EXISTENCE OF THE UNIQUE STRONG SOLUTION FOR A CLASS OF NON-NEWTONIAN FLUIDS WITH VACUUM

BY

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Abstract. The aims of this paper are to discuss local existence and uniqueness of solutions for a class of non-Newtonian fluids with vacuum in one-dimensional bounded intervals. The important point in this paper is that we allow the initial vacuum.

1. Introduction and main results. In this paper, we consider a class of compressible non-Newtonian fluids with vacuum in one-dimensional bounded intervals:

$$\begin{cases} \rho_t + (\rho u)_x = 0 & (x, t) \in (0, 1) \times (0, T), \\ (\rho u)_t + (\rho u^2)_x - \left[(u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \right]_x + \pi_x = \rho f, \end{cases} \quad (1.1)$$

with the initial and boundary conditions

$$\begin{cases} (\rho, u)|_{t=0} = (\rho_0, u_0) & x \in [0, 1], \\ u|_{x=0} = u|_{x=1} = 0 & t \in [0, T], \end{cases} \quad (1.2)$$

where ρ , u , and π denote the unknown density, velocity, and pressure, respectively. The motion of fluids is driven by an external force f , the initial density $\rho_0 \geq 0$, and $p > 2$, $\mu_0 > 0$ are both given constants. $\Omega_T = I \times (0, T)$, $I = (0, 1)$.

Fluid dynamics has attracted the attention of many mathematicians and engineers. The Navier-Stokes equations are generally accepted as the governing equations for the compressible or incompressible motion of viscous fluids, which is usually described by the principle of conservations of mass and momentum, and deduced as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\Gamma) + \nabla \pi = \rho f. \end{cases} \quad (1.3)$$

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In 1970, O. A. Ladyzhenskaya (see [1]) proposed a special form for Γ :

$$\Gamma_{ij} = (\mu_0 + \mu_1 |E_{ij}(\nabla u)|^{p-2}) E_{ij}(\nabla u).$$

These models are called

$$\left\{ \begin{array}{ll} \text{Newtonian} & \text{for } \mu_0 > 0, \mu_1 = 0, \\ \text{Ostwald – de Waele} & \text{for } \mu_0 = 0, \mu_1 > 0, \quad p > 1, \\ \text{Rabinowitsch} & \text{for } \mu_0, \mu_1 > 0, \quad p = 4, \\ \text{Eills} & \text{for } \mu_0, \mu_1 > 0, \quad p > 2, \\ \text{Bingham} & \text{for } \mu_0, \mu_1 > 0, \quad p = 1, \end{array} \right.$$

where the values of the parameters μ_1 , p in some of the pseudo-plastic Ostwald-de Waele models are given in Whitaker [2]. For example, for paper pulp, $\mu_1 = 0.418$, $p = 1.575$, and for carboxymethyl cellulose in water, $\mu_1 = 0.194$, $p = 1.566$. For more physical meaning, see [3, 4].

In this paper, we study local existence and uniqueness of solutions to the initial boundary value problem (1.1)–(1.2) with nonnegative initial densities. In the case that the data ρ_0 , u_0 , f are sufficiently regular and the initial density ρ_0 has a positive lower bound, there exists a unique measure-valued or weak solution to the problem (1.3), (1.2). For details, we refer the readers to the papers [5, 6, 7, 8, 9, 10].

On the other hand, for the Newtonian fluid, namely $p = 2$, there have been few existence results on the solutions for the general case of nonnegative initial densities. The first result was proved by R. Salvi and L. Straskraba. They showed in [11] that if Ω is a bounded domain in $\mathbb{R}^3$, $\pi = \pi(\cdot) \in C^2[0, \infty)$, $\rho_0 \in H^2(\Omega)$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and the compatibility condition

$$-\mu \Delta u_0 + \nabla \pi(\rho_0) = \rho_0^{1/2} g, \quad \text{for some } g \in L^2(\Omega) \quad (1.4)$$

is satisfied, then there exists a unique local strong solution (ρ, u) to the initial boundary value problem. Independently of their work, H. J. Choe and H. Kim in [12, 13] proved a similar existence result when Ω is either a bounded domain or the whole space and the compatibility condition (1.4) is satisfied.

However, for the non-Newtonian fluid with initial vacuums, there was no existence result for strong solutions. This is somewhat surprising because in the context of the compressible Euler equations, there have been many works concerning the existence of a strong solution with compactly supported initial data. For these results, see [14, 15, 16].

DEFINITION 1.1. The pair (ρ, u) is called a strong solution to the initial boundary value problem (1.1)–(1.2), if the following conditions are satisfied:

(i)

$$0 \leq \rho \in C([0, T]; H^1(I)), \quad \rho_t \in C([0, T]; L^2(I)), \quad u_t \in L^2(0, T; H_0^1(I)),$$

$$u \in C([0, T]; H_0^1(I)) \cap L^\infty(0, T; H^2(I)), \quad \sqrt{\rho} u_t \in L^\infty(0, T; L^2(I)).$$

(ii) For all $\varphi \in C([0, T]; H^1(I))$, $\varphi_t \in L^\infty(0, T; L^2(I))$, for a.e. $t \in (0, T)$, we have:

$$\int_0^1 \rho(x, t) \varphi(x, t) dx - \int_0^t \int_0^1 (\rho \varphi_t + \rho u \varphi_x) dx ds = \int_0^1 \rho_0 \varphi(x, 0) dx. \quad (1.5)$$

(iii) For all $\phi \in C([0, T]; H_0^1(I)) \cap L^\infty(0, T; H^2(I))$, $\phi_t \in L^2(0, T; H_0^1(I))$, for a.e. $t \in (0, T)$, we have:

$$\begin{aligned} & \int_0^1 \rho u \phi(x, t) dx - \int_0^t \int_0^1 \left\{ \rho u \phi_t + \rho u^2 \phi_x + \pi \phi_x - [|u_x|^2 + \mu_0]^{\frac{p-2}{2}} u_x \phi_x \right\} (x, s) dx ds \\ &= \int_0^1 \rho_0 u_0 \phi(x, 0) dx + \int_0^t \int_0^1 \rho f \phi(x, s) dx ds. \end{aligned} \quad (1.6)$$

Now we can state the main result of this paper.

THEOREM 1.2. Assume that ρ_0, u_0, f satisfy the following conditions:

$$0 \leq \rho_0 \in H^1(I), \quad u_0 \in H_0^1(I) \cap H^2(I), \quad f \in L^\infty(0, T; L^2(I)), \quad f_t \in L^\infty(0, T; L^2(I)),$$

and if there is a function $g \in L^2(I)$, such that the following identity holds:

$$- \left[(|u_{0x}|^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right]_x + \pi_x(\rho_0) = \rho_0^{\frac{1}{2}} g \quad \text{for a.e. } x \in I. \quad (1.7)$$

Then there exists a $T_* \in (0, +\infty)$, such that the initial and boundary problem (1.1)–(1.2) has a unique strong solution (ρ, u) in Ω_{T_*} , satisfying the following properties:

$$\begin{cases} \rho \in C([0, T_*]; H^1(I)), \quad \rho_t \in C([0, T_*]; L^2(I)), \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2(I)), \\ u \in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I)), \quad u_t \in L^2(0, T_*; H_0^1(I)), \\ \left((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \right)_x \in C([0, T_*]; L^2(I)). \end{cases} \quad (1.8)$$

The rest of this paper is organized as follows. In section 2, we first construct approximate solutions to the initial boundary value problem and obtain the uniform estimate on the approximate solutions. Finally, we obtain existence and uniqueness of solutions for the problem with positive densities. In section 3, we use the result of section 2, and finish the proof of the main Theorem 1.2.

2. Local existence and uniqueness for positive densities. In this section, we prove local existence and uniqueness of solutions to the original problem (1.1)–(1.2) with positive densities. Furthermore, we derive some uniform bounds that are independent of the lower bounds of the initial density. The bounds will be used in the next section to prove the existence of strong solutions with nonnegative densities. We state the following proposition:

PROPOSITION 2.1. Assume that ρ_0, f are both sufficiently smooth functions, $\rho_0 \geq \delta$ for some given constant $\delta > 0$, and $f(0, t) = f(1, t) = 0$, $u_0 \in H_0^1(I) \cap H^2(I)$. If there exists some $g(x) \in L^2(I)$, $g(0) = g(1) = 0$, such that

$$- \left[(u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right]_x + \pi_x(\rho_0) = \rho_0^{\frac{1}{2}} g,$$

then there exists a $T_* \in (0, +\infty)$ such that the initial and boundary problem (1.1)–(1.2) has a unique strong solution (ρ, u) in Ω_{T_*} , satisfying the following properties:

$$\begin{cases} \rho \in C([0, T_*]; H^1(I)), & u \in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I)), \\ \rho_t \in C([0, T_*]; L^2(I)), & u_t \in L^2(0, T_*; H_0^1(I)), \quad \sqrt{\rho}u_t \in L^\infty(0, T_*; L^2(I)), \\ \left((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \right)_x \in C([0, T_*]; L^2(I)). \end{cases}$$

2.1. *Construction and uniform estimates of approximate solutions.* Before constructing approximate solutions, we first prove the following lemma.

LEMMA 2.2 (Embedding inequality). Assume that $f = 0$ on $\partial\Omega$; here $\Omega \in \mathbb{R}^1$ is bounded and open, $f \in C^{2+\alpha}(\bar{\Omega})$. Then

$$|f'|_{L^\infty(\Omega)} \leq d^{\frac{1}{2}}(\Omega) |f''|_{L^2(\Omega)},$$

where $d(\Omega)$ denotes the length of Ω .

Proof. Since $f'(y)$ can be divided into

$$f'(y) = f'(\xi) + \int_\xi^y f''(\lambda) d\lambda \quad (2.9)$$

and $f \in C^{2+\alpha}(\bar{\Omega})$, then f attains its maximum (minimum) on $\bar{\Omega}$.

(i) If there exists a point ξ in Ω such that $f(\xi)$ attains its maximum (minimum), then $f'(\xi) = 0$.

(ii) If f can't attain the maximum (minimum) in Ω , then it attains either on boundary, and $f = 0$ on $\partial\Omega$, thus $f \equiv 0$.

Therefore, if we can find a point ξ in $\bar{\Omega}$ such that $f'(\xi) = 0$, then (2.9) may be rewritten $\forall y \in \bar{\Omega}$ as

$$\begin{aligned} f'(y) &= \int_\xi^y f''(\lambda) d\lambda \\ &\leq \left(\int_\xi^y (f''(\lambda))^2 d\lambda \right)^{\frac{1}{2}} \cdot (y - \xi)^{\frac{1}{2}} \\ &\leq d^{\frac{1}{2}}(\Omega) \left(\int_\Omega (f''(\lambda))^2 d\lambda \right)^{\frac{1}{2}} \\ &= d^{\frac{1}{2}}(\Omega) |f''(y)|_{L^2(\Omega)}, \end{aligned}$$

where $d(\Omega)$ denotes the length of Ω . Thus we have

$$|f'|_{L^\infty(\Omega)} \leq d^{\frac{1}{2}}(\Omega) |f''|_{L^2(\Omega)}. \quad \square$$

Now, we directly construct approximate solutions of (1.1)–(1.2) as follows.

Our first step is to define $u^0 = 0$ and substitute it into the following initial problem:

$$\begin{cases} \rho_t^1 + u^0 \rho_x^1 + u_x^0 \rho^1 = 0, \\ \rho^1(x, 0) = \rho_0. \end{cases}$$

Obviously, the initial value problem yields a unique smooth solution ρ^1 .

Our second step is to substitute ρ^1 into the following initial boundary value problem:

$$\begin{cases} \rho^1 u_t^1 + \rho^1 u^0 u_x^1 - \left[((u_x^1)^2 + \mu_0)^{\frac{p-2}{2}} u_x^1 \right]_x + \pi_x^1 = \rho^1 f & (x, t) \in \Omega_T, \\ u^1(0, t) = u^1(1, t) = 0, \\ u^1(x, 0) = u_0. \end{cases}$$

According to the classical existence theorem (see [17]), this initial boundary value problem yields a unique smooth solution u^1 .

In the third step, we substitute u^1 into the following initial problem:

$$\begin{cases} \rho_t^2 + u^1 \rho_x^2 + u_x^1 \rho^2 = 0, \\ \rho^2(x, 0) = \rho_0. \end{cases}$$

Using the existence theorem of the initial problem of the first-order linear hyperbolic equation, we can gain a unique smooth solution ρ^2 and substitute ρ^2 into the following initial boundary value problem:

$$\begin{cases} \rho^2 u_t^2 + \rho^2 u^1 u_x^2 - \left[((u_x^2)^2 + \mu_0)^{\frac{p-2}{2}} u_x^2 \right]_x + \pi_x^2 = \rho^2 f & (x, t) \in \Omega_T, \\ u^2(0, t) = u^2(1, t) = 0, \\ u^2(x, 0) = u_0. \end{cases}$$

Then we have a smooth solution u^2 . Using step-by-step iteration, we can thus obtain a number of smooth function sequences $\{(\rho^k, u^k)\}$, and (ρ^k, u^k) is a unique smooth solution of the following initial boundary value problem:

$$\rho_t^k + u^{k-1} \rho_x^k + u_x^{k-1} \rho^k = 0, \quad (2.10)$$

$$\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \mathbb{L}_p u^k + \pi_x^k = \rho^k f, \quad (x, t) \in \Omega_T, \quad (2.11)$$

$$\rho^k|_{t=0} = \rho_0, \quad u^k|_{t=0} = u_0, \quad x \in \bar{I}; \quad u^k(0, t) = u^k(1, t) = 0, \quad t \in [0, T], \quad (2.12)$$

where $\pi^k \equiv \pi(\rho^k) = A(\rho^k)^\gamma$, $A > 0$, $\gamma > 1$,

$$\mathbb{L}_p u^k = - \left[((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right]_x,$$

and $u_0 \in H_0^1(I) \cap H^2(I)$ is a smooth solution of the following boundary value problem:

$$\begin{cases} \left[((u_{0x})^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right]_x = \pi_x(\rho_0) - \rho_0^{\frac{1}{2}} g, \\ u_0(0) = u_0(1) = 0. \end{cases} \quad (2.13)$$

Furthermore, by (2.13), we have

$$\begin{aligned} & (u_{0x}^2 + \mu_0)^{\frac{p-4}{2}} ((p-1)u_{0x}^2 + \mu_0) |u_{0xx}| \\ & \geq (u_{0x}^2 + \mu_0)^{\frac{p-4}{2}} (u_{0x}^2 + \mu_0) |u_{0xx}| \\ & = (u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} |u_{0xx}| \\ & \geq \mu_0^{\frac{p-2}{2}} |u_{0xx}|. \end{aligned}$$

Then

$$\begin{aligned}
|u_{0xx}|_{L^2(I)} &\leq \mu_0^{\frac{2-p}{2}} |\pi_x(\rho_0) - \rho_0^{\frac{1}{2}} g|_{L^2(I)} \\
&\leq \mu_0^{\frac{2-p}{2}} \left(A\gamma |\rho_{0x}|_{L^2(I)} |\rho_0|_{L^\infty(I)}^{\gamma-1} + |\rho_0|_{L^\infty(I)}^{\frac{1}{2}} |g|_{L^2(I)} \right) \\
&\leq C,
\end{aligned}$$

where C is a positive constant, depending only on M^0 . Throughout the paper, we denote by

$$M^0 = 1 + \mu_0 + \mu_0^{-1} + |\rho_0|_{H^1(I)} + |g|_{L^2(I)} + |f|_{L^\infty(0,T;L^2(I))} + |f_t|_{L^\infty(0,T;L^2(I))}. \quad (2.14)$$

From now on, we derive uniform bounds on the approximate solutions, including $|u_{xx}^k|_{L^2(I)}$.

Let $K \geq 1$ be a fixed integer, and let us construct an auxiliary function

$$\Phi_K(t) = \max_{1 \leq k \leq K} \sup_{0 \leq s \leq t} \left(1 + |\rho^k(s)|_{H^1(I)} + |u^k(s)|_{W_0^{1,p}(I)} + |\sqrt{\rho^k} u_t^k(s)|_{L^2(I)} \right).$$

First, we estimate $|u^k(s)|_{W_0^{1,p}(I)}$ of $\Phi_K(t)$.

Multiplying (2.11) by u_t^k and integrating it over $(0, 1)$ on x , we deduce that

$$\begin{aligned}
&\int_0^1 \rho^k |u^k|^2 dx + \int_0^1 \left[((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right] u_{xt}^k dx \\
&= \frac{d}{dt} \int_0^1 \pi^k u_x^k dx + \int_0^1 [(\rho f - \rho u^{k-1} u_x^k) u_t^k - \pi_t^k u_x^k] dx,
\end{aligned} \quad (2.15)$$

and integrating over $(0, t)$ on the time variable for (2.15), we have

$$\begin{aligned}
&\int_0^t \int_0^1 \rho^k |u_t^k|^2 dx ds + \int_0^t \int_0^1 ((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k u_{xt}^k dx ds \\
&= - \int_0^1 \pi^k u_x^k(0) dx + \int_0^1 \pi^k u_x^k(t) dx + \int_0^t \int_0^1 [(\rho f - \rho u^{k-1} u_x^k) u_t^k - \pi_t^k u_x^k] dx ds.
\end{aligned} \quad (2.16)$$

We first compute the second term of (2.16) to get

$$\begin{aligned}
\int_0^1 ((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k u_{xt}^k dx &= \frac{1}{2} \int_0^1 ((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} (u_x^k)_t^2 dx \\
&= \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\int_0^{(u_x^k)^2} (s^2 + \mu_0)^{\frac{p-2}{2}} ds \right) dx,
\end{aligned} \quad (2.17)$$

and

$$\begin{aligned}
 & \int_0^{(u_x^k)^2} (s + \mu_0)^{\frac{p-2}{2}} ds \\
 &= \int_{\mu_0}^{(u_x^k)^2 + \mu_0} t^{\frac{p-2}{2}} dt \\
 &= \frac{2}{p} \left[((u_x^k)^2 + \mu_0)^{\frac{p}{2}} - \mu_0^{\frac{p}{2}} \right] \\
 &\geq \frac{2}{p} |u_x^k|^p - \frac{2}{p} \mu_0^{\frac{p}{2}}.
 \end{aligned} \tag{2.18}$$

Substituting (2.17), (2.18) into (2.16), we have

$$\begin{aligned}
 & \int_0^t \left| \sqrt{\rho^k} u_t^k(s) \right|_{L^2(I)}^2 ds + \frac{1}{p} \int_0^1 |u_x^k(t)|^p dx \\
 &\leq \frac{2}{p} |u_x^k(0)|_{L^p(I)}^p + |\pi^k(0)|_{L^2(I)}^2 + \frac{2}{p} \mu_0^{\frac{p}{2}} + \int_0^1 |\pi^k u_x^k(t)| dx \\
 &\quad + \int_0^t \int_0^1 |(\rho^k f - \rho^k u^{k-1} u_x^k) u_t^k - \pi_t^k u_x^k| dx ds \\
 &\leq C + \int_0^1 |\pi^k u_x^k(t)| dx + \int_0^t \int_0^1 |(\rho^k f - \rho^k u^{k-1} u_x^k) u_t^k - \pi_t^k u_x^k| dx ds,
 \end{aligned} \tag{2.19}$$

where C is a positive constant, depending only on M_0 .

By (2.10), we have

$$\pi_t^k = -\gamma \pi^k u_x^{k-1} - \pi_x^k u^{k-1}.$$

Substituting it into (2.19), we have

$$\begin{aligned}
& \int_0^t \left| \sqrt{\rho^k} u_t^k(s) \right|_{L^2(I)}^2 ds + |u_x^k(t)|_{L^p}^p \\
& \leq C + \int_0^t \int_0^1 (|\rho^k f u_t^k| + |\rho^k u^{k-1} u_x^k u_t^k| + |\pi_x^k u^{k-1} u_x^k| + \gamma |\pi u_x^{k-1} u_x^k|) dx ds \\
& \quad + C |\pi^k|_{L^2(I)}^{\frac{p}{p-1}} + \frac{1}{2} |u_x^k(t)|_{L^p(I)}^p \\
& \leq C + \eta \int_0^t \left| \sqrt{\rho^k} u_x^k(s) \right|_{L^2(I)}^2 ds + C_\eta \int_0^t \left| \sqrt{\rho^k} f(s) \right|_{L^2(I)}^2 ds \\
& \quad + C_\eta \int_0^t |\rho^k(s)|_{L^\infty(I)} |u^{k-1}(s)|_{L^\infty(I)}^2 |u_x^k(s)|_{L^p(I)}^2 ds \\
& \quad + \int_0^t \left(|\pi_x^k(s)|_{L^2(I)} |u^{k-1}(s)|_{L^\infty(I)} |u_x^k(s)|_{L^p(I)} \right. \\
& \quad \left. + \gamma |\pi(s)|_{L^\infty(I)} |u_x^{k-1}(s)|_{L^p(I)} |u_x^k(s)|_{L^p(I)} \right) ds \\
& \quad + C |\pi^k(t)|_{L^2(I)}^{\frac{p}{p-1}} + \frac{1}{2} |u_x^k(t)|_{L^p(I)}^p,
\end{aligned}$$

where $0 < \eta < 1$. The above inequality can be written as:

$$\begin{aligned}
& \int_0^t \left| \sqrt{\rho^k} u_t^k(s) \right|_{L^2(I)}^2 ds + |u_x^k(t)|_{L^p(I)}^p \\
& \leq C + \int_0^t \left(\left| \sqrt{\rho^k} f(s) \right|_{L^2(I)}^2 + |\rho^k(s)|_{L^\infty(I)} |u_x^{k-1}(s)|_{L^p(I)}^2 |u_x^k(s)|_{L^p(I)}^2 \right) ds \\
& \quad + \int_0^t |\pi_x^k(s)|_{L^2(I)} |u_x^{k-1}(s)|_{L^p(I)} |u_x^k(s)|_{L^p(I)} ds + C |\pi^k(t)|_{L^2(I)}^2 \\
& \leq C \int_0^t \Phi_K^{\gamma+4}(s) ds + C |\pi^k(t)|_{L^2(I)}^2.
\end{aligned} \tag{2.20}$$

To estimate the right of (2.20), we can first obtain

$$\begin{aligned}
& |\rho^k(t)|_{L^\infty(I)} + |\pi^k(t)|_{H^1(I)} \\
& \leq |\rho^k(t)|_{H^1(I)} + A |\rho^k(t)|_{L^\infty(I)}^{\gamma-1} |\rho^k(t)|_{H^1(I)} \\
& \leq C \Phi_K^\gamma(t).
\end{aligned}$$

By (2.10), we have

$$\begin{aligned}
 & \int_0^1 |\pi^k(t)|^2 dx \\
 &= \int_0^1 |\pi^k(0)|^2 dx + \int_0^t \frac{\partial}{\partial s} \left(\int_0^1 (\pi^k(s))^2 dx \right) ds \\
 &= \int_0^1 |\pi^k(0)|^2 dx + 2 \int_0^t \int_0^1 (\pi^k(\rho^k))' \pi^k (-\rho_x^k u^{k-1} - \rho^k u_x^{k-1}) dx ds \\
 &\leq C + C \int_0^t |\rho^k(s)|_{L^\infty(I)}^{\gamma-1} |\pi^k(s)|_{L^\infty(I)} |\rho_x^k(s)|_{L^2(I)} |u_x^{k-1}(s)|_{L^p(I)} ds \\
 &\leq C \left(1 + \int_0^t \Phi_K^{\gamma+2}(s) ds \right),
 \end{aligned}$$

where C is a positive constant, depending only on M^0 . Substituting it into (2.20), we have

$$\int_0^t \left| \sqrt{\rho^k} u_t^k(s) \right|_{L^2(I)}^2 ds + |u_x^k(t)|_{L^p(I)}^p \leq C \left(1 + \int_0^t \Phi_K^{\gamma+4}(s) ds \right) \quad (2.21)$$

for all k , $1 \leq k \leq K$, where C is a positive constant, depending only on M^0 .

Second, we will estimate the $\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)}$ of $\Phi_K(t)$.

We first estimate $|u_{xx}(t)|_{L^2(I)}$. By (2.11), we have

$$\left[((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right]_x = \rho^k u_t^k + \rho^k u^{k-1} u_x^k + \pi_x^k - \rho^k f.$$

Since

$$\left| \left(((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right)_x \right| \geq \mu_0^{\frac{p-2}{2}} |u_{xx}|,$$

then

$$|u_{xx}^k| \leq C |\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \pi_x^k - \rho^k f|.$$

Taking it by the L^2 norm, the inequality still holds:

$$\begin{aligned}
 & |u_{xx}^k(t)|_{L^2(I)} \\
 &\leq C |\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \pi_x^k - \rho^k f|_{L^2(I)} \\
 &\leq C \left(|\rho^k u_t^k(t)|_{L^2(I)} + |\rho^k u^{k-1} u_x^k(t)|_{L^2(I)} + |\pi_x^k(t)|_{L^2(I)} + |\rho^k f(t)|_{L^2(I)} \right) \\
 &\leq C \left(|\rho^k|^{\frac{1}{2}}_{L^\infty(I)} \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)} + |\rho^k|_{L^\infty(I)} |u^{k-1}|_{L^\infty(I)} |u_x^k(t)|_{L^p(I)} \right. \\
 &\quad \left. + |\pi_x^k(t)|_{L^2(I)} + |\rho^k f(t)|_{L^2(I)} \right).
 \end{aligned}$$

Hence, we deduce that

$$|u_{xx}^k(t)|_{L^2(I)} \leq C\Phi_K^{\gamma+2}(t). \quad (2.22)$$

We differentiate (2.11) with respect to t and multiply it by u_t^k , and by integrating it over $(0, 1)$ on x , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^k |u_t^k|^2(t) dx + \int_0^1 \left[((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right]_t u_{xt}^k(t) dx \\ &= \int_0^1 [(\rho^k u^{k-1})_x (u_t^k + u^{k-1} u_x^k - f) - \rho^k u_t^{k-1} u_x^k + \rho^k f_t] u_t^k dx. \end{aligned} \quad (2.23)$$

Since

$$\begin{aligned} & \left[((u_x^k)^2 + \mu_0)^{\frac{p-2}{2}} u_x^k \right]_t u_{xt}^k \\ &= ((u_x^k)^2 + \mu_0)^{\frac{p-4}{2}} ((p-1)(u_x^k)^2 + \mu_0) (u_{xt}^k)^2 \\ &\geq \mu_0^{\frac{p-2}{2}} (u_{xt}^k)^2, \end{aligned}$$

then (2.23) may be written as:

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \rho^k |u_t^k|^2(t) dx + \int_0^1 |u_{xt}^k|^2 dx \\ &\leq C \int_0^1 \pi_t^k u_{xt}^k dx + \int_0^1 [(\rho^k u^{k-1})_x (u_t^k + u^{k-1} u_x^k - f) - \rho^k u_t^{k-1} u_x^k + \rho^k f_t] u_t^k dx \\ &\leq C \int_0^1 \left(2\rho^k |u^{k-1}| |u_t^k| |u_{xt}^k| + \rho^k |u^{k-1}| |u_x^{k-1}| |u_x^k| |u_t^k| + |\rho_x^k| |u^{k-1}|^2 |u_t^k| |u_x^k| \right. \\ &\quad \left. + \rho^k |u_t^{k-1}| |u_t^k| |u_x^k| + |\pi_x^k| |u^{k-1}| |u_{xt}^k| + \gamma \pi^k |u_x^{k-1}| |u_{xt}^k| + |\rho_x^k| |u^{k-1}| |f| |u_t^k| \right. \\ &\quad \left. + \rho^k |u_x^{k-1}| |f| |u_t^k| + \rho^k |u^{k-1}| |f| |u_{xt}^k| + \rho^k |u_t^k| |f_t| \right) dx \\ &= C \sum_{j=1}^{10} I_j. \end{aligned} \quad (2.24)$$

By Sobolev inequality and Young's inequality, we obtain

$$\begin{aligned}
 I_1 &= \int_0^1 2\rho^k |u^{k-1}| |u_t^k| |u_{xt}^k| \, dx \\
 &\leq 2 |\rho^k|_{L^\infty(I)}^{\frac{1}{2}} |u^{k-1}|_{L^\infty(I)} \left| \sqrt{\rho^k} u_t^k \right|_{L^2(I)} |u_{xt}^k(t)|_{L^2(I)} \\
 &\leq C\Phi_K^5(t) + \frac{1}{8} |u_{xt}^k|_{L^2(I)}^2, \\
 I_2 &= \int_0^1 \rho^k |u^{k-1}| |u_x^{k-1}| |u_x^k| |u_t^k| \, dx \\
 &\leq |\rho^k|_{L^\infty(I)} |u^{k-1}|_{L^\infty(I)} |u_x^{k-1}|_{L^p(I)} |u_x^k|_{L^p} |u_t^k|_{L^\infty(I)} \\
 &\leq |\rho^k|_{L^\infty(I)} |u_x^{k-1}|_{L^p(I)}^2 |u_x^k|_{L^p(I)} |u_{xt}^k(t)|_{L^2(I)} \\
 &\leq C\Phi_K^8(t) + \frac{1}{8} |u_{xt}^k(t)|_{L^2(I)}^2, \\
 I_3 &= \int_0^1 |\rho_x^k| |u^{k-1}|^2 |u_t^k| |u_x^k| \, dx \\
 &\leq |\rho_x^k|_{L^2(I)} |u^{k-1}|_{L^\infty(I)}^2 |u_t^k|_{L^\infty(I)} |u_x^k|_{L^p(I)} \\
 &\leq |\rho_x^k|_{L^2(I)} |u_x^{k-1}|_{L^p(I)}^2 |u_{xt}^k|_{L^2(I)} |u_x^k|_{L^p(I)} \\
 &\leq C\Phi_K^8(t) + \frac{1}{8} |u_{xt}^k(t)|_{L^2(I)}^2, \\
 I_4 &= \int_0^1 \rho^k |u_t^{k-1}| |u_t^k| |u_x^k| \, dx \\
 &\leq |\rho^k|_{L^\infty(I)}^{\frac{3}{4}} |u_t^{k-1}|_{L^\infty(I)} \left| \sqrt{\rho^k} u_t^k \right|_{L^2(I)}^{\frac{1}{2}} |u_t^k|_{L^\infty(I)}^{\frac{1}{2}} |u_x^k|_{L^p(I)} \\
 &\leq |\rho^k|_{L^\infty(I)}^{\frac{3}{4}} |u_{xt}^{k-1}|_{L^2(I)} |u_{xt}^k|_{L^2(I)}^{\frac{1}{2}} \left| \sqrt{\rho^k} u_t^k \right|_{L^2(I)}^{\frac{1}{2}} |u_x^k|_{L^p(I)} \\
 &\leq C\Phi_K^{\frac{9}{2}}(t) |u_{xt}^k|_{L^2(I)} + \frac{1}{2} |u_{xt}^{k-1}|_{L^2(I)}^2 \\
 &\leq C\Phi_K^9(t) + \frac{1}{2} |u_{xt}^{k-1}(t)|_{L^2(I)}^2 + \frac{1}{8} |u_{xt}^k(t)|_{L^2(I)}^2,
 \end{aligned}$$

$$\begin{aligned}
I_5 &= \int_0^1 |\pi_x^k| |u^{k-1}| |u_{xt}^k| \, dx \\
&\leq |\pi_x^k|_{L^2(I)} |u^{k-1}|_{L^\infty(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq C \Phi_K^{2(\gamma+1)}(t) + \frac{1}{8} |u_{xt}^k|_{L^2(I)}^2,
\end{aligned}$$

$$\begin{aligned}
I_6 &= \int_0^1 \gamma \pi^k |u_x^{k-1}| |u_{xt}^k| \, dx \\
&\leq C |\pi^k|_{L^\infty(I)} |u_x^{k-1}|_{L^p(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq C |\pi_x^k|_{L^2(I)} |u_x^{k-1}|_{L^p(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq C \Phi_K^{2(\gamma+1)}(t) + \frac{1}{8} |u_{xt}^k(t)|_{L^2(I)}^2,
\end{aligned}$$

$$\begin{aligned}
I_7 &= \int_0^1 |\rho_x^k| |u^{k-1}| |f| |u_t^k| \, dx \\
&\leq |\rho_x^k|_{L^2(I)} |u^{k-1}|_{L^\infty(I)} |f|_{L^2(I)} |u_t^k|_{L^\infty(I)} \\
&\leq |\rho_x^k|_{L^2(I)} |u_x^{k-1}|_{L^p(I)} |f|_{L^2(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq C \Phi_K^4(t) + \frac{1}{8} |u_{xt}^k|_{L^2(I)}^2,
\end{aligned}$$

$$\begin{aligned}
I_8 &= \int_0^1 \rho^k |u_x^{k-1}| |f| |u_t^k| \, dx \\
&\leq |\rho^k|_{L^\infty(I)} |u_t^k|_{L^\infty(I)} |f|_{L^2(I)} |u_x^{k-1}|_{L^p(I)} \\
&\leq C \Phi_K^4(t) + \frac{1}{8} |u_{xt}^k|_{L^2(I)}^2,
\end{aligned}$$

$$\begin{aligned}
I_9 &= \int_0^1 \rho^k |u^{k-1}| |f| |u_{xt}^k| \, dx \\
&\leq |\rho^k|_{L^\infty(I)} |u^{k-1}|_{L^\infty(I)} |f|_{L^2(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq |\rho^k|_{L^\infty(I)} |u_x^{k-1}|_{L^p(I)} |f|_{L^2(I)} |u_{xt}^k|_{L^2(I)} \\
&\leq C \Phi_K^4(t) + \frac{1}{8} |u_{xt}^k|_{L^2(I)}^2,
\end{aligned}$$

$$\begin{aligned}
 I_{10} &= \int_0^1 \rho^k |u_t^k| |f_t| \, dx \\
 &\leq |\rho^k|_{L^\infty(I)}^{\frac{1}{2}} \left| \sqrt{\rho^k} u_t^k \right|_{L^2(I)} |f_t|_{L^2(I)} \\
 &\leq C \Phi_K^{\frac{3}{2}}(t).
 \end{aligned}$$

Substituting I_j ($j = 1, 2, \dots, 10$) into (2.24) and integrating it over (τ, t) on the time variable yields

$$\begin{aligned}
 &\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)}^2 + \int_\tau^t |u_{xt}^k|_{L^2}^2(s) \, ds \\
 &\leq C \int_\tau^t \Phi_K^{2\gamma+7}(s) \, ds + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 + \frac{1}{2} \int_\tau^t |u_{xt}^{k-1}|_{L^2(I)}^2(s) \, ds \\
 &\leq C \int_\tau^t \Phi_K^{2\gamma+7}(s) \, ds + \sup_{1 \leq k \leq K} \left(1 + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 \right) + \frac{1}{2} \int_\tau^t |u_{xt}^{k-1}|_{L^2(I)}^2(s) \, ds.
 \end{aligned} \tag{2.25}$$

From the above recursive relation, we obtain

$$\begin{aligned}
 &\int_\tau^t |u_{xt}^k|_{L^2(I)}^2(s) \, ds \\
 &\leq \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^K} \right) C \left[\int_\tau^t \Phi_K^{2\gamma+7}(s) \, ds + \sup_{0 \leq k \leq K} \left(1 + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 \right) \right] \\
 &\leq 2C \left[\int_\tau^t \Phi_K^{2\gamma+7}(s) \, ds + \sup_{0 \leq k \leq K} \left(1 + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 \right) \right]
 \end{aligned}$$

for $1 \leq k \leq K$. According to the above inequality, (2.25) may be written as

$$\begin{aligned}
 &\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)}^2 + \int_\tau^t |u_{xt}^k|_{L^2}^2(s) \, ds \\
 &\leq C \int_\tau^t \Phi_K^{2\gamma+7}(s) \, ds + \sup_{0 \leq k \leq K} \left(1 + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 \right),
 \end{aligned} \tag{2.26}$$

where C is a positive constant, depending only on M^0 .

To obtain the estimate of $\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)}^2$, we need to estimate

$$\lim_{\tau \rightarrow 0} \sup_{0 \leq k \leq K} \left(1 + \left| \sqrt{\rho^k} u_t^k(\tau) \right|_{L^2(I)}^2 \right).$$

By (2.11), we have

$$\int_0^1 \rho^k |u_t^k|^2(t) \, dx \leq 2 \int_0^1 \left(\rho^k |u^{k-1}|^2 |u_x^k|^2 + \rho^k |f|^2 + (\rho^k)^{-1} |\mathbb{L}_p u^k + \pi_x^k|^2 \right) \, dx.$$

According to the smoothing of (ρ^k, u^k) , we get

$$\begin{aligned}
& \lim_{t \rightarrow 0} \int_0^1 \left(\rho^k |u^{k-1}|^2 |u_x^k|^2 + \rho^k |f|^2 + (\rho^k)^{-1} |\mathbb{L}_p u^k + \pi_x^k|^2 \right) (x, t) dx \\
&= \int_0^1 \left(\rho_0 |u_0|^2 |u_{0x}|^2 + \rho_0 |f|^2 + (\rho_0)^{-1} |\mathbb{L}_p u_0 + \pi_x^k(\rho_0)|^2 \right) (x, t) dx \\
&= \int_0^1 \left(\rho_0 |u_0|^2 |u_{0x}|^2 + \rho_0 |f|^2 \right) dx + |g|_{L^2(I)}^2 \\
&\leq |\rho_0|_{L^\infty(I)} |u_0|_{L^\infty(I)} |u_{0x}|_{L^2(I)}^2 + |\rho_0|_{L^\infty(I)} |f|_{L^2(I)} + |g|_{L^2(I)}^2.
\end{aligned}$$

By (2.14), we obtain that

$$\limsup_{\tau \rightarrow 0} \int_0^1 \rho^k |u_t^k|^2(\tau) dx \leq C,$$

where C is a positive constant, depending only on M^0 .

Taking (2.26) the limit on τ , letting $\tau \rightarrow 0$, we have

$$\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)}^2 + \int_0^t |u_{xt}^k|_{L^2}^2(s) ds \leq C_1 + C_2 \int_0^t \Phi_K^{2\gamma+7}(s) ds. \quad (2.27)$$

Multiplying (2.10) by ρ^k and integrating over $(0, 1)$ with respect to x , we have

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |\rho^k(t)|^2 dx + \int_0^1 (\rho u^{k-1})_x \rho^k(t) dx = 0.$$

Integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |\rho^k(t)|^2 dx \leq \frac{1}{2} \int_0^1 |u_x^{k-1}| |\rho^k|^2(t) dx.$$

Using Sobolev inequality, we have

$$\frac{d}{dt} |\rho^k(t)|_{L^2(I)} \leq C |u_{xx}^{k-1}(t)|_{L^2(I)} |\rho^k(t)|_{L^2(I)}. \quad (2.28)$$

Then differentiating (2.10) with respect to x , multiplying it by ρ_x^k , and integrating over $(0, 1)$ on x , we deduce

$$\begin{aligned}
\frac{d}{dt} \int_0^1 |\rho_x^k|^2(t) dx &= \int_0^1 \left(\frac{1}{2} u_x^{k-1} (\rho_x^k)^2 + \rho^k \rho_x^k u_{xx}^{k-1} \right) (t) dx \\
&\leq C \int_0^1 (|u_x^{k-1}| |\rho_x^k|^2 + \rho^k |\rho_x^k| |u_{xx}^{k-1}|) (t) dx \\
&\leq C (|u_x^{k-1}(t)|_{L^2(I)} |\rho_x^k(t)|_{L^2(I)}^2 + |\rho^k(t)|_{L^\infty(I)} |\rho_x^k(t)|_{L^2(I)} |u_{xx}^{k-1}(t)|_{L^2(I)}).
\end{aligned}$$

Using Sobolev inequality, we obtain

$$\frac{d}{dt} |\rho_x^k(t)|_{L^2(I)} \leq C |\rho^k(t)|_{H^1(I)} |u_{xx}^{k-1}(t)|_{L^2(I)}. \quad (2.29)$$

Using (2.28) and (2.29), we have

$$\begin{aligned}
\frac{d}{dt} (|\rho^k(t)|_{L^2(I)} + |\rho_x^k(t)|_{L^2(I)}) &= \frac{d}{dt} |\rho^k(t)|_{H^1(I)} \\
&\leq C |\rho^k(t)|_{H^1(I)} |u_{xx}^{k-1}(t)|_{L^2(I)}.
\end{aligned}$$

By Gronwall's inequality, we obtain

$$\sup_{0 \leq t \leq T} |\rho^k(t)|_{H^1(I)} \leq |\rho_0^k|_{H^1(I)} \exp \left(C \int_0^t |u_{xx}^{k-1}(\cdot, s)|_{L^2(I)} ds \right). \quad (2.30)$$

Substituting (2.22) into (2.30), we have

$$\begin{aligned} |\rho^k(t)|_{H^1(I)} &\leq C \exp \left(C \int_0^t |u_{xx}^{k-1}(s)|_{L^2(I)} ds \right) \\ &\leq C \exp \left(C \int_0^t \Phi_K^{\gamma+2}(s) ds \right). \end{aligned} \quad (2.31)$$

By (2.10), we have

$$|\rho_t^k(t)|_{L^2(I)} \leq |\rho_x^k(t)|_{L^2(I)} |u^{k-1}(t)|_{L^\infty(I)} + |\rho^k(t)|_{L^\infty(I)} |u_x^{k-1}(t)|_{L^2(I)} \leq C \Phi_K^2(t).$$

Combining (2.21), (2.27), (2.31) with (2.22), we obtain

$$\begin{aligned} &|u_x^k(t)|_{L^p(I)} + |u_{xx}(t)|_{L^2(I)} + |\rho^k(t)|_{H^1(I)} + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2(I)} + \int_0^t |u_{xt}^k(s)|_{L^2(I)}^2 ds \\ &\leq C_1 \exp \left(C_2 \int_0^t \Phi_K^{2\gamma+7}(s) ds \right), \end{aligned} \quad (2.32)$$

where C_1, C_2 is a positive constant, depending only on M^0 .

If

$$\int_0^T \Phi_K^{2\gamma+7}(s) ds < 1,$$

then we may denote $T = T_1$. By inserting the above inequality into (2.32), we can directly obtain (2.34).

On the other hand, if

$$\int_0^T \Phi_K^{2\gamma+7}(s) ds \geq 1,$$

then we can find $t_0 \in (0, T)$ such that

$$\int_0^{t_0} \Phi_K^{2\gamma+7}(s) ds = 1.$$

Substituting the above identity into (2.32) yields

$$\sup_{0 \leq t \leq t_0} \Phi_K(t) \leq C_1 e^{C_2},$$

and

$$1 = \int_0^{t_0} \Phi_K^{2\gamma+7}(s) ds \leq \int_0^{t_0} C_1^{2\gamma+7} e^{(2\gamma+7)C_2} ds \leq C_1^{2\gamma+7} e^{(2\gamma+7)C_2} t_0,$$

then

$$1 \leq C_1^{2\gamma+7} e^{(2\gamma+7)C_2} t_0.$$

Here we may denote

$$T_1 = C_1^{-2\gamma-7} e^{-(2\gamma+7)C_2},$$

and thus we deduce

$$\sup_{0 \leq t \leq T_1} \Phi_K(t) \leq C_1 e^{C_2} \leq C, \quad (2.33)$$

where C is a positive constant, depending only on M_0 .

Therefore, for all $1 \leq k \leq K$, we deduce that

$$\begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T_1} \left(|\rho^k(t)|_{H^1(I)} + |u^k(t)|_{W_0^{1,p} \cap H^2(I)} + |\sqrt{\rho^k} u_t^k(t)|_{L^2(I)} + |\rho_t^k(t)|_{L^2(I)} \right) \\ & + \int_0^{T_1} |u_{xt}^k(s)|_{L^2(I)}^2 ds \leq C, \end{aligned} \quad (2.34)$$

where C is a positive constant, depending only on M^0 .

2.2. Convergence of approximate solutions. In this subsection, we will prove that the approximate solutions (ρ^k, u^k) strongly converge on the solution of the problem (1.1)–(1.2) with positive densities. For this, we denote:

$$\bar{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k. \quad (2.35)$$

By (2.11), we get

$$\begin{aligned} & \rho^{k+1} \bar{u}_t^{k+1} + \rho^{k+1} u^k \bar{u}_x^{k+1} + [\mathbb{L}_p u^{k+1} - \mathbb{L}_p u^k] + (\pi_x^{k+1} - \pi_x^k) \\ & = \bar{\rho}^{k+1} (f - u_t^k - u^k u_x^k) - \rho^k \bar{u}^k u_x^k. \end{aligned} \quad (2.36)$$

Multiplying (2.36) by $\bar{u}^{k+1}$, integrating over $(0, 1)$ on x , and using (2.10) and Young's inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \rho^{k+1} |\bar{u}^{k+1}|^2 dx + \int_0^1 [\mathbb{L}_p u^{k+1} - \mathbb{L}_p u^k] \bar{u}^{k+1} dx \\ & \leq C \int_0^1 (|\bar{\rho}^{k+1}| |f - u_t^k - u^k u_x^k| |\bar{u}^{k+1}| + \rho^k |\bar{u}^k| |u_x^k| |\bar{u}^{k+1}| + |\pi_x^{k+1} - \pi_x^k|^2) dx. \end{aligned} \quad (2.37)$$

Let

$$\Psi(s) = (s^2 + \mu_0)^{\frac{p-2}{2}} s.$$

Then

$$\int_0^1 [\mathbb{L}_p u^{k+1} - \mathbb{L}_p u^k] \bar{u}^{k+1} dx = \int_0^1 \left[\int_0^1 \Psi'(\theta u^{k+1} + (1-\theta)u^k) d\theta \right] (\bar{u}_x^{k+1})^2 dx. \quad (2.38)$$

Since

$$\begin{aligned} \Psi'(s) &= \left[(s^2 + \mu_0)^{\frac{p-2}{2}} s \right]' \\ &= (p-2)(s^2 + \mu_0)^{\frac{p-4}{2}} s^2 + (s^2 + \mu_0)^{\frac{p-2}{2}} \\ &= (s^2 + \mu_0)^{\frac{p-4}{2}} ((p-1)s^2 + \mu_0) \\ &\geq (s^2 + \mu_0)^{\frac{p-2}{2}} \\ &\geq (\mu_0)^{\frac{p-2}{2}}, \end{aligned}$$

(2.38) can be written as

$$\int_0^1 [\mathbb{L}_p u^{k+1} - \mathbb{L}_p u^k] \bar{u}^{k+1} dx \geq (\mu_0)^{\frac{p-2}{2}} \int_0^1 (\bar{u}_x^{k+1})^2 dx.$$

Substituting it into (2.37), we have

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \rho^{k+1} |\bar{u}^{k+1}|^2 dx + (\mu_0)^{\frac{p-2}{2}} \int_0^1 (\bar{u}_x^{k+1})^2 dx \\ & \leq C \int_0^1 (|\bar{\rho}^{k+1}| |f - u_t^k - u^k u_x^k| |\bar{u}^{k+1}| + \rho^k |\bar{u}^k| |u_x^k| |\bar{u}^{k+1}| + |\pi^{k+1} - \pi^k| |\bar{u}_x^{k+1}|) dx \\ & \leq C (|\bar{\rho}^{k+1}|_{L^2(I)} |u_{xt}^k|_{L^2(I)} |\bar{u}_x^{k+1}|_{L^2(I)} + |\bar{\rho}^{k+1}|_{L^2(I)} |u_x^k|_{L^p(I)} |u_{xx}^k|_{L^2(I)} |\bar{u}_x^{k+1}|_{L^2(I)} \\ & \quad + |\bar{\rho}^{k+1}|_{L^2(I)} |f|_{L^2(I)} |\bar{u}_x^{k+1}|_{L^2(I)} + |\rho^k|_{L^2(I)}^{\frac{1}{2}} |\sqrt{\rho^k} \bar{u}^k|_{L^2(I)} |u_{xx}^k|_{L^2(I)} |\bar{u}_x^{k+1}|_{L^2(I)} \\ & \quad + |\pi^{k+1} - \pi^k|_{L^2(I)} |\bar{u}_x^{k+1}|_{L^2(I)}), \end{aligned}$$

and hence,

$$\frac{d}{dt} \int_0^1 \rho^{k+1} |\bar{u}^{k+1}|^2 dx + \int_0^1 (\bar{u}_x^{k+1})^2 dx \leq B^k(t) |\bar{\rho}^{k+1}|_{L^2(I)} + C |\sqrt{\rho^k} \bar{u}^k|_{L^2(I)}^2, \quad (2.39)$$

where $B^k(t) = C(1 + |f(t)|_{L^2(I)}^2 + |u_{xt}^k(t)|_{L^2(I)}^2)$. By (2.35), for all $k \geq 1$, we have

$$\int_0^t B^k(s) ds \leq C + Ct,$$

where $t < T_1$ and C is a positive constant, depending only on M^0 .

By (2.10), we have

$$\bar{\rho}_t^{k+1} + (\bar{\rho}^{k+1} u^k)_x + (\rho^k \bar{u}^k)_x = 0.$$

Multiplying it by $\bar{\rho}^{k+1}$ and integrating over $(0, 1)$, we get

$$\begin{aligned} \frac{d}{dt} |\bar{\rho}^{k+1}|_{L^2(I)}^2 & \leq C |\bar{\rho}^{k+1}|_{L^2(I)}^2 |u_x^k|_{L^\infty(I)} + |\rho^k|_{H^1(I)} |\bar{u}_x^k|_{L^2(I)} |\bar{\rho}^{k+1}|_{L^2(I)} \\ & \leq C |\bar{\rho}^{k+1}|_{L^2(I)}^2 |u_{xx}^k|_{L^2(I)} + C_\eta |\bar{\rho}^{k+1}|_{L^2(I)} |\rho^k|_{H^1(I)}^2 + \eta |\bar{u}_x^k|_{L^2(I)}^2 \\ & \leq D_\eta^k(t) |\bar{\rho}^{k+1}|_{L^2(I)}^2 + \eta |\bar{u}_x^k|_{L^2(I)}^2, \end{aligned} \quad (2.40)$$

where $D_\eta^k(t) = C |u_{xx}^k(t)|_{L^2(I)} + C_\eta |\rho^k(t)|_{H^1(I)}^2$, for all $t \leq T_1$ and $k \geq 1$. By (2.35), we have

$$\int_0^t D_\eta^k(s) ds \leq C + C_\eta t.$$

Combining (2.39) with (2.40), we obtain

$$\begin{aligned} & \frac{d}{dt} (|\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2(I)}^2 + |\bar{\rho}^{k+1}(t)|_{L^2(I)}^2) + |\bar{u}_x^{k+1}(t)|_{L^2(I)}^2 \\ & \leq C |\sqrt{\rho^k} \bar{u}^k|_{L^2(I)}^2 + E_\eta(t) |\bar{\rho}^{k+1}|_{L^2(I)}^2 + \eta |\bar{u}^k|_{L^2(I)}^2, \end{aligned} \quad (2.41)$$

where $E_\eta(t)$ depends only on $B^k(t)$ and $D_\eta(t)$. By (2.35), for all $t \leq T_1$ and $k \geq 1$, we have

$$\int_0^t E_\eta^k(s) \, ds \leq C + C_\eta t.$$

Integrating (2.41) over $(0, t) \subset (0, T_1)$ on the time variable and using Gronwall's inequality, we have

$$\begin{aligned} & \left(|\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2(I)}^2 + |\bar{\rho}^{k+1}(t)|_{L^2(I)}^2 \right) + \int_0^t |\bar{u}_x^{k+1}(s)|_{L^2(I)}^2 \, ds \\ & \leq C \exp(C_\eta t) \int_0^t \left(|\sqrt{\rho^k} \bar{u}^k(s)|_{L^2(I)}^2 + |\bar{u}_x^k(s)|_{L^2(I)}^2 \right) \, ds, \end{aligned}$$

where C is a positive constant, depending only on M^0 .

Hence, we choose $\eta > 0$, then take enough small T_* such that $4C(T_* + \eta) < 1$, $T_* < T_1$, and $\exp(C_\eta T_*) < 2$; then

$$\begin{aligned} & \left(|\sqrt{\rho^1} \bar{u}^1(t)|_{L^2(I)}^2 + |\bar{\rho}^1(t)|_{L^2(I)}^2 \right) + \int_0^t |\bar{u}_x^1(s)|_{L^2(I)}^2 \, ds \\ & \leq \frac{1}{2} \int_0^t \left(|\sqrt{\rho^0} \bar{u}^0(s)|_{L^2(I)}^2 + \eta |\bar{u}_x^0(s)|_{L^2(I)}^2 \right) \, ds, \\ & \left(|\sqrt{\rho^2} \bar{u}^2(t)|_{L^2(I)}^2 + |\bar{\rho}^2(t)|_{L^2(I)}^2 \right) + \int_0^t |\bar{u}_x^2(s)|_{L^2(I)}^2 \, ds \\ & \leq \frac{1}{2} \int_0^t \left(|\sqrt{\rho^1} \bar{u}^1(s)|_{L^2(I)}^2 + \eta |\bar{u}_x^1(s)|_{L^2(I)}^2 \right) \, ds, \\ & \vdots \\ & \left(|\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2(I)}^2 + |\bar{\rho}^{k+1}(t)|_{L^2(I)}^2 \right) + \int_0^t |\bar{u}_x^{k+1}(s)|_{L^2(I)}^2 \, ds \\ & \leq \frac{1}{2} \int_0^t \left(|\sqrt{\rho^k} \bar{u}^k(s)|_{L^2(I)}^2 + \eta |\bar{u}_x^k(s)|_{L^2(I)}^2 \right) \, ds. \end{aligned}$$

Therefore, we combine the above inequalities, and using Gronwall's inequality, we deduce that

$$\sum_{k=1}^K \left[\sup_{0 \leq t \leq T_*} \left(|\bar{\rho}^{k+1}(t)|_{L^2(I)}^2 + |\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2(I)}^2 \right) + \int_0^{T_*} |\bar{u}_x^{k+1}(s)|_{L^2(I)}^2 \, ds \right] < C, \quad (2.42)$$

where C is a positive constant, depending only on M^0 .

For the smooth function u^{k-1} , we consider the following initial problem

$$\begin{cases} \rho_t^k + u^{k-1} \rho_x^k + u_x^{k-1} \rho^k = 0, \\ \rho^k|_{t=0} = \rho_0^\delta. \end{cases}$$

Obviously, there is a unique solution ρ^k on this above initial problem. Using the method of characteristics, we obtain

$$\frac{d\rho^k}{dt} = -\rho^k u_x^{k-1}, \quad (2.43)$$

$$\frac{dx}{dt} = u^{k-1}(x, t), \quad (2.44)$$

$$x|_{t=0} = x_0, \quad \rho^k|_{t=0} = \rho_0^\delta. \quad (2.45)$$

By (2.44) and (2.45), we have

$$x(t) = x_0 + \int_0^t u^{k-1}(x(s), s) ds = U(x_0, t).$$

Using (2.43), we have

$$d(\ln \rho^k) = -u_x^{k-1} dt;$$

then

$$\rho^k(x, t) = \rho_0(U(x_0, t)) \exp \left(- \int_0^t u_x^{k-1}(U(x_0, s), s) ds \right). \quad (2.46)$$

Using (2.46) and Sobolev inequality, we have

$$\rho^k(x, t) \geq \delta \exp \left[- \int_0^T |u_x^{k-1}(\cdot, s)|_{L^\infty(I)} ds \right]. \quad (2.47)$$

Then for all $t \in (0, T_*)$, we get

$$\rho^{k+1} \geq \delta C^{-1} > 0.$$

Hence, by (2.42), we get the following convergence: as $k \rightarrow +\infty$,

$$\begin{aligned} u^k &\rightarrow u \quad L^\infty(0, T_*; L^2(I)) \cap L^2(0, T_*; H_0^1(I)), \\ \rho^k &\rightarrow \rho \quad L^\infty(0, T_*; L^2(I)). \end{aligned} \quad (2.48)$$

By virtue of the lower semi-continuity of various norms, we deduce that (ρ, u) satisfies the following uniform estimate:

$$\begin{aligned} &\text{ess sup}_{0 \leq t \leq T_*} \left(|\rho(t)|_{H^1(I)} + |u(t)|_{W_0^{1,p}(I) \cap H^2(I)} + |\sqrt{\rho} u_t(t)|_{L^2(I)} + |\rho_t(t)|_{L^2(I)} \right) \\ &+ \int_0^{T_*} |u_{xt}(s)|_{L^2(I)}^2 ds \leq C, \end{aligned} \quad (2.49)$$

where C is a positive constant, depending only on M_0 .

2.3. The proof of Proposition 2.1. In this subsection, we will prove Proposition 2.1 in order to use (2.48) and (2.49); namely, under the assumption of Proposition 2.1, we will prove that (ρ, u) is a unique solution of the initial boundary value problem (1.1)–(1.2) with positive densities. For this, we only need to prove that (ρ, u) satisfies the following properties:

(i)

$$\begin{aligned} \rho &\in C([0, T_*]; H^1(I)), \quad \rho_t \in C([0, T_*]; L^2(I)), \quad \sqrt{\rho}u_t \in L^\infty(0, T_*; L^2(I)), \\ u &\in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I)), \quad u_t \in L^2(0, T_*; H_0^1(I)), \\ \left((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \right)_x &\in C([0, T_*]; L^2(I)). \end{aligned}$$

(ii) For all $\varphi \in C([0, T_*]; H^1(I))$, $\varphi_t \in L^\infty(0, T_*; L^2(I))$, for a.e. $t \in (0, T_*)$, the following identity holds:

$$\int_0^1 \rho(x, t) \varphi(x, t) dx - \int_0^t \int_0^1 (\rho \varphi_t + \rho u \varphi_x) dx ds = \int_0^1 \rho_0 \varphi(x, 0) dx. \quad (2.50)$$

(iii) For all $\phi \in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I))$, $\phi_t \in L^2(0, T_*; H_0^1(I))$, for a.e. $t \in (0, T_*)$, the following identity holds:

$$\begin{aligned} &\int_0^1 \rho u \phi(x, t) dx - \int_0^t \int_0^1 \left\{ \rho u \phi_t + \rho(u)^2 \phi_x + \pi \phi_x - (u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \phi_x \right\} (x, s) dx ds \\ &= \int_0^1 \rho_0 u_0 \phi(x, 0) dx + \int_0^t \int_0^1 \rho f \phi(x, s) dx ds. \end{aligned} \quad (2.51)$$

Since (ρ^k, u^k) is a smooth solution of (2.10)–(2.12), it naturally satisfies the following identities:

$$\int_0^1 \rho^k(x, t) \varphi(x, t) dx - \int_0^t \int_0^1 (\rho^k \varphi_t + \rho^k u^{k-1} \varphi_x) dx ds = \int_0^1 \rho_0 \varphi(x, 0) dx,$$

where $\varphi \in C([0, T_*]; H^1(I))$, $\varphi_t \in L^\infty(0, T_*; L^2(I))$, and

$$\begin{aligned} &\int_0^1 \rho^k u \phi(x, t) dx - \int_0^t \int_0^1 \left\{ \rho^k u^k \phi_t + \rho^k (u^k)^2 \phi_x + \pi^k \phi_x - (u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \phi_x \right\} (x, s) dx ds \\ &= \int_0^1 \rho_0 u_0 \phi(x, 0) dx + \int_0^t \int_0^1 \rho^k f \phi(x, s) dx ds, \end{aligned}$$

where $\phi \in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I))$, $\phi_t \in L^2(0, T_*; H_0^1(I))$. Hence, if we want to prove (2.50), (2.51), we only need to prove that, as $k \rightarrow \infty$,

$$\begin{aligned} \sum_{i=1}^3 \mathbf{I}_i^k &= \int_0^1 [\rho^k(x, t) - \rho(x, t)] \varphi(x, t) dx \\ &\quad - \int_0^t \int_0^1 [(\rho^k - \rho) \varphi_t + (\rho^k u^{k-1} - \rho u) \varphi_x] dx ds \rightarrow 0, \end{aligned} \quad (2.52)$$

and

$$\begin{aligned}
 \sum_{i=1}^6 \Pi_i^k &= \int_0^1 (\rho^k(x, t) u^k(x, t) - \rho(x, t) u(x, t)) \phi(x, t) dx \\
 &\quad - \int_0^t \int_0^1 \left\{ (\rho^k u^k - \rho u) \phi_t + (\rho^k (u^k)^2 - \rho (u)^2) \phi_x + (\pi^k - \pi) \phi_x \right. \\
 &\quad \left. - ((u^k)_x^2 + \mu_0)^{\frac{p-2}{2}} u_x^k - (u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \phi_x \right\} dx ds \\
 &\quad - \int_0^t \int_0^1 (\rho^k - \rho) f \phi(x, s) dx ds \rightarrow 0.
 \end{aligned} \tag{2.53}$$

We first calculate the fifth term of (2.53):

$$\Pi_5^k = \int_0^t \int_0^1 \left[((u^k)_x^2 + \mu_0)^{\frac{p-2}{2}} u_x^k - (u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x \right] \phi_x dx ds, \tag{2.54}$$

where $\phi \in C([0, T_*]; H_0^1(I)) \cap L^\infty(0, T_*; H^2(I))$, $\phi_t \in L^2(0, T_*; H_0^1(I))$.

Let

$$\omega(s) = (s^2 + \mu_0)^{\frac{p-2}{2}} s. \tag{2.55}$$

Then we obtain

$$\begin{aligned}
 \Pi_5^k &\leq \int_0^t \int_0^1 |\omega(u_x^k) - \omega(u_x)| |\phi_x| dx ds \\
 &= \int_0^t \int_0^1 \left| \int_0^1 \omega'(\theta u_x^k + (1-\theta)u_x) d\theta (u_x^k - u_x) \right| |\phi_x| dx ds \\
 &\leq \int_0^t \int_0^1 \left| \int_0^1 \omega'(\theta u_x^k + (1-\theta)u_x) d\theta \right| |u_x^k - u_x| |\phi_x| dx ds \\
 &\leq \int_0^t \int_0^1 \left| \int_0^1 (\mu_0 + (\theta u_x^k + (1-\theta)u_x)^2)^{\frac{p-2}{2}} d\theta \right| |u_x^k - u_x| |\phi_x| dx ds \\
 &\leq |\phi_x|_{L^\infty(0, T_*; L^\infty(I))} \left(\mu_0 + 2|u_x^k|_{L^\infty(0, T_*; L^\infty(I))}^2 + 2|u_x|_{L^\infty(0, T_*; L^\infty(I))}^2 \right)^{\frac{p-2}{2}} \\
 &\quad \cdot \int_0^t \left(\int_0^1 |u_x^k - u_x|^2 dx \right)^{\frac{1}{2}} ds \\
 &\leq C \int_0^t \left(\int_0^1 |u_x^k - u_x|^2 dx \right)^{\frac{1}{2}} ds,
 \end{aligned}$$

where C is a positive constant, depending only on M_* . Here and below, we denote

$$\begin{aligned}
 M_* &= 1 + M^0 + |\phi_x|_{L^\infty(0, T_*; L^\infty(I))} + |\phi_t|_{L^\infty(0, T_*; L^2(I))} \\
 &\quad + |\varphi_x|_{L^\infty(0, T_*; L^2(I))} + |\varphi_t|_{L^\infty(0, T_*; L^2(I))} + |f|_{L^\infty(0, T_*; L^2(I))}.
 \end{aligned}$$

We then calculate the second and third terms of (2.53):

$$\Pi_2^k = \int_0^t \int_0^1 (\rho^k u^k - \rho u) \phi_t dx ds \quad (2.56)$$

and

$$\Pi_3^k = \int_0^t \int_0^1 (\rho^k (u^k)^2 - \rho (u)^2) \phi_x dx ds. \quad (2.57)$$

For (2.56), we obtain

$$\begin{aligned} \Pi_2^k &\leq \int_0^t \int_0^1 |\rho^k u^k - \rho u| |\phi_t| dx ds \\ &\leq \int_0^t \int_0^1 |\rho^k - \rho| |u| |\phi_t| dx ds + \int_0^t \int_0^1 |\rho^k| |u^k - u| |\phi_t| dx ds \\ &\leq |u|_{L^\infty(0, t; L^\infty(I))} |\phi_t|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |\rho^k - \rho|^2 dx \right)^{1/2} ds \\ &\quad + |\rho^k|_{L^\infty(0, t; L^\infty(I))} |\phi_t|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |u^k - u|^2 dx \right)^{1/2} ds \\ &\leq C \left[\int_0^t \left(\int_0^1 |\rho^k - \rho|^2 dx \right)^{1/2} ds + \int_0^t \left(\int_0^1 |u^k - u|^2 dx \right)^{1/2} ds \right], \end{aligned} \quad (2.58)$$

where C is a positive constant, depending only on M_* .

Now we calculate (2.57):

$$\begin{aligned} \Pi_3^k &\leq \int_0^t \int_0^1 |\rho^k (u^k)^2 - \rho (u)^2| |\phi_x| dx ds \\ &\leq \int_0^t \int_0^1 |\rho^k u^k - \rho u| |u^k| |\phi_x| dx ds + \int_0^t \int_0^1 |u^k - u| |\rho u| |\phi_x| dx ds \\ &\leq |u^k|_{L^\infty(0, t; L^\infty(I))} |\phi_x|_{L^\infty(0, t; L^\infty(I))} \int_0^t \int_0^1 |\rho^k u^k - \rho u| dx ds \\ &\quad + |\rho u \phi_x|_{L^\infty(0, t; L^\infty(I))} \int_0^t \int_0^1 |u^k - u| dx ds \\ &\leq C \left[\int_0^t \left(\int_0^1 |\rho^k - \rho|^2 dx \right)^{1/2} ds + \int_0^t \left(\int_0^1 |u^k - u|^2 dx \right)^{1/2} ds \right], \end{aligned} \quad (2.59)$$

where C is a positive constant, depending only on M_* .

Finally, we calculate the remaining terms:

$$\begin{aligned}
 \mathbf{I}_1^k &\leq \int_0^1 [\rho^k(x, t) - \rho(x, t)] \varphi(x, t) \, dx \\
 &\leq |\varphi|_{L^\infty(I)} \int_0^1 |\rho^k(x, t) - \rho(x, t)| \, dx \\
 &\leq C \left(\int_0^1 |\rho^k(x, t) - \rho(x, t)|^2 \, dx \right)^{\frac{1}{2}}. \\
 \mathbf{I}_2^k + \mathbf{I}_3^k &\leq \int_0^t \int_0^1 (|\rho^k - \rho| |\varphi_t| + |\rho^k u^k - \rho u| |\varphi_x|) \, dx \, ds \\
 &\leq |\varphi_t|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |\rho^k - \rho|^2 \, dx \right)^{\frac{1}{2}} \, ds \\
 &\quad + |\varphi_x|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |\rho^k u^k - \rho u|^2 \, dx \right)^{\frac{1}{2}} \, ds \\
 &\leq C \left[\int_0^t \left(\int_0^1 |\rho^k - \rho|^2 \, dx \right)^{\frac{1}{2}} \, ds + \int_0^t \left(\int_0^1 |u^k - u|^2 \, dx \right)^{\frac{1}{2}} \, ds \right]. \\
 \mathbf{\Pi}_1^k &\leq \int_0^1 |\rho^k u^k - \rho u| |\phi(x, t)| \, dx \\
 &\leq |\phi(x, t)|_{L^2(I)} \left(\int_0^1 |\rho^k u^k - \rho u|^2 \, dx \right)^{\frac{1}{2}} \\
 &\leq C \left[\left(\int_0^1 |\rho^k - \rho|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_0^1 |u^k - u|^2 \, dx \right)^{\frac{1}{2}} \right]. \\
 \mathbf{\Pi}_4^k &\leq \int_0^t \int_0^1 |\pi^k - \pi| |\phi_x| \, dx \, ds \\
 &\leq |\phi_x|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |\pi^k - \pi|^2 \, dx \right)^{\frac{1}{2}} \, ds \\
 &\leq C \int_0^t \left(\int_0^1 |\pi^k - \pi|^2 \, dx \right)^{\frac{1}{2}} \, ds. \\
 \mathbf{\Pi}_6^k &\leq \int_0^t \int_0^1 |\rho^k - \rho| |f \phi(x, s)| \, dx \, ds \\
 &\leq |f|_{L^2(0, t; L^2(I))} |\phi(x, s)|_{L^\infty(0, t; L^2(I))} \int_0^t \left(\int_0^1 |\rho^k - \rho|^2 \, dx \right)^{\frac{1}{2}} \, ds \\
 &\leq C \int_0^t \left(\int_0^1 |\rho^k - \rho|^2 \, dx \right)^{\frac{1}{2}} \, ds.
 \end{aligned}$$

Since the initial value is independent of k , as $k \rightarrow +\infty$, (ρ_0, u_0) still satisfies the following equation:

$$\left((u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right)_x = \pi_x(\rho_0) - \rho_0^{\frac{1}{2}} g.$$

Hence, we deduce that

$$\begin{aligned} & \sum_{i=1}^3 |\mathbf{I}_i^k| + \sum_{i=1}^6 |\mathbf{II}_i^k| \\ & \leq C \left\{ \int_0^t \left(\int_0^1 |u_x^k - u_x|^2 dx \right)^{\frac{1}{2}} ds + \int_0^t \left(\int_0^1 |u_x^k - u_x|^2 dx \right)^{\frac{p-1}{2}} ds \right. \\ & \quad + \int_0^t \left(\int_0^1 |\rho^k - \rho|^2 dx \right)^{\frac{1}{2}} + \int_0^t \left(\int_0^1 |\pi^k - \pi|^2 dx \right)^{\frac{1}{2}} ds \\ & \quad \left. + \left(\int_0^1 |\rho^k - \rho|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 |u^k - u|^2 dx \right)^{\frac{1}{2}} \right\}, \end{aligned}$$

where C is a positive constant, depending only on M_* . By the convergence (2.48), we see, as $k \rightarrow +\infty$, that the right-hand side of the above inequality vanishes; namely, (ρ, u) satisfies the integration equations (2.50) and (2.51).

To finish the proof of existence, we still need to prove that (ρ, u) satisfies the following properties:

$$\begin{aligned} \rho & \in C([0, T_*]; H^1(I)), \quad \rho_t \in C([0, T_*]; L^2(I)), \\ u & \in C([0, T_*]; H_0^1(I)), \quad L_p u \in C([0, T_*]; L^2(I)). \end{aligned}$$

First, we prove

$$\rho \in C([0, T_*]; H^1(I)).$$

Since ρ satisfies the following regularity:

$$\operatorname{ess\,sup}_{0 \leq t \leq T_*} (|\sqrt{\rho} u_t(t)|_{L^2(I)}^2 + |\rho(t)|_{H^1(I)} + |\rho_t(t)|_{L^2(I)}) \leq C,$$

by the embedding theorem (see [3], Chapter I), we obtain

$$\rho \in C([0, T_*]; L^2(I)).$$

Now, we will prove

$$\rho \in C([0, T_*]; H^1(I)).$$

Using (2.10), we have

$$\frac{d}{dt} |\rho_x^k(t)|_{L^2(I)}^2 \leq C |\rho_x^k(0)|_{L^2(I)}^2 |u_x^{k-1}(t)|_{H^1(I)}$$

and

$$|\rho_x^k(t)|_{L^2(I)}^2 \leq |\rho_x(0)|_{L^2(I)}^2 + C t.$$

By the lower semi-continuity of various norms, we have

$$|\rho_x(t)|_{L^2(I)}^2 \leq |\rho_x(0)|_{L^2(I)}^2 + C t.$$

Taking limits on the above inequality, letting $t \rightarrow 0$, we have

$$\limsup_{t \rightarrow 0} |\rho_x(t)|_{L^2(I)}^2 \leq |\rho_x(0)|_{L^2(I)}^2.$$

Hence, using the strong convergence for the space L^2 (see [18]), we have

$$\lim_{t \rightarrow 0^+} |\rho_x(t) - \rho_x(0)|_{L^2(I)}^2 = 0.$$

For each fixed $t_0 \in [0, t]$, the function $\hat{\rho} = \hat{\rho}(x, t) = \hat{\rho}(x, \pm t + t_0)$ is a unique strong solution to the similar initial problem

$$\begin{cases} \hat{\rho}_t + (\hat{\rho}\hat{v})_x = 0, \\ \hat{\rho}(0) = \rho(t_0), \end{cases} \quad (2.60)$$

where $\hat{v} = \hat{v}(x, t) = \pm v(x, \pm t + t_0)$. Therefore, $\rho_x \in C([0, T_*]; L^2(I))$. Then we have $\rho \in C([0, T_*]; H^1(I))$.

By virtue of (2.49) and the embedding theorem (see [3]), we obtain

$$u \in C([0, T]; H_0^1(I)).$$

Hence, we only need to prove that u satisfies the following properties:

$$\mathbb{L}_p u \in C([0, T_*]; L^2(I)).$$

By the momentum equation of (2.11), $\forall \omega \in H_0^1(I)$, we get

$$\begin{aligned} (\rho u_t, \omega)_{L^2(I)} &= \left(-\rho u u_x + (|u_x|^2 + \mu_0)^{\frac{p-2}{2}} u_x - \pi_x + \rho f, \omega \right)_{L^2(I)} \\ &= (-\rho u u_x + \rho f, \omega)_{L^2(I)} - ((|u_x|^2 + \mu_0)^{\frac{p-2}{2}} u_x - \pi, \omega_x)_{L^2(I)}; \end{aligned}$$

then

$$\frac{d}{dt} (\rho u_t, \omega)_{L^2(I)} = ((-\rho u u_x + \rho f)_t, \omega)_{L^2(I)} - (((|u_x|^2 + \mu_0)^{\frac{p-2}{2}} u_x - \pi)_t, \omega_x)_{L^2(I)}. \quad (2.61)$$

By (2.49), we obtain $A(t)|\omega|_{H_0^1(I)}$ is the upper boundary for the right-hand side of (2.61), where $A(t) \in L^2(0, T_*)$. Then $(\rho u_t)_t \in L^2(0, T; H^{-1}(I))$ (see [20], third chapter lemma 1.1). Hence, $\rho u_t \in L^2(0, T; H_0^1(I))$. By the embedding theorem, we obtain $\rho u_t \in C([0, T_*]; L^2(I))$.

Therefore, we obtain, for all $t \in [0, T]$, that $u = u(t) \in H_0^1(I) \cap H^2(I)$ is a solution of the following elliptical equation (see [19]):

$$\mathbb{L}_p u = G - \rho u u_x,$$

where $G = \rho f - \rho u_t - \pi_x \in C([0, T]; L^2(I))$.

Since

$$\begin{aligned} & |\mathbb{L}_p u(t) - \mathbb{L}_p u(s)|_{L^2(I)} \\ &= |G(t) - \rho u u_x(t) - (G(s) - \rho u u_x(s))|_{L^2(I)} \\ &\leq |\rho u u_x(t) - \rho u u_x(s)|_{L^2(I)} + |G(s) - G(t)|_{L^2(I)}, \end{aligned} \quad (2.62)$$

the first term of (2.62) may be written as

$$\begin{aligned}
& |\rho u u_x(t) - \rho u u_x(s)|_{L^2(I)} \\
& \leq |(\rho(t) - \rho(s))u(t)u_x(t)|_{L^2(I)} + |\rho(s)(u(t) - u(s))u_x(t)|_{L^2(I)} \\
& \quad + |\rho(s)u(s) \cdot (u_x(t) - u_x(s))|_{L^2(I)} \\
& \leq (|\rho(t) - \rho(s)|_{L^2(I)}|u_x(t)|_{L^p(I)} + |\rho(s)|_{L^2(I)}|u_x(s) - u_x(t)|_{L^2(I)})|u_x(t)|_{L^\infty(I)} \\
& \quad + |\rho(s)|_{L^\infty(I)}|u_x(s)|_{L^\infty(I)}|u_x(t) - u_x(s)|_{L^2(I)} \\
& \leq C(|\rho(t) - \rho(s)|_{L^2(I)} + |u(t) - u(s)|_{H_0^1(I)}).
\end{aligned} \tag{2.63}$$

Hence,

$$\lim_{t \rightarrow s} |\mathbb{L}_p u(t) - \mathbb{L}_p u(s)|_{L^2(I)} = 0.$$

This proves the continuity of $\mathbb{L}_p u(t)$ in $L^2(I)$.

The proof of uniqueness is the same as Theorem 1.2; we omit it here. Hence, Proposition 2.1 is proved. $\square$

3. The proof of Theorem 1.2. In this section, we will prove the main Theorem 1.2. We allow the problem to have the state of vacuum in Theorem 1.2, and in the previous section, we assumed that the initial density is positive and we obtained a similar result, so we hope to use the result to prove Theorem 1.2. First, we regularize the known functions of the original problem (1.1)–(1.2) such that they satisfy the conditions of Proposition 2.1. Using Proposition 2.1, we can prove that the limit of solutions of the regulation problem is the solution to the original problem (1.1)–(1.2).

3.1. *The proof of existence.* Assume that ρ_0 is sufficiently smooth; then, for each small $\delta > 0$, let $\rho_0^\delta = J_\delta * \rho_0 + \delta$. J_δ is a mollifier on I , and $u_0^\delta \in H_0^1(I) \cap H^2(I)$ is a solution of the following boundary value problem (see [19], Chapter II):

$$- \left[((u_{0x}^\delta)^2 + \mu_0)^{\frac{p-2}{2}} u_{0x}^\delta \right]_x = -\pi_x(\rho_0^\delta) + (\rho_0^\delta)^{\frac{1}{2}} g_\delta \quad x \in (0, 1), \tag{3.64}$$

$$u_0^\delta(0) = u_0^\delta(1) = 0, \tag{3.65}$$

where $g_\delta \in C^\infty(I)$ and satisfies

$$|g_\delta|_{L^2(I)} \leq |g|_{L^2(I)}, \quad \lim_{\delta \rightarrow 0} |g_\delta - g|_{L^2(I)} = 0.$$

With $\rho_0^\delta = J_\delta * \rho_0 + \delta$, there exists a subsequence $\{\rho^{\delta_j}\}$ of $\{\rho^\delta\}$ that satisfies the following convergence: as $\delta_j \rightarrow 0$,

$$-\pi_x(\rho_0^{\delta_j}) + (\rho_0^{\delta_j})^{\frac{1}{2}} g_{\delta_j} \rightarrow -\pi_x(\rho_0) + (\rho_0)^{\frac{1}{2}} g \quad \text{in } L^2(I).$$

By (3.64), there exists a subsequence $\{u_0^{\delta_j}\}$ of $\{u_0^\delta\}$ such that as $\delta_j \rightarrow 0$,

$$\left[((u_{0x}^{\delta_j})^2 + \mu_0)^{\frac{p-2}{2}} u_{0x}^{\delta_j} \right]_x \rightarrow \left[((u_{0x})^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right]_x \quad \text{in } L^2(I).$$

Hence, (ρ_0, u_0) satisfies the following equation:

$$- \left[((u_{0x})^2 + \mu_0)^{\frac{p-2}{2}} u_{0x} \right]_x = -\pi_x(\rho_0) + (\rho_0)^{\frac{1}{2}} g \quad \text{for a.e. } x \in I. \quad (3.66)$$

Taking $f^\delta \in C^\infty(\bar{\Omega}_T)$, $f^\delta(0, t) = f^\delta(1, t) = 0$, satisfying

$$\begin{aligned} |f^\delta|_{L^\infty(0, T; L^2(I))} &\leq C |f|_{L^\infty(0, T; L^2(I))}, \\ |f_t^\delta|_{L^\infty(0, T; L^2(I))} &\leq C |f_t|_{L^\infty(0, T; L^2(I))}, \end{aligned}$$

and as $\delta \rightarrow 0$, there hold the following limits:

$$\begin{aligned} |f^\delta - f|_{L^\infty(0, T; L^2(I))} &\rightarrow 0, \\ |f_t^\delta - f_t|_{L^\infty(0, T; L^2(I))} &\rightarrow 0. \end{aligned}$$

By Proposition 2.1, there exists a $T_* \in (0, +\infty)$; therefore the initial boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - ((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x)_x + \pi_x = \rho f^\delta & (x, t) \in \Omega_T, \\ (\rho, u)|_{t=0} = (\rho_0^\delta, u_0^\delta) & x \in [0, 1], \\ u|_{x=0} = u|_{x=1} = 0 & t \in [0, T], \\ \pi \equiv \pi(\rho) = A\rho^\gamma, \quad A > 0, \quad \gamma > 1, \end{cases}$$

yields a unique solution (ρ^δ, u^δ) . Moreover, (ρ^δ, u^δ) satisfies the following uniform estimate:

$$\begin{aligned} \text{ess sup}_{0 \leq t \leq T_*} & \left(|\rho^\delta(t)|_{H^1(I)} + |u^\delta(t)|_{W_0^{1,p}(I) \cap H^2(I)} + |\sqrt{\rho^\delta} u_t^\delta(t)|_{L^2(I)} + |\rho_t^\delta(t)|_{L^2(I)} \right) \\ & + \int_0^{T_*} |u_{xt}^\delta(s)|_{L^2(I)}^2 ds \leq C, \end{aligned}$$

where C is a positive constant, depending only on M^0 . Then we obtain the following convergence:

$$\begin{aligned} u^\delta &\rightarrow u \quad \text{in } L^\infty(0, T_*; L^2(I)) \cap L^2(0, T_*; H_0^1(I)), \\ \rho^\delta &\rightarrow \rho \quad \text{in } L^\infty(0, T_*; L^2(I)). \end{aligned}$$

Hence, similar to the proof of Proposition 2.1, we may prove that (ρ, u) is the solution of the original problem (1.1)–(1.2); moreover, (ρ, u) satisfies the following uniform estimate:

$$\begin{aligned} \text{ess sup}_{0 \leq t \leq T_*} & \left(|\rho(t)|_{H^1(I)} + |u(t)|_{W_0^{1,p}(I) \cap H^2(I)} + |\sqrt{\rho} u_t(t)|_{L^2(I)} + |\rho_t(t)|_{L^2(I)} \right) \\ & + \int_0^{T_*} |u_{xt}(s)|_{L^2(I)}^2 ds \leq C, \end{aligned} \quad (3.67)$$

where C is a positive constant, depending only on M^0 .

The proof of continuity of the solution of (1.1)–(1.2) is similar to the proof of Proposition 2.1. Therefore, the proof of existence of the main Theorem 1.2 is finished.

3.2. *The proof of uniqueness.* Assume that (ρ, u) , $(\bar{\rho}, \bar{u})$ are both solutions of (1.1)–(1.2); then

$$\begin{aligned} & \int_0^1 [\rho(x, t)u(x, t) - \bar{\rho}(x, t)\bar{u}(x, t)]\phi(x, t) dx - \int_0^t \int_0^1 \left\{ (\rho u - \bar{\rho}\bar{u})\phi_t + (\rho u^2 - \bar{\rho}\bar{u}^2)\phi_x \right. \\ & \quad \left. - \left((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x - (\bar{u}_x^2 + \mu_0)^{\frac{p-2}{2}} \bar{u}_x \right) \phi_x + (\pi - \bar{\pi})\phi_x \right\} (x, s) dx ds \\ & = \int_0^t \int_0^1 (\rho - \bar{\rho})f\phi(x, s) dx ds. \end{aligned} \quad (3.68)$$

By definition (1.5), we have

$$\begin{aligned} 0 & = \int_0^t \int_0^1 \{ (u - \bar{u})\bar{u}_x((\rho - \bar{\rho})\bar{u} - (\rho u - \bar{\rho}\bar{u}) + \rho(u - \bar{u})) \} (x, s) dx ds \\ & = \int_0^t \int_0^1 \{ \bar{u}_t(\rho - \bar{\rho})(u - \bar{u}) - (\rho - \bar{\rho})\bar{u}\bar{u}_x(u - \bar{u}) - \rho(u - \bar{u})^2\bar{u}_x \\ & \quad + (\rho - \bar{\rho})\bar{u}(u - \bar{u})_t + (\rho u - \bar{\rho}\bar{u})\bar{u}(u - \bar{u})_x \} (x, s) dx ds + \int_0^1 (\rho - \bar{\rho})\bar{u}(u - \bar{u})(x, t) dx \\ & = \int_0^t \int_0^1 \{ (\rho - \bar{\rho})(\bar{u}_t - \bar{u}\bar{u}_x)(u - \bar{u}) + \rho(u - \bar{u})^2\bar{u}_x + (\rho - \bar{\rho})\bar{u}(u - \bar{u})_t \\ & \quad + (\rho u - \bar{\rho}\bar{u})\bar{u}(u - \bar{u})_x \} (x, s) dx ds + \int_0^1 (\rho - \bar{\rho})\bar{u}(u - \bar{u})(x, t) dx. \end{aligned} \quad (3.69)$$

Let

$$\Psi(s) = (s^2 + \mu_0)^{\frac{p-2}{2}} s.$$

Then

$$\int_0^1 [\mathbb{L}_p u - \mathbb{L}_p \bar{u}](u - \bar{u})_x dx = \int_0^1 \left[\int_0^1 \Psi'(\theta u + (1 - \theta)\bar{u}) d\theta \right] (u_x - \bar{u}_x)^2 dx, \quad (3.70)$$

and since

$$\begin{aligned} \Psi'(s) & = \left[(s^2 + \mu_0)^{\frac{p-2}{2}} s \right]' \\ & = (p-2)(s^2 + \mu_0)^{\frac{p-4}{2}} s^2 + (s^2 + \mu_0)^{\frac{p-2}{2}} \\ & = (s^2 + \mu_0)^{\frac{p-4}{2}} ((p-1)s^2 + \mu_0) \\ & \geq (s^2 + \mu_0)^{\frac{p-2}{2}} \\ & \geq \mu_0^{\frac{p-2}{2}}, \end{aligned}$$

then (3.70) may be written as

$$\int_0^1 [\mathbf{L}_p u - \mathbf{L}_p \bar{u}] (u - \bar{u})_x dx \geq C \int_0^1 (u_x - \bar{u}_x)^2 dx. \quad (3.71)$$

In (3.68), we take $\phi = (u - \bar{u}) \in C([0, T]; H_0^1(I))$. Substituting (3.71) into (3.68), and using (3.69), we have

$$\begin{aligned} & \frac{1}{2} \int_0^1 \rho (u - \bar{u})^2(x, t) dx + \int_0^t \int_0^1 \left((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x - (\bar{u}_x^2 + \mu_0)^{\frac{p-2}{2}} \bar{u}_x \right) (u - \bar{u})_x dx ds \\ &= \int_0^t \int_0^1 \{ (\rho - \bar{\rho})(f + \bar{u}_t - \bar{u} \bar{u}_x)(u - \bar{u}) + \rho (u - \bar{u})^2 \bar{u}_x + (\pi - \bar{\pi})(u - \bar{u})_x \}(x, t) dx ds \\ &\leq \int_0^t \int_0^1 [|\rho - \bar{\rho}| |h| |u - \bar{u}| + \rho |u - \bar{u}|^2 |\bar{u}_x| + |\pi - \bar{\pi}| |u_x - \bar{u}_x|] dx ds \\ &\leq \int_0^t \left\{ |\rho - \bar{\rho}|_{L^2(I)} |h|_{L^2(I)} |u - \bar{u}|_{L^\infty(I)} + |\sqrt{\rho}(u - \bar{u})|_{L^2(I)}^2 |\bar{u}_x|_{L^\infty(I)} \right. \\ &\quad \left. + |\pi - \bar{\pi}|_{L^p(I)} |u_x - \bar{u}_x|_{L^2(I)} \right\} ds \\ &\leq C \int_0^t \left\{ |(\rho - \bar{\rho})(s)|_{L^2(I)}^2 |h(s)|_{L^2(I)}^2 + \frac{1}{4} |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 \right. \\ &\quad \left. + |\sqrt{\rho}(u - \bar{u})(s)|_{L^2(I)}^2 |\bar{u}_x(s)|_{L^\infty(I)} + C |(\pi - \bar{\pi})(s)|_{L^2(I)}^2 + \frac{1}{4} |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 \right\} ds, \end{aligned}$$

where $h = (f + \bar{u}_t - \bar{u} \cdot \bar{u}_x) \in L^2(0, T; L^2(I))$.

Then we obtain

$$\begin{aligned} & |\sqrt{\rho}(u - \bar{u})(t)|_{L^2(I)}^2 + C \int_0^t |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 ds \\ &\leq \int_0^t \left[C |(\rho - \bar{\rho})(s)|_{L^2(I)}^2 |h(s)|_{L^2(I)}^2 + |\sqrt{\rho}(u - \bar{u})(s)|_{L^2(I)}^2 |\bar{u}_x(s)|_{L^\infty(I)} \right. \\ &\quad \left. + C |(\pi - \bar{\pi})(s)|_{L^2(I)}^2 \right] ds \\ &\leq \int_0^t A(s) \left[|\sqrt{\rho}(u - \bar{u})(s)|_{L^2(I)}^2 + |(\rho - \bar{\rho})(s)|_{L^2(I)}^2 + |(\pi - \bar{\pi})(s)|_{L^2(I)}^2 \right] ds, \end{aligned} \quad (3.72)$$

where the nonnegative function $A(t) = C(1 + |h(s)|_{L^2(I)}^2 + |\bar{u}_x(s)|_{L^\infty(I)}^2) \in L^1(0, T_*)$, and C is a positive constant, depending only on M^0 .

In (1.5), we take $\varphi = (\rho - \bar{\rho}) \in C([0, T]; H^1(I))$; then

$$\frac{1}{2} \int_0^1 (\rho - \bar{\rho})^2 dx = \int_0^t \int_0^1 (\rho - \bar{\rho})(\rho - \bar{\rho})_t dx ds + \int_0^t \int_0^1 (\rho u - \bar{\rho} \bar{u})(\rho - \bar{\rho})_x dx ds.$$

So, we have

$$\begin{aligned}
& \frac{1}{2} \int_0^1 (\rho - \bar{\rho})^2(t) dx \\
&= \int_0^t \int_0^1 (\rho u - \bar{\rho} \bar{u})_x (\rho - \bar{\rho}) dx ds \\
&= \int_0^t \int_0^1 (\rho u - \rho \bar{u} + \rho \bar{u} - \bar{\rho} \bar{u})_x (\rho - \bar{\rho}) dx ds \\
&= \int_0^t \int_0^1 \left(\rho_x (u - \bar{u}) (\rho - \bar{\rho}) + \rho (u - \bar{u})_x (\rho - \bar{\rho}) + \frac{1}{2} \bar{u}_x (\rho - \bar{\rho})^2 \right) dx ds \quad (3.73) \\
&\leq \int_0^t (|\rho_x|_{L^2(I)} |u - \bar{u}|_{L^\infty} |\rho - \bar{\rho}|_{L^2(I)} + |\rho|_{L^\infty(I)} |u_x - \bar{u}_x|_{L^2} |\rho - \bar{\rho}|_{L^2(I)} \\
&\quad + \frac{1}{2} |\bar{u}_x|_{L^\infty(I)} |\rho - \bar{\rho}|_{L^2(I)}^2) ds \\
&\leq \int_0^t \left(B(s) |(\rho - \bar{\rho})(s)|_{L^2(I)}^2 + \frac{1}{8} |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 \right) ds,
\end{aligned}$$

where the nonnegative function $B(t) = C(|\rho|_{H^1(I)} + |\bar{u}_x|_{L^\infty(I)}) \in L^1(0, T_*)$, and C is a positive constant, depending only on M^0 .

Similarly, we have

$$\int_0^t \frac{1}{2} \frac{d}{dt} \int_0^1 (\pi - \bar{\pi})^2 dx ds \leq \int_0^t \left(D(s) |(\pi - \bar{\pi})(s)|_{L^2(I)}^2 + \frac{1}{8} |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 \right) ds, \quad (3.74)$$

where the nonnegative function $D(t) = C(|\pi|_{H^1(I)} + |\bar{u}_x|_{L^\infty(I)}) \in L^1(0, T_*)$, and C is a positive constant, depending only on M^0 .

Combining (3.72)–(3.74), we obtain

$$\begin{aligned}
& \frac{1}{4} \int_0^t |(u_x - \bar{u}_x)(s)|_{L^2(I)}^2 ds + \left[|\sqrt{\rho}(u - \bar{u})(t)|_{L^2(I)}^2 + |(\rho - \bar{\rho})(t)|_{L^2(I)}^2 + |(\pi - \bar{\pi})(t)|_{L^2(I)}^2 \right] \\
&\leq \int_0^{T_*} (A(s) + B(s) + D(s)) \left(|\sqrt{\rho}(u - \bar{u})(s)|_{L^2(I)}^2 + |(\rho - \bar{\rho})(s)|_{L^2(I)}^2 \right. \\
&\quad \left. + |(\pi - \bar{\pi})(s)|_{L^2(I)}^2 \right) ds.
\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}
& \frac{1}{4} |(u_x - \bar{u}_x)(t)|_{L^2(I)}^2 + \operatorname{ess\,sup}_{0 \leq t \leq T_*} \left(|\sqrt{\rho}(u - \bar{u})(t)|_{L^2(I)}^2 + |(\rho - \bar{\rho})(t)|_{L^2(I)}^2 \right. \\
&\quad \left. + |(\pi - \bar{\pi})(t)|_{L^2(I)}^2 \right) \leq 0.
\end{aligned}$$

Then we get

$$\rho = \bar{\rho}, \quad \sqrt{\rho}(u - \bar{u}) = 0, \quad u_x = \bar{u}_x,$$

and $u(x, t) \in C([0, T_*]; H_0^1(I))$. So,

$$\rho = \bar{\rho}, \quad u = \bar{u}.$$

Hence, the proof of uniqueness of the main theorem 1.2 is finished.

Therefore, Theorem 1.2 is proved.

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