RECOVERY OF MULTIPLE OBSTACLES BY PROBE METHOD

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Abstract. We consider an inverse scattering problem for multiple obstacles $D = \bigcup_{j=1}^N D_j \subset R^3$ with different types of boundary for D_j . By constructing an indicator function from the far-field pattern of the scattered wave, we can firstly reconstruct the shape of all obstacles, then identify the type of boundary for each obstacle, as well as the boundary impedance in the case that obstacles have the Robin-type boundary condition. The novelty of our probe method compared with the existing probe method is that we succeeded in identifying the type of boundary condition for multiple obstacles by analyzing the behavior of both the imaginary part and the real part of the indicator function. The numerical realizations are given to show the performance of this inversion method.

1. Introduction. Let D be a bounded domain in R^3 such that $D = \bigcup_{j=1}^N D_j$, $\overline{D_i} \cap \overline{D_j} = \emptyset$ ($i \neq j$). Each D_j is a simply connected domain with C^2 boundary ∂D_j . The scattering of time-harmonic acoustic plane waves by the obstacle D with some boundary

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is modelled as an exterior boundary value problem for the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx \cdot d}$, $d \in S^2 = \{\xi \in R^3 : |\xi| = 1\}$, the total wave field $u = u^i + u^s \in H_{loc}^1(R^3 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^3 \setminus \overline{D}, \\ Bu(x, t) = 0, & \text{on } \partial D, \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{r}\right), & r = |x| \longrightarrow \infty, \end{cases} \quad (1.1)$$

where B is a boundary operator corresponding to different types of the obstacles D_j ($1 \leq j \leq N$), that is,

$$Bu = \begin{cases} u & \text{if } \partial D_j \text{ is sound-soft,} \\ \frac{\partial u}{\partial \nu} & \text{if } \partial D_j \text{ is sound-hard,} \\ \frac{\partial u}{\partial \nu} + i\sigma(x)u & \text{if } \partial D_j \text{ is Robin-type,} \end{cases} \quad (1.2)$$

where ν is the unit normal on ∂D_j directed into the exterior of D_j , and $\sigma(x) > 0$ is the boundary impedance coefficient. By the results in [5], we know that there exists a unique solution for the forward scattering problem (1.1).

For the incident field $u^i(x) = e^{ikx \cdot d}$, the far-field pattern $u^\infty(\theta, d)$ can be defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ u^\infty(\theta, d) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \longrightarrow \infty,$$

where $\theta, d \in S^2$.

Generally, the inverse scattering problem corresponding to (1.1) is to identify the boundary ∂D and also $\sigma(x)$ in the case of a Robin-type boundary, from a knowledge of the far-field pattern. If D is just one obstacle, then identifying ∂D for each kind of boundary conditions has been discussed thoroughly. For example, if D is sound-soft (Dirichlet boundary condition on ∂D) or sound-hard (Neumann boundary condition on ∂D), the problems have been studied by many researchers; see [3], [6], [10], [11], [13], [18], [22]. In the case of an obstacle with Robin-type boundary, the problem of reconstructing $\sigma(x)$, when ∂D is given, has also been studied; see [4], [6], [20], [21]. For the inverse scattering problem of determining both ∂D and the boundary impedance, an approximate determination (or reconstruction) of the shape of D and the boundary impedance was discussed in [24] by using the asymptotic behavior of the low frequency scattered waves associated with three different incident waves (or frequencies). In [16], one numerical method is proposed to determine both ∂D and the impedance $\sigma(x)$. In [1], the authors gave a uniqueness and reconstruction formula for identifying ∂D and the impedance for a Robin-type obstacle from the far-field pattern, by applying the probe method introduced by M. Ikehata (see [10], [11], [12], [13] and [14] for example). Moreover, it has also been noticed that the probe method, the singular sources method given in [23], as well as the factorization method [15], can be applied to determine the boundaries of multiple obstacles if their boundary types are the same (sound-soft or sound-hard). Now, we consider another problem: if there are many obstacles with different types of boundary such as sound-soft, sound-hard, as well as Robin-type, can we still identify their shapes and locations as well as the type of boundary for each obstacle? What is the numerical performance of the inversion scheme in this multiple-obstacle case? In [17] and [9], the theory of the factorization method was adapted and applied to

identify the shapes and locations of obstacles. Its numerical examples were given in [17]. However, the types of boundaries were not identified. So, the aim of this paper is to give a reconstruction formula for identifying not only the shapes and locations of obstacles, but also their boundary types at the same time.

Our result can be stated as follows.

THEOREM 1.1. Let D be a bounded domain consisting of finite obstacles D_j ($j = 1, 2, \dots, N$), namely, $D = \bigcup_{j=1}^N D_j$. We assume that each obstacle D_j is a simply connected bounded domain with C^2 boundary ∂D_j and $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$. For given incident plane waves $u^i(x, d) = e^{ikx \cdot d}$, consider the following scattering problem for the total wave field $u(x, d) = u^i(x, d) + u^s(x, d)$:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^3 \setminus \overline{D}, \\ B_j u(x, d) = 0, & \text{on } \partial D_j, j = 1, 2, \dots, N, \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{r}\right), & r = |x| \rightarrow \infty, \end{cases} \quad (1.3)$$

where B_j is one of the boundary operators in (1.2) for $j = 1, 2, \dots, N$. Assume that $\overline{D} \subset \Omega$ for some known sphere Ω and $0 < \sigma_j(x) \in C^1(\partial D_j)$ for Robin-type obstacles D_j . If there exists at least one Robin-type obstacle, then from the far-field pattern $u^\infty(\theta, d)$ for all $\theta, d \in S^2$, we can

- (1) determine the number of obstacles N ,
- (2) reconstruct ∂D_j for $j = 1, 2, \dots, N$,
- (3) identify the type of each obstacle D_j ,
- (4) reconstruct $\sigma_j(x)$ for the Robin-type obstacles D_j .

A similar result holds for the two dimensional case. The numerical performance of this method is given for the two dimensional case in the last section. Our main tool to deal with this problem is still the probe method. This method gives a reconstruction formula for the shape and location of an obstacle by using the indicator function and analyzing its behavior. However, there are some new ingredients in this paper. In the case of multiple obstacles, we not only have to determine the shape and location of each obstacle, but also we have to determine the number of obstacles and identify the type of each obstacle. This is the major and important difference between the multiple-obstacle inversion and the single-obstacle one. Especially, we have to extract some characteristics of the indicator function such that we can distinguish sound-hard obstacles and obstacles with Robin-type boundary, since in most cases, we can consider the Neumann boundary as a special case of the Robin boundary with $\sigma(x) = 0$. Then the most important ingredient of this paper is that we succeeded in providing a method distinguishing the sound-hard boundary and the Robin-type boundary. More precisely, we can determine the positions of obstacles and identify the sound-soft boundary from the real part of the indicator function, while distinguishing the sound-hard boundary from the Robin-type boundary is done by considering the imaginary part of the indicator function. In order to carry this out, the most important and difficult thing is to rewrite the indicator function in an appropriate form and analyze its behavior. The number of obstacles can be obtained immediately when we get the whole image of all obstacles.

We will give a mathematically rigorous reconstruction formula for recovering ∂D_j for $j = 1, 2, \dots, N$ and their boundary types. Then the uniqueness of identifying ∂D_j and the determination of the number of obstacles from $u^\infty(d, \theta)$ for all $d, \theta \in S^2$ becomes obvious from the reconstruction. Since our reconstruction procedure is pointwise, it is enough to consider the case that D consists of 3 obstacles with sound-soft, sound-hard and Robin-type boundary, respectively, and to illustrate the reconstruction procedure for identifying the location and type for each obstacle. This does not lose any generality. More precisely, we assume that D_1, D_2, D_3 are sound-soft, sound-hard and Robin-type, respectively. Once we have identified the shape, location and type of each obstacle, we determine $\sigma(x) := \sigma_3(x)$ on ∂D_3 by the moment method. So, henceforth we assume $N = 3$.

The numerical performance of our method for the two dimensional case was tested for 3 circle shaped obstacles with different types of boundaries using the numerical realization scheme of the probe method given in [8]. The numerical result shows that we can reconstruct a little more than a convex hull of multiple obstacles only; the neighboring parts of different obstacles are hard to recover. However, the center of the obstacles and the boundary types of the obstacles can be identified.

REMARK 1.2. For our problem, if $\sigma(x) \in C^1(\partial D_3)$, the well-posedness for this direct problem can be established from the standard scattering theory. That is, we can apply the radiation condition to get the uniqueness (Theorem 3.12, [5]) and use the combined single-layer and double-layer theory to get the existence of the solution.

REMARK 1.3. We can also identify D_1 and D_2 in case of $D_3 = \emptyset$ if we assume the unique solvability of the boundary value problem (2.2) given later.

REMARK 1.4. Our reconstruction method considers the inverse scattering problem with multiple obstacles with different types of boundary by firstly transforming the far-field pattern of the scattered wave to the Dirichlet-to-Neumann (D-to-N) map defined on the surface of a ball, and then recovering the obstacles from the D-to-N map by the probe method. The main difficulties are that we have to solve an integral equation of the second kind with hyper-singular kernel (see (2.4) in the sequel) from the far-field pattern to construct the D-to-N map and to provide a numerically feasible method for the Runge approximation. The first problem can be solved by the technique proposed in [19], and the second one has been solved recently by the optimization technique ([2]).

Our paper is organized as follows:

- Section 2: Preliminary results
- Section 3: Probe method
- Section 4: Moment method for determining $\sigma(x)$
- Section 5: Some estimates
- Section 6: Singularity analysis
- Section 7: Numerical performance

2. Preliminary results. In this section, we give some known results for the probe method, which are necessary for our paper.

Without loss of generality, we assume that $\overline{D} \subset B(0, \frac{R}{2})$ for some constant $R > 0$. We also assume that 0 is not a Dirichlet eigenvalue of $\Delta + k^2$ in $\Omega := B(0, R)$ for given $k > 0$.

PROPOSITION 2.1. The scattered solution $u^s(x, d)$ for $|x| > \frac{R}{2}$ can be determined uniquely from $u^\infty(d, \theta)$.

The physical background for this proposition is obvious; that is, the far-field pattern of the scattered wave determines the near-field pattern outside the obstacle completely. This procedure has nothing to do with the boundary conditions of scatterers. For the proof, see Theorem 3.6 and Corollary 3.8 in [5] or [1].

Let $G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ be the fundamental solution of the Helmholtz equation. For each $y \in R^3 \setminus \overline{D}$, we define $E(\cdot, y) \in H_{loc}^1(R^3 \setminus \overline{D})$ as the solution to

$$\begin{cases} \Delta E + k^2 E = 0, & \text{in } R^3 \setminus \overline{D}, \\ B_i E(x, y) = -B_i G(x, y), & \text{on } \partial D_i, \quad i = 1, 2, 3, \\ \frac{\partial E}{\partial r} - ikE = O\left(\frac{1}{r}\right), & r = |x| \rightarrow \infty. \end{cases} \quad (2.1)$$

PROPOSITION 2.2. For $x, y \in \partial\Omega$, $E(x, y)$, $\frac{\partial}{\partial \nu(x)} E(x, y)$ and $\frac{\partial}{\partial \nu(y)} E(x, y)$ can be determined from $u^\infty(d, \theta)$ for all $d, \theta \in S^2$.

The proof for $D = D_1 \cup D_2 \cup D_3$ given here is analogous to that given in [1] for $D = D_3$.

Proof. In fact, since we can choose R , so we assume that 0 is not the Dirichlet eigenvalue of $\Delta + k^2$. Therefore $\{e^{ikx \cdot d} \mid d \in S^2\}$ is dense in $L^2(\partial\Omega)$ ([6], Theorem 5.5). For any fixed $y \in \partial B(0, R_1)$ ($R_1 > R$), there exists a sequence $\{\alpha_j^n(y), d_j^n(y)\}$ such that

$$\sum_{1 \leq j \leq m_n(y)} \alpha_j^n(y) e^{ikx \cdot d_j^n(y)} \rightarrow G(x - y) \quad \text{in } L^2(\partial\Omega)$$

as $n \rightarrow \infty$. On the other hand, since both $\sum_{1 \leq j \leq m_n(y)} \alpha_j^n(y) e^{ikx \cdot d_j^n(y)}$ and $G(x - y)$ satisfy the Helmholtz equation in Ω , by the result in [6] (Theorem 5.4), we know that $\sum_{1 \leq j \leq m_n(y)} \alpha_j^n(y) e^{ikx \cdot d_j^n(y)} \rightarrow G(x - y)$ uniformly on any compact subset of Ω (together with all their derivatives). Therefore it follows that

$$B_i \sum_{1 \leq j \leq m_n(y)} \alpha_j^n(y) e^{ikx \cdot d_j^n(y)} \rightarrow B_i G(x - y)$$

in $L^2(\partial D_i)$, $i = 1, 2, 3$ as $n \rightarrow \infty$. Since $u^s(x, d) \in H^1(R^3 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & \text{in } R^3 \setminus \overline{D}, \\ B_i u^s = -B_i e^{ikx \cdot d}, & \text{on } \partial D_i, \quad i = 1, 2, 3, \\ \frac{\partial u^s}{\partial r} - ik u^s = O\left(\frac{1}{r}\right), & r = |x| \rightarrow \infty, \end{cases}$$

by the continuous dependence of the direct scattering problem, we have

$$\sum_{1 \leq j \leq m_n(y)} \alpha_j^n(y) u^s(x, d_j^n(y)) \rightarrow E(x, y) \quad \text{uniformly on } R/2 < |x| < 2R$$

for $y \in \partial B(0, R_1)$. Now we get from Proposition 1 that, for $y \in \partial B(0, R_1)$ and $R/2 < |x| < 2R$, $E(x, y)$, $\frac{\partial E(x, y)}{\partial \nu(x)}$, $\frac{\partial E(x, y)}{\partial \nu(y)}$ can be determined by $\{u^\infty(d, \theta) \mid \theta, d \in S^2\}$. Since R_1 is arbitrary, we complete the proof by letting $R_1 \rightarrow R$. $\square$

Consider a solution $u(x) \in H^1(\Omega \setminus \overline{D})$ to the following boundary value problem:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j u(x, t) = 0, & \text{on } \partial D_j, j = 1, 2, 3, \\ u(x) = f, & \text{on } \partial \Omega \end{cases} \quad (2.2)$$

for given $f \in H^{1/2}(\partial \Omega)$.

Since we have used $\overline{D}$ to indicate the closure of the domain D , we will use $\tilde{z}$ to indicate the complex conjugate of the complex number z in the sequel.

LEMMA 2.3. If $D_3 \neq \emptyset$, then there exists a unique solution to (2.2) for any $f \in H^{1/2}(\partial \Omega)$.

Proof. Firstly, we prove the uniqueness. It is enough to prove that $f = 0$ implies $u = 0$ in $\Omega \setminus \overline{D}$. For $f = 0$, it is easy to see from (2.2) that

$$\begin{aligned} 0 &= \int_{\Omega \setminus \overline{D}} (\Delta u + k^2 u) \tilde{u} dx = \int_{\partial D} \tilde{u} \frac{\partial u}{\partial \nu} ds - \int_{\Omega \setminus \overline{D}} (\nabla u \cdot \nabla \tilde{u} - k^2 u \tilde{u}) dx, \\ 0 &= \int_{\Omega \setminus \overline{D}} (\Delta \tilde{u} + k^2 \tilde{u}) u dx = \int_{\partial D} u \frac{\partial \tilde{u}}{\partial \nu} ds - \int_{\Omega \setminus \overline{D}} (\nabla \tilde{u} \cdot \nabla u - k^2 \tilde{u} u) dx \end{aligned}$$

due to $u = 0$ on $\partial \Omega$. Subtracting these two equalities and noticing the boundary conditions of u on ∂D_1 and ∂D_2 lead to

$$\int_{\partial D_3} \left(\tilde{u} \frac{\partial u}{\partial \nu} - u \frac{\partial \tilde{u}}{\partial \nu} \right) ds = 0.$$

Now the boundary condition in ∂D_3 leads to

$$\int_{\partial D_3} \sigma(x) |u(x)|^2 ds = 0$$

from which we get $u = 0$ on ∂D_3 and $\frac{\partial u}{\partial \nu} = 0$ on ∂D_3 from the boundary condition. Now the uniqueness of the Cauchy problem for the Helmholtz equations outside D_3 implies $u = 0$ in $\Omega \setminus \overline{D}$. On the other hand, by the integral equation method for the scattering problem ([5], [6]), we know that the direct problem (2.2) can be transformed into a Fredholm integral equation of the second kind; therefore the uniqueness implies the existence due to the Fredholm alternative theorem. $\square$

REMARK 2.4. The existence of D_3 is important to the proof of our uniqueness. If we do not have an obstacle with Robin-type boundary, i.e., $D = D_1 \cup D_2$, then we have to assume the uniqueness of a solution in $H^1(\Omega \setminus \overline{D})$ to

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \setminus (D_1 \cup D_2), \\ B_i u = 0 & \text{on } \partial D_i, i = 1, 2, \\ u(x) = f & \text{on } \partial \Omega. \end{cases} \quad (2.3)$$

Define the Dirichlet-to-Neumann map $\Lambda_{\partial D, \sigma}$ formally by

$$\Lambda_{\partial D, \sigma} : f \longrightarrow \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} \in H^{-1/2}(\partial \Omega),$$

where $u \in H^1(\Omega \setminus \overline{D})$ is the solution of (2.2) for $f \in H^{1/2}(\partial\Omega)$. In the next lemma, we show the relations between the far-field patterns and the Dirichlet-to-Neumann map.

LEMMA 2.5. Let u be the solution to (2.2) for $f \in H^{1/2}(\partial\Omega)$. Then, $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ can be obtained from $f(x)$ and $u^\infty(d, \theta)$ for $\theta, d \in S^2$.

Proof. Let $x_0 \in \partial B(0, R_0)$ for $R/2 < R_0 < R$. By Green's formula, we have that, for $G_D = G_D(x, x_0) = G(x, x_0) + E(x, x_0)$,

$$\begin{aligned} u(x_0) &= \int_{\partial\Omega} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds + \sum_{j=1}^3 \int_{\partial D_j} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds \\ &= \int_{\partial\Omega} \left(G_D \frac{\partial u}{\partial \nu_1} - f \frac{\partial G_D}{\partial \nu_1} \right) ds, \end{aligned}$$

because $B_j G_D = B_j u = 0$ on ∂D_j , where ν_1 is the outward normal to the boundary of the domain $\Omega \setminus \overline{D}$.

Taking the normal derivatives of u on $\partial B(0, R_0)$ in the above expression and letting $R_0 \rightarrow R$, by the properties of the single layer potential and the double layer potential ([5]), we have

$$\frac{1}{2} \frac{\partial u(x_0)}{\partial \nu_1(x_0)} = \int_{\partial\Omega} \frac{\partial G_D(x, x_0)}{\partial \nu_1(x_0)} \frac{\partial u(x)}{\partial \nu_1(x)} ds - \frac{\partial}{\partial \nu_1(x_0)} \int_{\partial\Omega} f(x) \frac{\partial G_D(x, x_0)}{\partial \nu_1(x)} ds \quad (2.4)$$

for $x_0 \in \partial B(0, R) = \partial\Omega$. The equation (2.4) is a Fredholm integral equation of the second kind with respect to $\frac{\partial u(x)}{\partial \nu(x)}|_{\partial\Omega}$. There exists a unique solution due to the unique solvability of (2.2).

By Proposition 2.2, we know that, for $x, y \in \partial\Omega$, $\frac{\partial E(x, y)}{\partial \nu(x)}$ and $\frac{\partial E(x, y)}{\partial \nu(y)}$ can be obtained from $u^\infty(d, \theta)$, $\theta, d \in S^2$. Therefore $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ can be obtained from $u^\infty(d, \theta)$, $\theta, d \in S^2$ and $f(x)$. The proof is complete. $\square$

From this lemma, we see that the original inverse problem can be restated as the problem of reconstructing the shapes of the 3 obstacles and the boundary impedance of D_3 from the Dirichlet-to-Neumann map $\Lambda_{\partial D, \sigma}$.

REMARK 2.6. The Dirichlet-to-Neumann map $\Lambda_{\partial D, \sigma}$ can be defined by the following weak form.

For any $g \in H^{1/2}(\partial\Omega)$, take any $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Then it follows that

$$\begin{aligned} \langle \Lambda_{\partial D, \sigma} f, g \rangle &= \int_{\Omega \setminus \overline{D}} (\nabla u \cdot \nabla v - k^2 uv) dx + \int_{\partial D} v \frac{\partial u}{\partial \nu} ds(x) \\ &= \int_{\Omega \setminus \overline{D}} (\nabla u \cdot \nabla v - k^2 uv) dx + \int_{\partial D_1} v \frac{\partial u}{\partial \nu} ds(x) - \int_{\partial D_3} i\sigma(x) uv ds(x) \quad (2.5) \end{aligned}$$

for any $f \in H^{1/2}(\partial\Omega)$, where u is the solution to (2.2) for $f \in H^{1/2}(\partial\Omega)$.

Corresponding to the case $D = \emptyset$, we can formally define the Dirichlet-to-Neumann map $\Lambda_{0,0} : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ by

$$\Lambda_{0,0} : f \rightarrow \frac{\partial u_1}{\partial \nu}|_{\partial\Omega},$$

where $u_1(x) \in H^1(\Omega)$ is the solution to

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in } \Omega, \\ u_1(x) = f \in H^{1/2}(\partial\Omega), & \text{on } \partial\Omega. \end{cases} \quad (2.6)$$

Here note that by the assumption that 0 is not the Dirichlet eigenvalue of the operator $\Delta + k^2$ in Ω , (2.6) is uniquely solvable.

The weak formula of $\Lambda_{0,0}$ is given by

$$\langle \Lambda_{0,0} f, g \rangle = \int_{\Omega} (\nabla u_1 \cdot \nabla v - k^2 u_1 v) dx, \quad (2.7)$$

where u_1 is the solution of (2.6) for $f \in H^{1/2}(\partial\Omega)$ and $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g$ for $g \in H^1(\partial\Omega)$. For the solution u of (2.2) and the solution u_1 of (2.6), we have

LEMMA 2.7. Let $u \in H^1(\Omega \setminus \overline{D})$ and $u_1 \in H^1(\Omega)$ be the solutions to (2.2) and (2.6), respectively. There exists a constant $C = C(k, R, \sigma_0)$ such that, for all $f \in H^{1/2}(\partial\Omega)$,

$$\|u - u_1\|_{H^1(\Omega \setminus \overline{D})} \leq C \|u_1\|_{H^1(D)},$$

where $\sigma_0 > 0$ is a constant satisfying $0 < \sigma(x) \leq \sigma_0$, $x \in \partial D_3$.

The proof of Lemma 2.7 is almost the same as that given in [12]. But for the reader's convenience we give the proof in Section 5.

3. Probe method.

DEFINITION 3.1. For any non-self-intersecting continuous curve $c = \{c(t) \mid 0 \leq t \leq 1\}$, if it satisfies $c(0), c(1) \in \partial\Omega$ and $c(t) \in \Omega$ ($0 < t < 1$), then we call c a needle in Ω .

DEFINITION 3.2. For any needle c in Ω , we call

$$t(c, D) = \sup\{0 < t < 1 \mid c(s) \in \Omega \setminus \overline{D} \text{ for all } 0 < s < t\}$$

the geometric impact parameter (GIP). It is obvious that $t(c, D) = 1$ if c does not touch any point on ∂D .

From this definition, we know that if a needle c touches $\overline{D}$, then $t(c, D) < 1$ and $t(c, D)$ is the first hitting time, i.e., $c(t(c, D)) \in \partial D$ and $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < t(c, D)$ if we consider t as a time.

Since $\Omega \setminus \overline{D}$ is connected, we have a reconstruction algorithm for ∂D in terms of the geometric impact parameter and the needle, i.e.,

$$\partial D = \{c(t) \mid t = t(c, D), c \text{ is a needle and } t(c, D) < 1\}. \quad (3.1)$$

Therefore, in order to reconstruct ∂D , it suffices to consider the problem of calculating the GIP for each needle from the Dirichlet-to-Neumann map.

LEMMA 3.3. Suppose that Γ is an arbitrary open set of $\partial\Omega$. For each $0 < t < 1$, there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $H^1(\Omega)$, which satisfies

$$\Delta v_n + k^2 v_n = 0$$

such that $\text{supp}(v_n|_{\partial\Omega}) \subset \Gamma$ and

$$v_n \longrightarrow G(\cdot - c(t)) \quad \text{in} \quad H_{loc}^1(\Omega \setminus c_t),$$

where $c_t := \{c(t'); 0 < t' \leq t\}$.

This result comes from the Runge approximation theorem; see [10], [11].

REMARK 3.4. Usually the Runge approximation is not constructive, because its proof is done by using the unique continuation and Hahn-Banach theorem. However, for the Helmholtz equation, it is possible to make the Runge approximation constructive by using the translation theory (see [7]).

It is obvious that $v_n|_{\partial\Omega}$ depends on $c(t)$. We denote it by $v_n|_{\partial\Omega} = f_n(\cdot, c(t))$, where $f_n(\cdot, c(t)) \in H^{1/2}(\partial\Omega)$ and $\text{supp}(f_n(\cdot, c(t))) \subset \Gamma$.

DEFINITION 3.5. For a given needle c in Ω and $0 < t < 1$, we define the indicator function

$$I(t, c) = \lim_{n \rightarrow \infty} \hat{I}_n(t, c) \quad (3.2)$$

with

$$I_n(t, c) := \langle \overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n(\cdot, c(t))}^{\sim}, f_n(\cdot, c(t)) \rangle \quad (3.3)$$

whenever it is defined, where $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, and

$\overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n(\cdot, c(t))}^{\sim}$ is the complex conjugate of $(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n(\cdot, c(t))$.

Next we show that $\Re I(t, c)$ and $\Im I(t, c)$ ($\Re, \Im$ denote the real part and imaginary part respectively) can be used to calculate GIP from which the shapes and locations of 3 obstacles can be determined, and we can also identify the type of each obstacle.

THEOREM 3.6. For a given needle $c(t)$ in Ω , it follows that

(A) $t(c, D) = 1$ if and only if $I(t, c)$ exists for all $0 \leq t \leq 1$ and

$$\sup_{0 \leq t \leq 1} |\Re I(t, c)| < +\infty.$$

(B) $T = t(c, D)$ can be characterized by

(1) $I(t, c)$ exists for all $0 \leq t < T$ and

$$\sup_{0 \leq s \leq t} |\Re(I(s, c))| < +\infty, \quad \text{for } 0 \leq t < T,$$

(2) $\lim_{t \rightarrow T-} |\Re I(t, c)| = +\infty$.

(C) We can identify ∂D_i for $i = 1, 2, 3$ by

$$\begin{aligned} \lim_{t \rightarrow t(c, D)-} \Re I(t, c) = +\infty &\iff c(t(c, D)) \in \partial D_1, \\ \lim_{t \rightarrow t(c, D)-} \Re I(t, c) = -\infty \text{ and } \lim_{t \rightarrow t(c, D)-} \Im I(t, c) < +\infty &\iff c(t(c, D)) \in \partial D_2, \\ \lim_{t \rightarrow t(c, D)-} \Re I(t, c) = -\infty \text{ and } \lim_{t \rightarrow t(c, D)-} \Im I(t, c) = +\infty &\iff c(t(c, D)) \in \partial D_3. \end{aligned}$$

REMARK 3.7. The results (A) and (B) give a criterion for the geometric impact parameter $t(c, D)$ in terms of the indicator function. It is easy to see that T with the properties (1), (2) is given by

$$T = \sup\{0 < t < 1 \mid \sup_{0 \leq s \leq t} |\Re I(s, c)| < +\infty\}.$$

Furthermore, since $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$, we can identify ∂D_i according to (C).

Proof. For a given needle $c(t)$, by Lemma 3.3, we know that there exists a sequence $\{v_n(x)\} \in H^1(\Omega)$ which satisfies

$$\begin{cases} \Delta v_n + k^2 v_n = 0, & \text{in } \Omega, \\ v_n = f_n(\cdot, c(t)), & \text{on } \partial\Omega, \end{cases} \quad \text{supp}(f_n(\cdot, c(t))) \subset \Gamma,$$

and

$$v_n \longrightarrow G(\cdot - c(t)) \quad \text{in } H_{loc}^1(\Omega \setminus c_t) \quad (n \longrightarrow \infty).$$

Let $u_n(x) \in H^1(\Omega \setminus \overline{D})$ satisfy

$$\begin{cases} \Delta u_n + k^2 u_n = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_i u_n = 0, & \text{on } \partial D_i, \quad i = 1, 2, 3, \\ u_n = f_n, & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

Then $w_n = u_n - v_n|_{\Omega \setminus \overline{D}} \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta w_n + k^2 w_n = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j w_n = -B_j v_n, & \text{on } \partial D_j, \quad j = 1, 2, 3, \\ w_n = 0, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.7 and Lemma 3.3, we know that, for $c_t \cap \overline{D} = \emptyset$, it follows that

$$w_n \longrightarrow w \quad \text{in } H^1(\Omega \setminus \overline{D}), \quad n \longrightarrow \infty, \quad (3.5)$$

where $w = w(x, c(t))$ satisfies

$$\begin{cases} \Delta w + k^2 w = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j w = -B_j G(\cdot - c(t)), & \text{on } \partial D_j, \quad j = 1, 2, 3, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.6)$$

On the other hand, by the calculation in Section 6, we have two kinds of expressions for

$\langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) f_n, f_n \rangle$, i.e.,

$$\begin{aligned} & \overbrace{\langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) f_n(\cdot, c(t)), f_n(\cdot, c(t)) \rangle} \\ &= - \int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\ & \quad + \int_{\partial D_3} (2i\sigma v_n \tilde{w}_n + i\sigma |v_n|^2 + i\sigma |w_n|^2) ds \\ & \quad + \int_{\partial D_1} \left[\left(v_n \frac{\partial \tilde{w}_n}{\partial \nu} - \tilde{w}_n \frac{\partial v_n}{\partial \nu} \right) + \left(v_n \frac{\partial \tilde{v}_n}{\partial \nu} - \tilde{v}_n \frac{\partial w_n}{\partial \nu} \right) \right] ds. \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \overbrace{\langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) f_n(\cdot, c(t)), f_n(\cdot, c(t)) \rangle} \\ &= \int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx + \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\ & \quad - \int_{\partial D_2} \left[\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} \right] ds - \int_{\partial D_3} \left[\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} - i\sigma |u_n|^2 \right] ds. \end{aligned} \quad (3.8)$$

Let n tend to infinity in (3.7). Then, by (3.5), we have

$$\begin{aligned}
-I(t, c) &= \int_D \{|\nabla G(\cdot - c(t))|^2 - k^2 |G(\cdot - c(t))|^2\} dx \\
&+ \int_{\Omega \setminus \overline{D}} \{|\nabla w|^2 - k^2 |w|^2\} dx \\
&- i \int_{\partial D_3} \sigma(x) \{ |G(\cdot - c(t))|^2 + |w|^2 \} ds - 2i \int_{\partial D_3} \sigma(x) \tilde{w} G ds \\
&- \int_{\partial D_1} \left[\left(G \frac{\partial \tilde{w}}{\partial \nu} - \tilde{w} \frac{\partial G}{\partial \nu} \right) + \left(G \frac{\partial \tilde{G}}{\partial \nu} - \tilde{w} \frac{\partial w}{\partial \nu} \right) \right] ds. \tag{3.9}
\end{aligned}$$

We first prove (A). If $t(c, D) = 1$, then, by the definition of $t(c, D)$, we know that $c(t)$ does not touch $\overline{D}$, i.e. $c(t) \in \Omega \setminus \overline{D}$ for $0 \leq t \leq 1$. It is easy to verify that

$$\sup_{0 \leq t \leq 1} |\Re I(t, c)| < +\infty.$$

The converse is true if we prove (B). So, let's go to the proof of (B). Let $t(c, D) < 1$ and $x_0 = c(t(c, D)) \in \partial D$. Then, it is easy to see that we only have to prove $\lim_{t \rightarrow t(c, D)-} |\Re I(t, c)| = +\infty$. This is included in the proof of (C). So the rest of the argument is devoted to the proof of (C).

Likewise before, since $c(t) \in \Omega \setminus \overline{D}$ ($0 \leq t < t(c, D)$), we have $\sup_{0 \leq s \leq t} |\Re I(s, c)| < +\infty$ for $0 \leq t < t(c, D)$. Now observe that

$$\begin{aligned}
&- \Re I(t, c) = \int_D [|\nabla G(\cdot - c(t))|^2 - k^2 |G(\cdot - c(t))|^2] dx \\
&+ \int_{\Omega \setminus \overline{D}} (|\nabla w|^2 - k^2 |w|^2) dx + 2 \int_{\partial D_3} \sigma(x) \Im(\tilde{w} G(\cdot - c(t))) ds \\
&- \Re \left[\int_{\partial D_1} \left(G \frac{\partial \tilde{w}}{\partial \nu} - \tilde{w} \frac{\partial G}{\partial \nu} + G \frac{\partial \tilde{G}}{\partial \nu} - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds \right] \\
&\geq \int_D |\nabla G(\cdot - c(t))|^2 dx - k^2 \int_D |G(\cdot - c(t))|^2 dx - k^2 \int_{\Omega \setminus \overline{D}} |w|^2 dx \\
&+ 2 \int_{\partial D_3} \sigma(x) \Im(\tilde{w} G(\cdot - c(t))) ds \\
&- \Re \left[\int_{\partial D_1} \left(G \frac{\partial \tilde{w}}{\partial \nu} - \tilde{w} \frac{\partial G}{\partial \nu} + G \frac{\partial \tilde{G}}{\partial \nu} - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds \right]. \tag{3.10}
\end{aligned}$$

According to the result of singularity analysis about $w(x, x_0)$ and $G(x - x_0)$ for $x_0 \in \partial D$ given in Theorem 5.1 and Theorem 5.2 below, we have from (3.10),

$$\lim_{t \rightarrow t(c, D_j)-} \Re I(t, c) = -\infty$$

if $j = 2, 3$. On the other hand, consider the real part of the limit of (3.8) as $n \rightarrow \infty$. It is easy to find that the real part will tend to $+\infty$ when $c(t) \rightarrow \partial D_1$, since $\int_D |\nabla G(\cdot - c(t))|^2 dx$ will blow up, while the integrals on the boundary are clearly bounded and the L^2 integral of w is bounded by Theorem 5.1. These facts imply that we can

distinguish the sound-soft boundary D_1 from the other two kinds of boundaries (sound-hard and Robin-type). Now we want to distinguish ∂D_2 and ∂D_3 furthermore. For this purpose, we need to consider the imaginary part of (3.8). In fact, it yields from (3.7) that

$$\Im(\langle \overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n}^{\sim}, f_n \rangle) = \int_{\partial D_3} \sigma(x) |u_n|^2 ds. \quad (3.11)$$

Now we estimate the behavior of the imaginary part of the indicator function. Recalling our previous notation, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \Im(\langle \overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n}^{\sim}, f_n \rangle) &= \lim_{n \rightarrow \infty} \int_{\partial D_3} \sigma(x) |u_n|^2 ds \\ &= \lim_{n \rightarrow \infty} \int_{\partial D_3} \sigma(x) |v_n + w_n|^2 ds \\ &= \int_{\partial D_3} \sigma(x) |(G(x - c(t))) + w(x, c(t))|^2 ds, \end{aligned} \quad (3.12)$$

where w is the function defined by (3.6). According to the singularity analysis in section 5 and section 6, we know that $|G(x - c(t)) + w(x, c(t))|$ is estimated by $|G(x - c(t))|$. Hence, from (3.12) and the estimate for G in section 5, we have

$$\begin{aligned} \lim_{t \rightarrow t(c, D_3)-} \Im I(t, c) &= \lim_{t \rightarrow t(c, D_3)-} \int_{\partial D_3} \sigma(x) |(G(x - c(t))) + w(x, c(t))|^2 ds = +\infty, \\ \lim_{t \rightarrow t(c, D_2)-} \Im I(t, c) &= \lim_{t \rightarrow t(c, D_2)-} \int_{\partial D_3} \sigma(x) |G(x - c(t)) + w(x, c(t))|^2 ds < \infty. \end{aligned}$$

Since D_2 and D_3 are separated, these behaviors of $\Im I(t, c)$ enable us to distinguish ∂D_3 and ∂D_2 .

The proof is complete. $\square$

Now we give the reconstruction procedure for the shape and type of each obstacle. It can be realized by the following steps:

- Calculate the Dirichlet-to-Neumann map $\Lambda_{\sigma, D}$ from the far-field patterns $u^\infty(d, \theta)$, $d, \theta \in S^2$.
- For any given needle $c(t)$, calculate the sequences v_n and $f_n(\cdot, c)$.
- Calculate $\langle \overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n(\cdot, c(t))}^{\sim}, f_n(\cdot, c(t)) \rangle$.
- Calculate $I(t, c)$ and

$$\partial D_1 = \{c(t_0) \mid \sup_{0 \leq s \leq t} |\Re I(s, c)| < +\infty \text{ for } 0 \leq t < t_0, \lim_{t \rightarrow t_0-} \Re I(t, c) = +\infty\},$$

$$\begin{aligned} \partial D_2 = \{c(t_0) \mid \sup_{0 \leq s \leq t} |\Re I(s, c)|, \sup_{0 \leq s \leq t} |\Im I(s, c)| < +\infty \text{ for } 0 \leq t < t_0, \\ \lim_{t \rightarrow t_0-} \Re I(t, c) = -\infty \text{ and } \lim_{t \rightarrow t_0-} \Im I(t, c) < +\infty\}, \end{aligned}$$

$$\begin{aligned} \partial D_3 = \{c(t_0) \mid \sup_{0 \leq s \leq t} |\Re I(s, c)|, \sup_{0 \leq s \leq t} |\Im I(s, c)| < +\infty \text{ for } 0 \leq t < t_0, \\ \lim_{t \rightarrow t_0-} \Re I(t, c) = -\infty \text{ and } \lim_{t \rightarrow t_0-} \Im I(t, c) = +\infty\}. \end{aligned}$$

The rest of the proof of Theorem 1.1 is to reconstruct the boundary impedance on D_3 . This will be given in the next section.

4. Moment method for determining $\sigma(x)$. In this section, we reconstruct the boundary impedance $\sigma(x)$. Since in the previous section, we have reconstructed ∂D from the far-field patterns $u^\infty(d, \theta)$, $d, \theta \in S^2$, therefore in this section we assume that $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$ is known.

Consider the boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j u(x) = 0, & \text{on } \partial D_j, \quad j = 1, 2, \\ \frac{\partial u}{\partial \nu} + i\sigma(x)u = 0, & \text{on } \partial D_3, \\ u(x) = f, & \text{on } \partial \Omega \end{cases} \quad (4.1)$$

for a given $f(x) \in H^{1/2}(\partial \Omega)$.

LEMMA 4.1. Suppose that $u_j(x) \in H^1(\Omega \setminus \overline{D})$, $j = 1, 2, \dots$ satisfy (4.1) with $f = f_j$. Put $\phi_j(x) = u_j(x)|_{\partial D_3}$. If

$$\overline{\text{span}\{f_j(x)\}} = H^{1/2}(\partial \Omega), \quad (4.2)$$

then we have

$$\overline{\text{span}\{\phi_j(x)\}} = H^{1/2}(\partial D_3).$$

Proof. Assume that $f(x) \in H^{-1/2}(\partial D_3)$ satisfies

$$\int_{\partial D_3} \phi_j \tilde{f} ds = 0, \quad j = 1, 2, \dots \quad (4.3)$$

We want to prove that $f(x) = 0$. Here $\int_{\partial D_3} \phi_j \tilde{f} ds$ denotes the pairing $\langle \tilde{f}, \phi_j \rangle$ between $H^{-1/2}(\partial D_3)$ and $H^{1/2}(\partial D_3)$.

Consider the following boundary value problem:

$$\begin{cases} \Delta v + k^2 v = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j v(x) = 0, & \text{on } \partial D_j, \quad j = 1, 2, \\ \frac{\partial v}{\partial \nu} + i\sigma(x)v = \tilde{f}, & \text{on } \partial D_3, \\ v = 0, & \text{on } \partial \Omega. \end{cases} \quad (4.4)$$

Since $\sigma(x) > 0$, similar to the proof of Lemma 2.3, we know there exists a unique solution $v \in H^1(\Omega \setminus \overline{D})$ for (4.4).

By Green's formula, we know that

$$\begin{aligned} 0 &= \int_{\Omega \setminus \overline{D}} (v \Delta u_j - u_j \Delta v) dx \\ &= \int_{\partial \Omega} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right) + \int_{\partial D} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right), \end{aligned} \quad (4.5)$$

where ν_1 is the outward normal of the domain $\Omega \setminus \overline{D}$.

Noticing $\nu_1 = -\nu$ on ∂D and $v|_{\partial \Omega} = 0$, we have

$$\int_{\partial \Omega} \frac{\partial v}{\partial \nu} u_j ds = \int_{\partial D_3} \left(i\sigma v u_j + \frac{\partial v}{\partial \nu} u_j \right) ds$$

due to the boundary conditions on ∂D_1 and ∂D_2 . Therefore, it follows that

$$\int_{\partial \Omega} f_j \frac{\partial v}{\partial \nu} ds = \int_{\partial D_3} \phi_j \tilde{f} ds = 0, \quad j = 1, 2, \dots$$

Since $\overline{\text{span}\{f_j(x)\}} = H^{1/2}(\partial\Omega)$, we obtain

$$\frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0.$$

By the uniqueness of the Cauchy problem for the Helmholtz equations in the domain $\Omega \setminus \overline{D}$, we have $v(x) = 0$ in $\Omega \setminus \overline{D}$. Then by (4.4), we know that $f(x) = 0$. The proof is complete. $\square$

On the other hand, we can obtain $u_j|_{\partial D}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D}$ by solving the following Cauchy problem:

$$\begin{cases} \Delta u_j + k^2 u_j = 0, & \text{in } \Omega \setminus \overline{D}, \\ u_j = f_j, \frac{\partial u_j}{\partial \nu} = \Lambda_{\partial D, \sigma} f_j, & \text{on } \partial\Omega \end{cases} \quad (4.6)$$

for a given $f_j(x)$; hence both $u_j|_{\partial D_3}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D_3}$ are obtained.

Now, by integrating the Robin-type boundary condition over ∂D_3 , we have that the impedance $\sigma(x)$ satisfies

$$\int_{\partial D_3} i\sigma(x) u_j ds = - \int_{\partial D_3} \frac{\partial u_j}{\partial \nu} ds, \quad j = 1, 2, \dots. \quad (4.7)$$

Here note that $\text{span}\{u_j|_{\partial D_3}\}$ is dense in $H^{1/2}(\partial D_3)$ by Lemma 4.1; hence $\sigma(x)$ can be determined uniquely from this moment problem.

Now the recovery of the impedance $\sigma(x)$ can be realized by the following steps:

- Choose f_j , $j = 1, 2, \dots$ such that $\overline{\text{span}\{f_j\}_{j=1}^\infty} = H^{1/2}(\partial\Omega)$.
- For every f_j , solve the Cauchy problem (4.6) and obtain $u_j|_{\partial D_3}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D_3}$.
- Solve the moment problem (4.7) to get $\sigma(x)$.

5. Some estimates. In this section we give the proof of Lemma 2.7 and an estimate of $\|w\|_{L^2(\Omega \setminus \overline{D})}$.

Proof of Lemma 2.7. Let $p(x) = u(x) - u_1(x)|_{\Omega \setminus \overline{D}}$. It is easy to verify that $p(x)$ satisfies

$$\begin{cases} \Delta p + k^2 p = 0, & \text{in } \Omega \setminus \overline{D}, \\ B_j p = -B_j u_1, & \text{on } \partial D_j, j = 1, 2, 3, \\ p(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

By the well-posedness of the boundary value problem (5.1), we know that the solution $p(x) \in H^1(\Omega \setminus \overline{D})$ depends continuously on the boundary data on $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$. Therefore, there exists a constant $C > 0$ such that

$$\|p\|_{H^1(\Omega \setminus \overline{D})} \leq C \{ \|B_1 u_1\|_{H^{1/2}(\partial D_1)} + \|B_2 u_1\|_{H^{-1/2}(\partial D_2)} + \|B_3 u_1\|_{H^{-1/2}(\partial D_3)} \}. \quad (5.2)$$

On the other hand, the trace theorem for $u_1(x)$ on the domain D yields

$$\|B_1 u_1\|_{H^{1/2}(\partial D_1)} + \|B_2 u_1\|_{H^{-1/2}(\partial D_2)} + \|B_3 u_1\|_{H^{-1/2}(\partial D_3)} \leq C \|u_1\|_{H^1(D)}$$

due to $0 < \sigma(x) < \sigma_0$, and the proof is complete. $\square$

THEOREM 5.1. There exists a constant C independent of D such that

$$\|w\|_{L^2(\Omega \setminus \overline{D})} \leq C.$$

Proof. We adapt the proof of [12] to our case. First we define a function $v(x)$ by

$$\begin{cases} \Delta v + k^2 v = w, & \text{in } \Omega \setminus \overline{D}, \\ B_j v = 0, & \text{on } \partial D_j, \quad j = 1, 2, \\ \frac{\partial v}{\partial \nu} + i\tilde{\sigma} v = 0, & \text{on } \partial D_3, \\ v(x) = 0, & \text{on } \partial \Omega. \end{cases} \quad (5.3)$$

Then, from the well-posedness of this boundary problem, we have

$$\|v\|_{H^2(\Omega \setminus \overline{D})} \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}. \quad (5.4)$$

Since $\Omega \setminus \overline{D}$ is a domain in R^3 with C^2 boundary, by the Sobolev embedding theorems, we know that $H^2(\Omega \setminus \overline{D})$ can be embedded into $B^{1/2}(\Omega \setminus \overline{D})$ (Hölder space with exponent $1/2$) and $\|v\|_{B^{1/2}} \leq C \|v\|_{H^2}$. So we have

$$\|v\|_{B^{1/2}} \leq C \|w\|_{L^2}.$$

From this inequality, we know that

$$\begin{cases} |v(x) - v(y)| \leq C|x - y|^{1/2} \|w\|_{L^2(\Omega \setminus \overline{D})}, & x, y \in \Omega \setminus \overline{D}, \\ \|v\|_{L^\infty(\Omega \setminus \overline{D})} \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}. \end{cases} \quad (5.5)$$

Recalling the definition of weak solutions w and v to (3.6) and (5.3) respectively, by the Green formula and the boundary conditions for v, w , we have

$$\begin{aligned} 0 &= \int_{\Omega \setminus \overline{D}} (\Delta + k^2) \tilde{w}(x) v(x) dx \\ &= - \int_{\partial D} v \frac{\partial \tilde{w}}{\partial \nu} ds - \int_{\Omega \setminus \overline{D}} (\nabla \tilde{w}(x) \nabla v(x) - k^2 \tilde{w}(x) v(x)) dx \\ &= \int_{\partial D_2} v \frac{\partial \tilde{G}(x - c(t))}{\partial \nu} ds + \int_{\partial D_3} v \left[\tilde{w} i \tilde{\sigma} + \left(\frac{\partial}{\partial \nu} + i \tilde{\sigma} \right) \tilde{G} \right] ds \\ &\quad - \int_{\Omega \setminus \overline{D}} (\nabla \tilde{w}(x) \nabla v(x) - k^2 \tilde{w}(x) v(x)) dx, \end{aligned}$$

which yields

$$\begin{aligned} &\int_{\Omega \setminus \overline{D}} |w(x)|^2 dx = \int_{\Omega \setminus \overline{D}} (\Delta + k^2) v \tilde{w} dx \\ &= \int_{\partial D_1} \tilde{G} \frac{\partial v}{\partial \nu} ds + \int_{\partial D_3} \tilde{w} i \tilde{\sigma} v ds - \int_{\Omega \setminus \overline{D}} (\nabla \tilde{w}(x) \cdot \nabla v(x) - k^2 \tilde{w}(x) v(x)) dx \\ &= \int_{\partial D_1} \tilde{G} \frac{\partial v}{\partial \nu} ds - \int_{\partial D_2} v \frac{\partial \tilde{G}}{\partial \nu} ds - \int_{\partial D_3} v \left(\frac{\partial}{\partial \nu} + i \tilde{\sigma} \right) \tilde{G} ds \\ &= \int_{\partial D_1} \tilde{G} \frac{\partial v}{\partial \nu} ds - \int_{\partial D_3} v i \tilde{\sigma} \tilde{G} ds - v(c(t)) \left[\int_{\partial D_2} + \int_{\partial D_3} \right] \frac{\partial \tilde{G}}{\partial \nu} ds \\ &\quad - \left[\int_{\partial D_2} + \int_{\partial D_3} \right] (v(x) - v(c(t))) \frac{\partial \tilde{G}(x - c(t))}{\partial \nu} ds. \end{aligned} \quad (5.6)$$

On the other hand, if $y \notin \overline{D}_j$, we have

$$\int_{\partial D_j} \frac{\partial}{\partial \nu} \tilde{G}(x - y) ds + k^2 \int_{D_j} \tilde{G}(x - y) dx = \int_{D_j} (\Delta + k^2) \tilde{G}(x - y) dx = 0.$$

Therefore (5.6) leads to

$$\begin{aligned}
& \|w\|_{L^2(\Omega \setminus \overline{D})}^2 \\
&= \int_{\partial D_1} \tilde{G} \frac{\partial v}{\partial \nu} ds - \int_{\partial D_3} v \widetilde{\sigma G} ds + k^2 v(c(t)) \left(\int_{D_2} + \int_{D_3} \right) \tilde{G}(x - c(t)) dx \\
&\quad - \left(\int_{\partial D_2} + \int_{\partial D_3} \right) (v(x) - v(c(t))) \frac{\partial \tilde{G}(x - c(t))}{\partial \nu} ds.
\end{aligned} \tag{5.7}$$

On the one hand, (5.5) tells us

$$\frac{|v(x) - v(y)|}{|x - y|} \leq C \frac{1}{|x - y|^{1/2}} \|w\|_{L^2(\Omega \setminus \overline{D})},$$

which implies

$$\left| \int_{\partial D_1} \tilde{G} \frac{\partial v}{\partial \nu} ds \right| \leq C \|w\|_{L^2(\Omega \setminus \overline{D})} \int_{\partial D_1} |\tilde{G}(x - c(t))| \frac{1}{|x - c(t)|^{1/2}} ds \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}$$

as $c(t) \rightarrow \partial D_1$. On the other hand, the integrals $\int_{\partial D} |\tilde{G}(x - c(t))| ds$, $\int_D |\tilde{G}(x - c(t))| dx$ and $\int_{\partial D} |x - c(t)|^{1/2} \left| \frac{\partial}{\partial \nu} \tilde{G}(x - c(t)) \right| ds$ are bounded as $c(t) \rightarrow \partial D$. Therefore by (5.5) and (5.7), we have

$$\|w\|_{L^2(\Omega \setminus \overline{D})}^2 \leq C \|w\|_{L^2(\Omega \setminus \overline{D})}.$$

The proof is complete. $\square$

THEOREM 5.2. Assume $x_0 \in \partial D$ and $c(t) \in (\Omega \setminus \overline{D}) \cap \partial B(x_0, \delta)$ for some $\delta > 0$, where $B(x_0, \delta)$ is an open ball centered at x_0 with radius δ , then there exists some constant $C > 0$ such that for δ small enough the following estimates hold:

$$\begin{aligned}
\int_D |\nabla G(x - c(t))|^2 dx &\geq \frac{C}{\delta}, & \int_D |G(x - c(t))|^2 dx &\leq C, \\
\int_{\partial D} |G(x - c(t))|^2 ds &\leq C |\ln \delta|, & \int_{\partial D} |w(x, c(t))|^2 ds &\leq C \int_{\partial D} |G(x - c(t))|^2 ds,
\end{aligned}$$

where D should be D_j for $j = 1, 2, 3$ in the former three estimates and for $j = 2, 3$ in the fourth estimate; the constants $C > 0$ may be different.

Proof. Except the fourth estimate, the proofs of the estimates are given in [1]. For the reader's convenience we repeat them. Denote the tangent plane of ∂D at point x_0 by $T(x_0, \partial D)$. From the expressions of Green's function, we have

$$|\nabla G(x - c(t))|^2 = O\left(\frac{1}{|x - c(t)|^4}\right), \quad |G(x - c(t))|^2 = O\left(\frac{1}{|x - c(t)|^2}\right). \tag{5.8}$$

We have that, for δ small enough,

$$\begin{aligned}
\int_D \frac{1}{|x - c(t)|^4} dx &\geq \int_{D \cap B(x_0, \delta)} \frac{1}{|x - c(t)|^4} dx \geq \int_{D \cap B(x_0, \delta)} \frac{1}{(2\delta)^4} dx \\
&= \frac{1}{(2\delta)^4} \int_{D \cap B(x_0, \delta)} dx \geq \frac{1}{(2\delta)^4} \frac{1}{4} \int_{B(x_0, \delta)} dx = \frac{C}{\delta}.
\end{aligned} \tag{5.9}$$

Hence we have obtained the first estimate. The second estimate is obvious.

On the other hand, let $c'(t) \in \Omega \setminus \overline{D}$ satisfy

$$c'(t) \in \partial B(x_0, \delta), \quad c'(t) - x_0 \text{ is perpendicular to } T(x_0, \partial D).$$

Then,

$$\int_{\partial D} |G(x - c(t))|^2 dx \leq C \left(\int_{\partial D_1} + \int_{\partial D_2} \right) \frac{1}{|x - c(t)|^2} dx,$$

where

$$\begin{aligned} \partial D_1 &:= \partial D \cap \{x \in R^3 \mid |x - c'(t)|^2 \geq \frac{1}{|\ln \delta|}\}, \\ \partial D_2 &:= \partial D \cap \{x \in R^3 \mid |x - c'(t)|^2 \leq \frac{1}{|\ln \delta|}\}. \end{aligned}$$

Since $\frac{1}{2} \frac{1}{|\ln \delta|^{1/2}} - 2\delta \geq 0$ for small enough δ and

$$|c(t) - c'(t)| \leq 2\delta, \quad |x - c(t)| \geq \frac{1}{2} \frac{1}{|\ln \delta|^{1/2}} \quad (x \in \partial D_1),$$

we have

$$\int_{\partial D_1} \frac{1}{|x - c(t)|^2} ds \leq 4 |\ln \delta| \int_{\partial D_1} ds \leq C |\ln \delta|. \quad (5.10)$$

For the second integral, since

$$\partial D_2' = \{x \in R^3 \mid x \in T(x_0, \partial D), |x - x_0|^2 \leq \frac{2}{|\ln \delta|} - \delta^2\}$$

approximates ∂D_2 for small $\delta > 0$, we know that

$$\begin{aligned} \int_{\partial D_2} \frac{1}{|x - c(t)|^2} ds &\leq 2 \int_{\partial D_2'} \frac{1}{|x - c(t)|^2} ds = \int_{\partial D_2'} \frac{2}{|x - x_0|^2 + \delta^2} ds \\ &= 2 \int_0^{2\pi} \int_0^{\sqrt{2|\ln \delta|^{-1} - \delta^2}} \frac{r dr d\theta}{r^2 + \delta^2} \\ &= 4\pi \left(|\ln \delta| + \frac{1}{2} \ln 2 - \frac{1}{2} \ln(|\ln \delta|) \right) \leq C |\ln \delta| \end{aligned} \quad (5.11)$$

for $\delta > 0$ small enough. Then the third estimate follows from (5.10)-(5.11).

The fourth estimate will be given in the next section. The proof is complete. $\square$

6. Singularity analysis.

6.1. *Expression of $\Lambda_{\partial D, \sigma} - \Lambda_{0,0}$.* Here we prove (3.7). Let $v(x) \in H^1(\Omega \setminus \overline{D})$. From the definition of the weak solution of u_n to (3.4), we have

$$0 = \int_{\partial \Omega} v \frac{\partial u_n}{\partial \nu_1} ds + \int_{\partial D} v \frac{\partial u_n}{\partial \nu_1} ds - \int_{\Omega \setminus \overline{D}} (\nabla u_n \nabla v - k^2 v u_n) dx.$$

Hence, recalling the boundary condition of u_n , we get

$$\int_{\partial \Omega} v \frac{\partial u_n}{\partial \nu_1} ds = \int_{\Omega \setminus \overline{D}} (\nabla u_n \nabla v - k^2 v u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} v ds - \int_{\partial D_3} i \sigma u_n v ds. \quad (6.1)$$

Firstly, taking $v = \tilde{w}_n$ in this expression, we get that

$$0 = \int_{\Omega \setminus \overline{D}} (\nabla u_n \nabla \tilde{w}_n - k^2 \tilde{w}_n u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} \tilde{w}_n ds - \int_{\partial D_3} i \sigma u_n \tilde{w}_n ds. \quad (6.2)$$

due to $w_n|_{\Omega} = 0$. On the other hand, by taking $v = \tilde{v}_n$ also in the above expression, we have

$$\langle \Lambda_{\partial D, \sigma} f_n, \tilde{f}_n \rangle = \int_{\Omega \setminus \overline{D}} (\nabla u_n \nabla \tilde{v}_n - k^2 \tilde{v}_n u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} \tilde{v}_n ds - \int_{\partial D_3} i \sigma u_n \tilde{v}_n ds. \quad (6.3)$$

Analogously, we have

$$\langle \Lambda_{0,0} f_n, \tilde{f}_n \rangle = \int_{\Omega} \frac{\partial v_n}{\partial \nu_1} \tilde{v}_n ds = \int_{\Omega} (\nabla v_n \nabla \tilde{v}_n - k^2 v_n \tilde{v}_n) dx. \quad (6.4)$$

From the above expression, we get that

$$\begin{aligned} \overbrace{\langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) f_n, \tilde{f}_n \rangle} &= \int_{\Omega \setminus \overline{D}} (\nabla \tilde{w}_n \nabla v_n - k^2 \tilde{w}_n v_n) dx + \int_{\partial D_1} \frac{\partial \tilde{u}_n}{\partial \nu} v_n ds \\ &\quad - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx - \int_{\partial D_3} i \tilde{\sigma} \tilde{u}_n v_n ds. \end{aligned} \quad (6.5)$$

Recall $w_n = u_n - v_n$, (6.2) and consider

$$\begin{aligned} J_n &:= \int_{\Omega \setminus \overline{D}} (\nabla w_n \nabla \tilde{w}_n - k^2 w_n \tilde{w}_n) dx - \int_{\partial D_3} i \sigma w_n \tilde{w}_n ds + \int_{\partial D_1} \tilde{w}_n \frac{\partial w_n}{\partial \nu} ds \\ &= - \int_{\Omega \setminus \overline{D}} (\nabla v_n \nabla \tilde{w}_n - k^2 v_n \tilde{w}_n) dx + \int_{\partial D_3} i \sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds. \end{aligned}$$

Then, we get

$$\int_{\Omega \setminus \overline{D}} (\nabla v_n \nabla \tilde{w}_n - k^2 v_n \tilde{w}_n) dx = -J_n + \int_{\partial D_3} i \sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds. \quad (6.6)$$

Inserting this expression into (6.5), and recalling (6.2), (6.3) leads to

$$\begin{aligned} &\overbrace{\langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) f_n, f_n \rangle} \\ &= -J_n + \int_{\partial D_3} i \sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\ &\quad + \int_{\partial D_1} \frac{\partial \tilde{u}_n}{\partial \nu} v_n ds - \int_{\partial D_3} i \tilde{\sigma} \tilde{u}_n v_n ds \\ &= - \int_{\Omega \setminus \overline{D}} (\nabla w_n \nabla \tilde{w}_n - k^2 w_n \tilde{w}_n) dx + \int_{\partial D_3} i \sigma w_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial w_n}{\partial \nu} ds \\ &\quad + \int_{\partial D_3} i \sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\ &\quad + \int_{\partial D_1} \frac{\partial (\tilde{v}_n + \tilde{w}_n)}{\partial \nu} v_n ds - \int_{\partial D_3} i \tilde{\sigma} (\tilde{v}_n + \tilde{w}_n) v_n ds \\ &= - \int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\ &\quad + \int_{\partial D_3} (2i \sigma v_n \tilde{w}_n + i \sigma |v_n|^2 + i \sigma |w_n|^2) ds \\ &\quad + \int_{\partial D_1} \left[\left(v_n \frac{\partial \tilde{w}_n}{\partial \nu} - \tilde{w}_n \frac{\partial v_n}{\partial \nu} \right) + \left(v_n \frac{\partial \tilde{v}_n}{\partial \nu} - \tilde{v}_n \frac{\partial w_n}{\partial \nu} \right) \right] ds. \end{aligned} \quad (6.7)$$

This expression will be used to identify ∂D_1 from ∂D . Now we prove the other expression

(3.8) for $\langle \widetilde{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n}, f_n \rangle$, which applies the value of v_n and w_n on ∂D_2 and ∂D_3 . This expression will be applied furthermore to distinguish ∂D_2 and ∂D_3 from ∂D .

By a straightforward calculation, we get

$$\begin{aligned} \int_{\Omega \setminus \overline{D}} |\nabla(u_n - v_n)|^2 dx &= \int_{\partial \Omega} u_n \frac{\partial(\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds - \int_{\partial D} u_n \frac{\partial(\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds \\ &\quad + \int_{\Omega \setminus \overline{D}} k^2 u_n (\tilde{u}_n - \tilde{v}_n) dx - \int_{\Omega \setminus \overline{D}} \nabla v_n \cdot \nabla(\tilde{u}_n - \tilde{v}_n) dx. \end{aligned}$$

From the definition of the Dirichlet-to-Neumann map, this generates

$$\begin{aligned} \int_{\Omega \setminus \overline{D}} |\nabla w_n|^2 dx &= \langle \widetilde{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})f_n}, f_n \rangle - \int_{\partial D} u_n \frac{\partial(\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds \\ &\quad + \int_{\Omega \setminus \overline{D}} k^2 u_n (\tilde{u}_n - \tilde{v}_n) dx - \int_{\Omega \setminus \overline{D}} \nabla v_n \cdot \nabla \tilde{u}_n dx + \int_{\Omega \setminus \overline{D}} |\nabla v_n|^2 dx. \end{aligned} \quad (6.8)$$

Applying $\nabla v_n \cdot \nabla \tilde{u}_n = \nabla \cdot (\tilde{u}_n \nabla v_n) - \tilde{u}_n \Delta v_n$ in this expression, we get

$$\begin{aligned} & - \int_{\Omega \setminus \overline{D}} \nabla v_n \cdot \nabla \tilde{u}_n dx + \int_{\Omega \setminus \overline{D}} |\nabla v_n|^2 dx = - \int_{\partial \Omega} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds + \int_{\partial D} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds \\ & - k^2 \int_{\Omega \setminus \overline{D}} \tilde{u}_n v_n dx + \int_{\Omega} |\nabla v_n|^2 dx - \int_D |\nabla v_n|^2 dx. \end{aligned} \quad (6.9)$$

Also by applying $|\nabla v_n|^2 = \nabla \cdot (\tilde{v}_n \nabla v_n) - \tilde{v}_n \Delta v_n$ and noticing $w_n = u_n - v_n = 0$ on $\partial \Omega$ in (6.9), we get

$$\begin{aligned} & - \int_{\Omega \setminus \overline{D}} \nabla v_n \cdot \nabla \tilde{u}_n dx + \int_{\Omega \setminus \overline{D}} |\nabla v_n|^2 dx = \int_{\partial D} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds - k^2 \int_{\Omega \setminus \overline{D}} \tilde{u}_n v_n dx \\ & - \int_{\Omega} \tilde{v}_n \Delta v_n dx - \int_D |\nabla v_n|^2 dx. \end{aligned} \quad (6.10)$$

Now inserting (6.10) into (6.8) yields

$$\begin{aligned} & \int_{\Omega \setminus \overline{D}} |\nabla w_n|^2 dx \\ &= \langle (\Lambda_{\partial D, \sigma} - \Lambda_{0,0}) \tilde{f}_n, f_n \rangle + \int_{\partial D} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} - u_n \frac{\partial(\tilde{u}_n - \tilde{v}_n)}{\partial \nu} \right) ds + k^2 \int_D |v_n|^2 dx \\ & \quad + k^2 \int_{\Omega \setminus \overline{D}} [u_n (\tilde{u}_n - \tilde{v}_n) - \tilde{u}_n v_n + v_n \tilde{v}_n] dx - \int_D |\nabla v_n|^2 dx, \end{aligned} \quad (6.11)$$

that is,

$$\begin{aligned}
& \langle \overbrace{(\Lambda_{\partial D, \sigma} - \Lambda_{0,0})} f_n, f_n \rangle \\
&= \int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx + \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\
&\quad - \int_{\partial D} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} - u_n \frac{\partial(\tilde{u}_n - \tilde{v}_n)}{\partial \nu} \right) ds \\
&= \int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx + \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx \\
&\quad - \int_{\partial D_2} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} \right) ds - \int_{\partial D_3} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} - i\sigma |u_n|^2 \right) ds. \quad (6.12)
\end{aligned}$$

6.2. *Estimate for w .* Here we prove the fourth estimate in Theorem 5.2.

For a given needle $c \in \Omega \setminus \overline{D}$, put $x_0 = c(t) \in \Omega \setminus \overline{D}$ and let $a \in \partial D$ be the point at which the needle c first hits ∂D . Since other cases can be handled analogously, we only consider the case $a \in \partial D_3$. Suppose x_0 is very close to a . Consider two families of functions $\{w(\cdot, x_0)\}$, $\{z(\cdot, x_0)\}$ depending on x_0 in some function space X . We denote by $w(\cdot, x_0) \sim z(\cdot, x_0)$ in X if $\{w(\cdot, x_0) - z(\cdot, x_0)\}$ is a bounded set in X for x_0 very close to a .

Let $G_0(x - x_0) = \frac{1}{4\pi|x - x_0|}$. Then it is easy to see that

$$(\partial_\nu + i\sigma)G(x - x_0) \sim (\partial_\nu + i\sigma)G_0(x - x_0)$$

in $L^2(\partial D)$; hence

$$w(\cdot, x_0) \sim w_0(\cdot, x_0) \quad \text{in } H^1(\Omega \setminus \overline{D}), \quad (6.13)$$

where $w = w(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ is the solution to (3.6) and $w_0 = w_0(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ is the solution to

$$\begin{cases} \Delta w_0 + k^2 w_0 = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\frac{\partial}{\partial \nu} + i\sigma) w_0 = -(\frac{\partial}{\partial \nu} + i\sigma) G_0(\cdot - x_0), & \text{on } \partial D_3, \\ w_0 = 0, & \text{on } \partial \Omega. \end{cases} \quad (6.14)$$

Here, we did not write the other boundary conditions on ∂D_j ($j = 1, 2$). Since x_0 is close to $a \in \partial D_3$, the other boundary conditions will not affect the behavior of $w(\cdot, x_0)$ as $x_0 \rightarrow a$. So, hereafter we will suppress the other boundary conditions.

By the Sobolev embedding $H^{1/2}(\partial D) \hookrightarrow L^r(\partial D)$ with $2 \leq r \leq 4$ and the Hölder inequality, for any q ($\frac{4}{3} \leq q \leq 2$), there exists a constant $C > 0$ such that

$$\begin{aligned}
\left| \int_{\partial D} i\sigma(x) G_0(x - x_0) \phi ds \right| &\leq \|i\sigma G_0(\cdot - x_0)\|_{L^q(\partial D)} \|\phi\|_{L^r(\partial D)} \\
&\leq C \|i\sigma G_0(\cdot - x_0)\|_{L^q(\partial D)} \|\phi\|_{H^{1/2}(\partial D)} \quad (6.15)
\end{aligned}$$

for $\phi \in H^{1/2}(\partial D)$, where $\frac{1}{r} = 1 - \frac{1}{q}$ with $\frac{4}{3} \leq q < 2$.

Hence $i\sigma G_0(\cdot - x_0) \sim 0$ in $H^{-1/2}(\partial D)$, and by the well-posedness of our boundary value problem, this implies

$$w_0(\cdot, x_0) \sim w_1(\cdot, x_0) \quad \text{in } H^1(\Omega \setminus \overline{D}), \quad (6.16)$$

FIG. 1. Results for the case of three different types of obstacles. (a): the real part of the indicator function cut by $C_1 = -0.75$ and $C_2 = 0.05$; (b): the imaginary part of the indicator function cut by $C_3 = -0.12$ and $C_4 = 0.12$. We take $k = 1.0$ and $\alpha = 1e - 8$ in the computations.

FIG. 2. Results for the three-obstacles case with the maximum 10% random error in the Dirichlet-to-Neumann map. (a): the real part; (b): the imaginary part. In this computation, we take $k = 1.0$ and $\alpha = 1e - 4$.

where $w_1 = w_1(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ is the solution to

$$\begin{cases} \Delta w_1 + k^2 w_1 = 0, & \text{in } \Omega \setminus \overline{D}, \\ \left(\frac{\partial}{\partial \nu} + i\sigma(x)\right) w_1 = -\frac{\partial}{\partial \nu} G_0(\cdot - x_0), & \text{on } \partial D_3, \\ w_1 = 0, & \text{on } \partial \Omega. \end{cases} \quad (6.17)$$

Now consider the solution $w_2 = w_2(\cdot, x_0) \in H^1(\Omega \setminus \overline{D})$ to

$$\begin{cases} \Delta w_2 = 0, & \text{in } \Omega \setminus \overline{D}, \\ \frac{\partial}{\partial \nu} w_2 = -\frac{\partial}{\partial \nu} G_0(\cdot - x_0), & \text{on } \partial D_3, \\ w_2 = 0, & \text{on } \partial \Omega. \end{cases} \quad (6.18)$$

For this problem, we have

Claim 1: $i\sigma(x)w_2(\cdot, x_0) \sim 0$ in $H^{-1/2}(\partial D)$, $w_2(x, x_0) \sim 0$ in $L^2(\Omega \setminus \overline{D})$.

Proof. The proof given here also gives a more precise estimate for w_2 , which will be used in the sequel.

Let $y = (y_1, y_2, y_3) = (y_1(x, x_0), y_2(x, x_0), y_3(x, x_0))$ be boundary normal coordinates near point a such that

$$y(a) = 0, \quad J(x) := \frac{\partial(y(x, x_0))}{\partial x} = I(\text{identity matrix})$$

at $x = x_0$ and $D = \{y_1 < 0\}$ locally near point a . Also, let $y(x_0) = y_0$,

$$\begin{aligned} A(x) &:= |\det J(x)|^{-1} J(x) (J(x))^T, & x(y(x, x_0); x_0) &= x, \\ \underline{A}(y) &:= A(x(y; x_0)), & \underline{u}(y) &:= u(x(y; x_0)). \end{aligned}$$

Then it is easy to see that

- (1) $\underline{A}(y) \in C^1$ near $y = 0$;
- (2) $\Delta u = 0$ near point $a \iff \nabla \cdot \underline{A} \nabla \cdot \underline{u} = 0$ near 0 ;
- (3) $\delta(x(y; x_0) - x_0) = \delta(y - y_0)$;
- (4) $\partial_\nu = \partial_{y_1}$.

In order to simplify the description of our argument, from now on we extend $x(y; x_0)$ and $\underline{A}(y)$ to an open ball $V \subset \mathbb{R}^3$ centered at $y = 0$ without destroying their regularities and the positivity of $\underline{A}(y)$. By a direct estimate, we can easily see that

$$\underline{G}_0(y; y_0) \sim G_0(y - y_0) \quad \text{in } H^1(V),$$

where we have adopted the convention $y_0 = y(x_0; x_0)$.

Now consider the solution $\underline{w}_2^0 \in H^1(V \cap \mathbb{R}_+^3)$ to

$$\begin{cases} \Delta \underline{w}_2^0 = 0, & \text{in } y_1 > 0, \\ \partial_{y_1} \underline{w}_2^0 = -\partial_{y_1} G_0(y - y_0), & \text{on } y_1 = 0, \\ \text{DC} \end{cases} \quad (6.19)$$

and put $\underline{w}_2(y) := w_2(x(y, x_0))$, where DC means that we put the Dirichlet condition on the rest of the part of the boundary. If we can prove

Claim 2: $\nabla \cdot (\underline{A}(y) - \underline{A}(y_0)) \nabla \underline{w}_2^0 \sim 0$ in W^* , where $W := \{\phi \in H^1(V \cap \mathbb{R}_+^3) \mid w = 0 \text{ on } \partial V \cap \mathbb{R}_+^3\}$ and W^* is its dual space.

Then we have

$$\underline{w}_2 \sim \underline{w}_2^0 \quad \text{in } H^1(V \cap \mathbb{R}_+^3) \quad (6.20)$$

by observing that $z := (\underline{w}_2 - \underline{w}_2^0) + (\underline{G}_0 - G_0(\cdot - y_0))$ satisfies

$$\begin{cases} \nabla \cdot \underline{A} \nabla z = -\nabla \cdot (\underline{A}(y) - \underline{A}(y_0)) \nabla (\underline{w}_2^0 + G_0(\cdot - y_0)), & \text{in } V \cap R_+^3, \\ \partial_{y_1} z = 0, & \text{on } y_1 = 0. \end{cases} \quad (6.21)$$

The proof for Claim 2 will be given in Section 6.3. Therein we also give a precise expression for $\underline{w}_2^0(y)$. Then, this expression and (6.20) imply Claim 1. The proof is complete.

Now we can see that

$$w_1(\cdot, x_0) \sim w_2(\cdot, x_0) \quad \text{in } H^1(\Omega \setminus \overline{D}) \quad (6.22)$$

from Claim 1 and the well-posedness of our boundary value problem.

Now summing up (6.13), (6.16), (6.20) and (6.22), as well as the expression of $\tilde{w}_2^0(y)$ given in the next subsection, we have

$$\int_{\partial D} |w(x, x_0)|^2 ds \leq C \left(\int_{\partial D} |G(x - c(t))|^2 ds + 1 \right),$$

which completes the proof of the fourth estimate in Theorem 5.2.

6.3. Proof for Claim 2.

Proof. Let $y_0 = (y_{01}, y_{02}, y_{03}) = (y_{01}, y'_0)$. Then it is well known that $H(y) = H(y; y_0) = G_0(y - y_0)$ can be given by

$$H(y) = \begin{cases} H_+(y) = H_+(y; y_0), & \text{in } y_1 > y_{01}, \\ H_-(y) = H_-(y; y_0), & \text{in } y_1 < y_{01}, \end{cases} \quad (6.23)$$

with the solution $H_{\pm}(y)$ to

$$\begin{cases} \Delta H_{\pm}(y) = 0, & \text{in } \pm(y_1 - y_{01}) > 0, \\ H_+(y)|_{y_1=y_{01}+0} = H_-(y)|_{y_1=y_{01}-0}, \\ \partial_{y_1} H_+(y)|_{y_1=y_{01}+0} - \partial_{y_1} H_-(y)|_{y_1=y_{01}-0} = -\delta(y' - y'_0). \end{cases} \quad (6.24)$$

Denote by $\Gamma_{\pm}(y_1, \eta')$ and $w(y_1, \eta')$ the Fourier transforms of $\hat{H}_{\pm}(y)$ and $\tilde{w}_2^0(y)$ with respect to y' , respectively. Then, $\Gamma'_{\pm} := e^{iy'_0 \cdot \eta'} \Gamma_{\pm}$ and $w' := e^{iy'_0 \cdot \eta'} w$ satisfy

$$\begin{cases} (\partial_{y_1}^2 - |\eta'|^2) \Gamma'_{\pm} = 0, & \text{in } \pm(y_1 - y_{01}) > 0, \\ \Gamma'_+|_{y_1=y_{01}+0} = \Gamma'_-|_{y_1=y_{01}-0}, \\ \partial_{y_1} \Gamma'_+|_{y_1=y_{01}+0} - \partial_{y_1} \Gamma'_-|_{y_1=y_{01}-0} = -1 \end{cases} \quad (6.25)$$

and

$$\begin{cases} (\partial_{y_1}^2 - |\eta'|^2) w' = 0, & \text{in } y_1 > 0, \\ \partial_{y_1} w' = -\partial_{y_1} \Gamma'_-, & \text{on } y_1 = 0, \end{cases} \quad (6.26)$$

respectively. $\Gamma'_{\pm} = \Gamma'_{\pm}(y_1)$ is given by

$$\Gamma'_{\pm}(y_1) = 2^{-1} |\eta'|^{-1} e^{\mp(y_1 - y_{01})|\eta'|}.$$

Hence $w' = w'(y_1) = 2^{-1} |\eta'|^{-1} e^{-(y_1 + y_{01})|\eta'|}$. Comparing these two formulas, we have

$$\tilde{w}_2^0(y) = H_+(y_1, y'; -y_{01}, y'_0) = \frac{1}{4\pi \sqrt{(y_1 + y_{01})^2 + |y' - y'_0|^2}}.$$

This completes the proof of Claim 2. $\square$

7. Numerical performance. For simplicity, we present the realizations of the probe method for multiple obstacles in $\mathbb{R}^2$. The numerical realization of the probe method was first given in [8] for one sound-soft obstacle and later given in [2] for one obstacle with impedance boundary.

In this section, we give the numerical performance of the probe method for multiple obstacles for which the theoretical analysis is given in this paper. We present the scheme for the reconstructions of multiple obstacles based on the scheme given in [8]. Although the theoretical analysis only gives the behavior of the indicator function $I(t, c)$ given by (3.3) when $c(t)$ approaches the boundaries of the obstacles from the outside, we compute $I_n(t, c)$ for fixed $n \gg 1$ and all $c(t)$.

Since our purpose is to show the efficiency of the probe method, in this section, we generate the Neumann data by simulation directly, rather than from the far-field pattern. That is, we obtain the solution to (2.2) by assuming that the obstacle D is known. Then we can solve this problem by potential theory to generate the Neumann data on $\partial\Omega$. The details for this procedure in the one-obstacle case may be found in [2, 8, 21].

The essential part of the numerical performance of the probe method for multiple obstacles is the Runge approximation. For this, we use the scheme proposed in [8] which is as follows. To start with, we consider the minimum norm solution of the equation

$$Hg_z(x) := \int_{\mathbb{S}} e^{ikx \cdot d} g_z(d) ds(d) = G(\cdot, z) \quad (7.1)$$

for a given $z \in \Omega$. Theoretically, the approximation domain is

$$E(z) = E(c(t)) := \Omega \setminus \{c(\tau) : 0 < \tau \leq t\}. \quad (7.2)$$

However, this approximation domain $E(z)$ is unstable in numerics due to the approximation of the singular function $G(\cdot, z)$ in $E(z)$ with strong non-convexity as $z \rightarrow \partial D$. So we choose relatively simple domains in numerical implementations.

More precisely, we consider approximation domains with fixed shape in the form

$$E(z) = \{M(z)x + z : x \in E_0, M(z) \in SO(2, \mathbb{R})\} \quad (7.3)$$

with a fixed approximation domain $E_0 := E(0)$ corresponding to $z = 0$, where we assumed $0 \in \Omega$ and $SO(2, \mathbb{R})$ represents the set of 2×2 unitary orthogonal matrices.

Then, for any $z \in \Omega$ with corresponding approximation domain $E(z)$ of the form (7.3), the density

$$g_{\alpha, z} = e^{-i\kappa z \cdot d} g_{\alpha, 0}(M(z)^t d) \quad (7.4)$$

satisfies a Herglotz wave function such that

$$\|Hg_{\alpha, z} - G(\cdot, z)\|_{L^2(\partial E(z))} < \delta.$$

This property reduces the complexity of calculating the densities $g_{\alpha, z}$ by choosing a fixed base domain E_0 , from which all approximation domains $E(z)$ are generated by a rotation and a translation.

The procedure of realization of the probe method is as follows:

- (1) Choose a test domain E_0 such that $0 \notin E_0$ large enough and calculate $g_{\alpha, 0}$ for a proper choice of α , and then compute g_{α, z_j} via (7.4) for evaluation points $z_j = c(t_j) \in \Omega$ with $j = 1, \dots, N$.

- (2) Fix $n \gg 1$ and calculate $f_{n,j} := v_{n,z_j}|_{\partial\Omega}$ for $j = 1, \dots, N$ with $v_{n,z_j} = Hg_{\alpha,z_j}$, where $\frac{1}{n}$ is the discrepancy of the minimum norm solution, and α is chosen by trial and error.
- (3) Solve the direct obstacle boundary value problem and the direct obstacle-free boundary value problem with $f = f_{n,j}$ to simulate $\Lambda_D f_{n,j}, \Lambda_0 f_{n,j}$ for $j = 1, \dots, N$.
- (4) Calculate the approximation $I(f_{n,j}) \approx \hat{I}_n(z_j)$ with $c(t) = z_j$ given by (3.3) for $j = 1, \dots, N$.
- (5) Observe the behavior of the indicator function as $c(t)$ moves along a needle.

Example: Three-obstacle case with different type of boundary.

We choose Ω as a circular domain with radius $r_\Omega = 1.5$ and the three obstacles are

$$\begin{aligned}\partial D_1 &:= \{(x, y) : (x, y) = (0.1 + 0.3 \cos t, -0.7 + 0.3 \sin t); t \in [0, 2\pi)\}, \\ \partial D_2 &:= \{(x, y) : (x, y) = (-0.8 + 0.3 \cos t, 0.1 + 0.3 \sin t); t \in [0, 2\pi)\}, \\ \partial D_3 &:= \{(x, y) : (x, y) = (0.5 + 0.3 \cos t, 0.6 + 0.3 \sin t); t \in [0, 2\pi)\}\end{aligned}$$

with sound-soft, sound-hard and impedance boundary respectively.

For the sake of computational simplicity, we choose a square $S = (-1.6, 1.6) \times (-1.6, 1.6)$ which includes Ω , and divide S into 60×60 non-overlapping elements. Then $\{z_j\}_{j=1}^{3600}$ is the set for the centers of all elements. Furthermore, we take the impedance coefficient $\sigma = 1.0$ for the obstacle D_3 .

Figure 1(a) shows the real part of the indicator function cut by the constants $C_1 = -0.75$ and $C_2 = 0.05$. Figure 1(b) shows the imaginary part cut by the constants $C_3 = -0.12$ and $C_4 = 0.12$. Here, for instance, C_1 and C_2 are the cutting values of the lower and upper bounds of the level sets of the real part of the indicator function, respectively. Hence, Figure 1(a) shows the level sets of the real part of the indicator function between the lower level C_1 and the upper level C_2 . The meanings of C_3, C_4 are likewise those of C_1, C_2 .

From the real part of the indicator function, the cutoff negative value in Figure 1(a) gives an enclosure of the two obstacles, the sound-hard obstacle and the impedance obstacle. We observe that the boundary of this enclosure is close to some parts of the exact boundaries of the two obstacles, and the cutoff positive value in Figure 1(a) gives a very rough shape of the sound-soft obstacle. From the imaginary part, the cutoff positive value in Figure 1(b) gives a rough shape of the obstacle with impedance boundary and the cutoff negative value in Figure 1(b) gives a rough shape of the sound-soft obstacle.

Note that the red circle represents the exact boundary of the obstacle with impedance boundary, the green circle is the exact boundary of the sound-hard obstacle and the blue circle is the exact boundary of the sound-soft obstacle in Figure 1 and Figure 2.

From Figure 1(a) and Figure 1(b), we easily recognize that there are three different types of obstacles and identify some part of the boundary for the sound-hard obstacle and the obstacle with impedance boundary. Moreover, we obtain the rough shapes for the three obstacles.

Figure 2 shows the numerical results when we add the maximum 10% random error to the Dirichlet-to-Neumann map. We observe that the sound-soft obstacle cannot be

identified from the real part of the indicator function, but the imaginary part can give a rough shape for this obstacle. Therefore, we have similar results as in the no error case.

On the other hand, it can be found from these examples that we can reconstruct a little more than a convex hull of multiple obstacles only; the neighboring parts of different obstacles are hard to recover. We guess that the reason is due to the approximation accuracy of the fundamental solution in the approximated domain corresponding to the needle points nearing this neighboring part. Roughly speaking, it is not so easy to approximate the fundamental solution to an arbitrary accuracy as the approximation domain becomes more and more like a cone-like domain ([2]). So it can be concluded from the numerical performance here that the probe method can be applied to generate a good initial guess for other iterative inversion schemes. It seems not easy to identify the shapes of the multiple obstacles exactly by this method itself from the numerical point of view.

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