
 = & I_1 + \cdots + I_8.
 \end{aligned} \tag{3.3}$$

In each integral, the trajectories $x^\sharp(t+\tau)$, $T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*)$ are straight lines; therefore differentiation of each integral with the aid of Lemma 3.2 yields

$$\begin{aligned}
 \frac{d}{dt} \mathcal{D}(f_\varepsilon(t)) &= \underbrace{\iiint\limits_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma (Q^\sharp(f_\varepsilon, f_\varepsilon) + P^\sharp(f, f_\varepsilon))}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} \\
 &\quad \times (x, \xi, t) f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t) d\tau d\xi_* d\xi dx \\
 &+ \underbrace{\iiint\limits_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) (Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon))}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} \\
 &\quad \times (T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t) d\tau d\xi_* d\xi dx \\
 &+ \underbrace{\iiint\limits_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) \partial_\tau [f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)]}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} d\tau d\xi_* d\xi dx \\
 &+ I_1^b + \cdots + I_8^b,
 \end{aligned}$$

where I_i^b denotes the boundary term according to differentiation of end points of I_i . Let

$$g(x, \xi, \xi_*, t, \tau) := f_\varepsilon^\sharp(x, \xi, t) f_\varepsilon(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)$$

for the sake of simplicity. Then I_i^b is as follows:

$$\begin{aligned}
I_1^b &= \int_{-\infty}^0 -\xi \left[\int_0^\infty \int_{x^\sharp(t+\tau)/\tau}^\infty g(x, \xi, \xi_*, t, \tau) d\xi_* d\tau \right]_{x=-t\xi} d\xi \\
&\quad - \int_{-\infty}^0 \int_0^{-t\xi} \int_0^\infty \partial_t(x^\sharp(t+\tau)/\tau) g(x, \xi, \xi_*, t, \tau) \Big|_{\xi_*=x^\sharp(t+\tau)/\tau} d\tau dx d\xi \\
&:= I_{11}^b + I_{12}^b, \\
I_2^b &= \int_{-\infty}^0 -\xi \left[\int_0^\infty \int_{-\infty}^{x^\sharp(t+\tau)/\tau} g(x, \xi, \xi_*, t, \tau) d\xi_* d\tau \right]_{x=-t\xi} d\xi - I_{12}^b, \\
I_3^b &= \int_{-\infty}^0 \int_{-t\xi}^\infty \left[\int_{x^\sharp(t+\tau)/\tau}^\infty -g(x, \xi, \xi_*, t, \tau) d\xi_* \right]_{\tau=-x/\xi-t} d\tau dx d\xi \\
&\quad - \int_{-\infty}^0 \int_{-t\xi}^\infty \int_0^{-x/\xi-t} \partial_t(x^\sharp(t+\tau)/\tau) g(x, \xi, \xi_*, t, \tau) \Big|_{\xi_*=x^\sharp(t+\tau)/\tau} d\tau dx d\xi, \\
I_4^b &= - \int_{-\infty}^0 \int_{-t\xi}^\infty \left[\int_0^{x^\sharp(t+\tau)/\tau} g(x, \xi, \xi_*, t, \tau) d\xi_* \right]_{\tau=-x/\xi-t} d\tau dx d\xi - I_{32}^b, \\
I_5^b &= -I_{11}^b - I_{31}^b - \int_{-\infty}^0 \int_{-t\xi}^\infty \int_{-x/\xi-t}^\infty \partial_t(x^\sharp(t+\tau)/\tau) g(x, \xi, \xi_*, t, \tau) \Big|_{\xi_*=x^\sharp(t+\tau)/\tau} d\tau dx d\xi, \\
I_6^b &= -I_{21}^b - I_{41}^b - I_{53}^b, \\
I_7^b &= - \int_0^\infty \int_0^\infty \int_0^\infty \partial_t(x^\sharp(t+\tau)/\tau) g(x, \xi, \xi_*, t, \tau) \Big|_{\xi_*=x^\sharp(t+\tau)/\tau} d\tau dx d\xi, \\
I_8^b &= -I_7^b.
\end{aligned}$$

Here I_{ij}^b is the j -th term in I_i^b . From the above, the boundary terms $\sum_{i=1}^8 I_i^b = 0$.

To evaluate the integral $\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma \partial_\tau(g(x, \xi, \xi_*, t, \tau)) d\tau d\xi_* d\xi dx$, we decompose the integral domain as in (3.3) to write the integral as $\sum_{i=1}^8 I_i$. For instance, consider the first two integrals:

$$\begin{aligned}
I_1 + I_2 &= \int_{-\infty}^0 \int_0^{-t\xi} \int_0^\infty \int_{-\infty}^\infty |\xi^\sharp(t) - \xi_*|^\gamma \partial_\tau(g(x, \xi, \xi_*, t, \tau)) d\tau d\xi_* d\xi dx \\
&= \int_{-\infty}^0 \int_0^{-t\xi} \int_{-\infty}^\infty \int_0^\infty |\xi^\sharp(t) - \xi_*|^\gamma \partial_\tau(g(x, \xi, \xi_*, t, \tau)) d\tau d\xi_* d\xi dx.
\end{aligned}$$

The particle trajectory is bent at $\tau = \tau_0$ if $\tau_0 = x^\sharp(t+\tau)/\xi_* = -\frac{x+t\xi}{\xi+\xi_*}$ is nonnegative. Suppose $\tau_0 > 0$. Then we divide the integral $\int_0^\infty g d\tau = \int_0^{\tau_0} g d\tau + \int_{\tau_0}^\infty g d\tau$ to show

$$\int_0^\infty \partial_\tau(g(x, \xi, \xi_*, t, \tau)) d\tau = \lim_{\tau \rightarrow \infty} g(x, \xi, \xi_*, t, \tau) - g(x, \xi, \xi_*, t, 0),$$

due to the specular boundary condition and the fact that f_ε is continuous up to boundary. We treat $I_3 + I_4$, $I_5 + I_6$, $I_7 + I_8$ similarly. Then it follows that

$$\begin{aligned} & \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma \partial_\tau (g(x, \xi, \xi_*, t, \tau)) d\tau d\xi_* d\xi dx \\ &= \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} \lim_{\tau \rightarrow \infty} |\xi^\sharp(t) - \xi_*|^\gamma f(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) f^\sharp(x, \xi_*, t) d\xi_* d\xi dx \\ &\quad - \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma f(x^\sharp(t), \xi_*, t) f^\sharp(x, \xi_*, t) d\xi_* d\xi dx \\ &= - \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma f(x^\sharp(t), \xi_*, t) f^\sharp(x, \xi_*, t) d\xi_* d\xi dx. \end{aligned}$$

The first integral vanishes due to assumption $\mathcal{A}_2$. The proof is complete. $\square$

According to the next lemma, $\mathcal{J}_\varepsilon^i(t)$ is controlled by the production rate $\Lambda_\varepsilon(t)$ and the error term $P(f, f_\varepsilon)$.

LEMMA 3.4. $\mathcal{J}_\varepsilon^i$ ($i = 1, 2$) satisfies the following estimate:

$$\mathcal{J}_\varepsilon^i(t) \leq \mathcal{O}(1) \delta \left(\frac{1}{\kappa} \Lambda_\varepsilon(f(t)) + \iint_{\Omega \times \mathbb{R}^3} |P^\sharp(f, f_\varepsilon)| d\xi dx \right).$$

Proof. We start with $\mathcal{J}_\varepsilon^1(t)$. By the definition of $\mathcal{J}_\varepsilon^1(t)$ and Lemma 3.1, we have

$$\begin{aligned} \mathcal{J}_\varepsilon^1(t) &= \iint_{\Omega \times \mathbb{R}^3} (Q^\sharp(f_\varepsilon, f_\varepsilon) + P^\sharp(f, f_\varepsilon))(x, \xi, t) \\ &\quad \times \left(\iint_{\mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* \right) d\xi dx \\ &\leq \iint_{\Omega \times \mathbb{R}^3} (Q_+^\sharp(f_\varepsilon, f_\varepsilon) + |P^\sharp(f, f_\varepsilon)|)(x, \xi, t) \\ &\quad \times \left(\iint_{\mathbb{R}^3 \times \mathbb{R}_+} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* \right) d\xi dx \\ &\leq \mathcal{O}(1) \delta \iint_{\Omega \times \mathbb{R}^3} (Q_+^\sharp(f_\varepsilon, f_\varepsilon) + |P^\sharp(f, f_\varepsilon)|)(x, \xi, t) d\xi dx. \end{aligned}$$

We claim that

$$\iint_{\Omega \times \mathbb{R}^3} Q_+^\sharp(f_\varepsilon, f_\varepsilon)(x, \xi, t) d\xi dx \leq \mathcal{O}(1) \Lambda_\varepsilon(t).$$

In what follows, we use the notation $\xi = (\xi_1, \bar{\xi})$, $\bar{\xi} = (\xi_2, \xi_3)$. Recall that

$$\begin{aligned} \Lambda_\varepsilon(t) &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-x_1/t}^\infty |\xi - \xi_*|^\gamma f_\varepsilon(x + t\xi, \xi, t) f_\varepsilon(x + t\xi, \xi_*, t) d\xi_1 d\bar{\xi} d\xi_* dx \\ &\quad + \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-\infty}^{-x_1/t} |R(\xi) - \xi_*|^\gamma f_\varepsilon(x + t_0\xi + (t - t_0)R(\xi), R(\xi), t) \\ &\quad \times f_\varepsilon(x + t_0\xi + (t - t_0)R(\xi), \xi_*, t) d\xi_1 d\bar{\xi} d\xi_* dx. \end{aligned}$$

We now return to the left hand side of the above claim. Recall that

$$Q_+^\sharp(f_\varepsilon, f_\varepsilon) = \iint_{\mathbb{R}^3 \times \mathbf{S}^2} B(\xi^\sharp(t) - \xi_*, \omega) f_\varepsilon(t, x^\sharp(t), \xi^\sharp(t)') f_\varepsilon(t, x^\sharp(t), \xi_*') d\omega d\xi_*,$$

where

$$\xi^\sharp(t)' = \xi^\sharp(t) - [(\xi^\sharp(t) - \xi_*) \cdot \omega] \omega \quad \text{and} \quad \xi_*' = \xi_* + [(\xi^\sharp(t) - \xi_*) \cdot \omega] \omega.$$

Since

$$\begin{aligned} & \iint Q_+^\sharp(f_\varepsilon, f_\varepsilon) dx d\xi \\ &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\substack{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}} \int_{-x_1/t}^\infty \int_{\mathbf{S}^2} B(\xi - \xi_*, \omega) f_\varepsilon(t, x + t\xi, \xi', t) f_\varepsilon(x + t\xi, \xi_*', t) d\omega d\xi_1 d\bar{\xi} d\xi_* dx \\ &+ \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\substack{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}} \int_{-\infty}^{-x_1/t} \int_{\mathbf{S}^2} B(R(\xi) - \xi_*, \omega) f_\varepsilon(t, x + t_0\xi + (t - t_0)R(\xi), R(\xi)', t) \\ &\times f_\varepsilon(t, x + t_0\xi + (t - t_0)R(\xi), \xi_*', t) d\omega d\xi_1 d\bar{\xi} d\xi_* dx \\ &= A_1 + A_2 \end{aligned}$$

where $t_0 = -\frac{x_1}{\xi_1}$. Next we decompose $\mathbf{S}^2$:

$$\mathbf{S}^2 = \begin{cases} \mathbf{S}_1^+ \cup \mathbf{S}_1^-, & \text{if } \xi_1 \leq -\frac{x_1}{t}, \\ \mathbf{S}_2^+ \cup \mathbf{S}_2^-, & \text{if } \xi_1 > -\frac{x_1}{t}, \end{cases}$$

where

$$\mathbf{S}_1^\pm := \{\omega | [(\xi - \xi_*) \cdot \omega] \omega_1 \gtrless -x_1/t\} \quad \text{and} \quad \mathbf{S}_2^\pm := \{\omega | [(R(\xi) - \xi_*) \cdot \omega] \omega_1 \gtrless x_1/t\}.$$

We denote by $A_1^\pm$ the integral on $(x, \xi_*) \in \Omega \times \mathbb{R}^3$, $(\xi, \omega) \in \{\xi | \xi_1 > -\frac{x_1}{t}\} \times \mathbf{S}_1^\pm$ and by $A_2^\pm$ the integral on $(x, \xi_*) \in \Omega \times \mathbb{R}^3$, $(\xi, \omega) \in \{\xi | \xi_1 < -\frac{x_1}{t}\} \times \mathbf{S}_2^\pm$. Then we have $A_1 = A_1^+ + A_1^-$, $A_2 = A_2^+ + A_2^-$. Thus

$$\iint_{\Omega \times \mathbb{R}^3} Q_+^\sharp(f_\varepsilon, f_\varepsilon)(x, \xi, t) dx d\xi = A_1^+ + A_1^- + A_2^+ + A_2^-.$$

To estimate $A_j^\pm$, we make the change of variable $x \rightarrow x^*$ by

$$x^* = \mathbb{P}_x T^{-1}(t)(x^\sharp(t), \xi'). \quad (3.4)$$

Below, we present the estimation of the integral $A_1^-(t)$; other terms can be treated similarly.

Since the condition $\omega \in \{\omega | [(\xi - \xi_*) \cdot \omega] \omega_1 < -x_1/t\}$ implies

$$\xi'_1 = \xi_1 - [(\xi - \xi_*) \cdot \omega] \omega_1 > \xi_1 + \frac{x_1}{t} = \frac{x_1^\sharp(t)}{t},$$

we locate

$$x^* = x + t\xi - t_0\xi' - (t - t_0)R(\xi'), \quad t_0 = \frac{x_1 + t\xi_1}{\xi'_1}, \quad (3.5)$$

$$x_1^* + tR(\xi')_1 = -x_1 - t\xi_1. \quad (3.6)$$

Note that $x_1^* = -x_1 - t[(\xi - \xi_*) \cdot \omega]\omega_1$ is nonnegative. Now we write

$$\begin{aligned}
 A_1^-(t) &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-x_1/t}^{\infty} \int_{\mathbf{S}_1^-} B(\xi - \xi_*, \omega) f_{\varepsilon}(x + t\xi, \xi', t) f_{\varepsilon}(x + t\xi, \xi'_*, t) d\omega d\xi_1 d\bar{\xi} d\xi_* dx \\
 &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-x_1/t}^{\infty} \int_{\mathbf{S}_1^-} B(\xi - \xi_*, \omega) f_{\varepsilon}(x^* + t_0\xi' + (t - t_0)R(\xi'), \xi', t) \\
 &\quad \times f_{\varepsilon}(x^* + t_0\xi' + (t - t_0)R(\xi'), \xi'_*, t) d\omega d\xi_1 d\bar{\xi} d\xi_* dx^* \\
 &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{x_1^*/t}^{\infty} \int_{\mathbf{S}_1^-} B(\xi' - \xi'_*, \omega) f_{\varepsilon}(x^* + t_0\xi' + (t - t_0)R(\xi'), \xi', t) \\
 &\quad \times f_{\varepsilon}(x^* + t_0\xi' + (t - t_0)R(\xi'), \xi'_*, t) d\omega d\xi'_1 d\bar{\xi}' d\xi'_* dx^*.
 \end{aligned}$$

Here we used

$$\xi_1 > -\frac{x_1}{t} \Leftrightarrow R(\xi')_1 < -\frac{x^*}{t}$$

from (3.6) and the fact $|\xi - \xi_*| = |\xi' - \xi'_*|$. Also we replace

$$t_0 = \frac{x_1 + t\xi_1}{\xi'_1} = -\frac{x_1^* + tR(\xi')_1}{\xi'_1} = \frac{x_1^* + tR(\xi')_1}{R(\xi')_1},$$

so that $t - t_0 = -\frac{x_1^*}{R(\xi')_1}$. Thus if we let $t - t_0 = s_0$, we find

$$\begin{aligned}
 A_1^-(t) &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-\infty}^{-x_1^*/t} \int_{\mathbf{S}_1^-} B(\xi' - \xi'_*, \omega) f(x^* + s_0R(\xi') + (t - s_0)\xi', \xi', t) \\
 &\quad \times f(x^* + s_0R(\xi') + (t - s_0)\xi', \xi'_*, t) d\omega dR(\xi')_1 d\overline{R(\xi')} d\xi'_* dx^* \\
 &= \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2}}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^2} \int_{-\infty}^{-x_1^*/t} \int_{\mathbf{S}_1^-} B(\xi - \xi_*, \omega) f_{\varepsilon}(x^* + s_0\xi + (t - s_0)R(\xi), R(\xi), t) \\
 &\quad \times f_{\varepsilon}(x^* + s_0\xi + (t - s_0)R(\xi), \xi_*, t) d\omega d\xi_1 d\bar{\xi} d\xi_* dx \\
 &\leq 4\pi B_* \Lambda_{\varepsilon}(t), \quad s_0 = -\frac{x_1}{\xi_1},
 \end{aligned}$$

where we take off primes in $(R(\xi'), \xi'_*)$ and finally exchange $(x^*, R(\xi), \xi_*)$ with (x, ξ, ξ_*) .

Hence we have

$$\mathcal{J}_{\varepsilon}^1(t) \leq \mathcal{O}(1)\delta \left(\Lambda_{\varepsilon}(t) + \iint_{\Omega \times \mathbb{R}^3} |P^{\sharp}(f, f_{\varepsilon})|(x, \xi, t) d\xi dx \right). \quad (3.7)$$

FIG. 3. Trajectories for $\xi \in \mathcal{C}_i$ and $\xi_* \in \mathcal{C}_j^*$

Next we turn to estimate $\mathcal{J}_\varepsilon^2(t)$. We divide the integral according to the range of (ξ_1, ξ_{*1}) :

$$\begin{aligned} \mathcal{J}_\varepsilon^2(t) &= \sum_{i,j=1,2,3} \mathcal{J}_{\varepsilon,ij}^2(t), \\ \mathcal{J}_{\varepsilon,ij}^2(t) &:= \int_{\star} \int_{(\xi_1, \xi_{*1}) \in \mathcal{C}_i \times \mathcal{C}_j^*} |\xi^\sharp(t) - \xi_*|^\gamma f_\varepsilon(x^\sharp(t), \xi^\sharp(t), t) \\ &\quad \times (Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon))(T^{-1}(x^\sharp(t+\tau), \xi_*), t) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx, \end{aligned}$$

where

$$\int_{\star} = \underbrace{\iiint}_{\Omega \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2}$$

and $\mathcal{C}_i, \mathcal{C}_j^*$ are explained in the following. For given (t, τ, x) let us define sets $\mathcal{C}_i, \mathcal{C}_i^*$, $i = 1, 2, 3$, as

$$\begin{aligned} \mathcal{C}_1 &= \{\xi | \xi_1 > -\frac{x_1}{t+\tau}\}, & \mathcal{C}_1^* &= \{\xi_* | \xi_{*1} > \frac{x^\sharp(t+\tau)_1}{\tau}\}, \\ \mathcal{C}_2 &= \{\xi | -\frac{x_1}{t} < \xi_1 < -\frac{x_1}{t+\tau}\}, & \mathcal{C}_2^* &= \{\xi_* | \frac{x^\sharp(t+\tau)_1}{t+\tau} < \xi_{*1} < \frac{x^\sharp(t+\tau)_1}{\tau}\}, \\ \mathcal{C}_3 &= \{\xi | \xi_1 < -\frac{x_1}{t}\}, & \mathcal{C}_3^* &= \{\xi_* | \xi_{*1} < \frac{x^\sharp(t+\tau)_1}{t+\tau}\}. \end{aligned}$$

We observe that the set $\mathcal{C}_i$ is equivalent to the set $\mathcal{C}_{[i+2]}^*$, where the notation $[n]$ is defined by

$$[n] = \begin{cases} n, & n = 1, 2, 3, \\ n-3, & n \geq 4; \end{cases}$$

the roles of the sets $\mathcal{C}_i$ and $\mathcal{C}_i^*$ can be explained as follows (see Figure 3).

(i) When $(\xi, \xi_*) \in \mathcal{C}_1 \times \mathcal{C}_3^*$, neither $x^\sharp(s; 0, x, \xi)$ nor $x^\flat(s; t+\tau, x^\sharp(t+\tau), \xi_*)$ reflects on the wall on $0 < s < t+\tau$.

(ii) When $(\xi, \xi_*) \in \mathcal{C}_2 \times \mathcal{C}_1^*$, both $x^\sharp(s; 0, x, \xi)$ and $x^\flat(s; t+\tau, x^\sharp(t+\tau), \xi_*)$ reflect on the wall on $t < s < t+\tau$.

(iii) When $(\xi, \xi_*) \in \mathcal{C}_3 \times \mathcal{C}_2^*$, both $x^\sharp(s; 0, x, \xi)$ and $x^\flat(s; t+\tau, x^\sharp(t+\tau), \xi_*)$ reflect on the wall on $0 < s < t$.

Now we will show the following estimate by a suitable change of variables

$$\begin{aligned}
 & \int_{\star} \int_{\mathcal{C}_i \times \mathcal{C}_j^*} |\xi^{\sharp}(t) - \xi_*|^{\gamma} f_{\varepsilon}^{\sharp}(x, \xi, t) g(T^{-1}(\tau)(x^{\sharp}(t + \tau), \xi_*), t) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx \\
 & \leq \int_{\star} \int_{\mathcal{C}_{[j+1]} \times \mathcal{C}_{[i+2]}^*} (|\xi - \xi_*|^{\gamma} + |\xi - R(\xi_*)|^{\gamma}) f_{\varepsilon}(T^{-1}(\tau)(x^{\sharp}(t + \tau), \xi_*), t) \\
 & \quad \times g^{\sharp}(x, \xi, t) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx,
 \end{aligned} \tag{3.8}$$

where we let $g = Q(f_{\varepsilon}, f_{\varepsilon}) + P(f, f_{\varepsilon})$. The equality holds if $\gamma = 0$.

Assuming the above, we have

$$\begin{aligned}
 \mathcal{J}_{\varepsilon, ij}^2(t) & \leq \int_{\star} \int_{\mathcal{C}_{[j+1]} \times \mathcal{C}_{[i+2]}^*} (|\xi - \xi_*|^{\gamma} + |\xi - R(\xi_*)|^{\gamma}) (Q^{\sharp}(f_{\varepsilon}, f_{\varepsilon}) + P^{\sharp}(f, f_{\varepsilon}))(x, \xi, t) \\
 & \quad \times f_{\varepsilon}(T^{-1}(x^{\sharp}(t + \tau), \xi_*), t) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx.
 \end{aligned}$$

Thus we find

$$\begin{aligned}
 \mathcal{J}_{\varepsilon}^2(t) & = \sum_{i,j=1,2,3} \mathcal{J}_{\varepsilon, ij}^2(t) \\
 & \leq \int_{\star} \int \int (|\xi - \xi_*|^{\gamma} + |\xi - R(\xi_*)|^{\gamma}) (Q^{\sharp}(f_{\varepsilon}, f_{\varepsilon}) + P^{\sharp}(f, f_{\varepsilon}))(x, \xi, t) \\
 & \quad \times f_{\varepsilon}(T^{-1}(x^{\sharp}(t + \tau), \xi_*), t) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx \\
 & \leq O(1) \delta \int \int_{\Omega \times \mathbb{R}^3} (Q^{\sharp}(f_{\varepsilon}, f_{\varepsilon}) + P^{\sharp}(f, f_{\varepsilon}))(x, \xi, t) d\xi dx \\
 & \leq O(1) \delta \left(\Lambda_{\varepsilon}(t) + \int \int_{\Omega \times \mathbb{R}^3} |P^{\sharp}(f, f_{\varepsilon})| d\xi dx \right) \quad \text{by (3.7),}
 \end{aligned}$$

and the proof is complete. $\square$

Proof of (3.8). Note that

$$\begin{aligned}
 \xi \in \mathcal{C}_1 & \Rightarrow x^{\sharp}(t + \tau) = x + (t + \tau)\xi, \\
 \xi \in \mathcal{C}_2 & \Rightarrow x^{\sharp}(t + \tau) = x + t_0\xi + (t + \tau - t_0)R(\xi), \\
 \xi \in \mathcal{C}_3 & \Rightarrow x^{\sharp}(t + \tau) = x + t_0\xi + (t + \tau - t_0)R(\xi); \text{ and} \\
 \xi_* \in \mathcal{C}_1^* & \Rightarrow T^{-1}(t + \tau)(x^{\sharp}(t + \tau), \xi_*) = (x^{\sharp}(t + \tau) - \tau_0\xi_* - (t + \tau - \tau_0)R(\xi_*), R(\xi_*)), \\
 \xi_* \in \mathcal{C}_2^* & \Rightarrow T^{-1}(t + \tau)(x^{\sharp}(t + \tau), \xi_*) = (x^{\sharp}(t + \tau) - (t + \tau - \tilde{t}_0)\xi_* - \tilde{t}_0R(\xi_*), R(\xi_*)), \\
 \xi_* \in \mathcal{C}_3^* & \Rightarrow T^{-1}(t + \tau)(x^{\sharp}(t + \tau), \xi_*) = (x^{\sharp}(t + \tau) - (\tau + t)\xi_*, \xi_*),
 \end{aligned} \tag{3.9}$$

where $t_0 = -\frac{x_1}{\xi_1}$, $\tau_0 = \frac{x^{\sharp}(t+\tau)}{\xi_{*1}}$, $\tilde{t}_0 = t + \tau - \frac{x^{\sharp}(t+\tau)}{\xi_{*1}}$.

We perform a change of variable $x \rightarrow x^*$:

$$x^* := \mathbb{P}_x T^{-1}(t + \tau)(x^{\sharp}(t + \tau), \xi_*).$$

Below we present the proof of the case $(\xi, \xi_*) \in \mathcal{C}_1 \times \mathcal{C}_1^*$; the other cases can be treated similarly.

- $(\xi, \xi_*) \in \mathcal{C}_1 \times \mathcal{C}_1^*$. It follows from (3.9) that

$$\begin{aligned}
 x^* &= T^{-1}(t+\tau)(x^\sharp(t+\tau), \xi_*) \\
 &= x + (t+\tau)\xi - \tau_0\xi_* - (t+\tau-\tau_0)R(\xi_*), \text{ for } \tau_0 = \frac{x_1 + (t+\tau)\xi_1}{\xi_{*1}}, \\
 x_1^* &= (t+\tau)\xi_{*1} - (x_1 + (t+\tau)\xi_1) \\
 &= t\xi_{*1} - (x_1 + (t+\tau)\xi_1 - \tau\xi_{*1}).
 \end{aligned} \tag{3.10}$$

Since $(\xi, \xi_*) \in \mathcal{C}_1 \times \mathcal{C}_1^*$ implies $x_1 + (t+\tau)\xi_1 > 0$ and $x_1 + (t+\tau)\xi_1 - \tau\xi_{*1} < 0$ from (3.10), we have

$$\frac{x_1^*}{t+\tau} < \xi_{*1} < \frac{x_1^*}{t}. \tag{3.11}$$

On the other hand if we write (3.10) as

$$x_1^* + (t+\tau)\xi_1 - (t+\tau)\xi_{*1} = -x_1 < 0,$$

we find

$$\xi_1 < \frac{-x_1^* + (t+\tau)\xi_{*1}}{t+\tau}. \tag{3.12}$$

Note that $\tau_0 = \frac{-x_1^* + (t+\tau)\xi_{*1}}{\xi_{*1}}$ if written in terms of (x^*, ξ_*) .

We next make the change of variables $(\xi, \xi_*) \rightarrow (\eta, \eta_*)$ by

$$(\eta, \eta_*) = (R(\xi_*), \xi), \quad i.e. \quad (\xi, \xi_*) = (\eta_*, R(\eta)).$$

Then (3.11) and (3.12) become

$$\begin{aligned}
 -\frac{x_1^*}{t} &< \eta_1 < -\frac{x_1^*}{t+\tau}, \\
 \eta_1^* &< \frac{-x_1^* + (t+\tau)R(\eta)_1}{t+\tau} = \frac{x_1^\sharp(t+\tau; 0, x^*, \eta)}{t+\tau} := \frac{x^{*\sharp}(t+\tau)_1}{t+\tau}
 \end{aligned}$$

using

$$-x_1^* + (t+\tau)R(\eta)_1 = x_1^* + t_0\eta_1 + (t+\tau-t_0)R(\eta)_1$$

for $t_0 = -\frac{x_1^*}{\eta_1}$, which implies $(\eta, \eta_*) \in \mathcal{C}_2 \times \mathcal{C}_3^*$. With new variables (x^*, η, η_*) , we have

$$\begin{aligned}
 &\int_{\star} \int_{\mathcal{C}_1 \times \mathcal{C}_1^*} |\xi - \xi_*|^\gamma f_\varepsilon^\sharp(x, \xi, t) g(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*, t)) d\xi_1 d\xi_{*1} d\bar{\xi} d\bar{\xi}_* d\tau dx \\
 &= \int_{\star} \int_{\mathcal{C}_2 \times \mathcal{C}_3^*} |\eta - R(\eta_*)|^\gamma f_\varepsilon(T^{-1}(\tau)(x^{*\sharp}(t+\tau), \eta_*, t)) g^\sharp(x^*, \eta, t) d\eta_1 d\eta_{*1} d\bar{\eta} d\bar{\eta}_* d\tau dx^*.
 \end{aligned}$$

□

We close this section with the following proposition.

PROPOSITION 3.1. Suppose the main assumptions $\mathcal{A}$ hold, and let f be a mild solution in $\mathcal{S}(\delta, \alpha, \beta)$ corresponding to initial datum f_0 . Then there exists a collision potential $\mathcal{D}(f(t))$ satisfying

$$\mathcal{D}(f(t)) + C \int_0^t \Lambda(f(s)) ds \leq \mathcal{D}(f_0), \quad t \geq 0,$$

where C is a generic constant independent of t .

Proof. Summing up the estimates in Lemma 3.3 and Lemma 3.4, we have

$$\mathcal{D}(f_\varepsilon(t)) + C_1 \int_0^t \Lambda_\varepsilon(f_\varepsilon(s)) ds \leq \mathcal{D}(f_\varepsilon(0)) + \mathcal{O}(1)\delta \int_0^t \iint |P^\sharp(f, f_\varepsilon)(x, \xi, t)| d\xi dx ds$$

for sufficiently small δ and a positive constant C_1 . We will prove that the error term in the right hand side of the inequality goes to zero as $\varepsilon \rightarrow 0$ in Lemma 4.1. Then we pass $\varepsilon \rightarrow 0$ to get the desired result. $\square$

4. Uniform L^1 -stability. In this section, we study the L^1 -stability of mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$. We first apply the nonlinear functional approach in [14] for approximate Boltzmann equations (2.5). Then as $\varepsilon \rightarrow 0$, we recover the L^1 -stability estimate for continuous mild solutions.

4.1. *Nonlinear functionals.* Let f and $\bar{f}$ be two mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$ of (1.1) corresponding to initial data f_0 and $\bar{f}_0$, respectively. Recall that f and f_ε satisfy

$$\partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon),$$

where the error term $P(f, f_\varepsilon)$ is given by

$$P(f, f_\varepsilon) := Q_\varepsilon(f, f) - Q(f_\varepsilon, f_\varepsilon).$$

We next define nonlinear functionals for the mollifications f_ε and $\bar{f}_\varepsilon$:

$$\begin{aligned} \mathcal{D}_d^\varepsilon(t) &:= \iint_{\Omega \times \mathbb{R}^3} |f_\varepsilon - \bar{f}_\varepsilon|(T(t)(x, \xi), t) \\ &\quad \times \left[\iint_{\mathbb{R}_+ \times \mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma (f_\varepsilon + \bar{f}_\varepsilon)(T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_* \right] d\xi dx, \\ \mathcal{H}^\varepsilon(t) &:= \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} + K \mathcal{D}_d^\varepsilon(t), \\ \Lambda_d^\varepsilon(t) &:= \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma |f_\varepsilon - \bar{f}_\varepsilon|(T(t)(x, \xi), t) \\ &\quad \times (f_\varepsilon + \bar{f}_\varepsilon)(\mathbb{P}_x T(t)(x, \xi), \xi_*, t) d\xi_* d\xi dx, \end{aligned}$$

where K is a positive constant to be determined later. Note that the functional $\mathcal{H}^\varepsilon$ can be rewritten as

$$\begin{aligned} \mathcal{H}^\varepsilon(t) &= \iint_{\Omega \times \mathbb{R}^3} |f_\varepsilon - \bar{f}_\varepsilon|(T(t)(x, \xi), t) \\ &\quad \times \left[1 + K \iint_{\mathbb{R}_+ \times \mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma (f_\varepsilon^\sharp + \bar{f}_\varepsilon^\sharp)(T^{-1}(t + \tau)(x^\sharp(t + \tau), \xi_*), t) d\xi_* d\tau \right] d\xi dx. \end{aligned}$$

Lemma 3.1 yields the following equivalence relation:

$$\|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} \leq \mathcal{H}^\varepsilon(t) \leq C \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1},$$

where C is a positive constant independent of t and ε . On the other hand, note that the difference $|f_\varepsilon - \bar{f}_\varepsilon|(T(t)(x, \xi), t)$ satisfies a differential inequality:

$$\partial_t \left(|f_\varepsilon - \bar{f}_\varepsilon|(T(t)(x, \xi), t) \right) \leq \left(\mathcal{R}_2(f_\varepsilon, \bar{f}_\varepsilon) + |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)| \right) (T(t)(x, \xi), t), \quad (4.1)$$

where

$$\begin{aligned} \mathcal{R}_2(f_\varepsilon, \bar{f}_\varepsilon)(T(t)(x, \xi), t) \\ := \frac{1}{2\kappa} \iint_{\Omega \times \mathbf{S}_+^2} |\xi^\sharp(t) - \xi_*|^\gamma \Big(|f'_\varepsilon - \bar{f}'_\varepsilon| (f'_{\varepsilon*} + \bar{f}'_{\varepsilon*}) + |f'_{\varepsilon*} - \bar{f}'_{\varepsilon*}| (f'_\varepsilon + \bar{f}'_\varepsilon) \\ + |f_\varepsilon - \bar{f}_\varepsilon| (f_{\varepsilon*} + \bar{f}_{\varepsilon*}) + |f_{\varepsilon*} - \bar{f}_{\varepsilon*}| (f_\varepsilon + \bar{f}_\varepsilon) \Big). \end{aligned}$$

Here we used abbreviated notation:

$$\begin{aligned} f_\varepsilon &= f_\varepsilon(T(t)(x, \xi), t), & f'_\varepsilon &= f_\varepsilon(x^\sharp(t), \xi^\sharp, t), \\ f_{\varepsilon*} &= f_\varepsilon(x^\sharp(t), \xi_*, t), & f'_{\varepsilon*} &= f_\varepsilon(x^\sharp(t), \xi'_*, t). \end{aligned}$$

For later use, we introduce two auxiliary functionals:

$$\begin{aligned} \mathcal{E}(f_\varepsilon)(t) &:= \sup_{x, \xi} \iint_{\mathbb{R}^3 \times \mathbb{R}_+} |Q(f_\varepsilon, f_\varepsilon)| (T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_*, \\ \mathcal{G}(f, f_\varepsilon)(t) &:= \sup_{x, \xi} \iint_{\mathbb{R}^3 \times \mathbb{R}_+} |P(f, f_\varepsilon)| (T^{-1}(\tau)(x^\sharp(t + \tau), \xi_*), t) d\tau d\xi_*. \end{aligned}$$

The following lemma deals with the estimates of the nonlinear collision term and the mollification error term.

LEMMA 4.1. (1) Let $f \in \mathcal{S}(\delta, \alpha, \beta)$. Then $\mathcal{E}(f_\varepsilon)$ and $\mathcal{G}(f, f_\varepsilon)$ are integrable in t and satisfy

$$\int_0^\infty \mathcal{E}(f_\varepsilon)(t) dt + \int_0^\infty \mathcal{G}(f, f_\varepsilon)(t) dt \leq \mathcal{O}(1) \delta^2,$$

where the bounded function $\mathcal{O}(1)$ does not depend on ε and t .

(2) The error term $P(f, f_\varepsilon)$ goes to zero as $\varepsilon \rightarrow 0$, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \iiint_{\mathbb{R}_+ \times \Omega \times \mathbb{R}^3} |P(f, f_\varepsilon)| (T(t)(x, \xi), t) d\xi dx dt = 0.$$

Proof. Suppose $f \in \mathcal{S}(\delta, \alpha, \beta)$.

(1) We set

$$(X, \Xi) := T^{-1}(t + \tau)(x^\sharp(t + \tau), \xi_*).$$

As in the proof of Lemma 2.3, we have

$$\Xi \in \{\xi_*, R(\xi_*)\}$$

and

$$\begin{aligned} X \in \{x + (t + \tau)(\xi - \xi_*), x + (t + \tau)(R(\xi) - R(\xi_*)), \\ x + (t + \tau)(R(\xi) - \xi_*), x + (t + \tau)(\xi - R(\xi_*))\}. \end{aligned}$$

On the other hand, it follows from Lemma 2.4 in [14] that

$$\begin{aligned} & \iint_{\mathbb{R}^3 \times \mathbb{R}_+} |Q(f_\varepsilon, f_\varepsilon)|(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*, t)) d\tau d\xi_* \\ & \leq \mathcal{O}(1)\delta^2(1+t)^{\gamma+3} \sup_{(x, \xi)} \int_0^\infty \int_{\mathbb{R}^3} h_{\frac{1}{2}\alpha}(X)m_{\beta-3}(\xi_*) d\xi_* d\tau \\ & \leq \mathcal{O}(1)\delta^2(1+t)^{\gamma+3}. \end{aligned} \quad (4.2)$$

We now take supremum over x, ξ and then integrate in t to see

$$\int_0^\infty \mathcal{E}(f_\varepsilon)(t) dt \leq \mathcal{O}(1)\delta^2.$$

Since

$$|P(f, f_\varepsilon)(T^{-1}(\tau)(x^\sharp(t+\tau), \xi_*), t)| \leq (|Q_\varepsilon(f, f)| + |Q(f_\varepsilon, f_\varepsilon)|)(T(t)(X, \Xi), t)$$

and by Lemma 2.1 and Lemma 2.3, we can treat $\mathcal{G}(f, f_\varepsilon)(t)$ similarly to (4.2).

(2) For notational simplicity, we introduce

$$\begin{aligned} q_+^\sharp(f, f) &:= f(x^\sharp(t), \xi^{\sharp'}(t), t) f(x^\sharp(t), \xi_*, t), \\ q_-^\sharp(f, f) &:= f(T(t)(x, \xi), t) f_*(x^\sharp(t), \xi_*, t), \\ q_{\varepsilon\pm}(f, f) &:= q_\pm(f, f) *_{\varepsilon, \nu} \phi_\varepsilon. \end{aligned}$$

We **claim**: for given $t \in \mathbb{R}_+$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |P_\pm(f, f_\varepsilon)|(T(s)(x, \xi), s) dx d\xi ds = 0.$$

Proof of claim : Since $P_\pm^\sharp(f, f_\varepsilon) = Q_{\varepsilon\pm}^\sharp(f, f) - Q_\pm^\sharp(f_\varepsilon, f_\varepsilon)$, we have

$$\begin{aligned} & \int_0^t \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |P_\pm^\sharp(f, f_\varepsilon)| dx d\xi ds \\ & \leq \int_0^t \iiint_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi^\sharp(s) - \xi_*|^\gamma |q_{\varepsilon\pm}(f, f) - q_\pm(f_\varepsilon, f_\varepsilon)|^\sharp d\omega d\xi_* d\xi dx ds \quad (4.3) \\ & := \int_0^t p_\varepsilon(s) ds. \end{aligned}$$

It suffices to prove that the right hand side of (4.3) goes to zero as $\varepsilon \rightarrow 0$. We have

$$\begin{aligned} p_\varepsilon(t) & \leq \underbrace{\iiint_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega}}_{\mathcal{I}_1^\varepsilon(t)} |\xi^\sharp(t) - \xi_*|^\gamma q_{\varepsilon\pm}^\sharp(f, f) d\omega d\xi_* d\xi dx dt + \int_\Omega \int_{\mathbb{R}^3} |Q_{\varepsilon\pm}^\sharp(f_\varepsilon, f_\varepsilon)| d\xi dx dt \\ & := \mathcal{I}_1^\varepsilon(t) + \mathcal{I}_2^\varepsilon(t). \end{aligned}$$

It follows from Lemma 2.3 that

$$\mathcal{I}_2^\varepsilon(t) \leq \mathcal{O}(1)(1+t)^{-(3+\gamma)}. \quad (4.4)$$

For the term $\mathcal{I}_2^\varepsilon$, we have

$$\begin{aligned}
& \underbrace{\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2}}_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2} \int_{\mathbb{R}^3} |\xi^\sharp(t) - \xi_*|^\gamma q_\pm^\sharp(f, f)(x - \varepsilon\nu - y, \xi, \xi_*, t) \phi_\varepsilon(y) dy d\omega d\xi_* d\xi dx \\
& \leq \varepsilon^{-3} \underbrace{\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2}}_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2} \int_{|y| \leq \varepsilon} |\xi^\sharp(t) - \xi_*|^\gamma q_\pm^\sharp(f, f)(x - \varepsilon\nu - y, \xi, \xi_*, t) dy d\omega d\xi_* d\xi dx \\
& = \varepsilon^{-3} \iint \int_{|y| \leq \varepsilon} Q_\pm^\sharp(f, f)(x - \varepsilon\nu - y, \xi, t) dy d\xi dx \\
& \leq \mathcal{O}(1) \varepsilon^{-3} \iiint_{|y| \leq \varepsilon} (1+t)^{-(3+\gamma)} h_{0.5\alpha-\gamma}(x - \varepsilon\nu - y) m_{\beta-\gamma-3}(\xi) dy d\xi dx \\
& \leq \mathcal{O}(1) \varepsilon^{-3} \iiint_{|y| \leq \varepsilon} (1+t)^{-(3+\gamma)} h_{0.5\alpha-\gamma}(x - \varepsilon\nu - y) dy dx \\
& \leq \mathcal{O}(1) \varepsilon^{-3} \int_{|y| \leq \varepsilon} \int_{\mathbb{R}^3} (1+t)^{-(3+\gamma)} (1+|x - \varepsilon\nu - y|^2)^{-\frac{\alpha-2\gamma}{4}} dy \\
& \leq \mathcal{O}(1) (1+t)^{-(3+\gamma)}.
\end{aligned} \tag{4.5}$$

Hence $p_\varepsilon(t)$ is dominated by the integrable function $\mathcal{O}(1)(1+t)^{-(3+\gamma)}$.

We note that

$$\begin{aligned}
p_\varepsilon(t) & \leq \underbrace{\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2}}_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2} |\xi^\sharp(t) - \xi_*|^\gamma |q_{\varepsilon\pm}(f, f) - q(f, f)|^\sharp d\omega d\xi_* d\xi dx \\
& \quad + \underbrace{\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2}}_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbf{S}^2} |\xi^\sharp(t) - \xi_*|^\gamma |q_\pm(f_\varepsilon, f_\varepsilon) - q(f, f)|^\sharp d\omega d\xi_* d\xi dx.
\end{aligned}$$

Then the first integral converges to zero by L^1 convergence of convolution. The second integral also vanishes by the convolution property and a similar estimate to (4.5), replacing f_ε with the bounding function $h_{\alpha} m_\beta$. Now the Lebesgue dominated convergence theorem concludes the proof. $\square$

LEMMA 4.2. Suppose the main assumptions $\mathcal{A}$ in Section 1 hold. Let f and $\bar{f}$ be two mild solutions in $\mathcal{S}(\delta, \alpha, \beta)$ corresponding to initial data f_0 and $\bar{f}_0$, respectively, and let $f_\varepsilon, \bar{f}_\varepsilon$ be their mollifications. Then nonlinear functionals satisfy

$$\begin{aligned}
(1) \quad & \frac{d}{dt} \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} \leq \mathcal{O}(1) \Lambda_d^\varepsilon(t) + \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp d\xi dx, \\
(2) \quad & \frac{d\mathcal{D}_d^\varepsilon(t)}{dt} \leq -\bar{C}_1 \Lambda_\varepsilon^\varepsilon(t) \\
& \quad + \left(\mathcal{E}(f_\varepsilon)(t) + \mathcal{E}(\bar{f}_\varepsilon)(t) + \mathcal{G}(f, f_\varepsilon)(t) + \mathcal{G}(\bar{f}, \bar{f}_\varepsilon)(t) \right) \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} \\
& \quad + \mathcal{O}(1) \delta \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp d\xi dx,
\end{aligned}$$

where $\bar{C}_1$ is a positive constant independent of t and $\mathcal{O}(1)$ is a bounded function depending only on α, β and γ .

Proof. (1) We integrate (4.1) to get

$$\frac{d}{dt} \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} \leq \iint_{\Omega \times \mathbb{R}^3} \mathcal{R}_2^\#(f_\varepsilon, \bar{f}_\varepsilon) d\xi dx + \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\# d\xi dx.$$

The first term of the above inequality can be estimated as follows:

$$\begin{aligned} & \iint_{\Omega \times \mathbb{R}^3} \mathcal{R}_2^\#(f_\varepsilon, \bar{f}_\varepsilon) d\xi dx \\ & \leq \frac{2}{k} \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} |\xi^\#(t) - \xi_*|^\gamma |f_\varepsilon - \bar{f}_\varepsilon|^\#(x, \xi, t) (f_\varepsilon + \bar{f}_\varepsilon)(x^\#(t), \xi_*, t) d\omega d\xi_* d\xi dx}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \\ & = \frac{\mathcal{O}(1)}{\kappa} \Lambda_d^\varepsilon(t). \end{aligned} \tag{4.6}$$

(2) Recall that

$$\mathcal{D}_d^\varepsilon(t) = \iint_{\Omega \times \mathbb{R}^3} |f_\varepsilon - \bar{f}_\varepsilon|^\#(x, \xi, t) (I(f_\varepsilon) + I(\bar{f}_\varepsilon))(x, \xi, t) d\xi dx.$$

Following the same procedure as for Lemma 3.3, we have

$$\begin{aligned} & \frac{d\mathcal{D}_d^\varepsilon(t)}{dt} \\ & \leq - \iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} |f_\varepsilon - \bar{f}_\varepsilon|^\#(x, \xi, t) (f_\varepsilon + \bar{f}_\varepsilon)(x^\#(t), \xi_*, t) d\xi_* d\xi dx \\ & + \iint_{\Omega \times \mathbb{R}^3} \left(\mathcal{R}_2^\#(f_\varepsilon, \bar{f}_\varepsilon) + |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\# \right) (x, \xi, t) (I(f_\varepsilon) + I(\bar{f}_\varepsilon))(x, \xi, t) d\xi dx \\ & + \underbrace{\iiint_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} |f_\varepsilon - \bar{f}_\varepsilon|^\#(x, \xi, t) \left(Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon) \right.}_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \\ & \quad \left. + Q(\bar{f}_\varepsilon, \bar{f}_\varepsilon) + P(\bar{f}, \bar{f}_\varepsilon) \right) (T^{-1}(x^\#(t + \tau), \xi_*, t) d\tau d\xi_* d\xi dx. \end{aligned}$$

We use Lemma 3.1 and (4.6) to find

$$\begin{aligned} \frac{d\mathcal{D}_d^\varepsilon(t)}{dt} & \leq \left(\mathcal{O}(1) \frac{\delta}{\kappa} - 1 \right) \Lambda_d^\varepsilon(t) + \mathcal{O}(1) \delta \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\#(x, \xi, t) d\xi dx \\ & + \left(\mathcal{E}(f_\varepsilon)(t) + \mathcal{E}(\bar{f}_\varepsilon)(t) \right. \\ & \quad \left. + \mathcal{G}(f, f_\varepsilon)(t) + \mathcal{G}(\bar{f}, \bar{f}_\varepsilon)(t) \right) \|f_\varepsilon - \bar{f}_\varepsilon\|_{L^1}. \end{aligned}$$

We now choose δ sufficiently small so that

$$1 - \mathcal{O}(1) \frac{\delta}{\kappa} \geq C_1, \quad \text{for some constant } C_1 > 0.$$

□

The proof of Theorem 1.1. By the definition of $\mathcal{H}^\varepsilon(t)$ and Lemma 4.2, we have

$$\begin{aligned} \frac{d\mathcal{H}^\varepsilon(t)}{dt} &= \frac{d}{dt} \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} + K \frac{d\mathcal{D}_d^\varepsilon(t)}{dt} \\ &\leq (\mathcal{O}(1) - KC_1) \Lambda_d^\varepsilon(t) + \mathcal{O}(1) \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp(x, \xi, t) d\xi dx \\ &\quad + K \left(\mathcal{E}(f_\varepsilon)(t) + \mathcal{E}(\bar{f}_\varepsilon)(t) + \mathcal{G}(f, f_\varepsilon)(t) + \mathcal{G}(\bar{f}, \bar{f}_\varepsilon)(t) \right) \|f(t) - \bar{f}(t)\|_{L^1}. \end{aligned}$$

We now choose K sufficiently large such that

$$\mathcal{O}(1) - KC_1 \leq -C_2 \quad \text{for some constant } C_2 > 0.$$

For such K and C_2 , we have

$$\begin{aligned} \frac{d\mathcal{H}^\varepsilon(t)}{dt} + C_2 \Lambda_d^\varepsilon(t) &\leq K_2 \underbrace{\left(\mathcal{E}(f_\varepsilon)(t) + \mathcal{E}(\bar{f}_\varepsilon)(t) + \mathcal{G}(f, f_\varepsilon)(t) + \mathcal{G}(\bar{f}, \bar{f}_\varepsilon)(t) \right)}_{\text{integrable in } t} \mathcal{H}^\varepsilon(t) \\ &\quad + \mathcal{O}(1) \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp(x, \xi, t) d\xi dx. \end{aligned}$$

The Gronwall's inequality and time-integrability of $\mathcal{E}(f)$, $\mathcal{G}(f, f_\varepsilon)$ yield

$$\mathcal{H}^\varepsilon(t) + C_2 \int_0^t \Lambda_d^\varepsilon(s) ds \leq \bar{C}_2 \mathcal{H}^\varepsilon(0) + \mathcal{O}(1) \int_0^t \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp(x, \xi, s) d\xi dx ds,$$

where C_2 and $\bar{C}_2$ are positive constants independent of t and ε . The equivalence relation between $\mathcal{H}^\varepsilon(t)$ and L^1 -distance implies

$$\begin{aligned} \|f_\varepsilon(t) - \bar{f}_\varepsilon(t)\|_{L^1} + G_0 \int_0^t \Lambda_d^\varepsilon(s) ds \\ \leq G_1 \|f_{0\varepsilon} - \bar{f}_{0\varepsilon}\|_{L^1} + \mathcal{O}(1) \int_0^t \iint_{\Omega \times \mathbb{R}^3} |P(f, f_\varepsilon) - P(\bar{f}, \bar{f}_\varepsilon)|^\sharp(x, \xi, s) d\xi dx ds. \end{aligned}$$

We let $\varepsilon \rightarrow 0$ and use Lemma 4.1 to obtain the desired result. This completes the proof. $\square$

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