GENERALIZATION OF EULER'S EQUATIONS

By

NAIL H. IBRAGIMOV

*Department of Mathematics and Science, Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology, SE-371 79 Karlskrona, Sweden*

Abstract. A wide class of linear ordinary differential equations reducible to algebraic equations is studied. The method for solving all these equations is given. The new class depends essentially on two arbitrary functions and contains the constant coefficient equations and Euler's equations as particular cases.

1. Introduction. The linear ordinary differential equations with constant coefficients

$$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.}, \quad (1.1)$$

and Euler's equations

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} x y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.}, \quad (1.2)$$

are discussed in almost all textbooks on differential equations. They are useful in applications. Their most remarkable property is that both equations (1.1) and (1.2) are *reducible to algebraic equations*. Namely, their fundamental systems of solutions, and hence the general solutions, can be obtained by solving algebraic equations. One can then integrate the corresponding non-homogeneous linear equations by using the method of variation of parameters. It is important to understand the nature of the reducibility and to extend the class of linear equations reducible to algebraic equations.

In the present paper the author investigates this problem and finds a wide class of linear ordinary differential equations that are reducible to algebraic equations. The new class of homogeneous linear equations depends on two arbitrary functions of x and contains the equations (1.1) and (1.2) as particular cases. The method for solving all these equations is given in Sections 4 and 6, and illustrated by examples in Section 5.

The main statement for the second-order equations is as follows (see Section 4).

THEOREM 1.1. The linear second-order equations

$$P(x)y'' + Q(x)y' + R(x)y = F(x),$$

Received February 4, 2008.

2000 *Mathematics Subject Classification.* Primary 34A30.

Key words and phrases. Equations reducible to algebraic equations, symmetries, solution method, Euler's equations.

whose general solution can be obtained by solving algebraic equations and by quadratures, have the form

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = F(x), \quad (1.3)$$

where $\phi = \phi(x)$, $\sigma = \sigma(x)$ and $F(x)$ are arbitrary (smooth) functions, and $A, B = \text{const.}$ The homogeneous equation (1.3),

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0, \quad (1.4)$$

has solutions of the form

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad \lambda = \text{const.}, \quad (1.5)$$

where λ satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \quad (1.6)$$

If the characteristic equation (1.6) has distinct real roots $\lambda_1 \neq \lambda_2$, the general solution to Equation (1.4) is given by

$$y(x) = K_1 e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} dx} + K_2 e^{\int \frac{\sigma(x) + \lambda_2}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (1.7)$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Equation (1.4) is given by

$$y(x) = \left[K_1 \cos\left(\theta \int \frac{dx}{\phi(x)}\right) + K_2 \sin\left(\theta \int \frac{dx}{\phi(x)}\right) \right] e^{\int \frac{\sigma(x) + \gamma}{\phi(x)} dx}. \quad (1.8)$$

If the characteristic equation (1.6) has equal roots $\lambda_1 = \lambda_2$, the general solution to Equation (1.4) is given by

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)} \right] e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (1.9)$$

The solution of the non-homogeneous equation (1.3) can be obtained by the method of variation of parameters.

REMARK 1.1. The equations with constant coefficients and Euler's equation are the simplest representatives of Equation (1.4). Namely, setting $\phi(x) = 1$, $\sigma(x) = 0$, we obtain the second-order equation with constant coefficients

$$y'' + Ay' + By = 0,$$

and Equation (1.5) yields the well-known formula

$$y = e^{\lambda x},$$

where λ is determined by the characteristic equation (1.6).

If we set $\phi(x) = x$, $\sigma(x) = 0$, we obtain the second-order Euler's equation (1.2) written in the form

$$x^2 y'' + (A + 1)xy' + By = 0.$$

Then Equation (1.5) yields particular solutions to Euler's equations:

$$y = x^\lambda,$$

where λ is determined again by the characteristic equation (1.6). For details, see Section 5. For other functions $\phi(x)$ and $\sigma(x) = 0$, Equation (1.4) is new.

2. Constant coefficient and Euler's equations from the group standpoint.

For simplicity, we will consider in this section second-order equations

$$y'' + f(x)y' + g(x)y = 0. \quad (2.1)$$

2.1. *Equations with constant coefficients.* The equation with constant coefficients

$$y'' + Ay' + By = 0, \quad A, B = \text{const.}, \quad (2.2)$$

is invariant under the one-parameter group of translations in x and dilations in y , since it does not involve the independent variable x explicitly (the coefficients A and B are constant) and is homogeneous in the dependent variable y . In other words, Equation (2.2) admits the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}, \quad (2.3)$$

of the translations in x and dilations in y . They were used [1] as follows.

Let us find the invariant solution for $X = X_1 + \lambda X_2$, i.e.,

$$X = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \quad \lambda = \text{const.} \quad (2.4)$$

The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

of the equation

$$X(J) \equiv \frac{\partial J}{\partial x} + \lambda y \frac{\partial J}{\partial y} = 0$$

for the invariants $J(x, y)$ yields one functionally independent invariant

$$J = y e^{-\lambda x}.$$

According to the general theory, the invariant solution is given by $J = C$ with an arbitrary constant C . Thus, the general form of the invariant solutions for the operator (2.4) is

$$y = C e^{\lambda x}, \quad C = \text{const.}$$

Since Equation (2.2) is homogeneous one can set $C = 1$ and obtain *Euler's substitution*:

$$y = e^{\lambda x}. \quad (2.5)$$

As is well known, the substitution (2.5) reduces Equation (2.2) to the quadratic equation (the *characteristic equation*)

$$\lambda^2 + A\lambda + B = 0. \quad (2.6)$$

If Equation (2.6) has two distinct roots, $\lambda_1 \neq \lambda_2$, then Equation (2.5) provides two linearly independent solutions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x},$$

and hence, a fundamental set of solutions. If the roots are real, the general solution to Equation (2.2) is

$$y(x) = K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}, \quad K_1, K_2 = \text{const.} \quad (2.7)$$

If the roots are complex, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Equation (2.2) is given by

$$y(x) = [K_1 \cos(\theta x) + K_2 \sin(\theta x)] e^{\gamma x}, \quad K_1, K_2 = \text{const.} \quad (2.8)$$

In the case of equal roots $\lambda_1 = \lambda_2$, standard texts in differential equations make a guess, without motivation, that the general solution has the form

$$y(x) = (K_1 + K_2 x) e^{\lambda_1 x}, \quad K_1, K_2 = \text{const.} \quad (2.9)$$

The motivation is given in [1], Section 13.2.2, and states the following.

LEMMA 2.1. Equation (2.2) can be mapped to the equation $z'' = 0$ by a linear change of the dependent variable

$$y = \sigma(x) z, \quad \sigma(x) \neq 0, \quad (2.10)$$

if and only if the characteristic equation (2.6) has equal roots. Specifically, if $\lambda_1 = \lambda_2$, the substitution

$$y = z e^{\lambda_1 x} \quad (2.11)$$

carries Equation (2.2) to the equation $z'' = 0$. Substituting in (2.11) the solution $z = K_1 + K_2 x$ of the equation $z'' = 0$, we obtain the general solution (2.9) to Equation (2.2) whose coefficients satisfy the condition of equal roots for Equation (2.6):

$$A^2 - 4B = 0. \quad (2.12)$$

Proof. The reckoning shows that after the substitution (2.10), Equation (2.1) becomes (see, e.g., [2], Section 3.3.2)

$$z'' + I(x)z = 0,$$

where

$$I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)$$

and the function $\sigma(x)$ in the transformation (2.10) has the form

$$\sigma(x) = e^{-\frac{1}{2} \int f(x) dx}. \quad (2.13)$$

Hence, Equation (2.1) is carried into equation $z'' = 0$ if and only if $I(x) = 0$, i.e.,

$$f^2(x) + 2f'(x) - 4g(x) = 0. \quad (2.14)$$

In the case of Equation (2.2), the condition (2.14) is identical with Equation (2.12), and the function $\sigma(x)$ given by Equation (2.13) becomes

$$\sigma(x) = e^{-\frac{A}{2} x}. \quad (2.15)$$

Furthermore, under the condition (2.12), the repeated root of Equation (2.6) is equal to $\lambda_1 = -A/2$. Therefore Equation (2.15) can be written

$$\sigma(x) = e^{\lambda_1 x}.$$

We have arrived at the substitution (2.11), and hence at the solution (2.9), thus proving the lemma. $\square$

2.2. *Euler's equation.* Consider Euler's equation

$$x^2 y'' + A x y' + B y = 0, \quad A, B = \text{const.} \quad (2.16)$$

It is double homogeneous (see [2], Section 6.6.1), i.e., it admits the dilation groups in x and in y with the generators

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}. \quad (2.17)$$

We proceed as in Section 2.1 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const.} \quad (2.18)$$

The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

of the equation $X(J) = 0$ for the invariants $J(x, y)$ yields the invariant

$$J = y x^{-\lambda}$$

for the operator (2.18). The invariant solutions are given by $J = C$; therefore,

$$y = C x^\lambda, \quad C = \text{const.}$$

Due to the homogeneity of Equation (2.2) we can set $C = 1$ and obtain

$$y = x^\lambda. \quad (2.19)$$

Differentiating and multiplying by x , we have:

$$x y' = \lambda x^\lambda, \quad x^2 y'' = \lambda(\lambda - 1) x^\lambda.$$

Substituting in Equation (2.18) and dividing by the common factor $C x^\lambda$ we obtain the following *characteristic equation* for Euler's equation (2.16):

$$\lambda^2 + (A - 1) \lambda + B = 0. \quad (2.20)$$

REMARK 2.1. According to Equations (2.6), (2.20), the characteristic equations for Equation (2.2) with constant coefficients and for Euler's equation written in the form

$$x^2 y'' + (A + 1) x y' + B y = 0 \quad (2.21)$$

are identical.

3. New examples of reducible equations.

3.1. *First example.* Let us find the linear second-order equations (2.1),

$$y'' + f(x)y' + g(x)y = 0, \quad (2.1)$$

admitting the operator

$$X_1 = x^\alpha \frac{\partial}{\partial x}, \quad (3.1)$$

where α is any real-valued parameter. Taking the second prolongation of X_1 ,

$$X_1 = x^\alpha \frac{\partial}{\partial x} - \alpha x^{\alpha-1} y' \frac{\partial}{\partial y'} - [\alpha(\alpha-1)x^{\alpha-2}y' + 2x^{\alpha-1}y''] \frac{\partial}{\partial y''},$$

we write the invariance condition of Equation (2.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(2.1)} = 0,$$

and obtain:

$$x^{\alpha-2}[x^2 f' + \alpha x f - \alpha(\alpha-1)]y' + x^{\alpha-1}[xg' + 2\alpha g]y = 0. \quad (3.2)$$

Since Equation (3.2) should be satisfied identically in the variables x, y, y' , it splits into two equations:

$$x^2 f' + \alpha x f - \alpha(\alpha-1) = 0, \quad xg' + 2\alpha g = 0. \quad (3.3)$$

Solving the first-order linear differential equations (3.3) for the unknown functions $f(x)$ and $g(x)$, we obtain:

$$f(x) = \frac{\alpha}{x} + Ax^{-\alpha}, \quad g(x) = Bx^{-2\alpha}, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (3.1):

$$x^{2\alpha}y'' + (Ax^\alpha + \alpha x^{2\alpha-1})y' + By = 0, \quad A, B = \text{const.} \quad (3.4)$$

REMARK 3.1. When $\alpha = 0$, Equation (3.4) yields Equation (2.2) with constant coefficients. When $\alpha = 1$, Equation (3.4), upon setting $A+1$ as a new coefficient A , coincides with Euler's equation (2.16).

Since Equation (3.4) is linear homogeneous, it admits, along with (3.1), the operator

$$X_2 = y \frac{\partial}{\partial y}.$$

Now we proceed as in Section 2 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x^\alpha \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const.}$$

Let $\alpha \neq 1$. The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x^\alpha}$$

of the equation $X(J) = 0$ for the invariants $J(x, y)$ yields the invariant

$$J = y e^{\frac{\lambda}{\alpha-1} x^{1-\alpha}}.$$

Hence, the invariant solutions are obtained by setting $J = C$; therefore, letting $C = 1$ we have

$$y = e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}. \quad (3.5)$$

Differentiating we have:

$$y' = \lambda x^{-\alpha} e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}, \quad y'' = [\lambda^2 x^{-2\alpha} - \lambda \alpha x^{-\alpha-1}] e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}.$$

Substituting in Equation (3.4) and dividing by the non-vanishing factor $e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}$, we obtain the following *characteristic equation* for Equation (3.4):

$$\lambda^2 + A\lambda + B = 0. \quad (3.6)$$

Equation (3.6) is identical with the characteristic equation (2.6) for Equation (2.2) with constant coefficients.

REMARK 3.2. Equation (3.5) contains Euler's substitution (2.5) for Equation (2.2) with constant coefficients as a particular case with $\alpha = 0$.

REMARK 3.3. Using the statement that Equation (2.1) is mapped to the equation $z'' = 0$ if and only if the function

$$I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)$$

vanishes (see Lemma 2.1), one can verify that Equation (3.5) is equivalent by function to the equation $z'' = 0$ if and only if

$$A^2 - 4B = 0 \quad \text{and} \quad \alpha = 0 \text{ or } \alpha = 2. \quad (3.7)$$

The first equation in (3.7) means that the characteristic equation (3.6) has a repeated root, and hence there is only one solution of the form (3.5). Then, using the reasoning of Lemma 2.1 we can show that the general solution to Equation (3.4) with $A^2 - 4B = 0$, $\alpha = 2$, i.e., of the equation

$$y'' + \left(\frac{A}{x^2} + \frac{2}{x} \right) y' + \frac{B}{x^4} y = 0, \quad A^2 - 4B = 0,$$

is given by

$$y = \left(K_1 + \frac{K_2}{x} \right) e^{-\frac{\lambda}{x}}, \quad (3.8)$$

where K_1, K_2 are arbitrary constants and λ is the repeated root of the characteristic equation (3.6). For a more general statement, see Section 5.

3.2. *Second example.* Let us find the linear second-order equations (2.1) admitting the projective group with the generator

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \quad (3.9)$$

Taking the second prolongation of the operator (3.9),

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'} - 3xy'' \frac{\partial}{\partial y''},$$

and writing the invariance condition of Equation (2.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(2.1)} = 0,$$

we obtain

$$x(xf' + 2f)y' + (x^2g' + 4xg + f)y = 0.$$

Therefore, upon splitting with respect to y' and y ,

$$xf' + 2f = 0, \quad x^2g' + 4xg + f = 0. \quad (3.10)$$

Equations (3.10) yield:

$$f(x) = \frac{A}{x^2}, \quad g(x) = \frac{B}{x^4} - \frac{A}{x^3}, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (3.9):

$$x^4y'' + Ax^2y' + (B - Ax)y = 0, \quad A, B = \text{const.} \quad (3.11)$$

Equation (3.11) is linear homogeneous, and hence admits, along with (3.9), the operator

$$X_2 = y \frac{\partial}{\partial y}.$$

Now we proceed as in Section 2 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x^2 \frac{\partial}{\partial x} + (x + \lambda)y \frac{\partial}{\partial y}, \quad \lambda = \text{const.}$$

Writing the characteristic equation of the equation $X(J) = 0$ for the invariants $J(x, y)$ in the form

$$\frac{dy}{y} = \frac{x + \lambda}{x^2} dx,$$

we obtain the invariant

$$J = \frac{y}{x} e^{\frac{\lambda}{x}}.$$

Setting $J = C$ and letting $C = 1$ we obtain the following form of the invariant solutions:

$$y = x e^{-\frac{\lambda}{x}}. \quad (3.12)$$

Substituting (3.12) in Equation (3.11) we reduce the differential equation (3.11) to the algebraic equation

$$\lambda^2 + A\lambda + B = 0,$$

which is identical with the characteristic equation (2.6) for Equation (2.2) with constant coefficients.

EXAMPLE 3.1. Solve the equation

$$y'' + \frac{\omega^2}{x^4} = 0, \quad \omega = \text{const.} \quad (3.13)$$

This is an equation of the form (3.11) with $A = 0$, $B = \omega^2$. The algebraic equation (2.6) yields $\lambda_1 = -i\omega$, $\lambda_2 = i\omega$, and hence we have two independent invariant solutions (3.12):

$$y_1 = x e^{i\frac{\omega}{x}}, \quad y_2 = x e^{-i\frac{\omega}{x}}.$$

Taking their real and imaginary parts, as in the case of constant coefficient equations, we obtain the following fundamental system of solutions:

$$y_1 = x \cos\left(\frac{\omega}{x}\right), \quad y_2 = x \sin\left(\frac{\omega}{x}\right). \quad (3.14)$$

Hence, the general solution to Equation (3.12) is given by

$$y = x \left[K_1 \cos\left(\frac{\omega}{x}\right) + K_2 \sin\left(\frac{\omega}{x}\right) \right].$$

We can also solve the non-homogeneous equation

$$y'' + \frac{\omega^2}{x^4} = F(x), \quad (3.15)$$

e.g., by the method of *variation of parameters*, and obtain

$$y = x \left[K_1 \cos\left(\frac{\omega}{x}\right) + K_2 \sin\left(\frac{\omega}{x}\right) \right] + \frac{x}{\omega} \left[\cos\left(\frac{\omega}{x}\right) \int xF(x) \sin\left(\frac{\omega}{x}\right) dx - \sin\left(\frac{\omega}{x}\right) \int xF(x) \cos\left(\frac{\omega}{x}\right) dx \right]. \quad (3.16)$$

4. The general result for second-order equations.

4.1. *Main statements.* Note that the operators given in Equations (2.3), (2.17), (3.1), and (3.9) are particular cases of the generator

$$X_1 = \phi(x) \frac{\partial}{\partial x} + \sigma(x)y \frac{\partial}{\partial y} \quad (4.1)$$

of the general equivalence group of all linear ordinary differential equations. We will now find the linear second-order equations (2.1) admitting the operator (4.1) with any *fixed* functions $\phi(x)$ and $\sigma(x)$.

Taking the second prolongation of the operator (4.1),

$$X_1 = \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + [\sigma' y + (\sigma - \phi')y'] \frac{\partial}{\partial y'} + [\sigma'' y + (2\sigma' - \phi'')y' + (\sigma - 2\phi')y''] \frac{\partial}{\partial y''},$$

and writing the invariance condition of Equation (2.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(2.1)} = 0,$$

we obtain:

$$(\phi f' + f\phi' + 2\sigma' - \phi'')y' + (\phi g' + 2\phi'g + f\sigma' + \sigma'')y = 0.$$

It follows that:

$$\begin{aligned} \phi f' + f\phi' + 2\sigma' - \phi'' &= 0, \\ \phi g' + 2\phi'g + f\sigma' + \sigma'' &= 0. \end{aligned} \quad (4.2)$$

The first equation in (4.2) is written

$$(\phi f)' = (\phi' - 2\sigma)'$$

and yields:

$$f(x) = \frac{1}{\phi} [A + \phi' - 2\sigma], \quad A = \text{const.} \quad (4.3)$$

Substituting this in the second equation in (4.2), we obtain the following non-homogeneous linear first-order equation for determining $g(x)$:

$$\phi g' + 2\phi'g = -\sigma'' - \frac{\sigma'}{\phi} [A + \phi' - 2\sigma]. \quad (4.4)$$

The homogeneous equation

$$\phi(x)g' + 2\phi'(x)g = 0$$

with a given function $\phi(x)$ yields

$$g = \frac{C}{\phi^2(x)}.$$

By variation of the parameter C , we set

$$g = \frac{u(x)}{\phi^2(x)},$$

substitute it in Equation (4.4) and obtain:

$$u' = -A\sigma' + 2\sigma\sigma' - \phi'\sigma' - \phi\sigma'' \equiv -(A\sigma)' + (\sigma^2)' - (\phi\sigma)';$$

where

$$u = B - A\sigma + \sigma^2 - \phi\sigma', \quad B = \text{const.}$$

Therefore,

$$g = \frac{1}{\phi^2(x)} [B - A\sigma + \sigma^2 - \phi\sigma']. \quad (4.5)$$

Thus, we have arrived at the following result.

THEOREM 4.1. The homogeneous linear second-order equations (2.1) admitting the operator (4.1) with any given functions $\phi = \phi(x)$ and $\sigma = \sigma(x)$ have the form

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0. \quad (4.6)$$

Now we use the homogeneity of Equation (4.6) characterized by the generator

$$X_2 = y \frac{\partial}{\partial y}.$$

Namely, we look for the invariant solutions with respect to the linear combination $X = X_1 + \lambda X_2$:

$$X = \phi(x) \frac{\partial}{\partial x} + (\sigma(x) + \lambda)y \frac{\partial}{\partial y}, \quad \lambda = \text{const.},$$

and arrive at the following statement reducing the problem of integration of the differential equation (4.6) to the solution of the quadratic equation, namely, the characteristic equation as in the case of equations with constant coefficients.

THEOREM 4.2. Equation (4.6) has the invariant solutions of the form

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad \lambda = \text{const.}, \quad (4.7)$$

where λ satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \quad (4.8)$$

Proof. We solve the equation $X(J) = 0$ for the invariants $J(x, y)$, i.e., integrate the equation

$$\frac{dy}{y} = \frac{\sigma(x) + \lambda}{\phi(x)} dx$$

and obtain the invariant

$$J = y e^{-\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}.$$

Setting $J = C$ and letting $C = 1$ we obtain Equation (4.7) for the invariant solutions. Thus, we have:

$$y = e^{\int \frac{\sigma(x)+\lambda}{\phi(x)} dx}, \quad y' = \frac{\sigma + \lambda}{\phi} e^{\int \frac{\sigma(x)+\lambda}{\phi(x)} dx}, \quad (4.9)$$

$$y'' = \frac{1}{\phi^2} [(\sigma + \lambda)^2 - (\sigma + \lambda)\phi' + \phi\sigma'] e^{\int \frac{\sigma(x)+\lambda}{\phi(x)} dx}.$$

Substituting (4.9) in Equation (4.6) we obtain Equation (4.8), thus completing the proof. $\square$

4.2. *Distinct roots of the characteristic equation.* It is manifest that if the characteristic equation (4.8) has distinct real roots $\lambda_1 \neq \lambda_2$, the general solution to Equation (4.6) is given by

$$y(x) = K_1 e^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)} dx} + K_2 e^{\int \frac{\sigma(x)+\lambda_2}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (4.10)$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Equation (4.6) is given by

$$y(x) = \left[K_1 \cos\left(\theta \int \frac{dx}{\phi(x)}\right) + K_2 \sin\left(\theta \int \frac{dx}{\phi(x)}\right) \right] e^{\int \frac{\sigma(x)+\gamma}{\phi(x)} dx}. \quad (4.11)$$

4.3. *The case of repeated roots.*

THEOREM 4.3. If the characteristic equation (4.8) has equal roots $\lambda_1 = \lambda_2$, the general solution to Equation (4.6) is given by

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)} \right] e^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (4.12)$$

Proof. We introduce the new variables t and z defined by the equations

$$X_1(t) \equiv \phi(x) \frac{\partial t}{\partial x} + \sigma(x) y \frac{\partial t}{\partial y} = 1, \quad X_2(t) \equiv y \frac{\partial t}{\partial y} = 0, \quad (4.13)$$

and

$$X_1(z) \equiv \phi(x) \frac{\partial z}{\partial x} + \sigma(x) y \frac{\partial z}{\partial y} = 0, \quad X_2(z) \equiv y \frac{\partial z}{\partial y} = z, \quad (4.14)$$

respectively. Equations (4.13) are easily solved and yield

$$t = \int \frac{dx}{\phi(x)}. \quad (4.15)$$

Integration of the second equation in (4.14) with respect to y gives

$$z = v(x)y.$$

Substituting this in the first equation in (4.14) we obtain

$$\phi(x) \frac{dv}{dx} + \sigma(x) v = 0, \quad \text{whence} \quad v = e^{-\int \frac{\sigma(x)}{\phi(x)} dx}.$$

Thus,

$$z = y e^{-\int \frac{\sigma(x)}{\phi(x)} dx}. \quad (4.16)$$

$\square$

The passage to the new variables (4.15), (4.16) converts the operator X_1 given by (4.1) to the translation generator without changing the form of the dilation generator X_2 . In other words, upon introducing the new independent and dependent variables t and z given by (4.15) and (4.16), respectively, we arrive at the operators (2.3). Hence, in the variables t and z , Equation (4.16) becomes an equation with constant coefficients. Noting that the Equations (4.6) and (2.2) have Equation (4.8) as their common characteristic equation, we use Lemma 2.1 and write

$$z = (K_1 + K_2 t) e^{\lambda_1 t}.$$

Substituting this in Equation (4.16) and replacing t and z by their expressions (4.15) and (4.16), respectively, and solving for y , we finally arrive at Equation (4.12).

REMARK 4.1. The non-homogeneous Equation (4.6),

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = F(x), \quad (4.17)$$

can be integrated by the method of variation of parameters. Namely, we rewrite Equation (4.6) in the form

$$y'' + a(x)y' + b(x)y = P(x) \quad (4.18)$$

and employ the representation

$$y = K_1 y_1(x) + K_2 y_2(x) - y_1(x) \int \frac{y_2(x)P(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)P(x)}{W(x)} dx \quad (4.19)$$

of the solution to Equation (4.18). See [2], Section 3.3.4. Here

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is the Wronskian of a fundamental system of solutions $y_1(x)$, $y_2(x)$ for the homogeneous equation (4.18):

$$y'' + a(x)y' + b(x)y = 0.$$

5. Examples. Euler's substitution (2.5) as well as the solutions (2.19), (3.5) and (3.12) are encapsulated in Equation (4.7). We will now consider these and several other examples.

EXAMPLE 5.1. Let us take $\phi(x) = 1$, $\sigma(x) = 0$. Then Equations (4.6), (4.7) and (4.12) coincide with Equations (2.2), (2.5) and (2.9), respectively. Equation (4.11) becomes (2.8).

EXAMPLE 5.2. Let us take $\phi(x) = x$, $\sigma(x) = 0$. Then Equation (4.6) becomes Euler's equation written in the form (2.21), Equation (4.7) yields Equation (2.19) for invariant solutions, whereas Equation (4.12) provides the general solution

$$y(x) = (K_1 + K_2 \ln x) x^{\lambda_1} \quad K_1, K_2 = \text{const.}, \quad (5.1)$$

to Euler's equation (2.21) whose characteristic equation (2.6) has equal roots. Equation (4.11) leads to the following solution for complex roots $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$:

$$y(x) = [K_1 \cos(\theta \ln x) + K_2 \sin(\theta \ln x)] x^\gamma. \quad (5.2)$$

EXAMPLE 5.3. Let us take $\phi(x) = x^\alpha$, $\sigma(x) = 0$. Then Equations (4.6) and (4.7) coincide with Equations (3.4) and (3.5), respectively, whereas Equation (4.12) provides the general solution

$$y(x) = (K_1 + K_2 x^{1-\alpha}) e^{\frac{\lambda_1}{1-\alpha} x^{1-\alpha}}, \quad K_1, K_2 = \text{const.}, \quad (5.3)$$

to Equation (3.4) whose characteristic equation (3.6) has equal roots $\lambda_1 = \lambda_2$. Equation (5.3) extends the solution (3.8) to all equations (3.4) with the coefficients A, B, C satisfying the condition (2.12) of equal roots for the characteristic equation (3.6).

EXAMPLE 5.4. Let us take $\phi(x) = 1 + x^2$, $\sigma(x) = x$. Then Equation (4.6) becomes

$$(1 + x^2)^2 y'' + (1 + x^2) A y' + (B - Ax - 1) y = 0. \quad (5.4)$$

Working out the integral in Equation (4.7),

$$\int \frac{\sigma(x) + \lambda}{\phi(x)} dx = \int \frac{x}{1 + x^2} dx + \int \frac{\lambda}{1 + x^2} dx = \ln \sqrt{1 + x^2} + \lambda \arctan x,$$

we obtain the following expression for the invariant solutions:

$$y = \sqrt{1 + x^2} e^{\lambda \arctan x}, \quad (5.5)$$

where λ satisfies the characteristic equation (4.8):

$$\lambda^2 + A\lambda + B = 0. \quad (5.6)$$

If the characteristic equation (5.6) has distinct real roots, $\lambda_1 \neq \lambda_2$, the general solution to Equation (5.4) is given by

$$y(x) = \sqrt{1 + x^2} [K_1 e^{\lambda_1 \arctan x} + K_2 e^{\lambda_2 \arctan x}]. \quad (5.7)$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Equation (5.4) is given by

$$y(x) = [K_1 \cos(\theta \arctan x) + K_2 \sin(\theta \arctan x)] \sqrt{1 + x^2} e^{\gamma \arctan x}. \quad (5.8)$$

Finally, if the characteristic equation (5.6) has equal roots $\lambda_1 = \lambda_2$, the general solution to Equation (5.4) is given by

$$y(x) = (K_1 + K_2 \arctan x) \sqrt{1 + x^2} e^{\lambda \arctan x}. \quad (5.9)$$

EXAMPLE 5.5. Consider Equation (5.4) with $A = 0$, $B = \omega^2$. Then, according to Example 5.4, the equation

$$(1 + x^2)^2 y'' + (\omega^2 - 1)y = 0 \quad (5.10)$$

has the following general solution:

$$y(x) = [K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x)] \sqrt{1 + x^2}. \quad (5.11)$$

EXAMPLE 5.6. Let us solve the non-homogeneous equation

$$(1 + x^2)^2 y'' + (\omega^2 - 1)y = F(x). \quad (5.12)$$

Example 5.5 provides the fundamental system of solutions

$$y_1 = \sqrt{1 + x^2} \cos(\omega \arctan x), \quad y_2 = \sqrt{1 + x^2} \sin(\omega \arctan x)$$

for the homogeneous equation (5.10). We have:

$$\begin{aligned} y_1' &= \frac{1}{\sqrt{1+x^2}} [x \cos(\omega \arctan x) - \omega \sin(\omega \arctan x)], \\ y_2' &= \frac{1}{\sqrt{1+x^2}} [x \sin(\omega \arctan x) + \omega \cos(\omega \arctan x)]. \end{aligned}$$

Hence the Wronskian is $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \omega$. Now we rewrite Equation (5.12), in accordance with Remark 4.1, in the form

$$y'' + \frac{\omega^2 - 1}{(1+x^2)^2} y = \frac{F(x)}{(1+x^2)^2}, \quad (5.13)$$

use Equation (4.19) and obtain the following general solution to Equation (5.12):

$$\begin{aligned} y(x) &= \sqrt{1+x^2} \left[K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x) \right. \\ &\quad - \frac{1}{\omega} \cos(\omega \arctan x) \int \frac{F(x)}{(1+x^2)^{3/2}} \sin(\omega \arctan x) dx \\ &\quad \left. + \frac{1}{\omega} \sin(\omega \arctan x) \int \frac{F(x)}{(1+x^2)^{3/2}} \cos(\omega \arctan x) dx \right]. \end{aligned} \quad (5.14)$$

6. Third-order equations. The previous results can be extended to higher-order linear equations. I will discuss here the third-order equation

$$y''' + f(x)y'' + g(x)y' + h(x)y = 0. \quad (6.1)$$

THEOREM 6.1. The homogeneous linear third-order equation (6.1) admitting the operator (4.1) with any given $\phi = \phi(x)$ and $\sigma = \sigma(x)$ has the form

$$\begin{aligned} &\phi^3 y''' + [A + 3(\phi' - \sigma)] \phi^2 y'' \\ &\quad + [B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2] \phi y' \\ &\quad + [C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'] y = 0. \end{aligned} \quad (6.2)$$

Proof. We take the third prolongation of the operator (4.1):

$$\begin{aligned} X_1 &= \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + [\sigma' y + (\sigma - \phi') y'] \frac{\partial}{\partial y'} \\ &\quad + [\sigma'' y + (2\sigma' - \phi'') y' + (\sigma - 2\phi') y''] \frac{\partial}{\partial y''} \\ &\quad + [\sigma''' y + (3\sigma'' - \phi''') y' + 3(\sigma' - \phi'') y'' + (\sigma - 3\phi') y'''] \frac{\partial}{\partial y'''} , \end{aligned}$$

and write the invariance condition of Equation (6.1):

$$X_1(y''' + f(x)y'' + g(x)y' + h(x)y) \Big|_{(6.1)} = 0.$$

We nullify the coefficients for y'' , y' and y of the left-hand side of the above equation and split it into the following three equations:

$$\phi f' + \phi' f + 3(\sigma' - \phi'') = 0, \quad (6.3)$$

$$\phi g' + 2\phi' g + (2\sigma' - \phi'')f - \phi''' + 3\sigma'' = 0, \quad (6.4)$$

$$\phi h' + 3\phi' h + \sigma'' f + \sigma' g + \sigma''' = 0. \quad (6.5)$$

Equation (6.3) is written

$$(\phi f)' = 3(\phi' - \sigma)'$$

and yields:

$$f = \frac{1}{\phi} [A + 3(\phi' - \sigma)], \quad A = \text{const.} \quad (6.6)$$

We substitute (6.6) in Equation (6.4), integrate the resulting non-homogeneous linear first-order equation for g and obtain:

$$g = \frac{1}{\phi^2} [B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2], \quad (6.7)$$

where B is an arbitrary constant. Now we substitute (6.6), (6.7) in Equation (6.5), integrate the resulting first-order equation for h and obtain:

$$h = \frac{1}{\phi^3} [C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'], \quad (6.8)$$

where C is an arbitrary constant. Finally, substituting (6.6), (6.7) and (6.8) in Equation (6.1), we arrive at Equation (6.2). $\square$

THEOREM 6.2. Equation (6.2) has the invariant solutions of the form (4.7),

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad \lambda = \text{const.}, \quad (4.7)$$

where λ satisfies the *characteristic equation*

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0. \quad (6.9)$$

Proof. Adding to Equations (4.9) the expression for the third derivative y''' and substituting in Equation (6.2), we obtain Equation (6.9). $\square$

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