EXISTENCE AND REGULARITY OF PULLBACK ATTRACTORS FOR AN INCOMPRESSIBLE NON-NEWTONIAN FLUID WITH DELAYS

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Abstract. This paper studies an incompressible non-Newtonian fluid with delays in two-dimensional bounded domains. We first prove the existence and uniqueness of solutions. Then we establish the existence of pullback attractors $\{\mathcal{A}_{c_H}(t)\}_{t \in \mathbb{R}}$ (has L^2 -regularity), $\{\mathcal{A}_{c_W}(t)\}_{t \in \mathbb{R}}$ (has H^2 -regularity), and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$ (has L^2 -regularity), $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}}$ (has H^2 -regularity) corresponding to two different processes associated to the fluid, respectively. Meanwhile, we verify the regularity of the pullback attractors by proving

$$\mathcal{A}_{c_H}(t) = \mathcal{A}_{c_W}(t), \quad \mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R},$$

and

$$\mathcal{A}_{E_H^2}(t) = J(\mathcal{A}_{c_H}(t)) = J(\mathcal{A}_{c_W}(t)) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R},$$

where J is a linear operator. By the regularity we reveal the pullback asymptotic smoothing effect of the fluid in the sense that the solutions become eventually more regular than the initial data. This effect implies, in the case of delays, that the regularity of the fluid in its history state does not play an important role on the regularity of its eventual state. Finally, we give some remarks.

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1. Introduction. In this paper, we prove the unique existence of solutions and the existence and regularity of pullback attractors for the following incompressible non-Newtonian fluid with delays:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \tau(e(u)) = g(x) + f(t, u_t), \quad x = (x_1, x_2) \in \Omega, \quad (1.1)$$

$$\nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \quad (1.2)$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$, the unknown vector function $u = u(x, t) = (u_1, u_2)$ denotes the velocity of the fluid, $g(x) = (g_1, g_2)$ is the external force function, the scalar function p represents the pressure and $f(t, u_t)$ denotes the delay term with delay function u_t (see notation in Section 2). Equations (1.1)–(1.2) describe the motion of an isothermal incompressible viscous fluid in which the forcing term contains some delay features. These situations may occur, for example, when we want to control the system via applying a force which considers not only the present state but also the history state of the system. In (1.1), $\tau(e(u)) = (\tau_{ij}(e(u)))_{2 \times 2}$, which is usually called the extra stress tensor of the fluid, is a matrix of order 2×2 defined as

$$\tau_{ij}(e(u)) = 2\mu_0(\varepsilon + |e|^2)^{-\alpha/2}e_{ij} - 2\mu_1\Delta e_{ij}, \quad i, j = 1, 2, \quad (1.3)$$

where

$$e_{ij} = e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad |e|^2 = \sum_{i,j=1}^2 |e_{ij}|^2,$$

and $\mu_0, \mu_1, \alpha, \varepsilon$ are parameters which generally depend on the temperature and pressure of the fluid. In this paper, we assume that $\mu_0, \mu_1, \varepsilon$ are positive constants and $0 < \alpha < 1$. The extra stress tensor (1.3) was first formulated by Ladyzhenskaya [36] in the context of a non-Newtonian fluid. There are many works concerning the unique existence, regularity and long-term behavior of solutions to the addressed non-Newtonian fluid or its associated versions (see e.g. [4, 6, 7, 8, 9, 27, 32, 36, 41, 42, 57, 58]).

The long-term behavior of dynamical systems is an interesting and challenging problem because it can provide important information on the future evolution of the system. The attractor is an important concept in the study of the long-term behavior of dynamical systems. There are many works concerning this subject ([3, 24, 29, 37, 47, 52, 53]). For physical reasons (e.g. biological motivations (e.g., [35]) like species' growth), retarded differential equations have become an important area of applied mathematics (see [28, 30, 54]). There are many papers on retarded ODEs, PDEs and functional differential equations (see e.g. [43, 44, 45, 46]). For instance, Navier-Stokes equations with delays were systematically studied in [17, 18, 19]. Many important and general results on delay dynamical systems on $\mathcal{C}([-\nu, 0], X)$ (X is a Hilbert space or Banach space) were presented in [54]. Functional differential equations with delays were systematically studied in [28, 30, 54].

Recently, a new type of attractor, called the pullback attractor, has drawn much attention from mathematicians and physicists (see e.g. [25, 26, 33, 34, 38, 39, 48, 50, 51]). Actually, the theory of pullback attractors has been shown to be very useful in the understanding of the dynamics of non-autonomous and random dynamical systems,

including those with delays (see e.g. [1, 19, 21, 22, 23, 48, 50, 51]). There are several versions of the concept of pullback attractors; see [48] for details. One of the advantages of the theory of pullback attractors is that it allows one to handle more general non-autonomous terms and it works under random environments as well (see [13]). In fact, the theory of pullback attractors has been widely used and there are many works concerning this subject; one can refer to [10], [13]–[16], [20, 23, 33, 34, 38, 39, 48, 55, 59].

We would like to point out that Caraballo and Real [19] started an investigation involving Navier-Stokes models in which the forcing term contains delays. The idea of the current paper originates from [19]. Caraballo and Real [19] proved the unique existence of pullback attractors $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$ and $\{\mathcal{A}_{M_H^2}(t)\}_{t \in \mathbb{R}} \subset H \times L_H^2$ (see notation in Section 2), respectively, for two different processes $\{U(t, \tau)\}_{t \geq \tau}$ and $\{S(t, \tau)\}_{t \geq \tau}$ associated to the Navier-Stokes models with delays. The main technique for the investigation in [19] involves two mappings $\tilde{U}(\cdot, \cdot)$ and j which give some connection between the processes $\{U(t, \tau)\}_{t \geq \tau}$ and $\{S(t, \tau)\}_{t \geq \tau}$. Moreover, an interesting relation between $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ and $\{\mathcal{A}_{M_H^2}(t)\}_{t \in \mathbb{R}}$ is given via the mapping j . These techniques and arguments will also be applied by us. As will be shown below, the works of Caraballo and Real [19] greatly influence the presentation of this paper.

In this paper, we first aim to establish the existence of pullback attractors $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$, $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}} \subset H \times L_H^2$, and $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_W$, $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}} \subset W \times L_W^2$ for the corresponding processes defined on different state spaces. For the existence of $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$, we use the technique and argument of [19] with some slight differences on calculations. This result seems a trivial generalization of [19] to the non-Newtonian fluid, but it make sense because it is the base of the regularity of the pullback attractor and furthermore it will provide us the existence of a family of uniformly bounded pullback absorbing sets in the space $W \times L_W^2$. Indeed, we will prove, using the argument of energy equality (see e.g. Ball [5]) that the obtained uniformly bounded pullback absorbing set in $W \times L_W^2$ is in fact pre-compact in $W \times L_W^2$, and then we prove the existence of the pullback attractor $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}} \subset W \times L_W^2$. We would like to point out that it seems difficult to obtain the existence of the pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_W$ via the energy method. The main reason is that $\mathcal{C}_W$ is not a Hilbert space. One can see Remark 4.1 for details.

The second purpose of the current paper is to prove the regularity of the pullback attractors. We first establish

$$\mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R}. \quad (1.4)$$

The main step is to utilize the Uniform Gronwall Lemma to prove that the solutions of the addressed delay system with initial data in any bounded set of $H \times L_H^2$ will enter a bounded set of $W \times L_W^2$ after large enough time. Then we use (1.4) and the relation between $\mathcal{A}_{\mathcal{C}_H}(t)$ and $\mathcal{A}_{E_H^2}(t)$ to prove that $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ is a family of compact sets of $\mathcal{C}_W$. This fact allows us to establish the existence of the pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_W$. Meanwhile, the following regularity,

$$\mathcal{A}_{\mathcal{C}_H}(t) = \mathcal{A}_{\mathcal{C}_W}(t), \quad \forall t \in \mathbb{R}, \quad (1.5)$$

follows from the uniqueness of the uniformly bounded pullback attractor.

By (1.4) and (1.5), we can claim two conclusions. The first one is that the pullback attractors associated to (1.1)–(1.2) do not depend on the energy space chosen for the mathematical studying; the second one is the pullback asymptotic smoothing effect of the fluid in the sense that the solutions become eventually more regular (possessing H^2 -regularity) than the initial data (possessing L^2 -regularity), which implies, in the case of delays, that the regularity of the fluid in its history state does not play an important role on the regularity of its eventual state. Also by (1.4), (1.5) and the definition of the linear operator J (see (3.38)), we have

$$\mathcal{A}_{E_H^2}(t) = J(\mathcal{A}_{C_H}(t)) = J(\mathcal{A}_{C_W}(t)) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R}.$$

Compared with the work of [59], the new problem encountered in this paper is that C_H and C_W are neither Hilbert spaces nor uniform convex Banach spaces. In [59], the authors of the present paper proved the existence and regularity of the cocycle attractor for the non-Newtonian fluid without delays, by establishing that the associated cocycle satisfies the pullback condition (PC) (see Lemma 2.3 in [59]) and by using the regularity of solutions and embedding theorems. Since C_H and C_W are not uniform convex Banach spaces, the argument used in Section 3 of [59] seems difficult to apply. Also, the method of energy equality is not applicable because C_H and C_W are not Hilbert spaces. At the same time, if the spatial domain is unbounded, the embedding $W \hookrightarrow H$ is no longer compact and the arguments used both in Section 3 and Section 4 of [59] are inadequate. We note that the approach presented in this paper is still valid for the case when the spatial domain is unbounded; see Remark 6.2 for details.

The paper is organized as follows. In the next section, we first introduce some notation and prove the unique existence of solutions to our problem. Then we recall some definitions and already known results concerning pullback attractors. In Section 3, we prove the existence of pullback attractors $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}} \subset C_H$ and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}} \subset H \times L_H^2$ for two different processes defined on C_H and $H \times L_H^2$, respectively. In Section 4, we establish the existence of a pullback attractor $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}} \subset W \times L_W^2$. In Section 5, we first prove $\mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t)$ for all $t \in \mathbb{R}$. Then we establish the existence of the pullback attractor $\{\mathcal{A}_{C_W}(t)\}_{t \in \mathbb{R}}$ in C_W and show $\mathcal{A}_{C_H}(t) = \mathcal{A}_{C_W}(t)$ for all $t \in \mathbb{R}$. We end the paper with some remarks in Section 6.

2. Preliminaries.

In this paper, we use the following notation:

$L^p(\Omega)$ = the 2D vector Lebesgue space with norm $\|\cdot\|_{L^p(\Omega)}$; in particular, $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$,
 $H^m(\Omega)$ = the 2D vector Sobolev space $\{\phi = (\phi_1, \phi_2) \in L^2(\Omega), \nabla^k \phi \in L^2(\Omega), k \leq m\}$
 with norm $\|\cdot\|_{H^m(\Omega)}$ (see [2]),

$H_0^1(\Omega)$ = closure of $\{\phi : \phi = (\phi_1, \phi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)\}$ in $H^1(\Omega)$,

$\mathcal{V} = \{\phi \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) : \phi = (\phi_1, \phi_2), \nabla \cdot \phi = 0\}$,

H = closure of $\mathcal{V}$ in $L^2(\Omega)$ with norm $\|\cdot\|$, H' = dual space of H ,

W = closure of $\mathcal{V}$ in $H^2(\Omega)$ with norm $\|\cdot\|_W$, W' = dual space of W ,

$(\cdot, \cdot)$ = the inner product in H , $\langle \cdot, \cdot \rangle$ = the dual pairing between W and W' ,

$C_H = \mathcal{C}([-\nu, 0]; H)$, $C_W = \mathcal{C}([-\nu, 0]; W)$ (ν is a fixed positive number),

$L_H^2 = L^2(-\nu, 0; H)$, $L_W^2 = L^2(-\nu, 0; W)$,
 $E_H^2 = H \times L_H^2$, $E_W^2 = W \times L_W^2$,
 $(\cdot, \cdot)_{E_H^2}$ = the inner product in E_H^2 defined as

$$((u_0, \psi), (v_0, \varphi))_{E_H^2} = (u_0, v_0) + \int_{-\nu}^0 \psi(\theta) \varphi(\theta) d\theta, \quad (u_0, \psi), (v_0, \varphi) \in E_H^2,$$

$\|\cdot\|_{E_H^2}$ = the norm of E_H^2 induced by $(\cdot, \cdot)_{E_H^2}$,
 $(\cdot, \cdot)_{E_W^2}$ = the inner product in E_W^2 defined as

$$\begin{aligned} & ((u_0, \psi), (v_0, \varphi))_{E_W^2} \\ &= \sum_{k=0}^2 (\nabla^k u_0, \nabla^k v_0) + \sum_{k=0}^2 \int_{-\nu}^0 \nabla^k \psi(\theta) \nabla^k \varphi(\theta) d\theta, \quad (u_0, \psi), (v_0, \varphi) \in E_W^2, \end{aligned}$$

$\|\cdot\|_{E_W^2}$ = the norm of E_W^2 induced by $(\cdot, \cdot)_{E_W^2}$,
 $\mathbb{R}$ = the set of real numbers, $\mathbb{N}$ = the set of positive integers,
 $\mathbb{R}_\tau = [\tau, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$,

$\text{Dist}_E(X, Y)$ = the Hausdorff semidistance between $X \subset E$ and $Y \subset E$ defined by

$$\text{Dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_E,$$

“ $\longrightarrow$ ” and “ $\rightharpoonup$ ” denote convergence in the strong topology and the weak topology, respectively,

“ $\hookrightarrow$ ” denotes embedding between spaces.

We next introduce some operators so that we can put our problem into an abstract form. Set

$$a(u, v) = \sum_{i,j,k=1}^2 \left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(v)}{\partial x_k} \right) = \sum_{i,j,k=1}^2 \int_{\Omega} \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} dx, \quad u, v \in W. \quad (2.6)$$

LEMMA 2.1 (Bloom and Hao [8]). There exist two positive constants c_1 and c_2 which depend only on Ω such that

$$c_1 \|u\|_W^2 \leq a(u, u) \leq c_2 \|u\|_W^2, \quad \forall u \in W. \quad (2.7)$$

From the definition of $a(\cdot, \cdot)$ and Lemma 2.1 we see that $a(\cdot, \cdot)$ defines a positive definite symmetric bilinear form on W . As a consequence of the Lax-Milgram Lemma, we obtain an isometric operator $A \in \mathcal{L}(W, W')$, via $\langle Au, v \rangle = a(u, v)$, $\forall u, v \in W$. Moreover, let $D(A) = \{u \in W : Au \in H\}$. Then $D(A)$ is a Hilbert space and A is also an isometry map from $D(A)$ to H . Indeed, $A = P\Delta^2$, where P is the Leray projector from $L^2(\Omega)$ to H . For any $u \in D(A)$, we have

$$c_1 \|u\|_W \leq \|Au\|, \quad (2.8)$$

because by Lemma 2.1 we get

$$c_1 \|u\|_W^2 \leq a(u, u) = \langle Au, u \rangle = (Au, u) \leq \|Au\| \|u\|_W.$$

We also define a continuous trilinear form on $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ as follows:

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad u, v, w \in H_0^1(\Omega).$$

Since $W \subset H_0^1(\Omega)$, $b(\cdot, \cdot, \cdot)$ is continuous on $W \times W \times W$ and one can check

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad \forall u, v, w \in W. \quad (2.9)$$

Now for any $u \in W$,

$$\langle B(u), w \rangle = b(u, u, w), \quad \forall w \in W, \quad (2.10)$$

defines a continuous functional $B(u)$ from $W \times W$ to W' . Finally, for $u \in W$, we set

$$\mu(u) = 2\mu_0(\varepsilon + |e(u)|^2)^{-\alpha/2}$$

and define $N(u)$ as

$$\langle N(u), v \rangle = \sum_{i,j=1}^2 \int_{\Omega} \mu(u) e_{ij}(u) e_{ij}(v) dx, \quad \forall v \in W. \quad (2.11)$$

Then the functional $N(u)$ is continuous from W to W' . When $u \in D(A)$, $N(u)$ can be extended to H via

$$\langle N(u), v \rangle = (N(u), v) = - \int_{\Omega} \{ \nabla \cdot [\mu(u) e(u)] \} \cdot v dx, \quad \forall v \in H. \quad (2.12)$$

Given $T > \tau$ and $u : [\tau - \nu, T] \mapsto H$, for each $t \in [\tau, T]$, we denote by $u_t = u_t(\cdot)$ the function defined on $[-\nu, 0]$ according to the relation $u_t(\theta) = u(t + \theta)$, $\theta \in [-\nu, 0]$. In order to properly state the problem, we need some suitable hypotheses on the term containing delays. Let $f(\cdot, \cdot) : \mathbb{R} \times \mathcal{C}_H \mapsto H$ satisfy the following assumptions:

(A₁) $\forall u \in \mathcal{C}_H, \mathbb{R} \ni t \mapsto f(t, u) \in H$ is measurable.

(A₂) $f(t, 0) = 0, \quad \forall t \in \mathbb{R}$.

(A₃) There exists a positive constant C_f such that $\forall t \in \mathbb{R}, \forall u, v \in \mathcal{C}_H$, we have

$$\|f(t, u) - f(t, v)\| \leq C_f \|u - v\|_{\mathcal{C}_H}.$$

(A₄) There exist two positive constants β_0 and L_f such that for any $\beta \in [0, \beta_0)$, any $t > \tau$ and any $u, v \in \mathcal{C}([\tau - \nu, t]; H)$, it follows that

$$\int_{\tau}^t e^{\beta\xi} \|f(t, u_{\xi}) - f(t, v_{\xi})\|^2 d\xi \leq L_f^2 \int_{\tau-\nu}^t e^{\beta\xi} \|u(\xi) - v(\xi)\|^2 d\xi.$$

(A₅) $L_f < c_1 \mu_1$.

REMARK 2.1. Caraballo and Real [19] presented a specific function which satisfies all assumptions (A₁)–(A₄); see pp. 292, 293 in [19] for details. If we let the constant μ_1 be sufficiently large, then (A₁)–(A₅) are satisfied.

Indeed, (A₁)–(A₃) imply that for a given $u \in \mathcal{C}([\tau - \nu, T]; H)$ the function $f(t, u_t) : [\tau, T] \ni t \mapsto H$ is measurable (see [19]) and belongs to $L^\infty(\tau, T; H)$. Then (A₄) implies that the mapping $\mathcal{F} : \mathcal{C}([\tau - \nu, T]; H) \ni u \mapsto f(t, u_t) \in L^2(\tau, T; H)$ can be

extended uniquely to a mapping which is uniformly continuous from $L^2(\tau - \nu, T; H)$ into $L^2(\tau, T; H)$. Thus, for any $u, v \in L^2(\tau, T; H)$, it follows that

$$\int_{\tau}^t \|f(\xi, u_{\xi}) - f(\xi, v_{\xi})\|^2 d\xi \leq L_f^2 \int_{\tau-\nu}^t \|u(\xi) - v(\xi)\|^2 d\xi. \quad (2.13)$$

By the above notation and analysis, we consider problem (1.1)–(1.3), excluding the pressure p , in the solenoidal vector field as

$$\begin{cases} \text{For each } \tau \in \mathbb{R} \text{ and } \forall T > \tau, \text{ find} \\ u \in L^2(\tau - \nu, T; H) \cap L^2(\tau, T; W) \cap L^{\infty}(\tau, T; H), \\ \text{such that } \frac{\partial}{\partial t} u(t) + 2\mu_1 A u(t) + B(u(t)) + N(u(t)) = g(x) + f(t, u_t) \\ \text{in } \mathcal{D}(\tau, +\infty; W'), \quad u(\tau) = u_0, \quad u(t) = \psi(t - \tau), \quad t \in [\tau - \nu, \tau]. \end{cases} \quad (2.14)$$

For problem (2.14), we have the following existence and uniqueness result.

THEOREM 2.1. Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then for each $\tau \in \mathbb{R}$, it follows that

(1) For any $(u_0, \psi) \in E_H^2$, there exists a unique solution u to problem (2.14); moreover, $u \in \mathcal{C}([\tau, +\infty); H)$.

(2) For any $(u_0, \psi) \in E_w^2$, problem (2.14) admits a unique solution u satisfying

$$u \in L^2(\tau - \nu, T; H) \cap L^2(\tau, T; D(A)) \cap \mathcal{C}([\tau, T]; W) \text{ and } \frac{\partial u}{\partial t} \in L^2(\tau, T; H). \quad (2.15)$$

In particular, if $\psi \in \mathcal{C}_w$ and $u_0 = \psi(0)$, then $u \in \mathcal{C}([\tau - \nu, +\infty); W)$.

Proof. For the existence of solutions, we can use the Galerkin approximation scheme similar to that in Constantin and Foias [12], and Caraballo and Real [19]. Here we only emphasize the different places involving the new term $N(\cdot)$ and the higher-order term $A(\cdot)$. Indeed, by the definition of A and the classical spectral theory of elliptic operators (see e.g. [41]), we see that the operator A possesses a family of eigenfunctions $\{w_n\}_{n=1}^{\infty} \subset W \cap D(A)$, which form a basis of W and are orthonormal in H . We consider the subspace W_m of W spanned by $w_1, w_2, \dots, w_m$, and the projector $P_m : H \mapsto W_m$ defined as

$$P_m u = \sum_{i=1}^m (u, w_i) w_i, \quad u \in H.$$

Define

$$u^m(t) = \sum_{i=1}^m \gamma_{mi}(t) w_i$$

and consider the following problem:

$$\begin{cases} \text{Find } u \in L^2(\tau - \nu, T; W_m) \cap \mathcal{C}([\tau, T]; W_m) \text{ such that} \\ \frac{\partial}{\partial t} (u^m(t), w_i) + 2\mu_1 (A u(t), w_i) + \langle B(u(t)), w_i \rangle + \langle N(u(t)), w_i \rangle \\ \quad = (g, w_i) + (f(t, u_t), w_i) \quad \text{in } \mathcal{D}'(\tau, T), \quad 1 \leq i \leq m, \\ u^m(\tau) = P_m u_0 = u_0^m, \quad u^m(t) = P_m \psi(t - \tau), \quad t \in [\tau - \nu, \tau]. \end{cases} \quad (2.16)$$

Note that problem (2.16) is a system of ordinary differential equations with respect to the unknown functions $\gamma^m(t) = (\gamma_{m1}(t), \dots, \gamma_{mm}(t))$. We can obtain the existence of solutions to problem (2.16) defined on an interval $[\tau, t_*](\tau < t_* \leq T)$ by applying

essentially the same proof as that in Caraballo and Real [17]. We next derive the a priori estimates which show that one can take $t_* = T$. Indeed, multiplying the equation in (2.16) by $\gamma_{mi}(t)$ and summing in i , we obtain by using Lemma 2.1 that for all $t \in [\tau, t_*]$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u^m(t)\|^2 + 2c_1\mu_1 \|u^m(t)\|_w^2 + \langle B(u^m(t)), u^m(t) \rangle + \langle N(u^m(t)), u^m(t) \rangle \\ & \leq \frac{1}{2} \frac{d}{dt} \|u^m(t)\|^2 + 2\mu_1 a(u^m(t), u^m(t)) + \langle B(u^m(t)), u^m(t) \rangle + \langle N(u^m(t)), u^m(t) \rangle \\ & = (g, u^m(t)) + (f(t, u_t^m), u^m(t)), \end{aligned}$$

integrating which over $[\tau, t]$, we have

$$\begin{aligned} & \frac{1}{2} \|u^m(t)\|^2 + 2c_1\mu_1 \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi \\ & + \int_{\tau}^t \langle B(u^m(\xi)), u^m(\xi) \rangle d\xi + \int_{\tau}^t \langle N(u^m(\xi)), u^m(\xi) \rangle d\xi \\ & = \frac{1}{2} \|u_0^m\|^2 + \int_{\tau}^t (g, u^m(\xi)) d\xi + \int_{\tau}^t (f(\xi, u_{\xi}^m), u^m(\xi)) d\xi. \end{aligned} \quad (2.17)$$

Now by (2.9) and the definition of the operator $N(\cdot)$, we get

$$\int_{\tau}^t \langle B(u^m(\xi)), u^m(\xi) \rangle d\xi = 0, \quad (2.18)$$

$$\int_{\tau}^t \langle N(u^m(\xi)), u^m(\xi) \rangle d\xi \geq 0. \quad (2.19)$$

Also, we have

$$\int_{\tau}^t (g, u^m(\xi)) d\xi \leq \frac{1}{2c_1\mu_1} \int_{\tau}^t \|g\|^2 d\xi + \frac{c_1\mu_1}{2} \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi, \quad (2.20)$$

and by $(\mathbf{A}_2)$, $(\mathbf{A}_4)$ and $(\mathbf{A}_5)$,

$$\begin{aligned} \int_{\tau}^t (f(\xi, u_{\xi}^m), u^m(\xi)) d\xi &= \int_{\tau}^t (f(\xi, u_{\xi}^m) - f(\xi, 0), u^m(\xi)) d\xi \\ &\leq \int_{\tau}^t \|f(\xi, u_{\xi}^m) - f(\xi, 0)\| \|u^m(\xi)\| d\xi \\ &\leq \frac{1}{2c_1\mu_1} \int_{\tau}^t \|f(\xi, u_{\xi}^m) - f(\xi, 0)\|^2 d\xi \\ &\quad + \frac{c_1\mu_1}{2} \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi \\ &\leq \frac{L_f^2}{2c_1\mu_1} \int_{\tau-\nu}^{\tau} \|u^m(\xi)\|^2 d\xi + \frac{L_f^2 + c_1^2\mu_1^2}{2c_1\mu_1} \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi \\ &= \frac{L_f^2}{2c_1\mu_1} \|P_m\psi(\xi)\|_{L_H^2}^2 + \frac{L_f^2 + c_1^2\mu_1^2}{2c_1\mu_1} \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi \\ &\leq \frac{L_f^2}{2c_1\mu_1} \|\psi\|_{L_H^2}^2 + c_1\mu_1 \int_{\tau}^t \|u^m(\xi)\|_w^2 d\xi. \end{aligned} \quad (2.21)$$

It then follows from (2.17)–(2.21) that

$$\|u^m(t)\|^2 + c_1\mu_1 \int_{\tau}^t \|u^m(\xi)\|_W^2 d\xi \leq \|u_0\|^2 + \frac{1}{c_1\mu_1} \int_{\tau}^t \|g\|^2 d\xi + \frac{L_f^2}{c_1\mu_1} \|\psi\|_{L_H^2},$$

by which we can take constants (depending on u_0 , c_1 , L_f , ψ , g , T , but not on m and t_*) M_1 and M_2 such that

$$\sup_{t \in [\tau, t_*]} \|u^m(t)\|^2 \leq M_1, \quad \int_{\tau}^{t_*} \|u^m(\xi)\|_W^2 d\xi \leq M_2.$$

Thus, we can take $t_* = T$ and conclude that the sequence $\{u^m\}_{m=1}^{\infty}$ is bounded in $L^2(\tau, T; W) \cap L^{\infty}(\tau, T; H)$. Note that $u^m(t) = P_m\psi(t - \tau)$ for $t \in [\tau - \nu, \tau)$ and by the choice of the basis $\{w_i\}_{i=1}^{\infty}$, we find that for $t \in [\tau - \nu, \tau)$ the sequence $\{u^m\}_{m=1}^{\infty}$ converges to ψ in L_W^2 . Consequently, $f(\cdot, u^m)$ is bounded in $L^2(\tau, T; W')$, and by Lemma 2.2 of [58], $B(u^m)$ and $N(u^m)$ are bounded in $L^2(\tau, T; W')$. Then using the same reasoning as in [58] we can prove that the sequence $\{\frac{\partial u^m}{\partial t}\}_{m=1}^{\infty}$ is bounded in $L^2(\tau, T; W')$. By Lemma 2.2 below and the assumptions on f , we see that there exist a subsequence (extract if necessary) of $\{u^m\}_{m=1}^{\infty}$ and $u \in L^2(\tau - \nu, T; W)$ such that

$$\begin{cases} u^m \rightharpoonup u \text{ weakly in } L^2(\tau - \nu, T; W), \\ u^m \longrightarrow u \text{ strongly in } L^2(\tau, T; H), \\ u^m \longrightarrow u \text{ weakly-star in } L^{\infty}(\tau - \nu, T; H), \\ f(\cdot, u^m) \rightharpoonup f(\cdot, u) \text{ weakly in } L^2(\tau - \nu, T; W'). \end{cases} \quad (2.22)$$

By (2.22), we can take limits (similar to that of Bloom and Hao [8]) in (2.16) after integrating over (τ, t) (for $t \in (\tau, T)$), and get that u is the solution to problem (2.14). The existence of solutions has been proved. The uniqueness of solutions will follow directly from Lemma 3.1 in Section 3. We end the proof of (1). The proof of (2) is similar to that of (1) and we omit it here. This concludes the proof of Theorem 2.1. $\square$

We next recall some definitions and results concerning the pullback attractors. These definitions and results can be found in [19, 24].

Let (E, d) be a complete metric space and $\mathcal{B}(E)$ the collection of all bounded sets of E .

DEFINITION 2.1. A two-parameter mapping $\{V(t, \tau)\}_{t \geq \tau}$ is said to be a process on E if

- (i) $V(t, \tau) : E \mapsto E$ for all $t \geq \tau$, $\tau \in \mathbb{R}$;
- (ii) $V(\tau, \tau) = \text{Id}$ (identity operator in E), $\tau \in \mathbb{R}$;
- (iii) $V(t, s)V(s, \tau) = V(t, \tau)$ for all $t \geq s \geq \tau$.

If, in addition,

- (iv) $V(t, \tau)x$ is jointly continuous in (t, τ, x) for all $t \geq \tau$, $\tau \in \mathbb{R}$, $x \in E$,

then $\{V(t, \tau)\}_{t \geq \tau}$ is said to be a continuous process on E .

DEFINITION 2.2. Let $\{V(t, \tau)\}_{t \geq \tau}$ be a process on E . A family of sets $\{\mathcal{B}(t)\}_{t \in \mathbb{R}} \subset E$ is said to be pullback absorbing for the process V if for any $t \in \mathbb{R}$ and any $\mathcal{B} \in \mathcal{B}(E)$, there exists $T_{\mathcal{B}}(t) > 0$ such that $V(t, t-s)\mathcal{B} \subset \mathcal{B}(t)$, $\forall s \geq T_{\mathcal{B}}(t)$. The absorption is said to be uniform if $T_{\mathcal{B}}(t)$ does not depend on t .

DEFINITION 2.3. Let $\{V(t, \tau)\}_{t \geq \tau}$ be a process on E . A family of compact sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be a (global) pullback attractor for $\{V(t, \tau)\}_{t \geq \tau}$ if it satisfies

- (i) Invariance: $V(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, $\forall t \geq \tau$, and
(ii) Pullback attracting: $\lim_{s \rightarrow +\infty} \text{Dist}_E(V(t, t-s)\mathcal{B}, \mathcal{A}(t)) = 0$, $\forall \mathcal{B} \in \mathcal{B}(E)$, $\forall t \in \mathbb{R}$.

REMARK 2.2. The pullback attractor given by Definition 2.2 is indeed global. For other versions of the definition of pullback attractor, e.g., global pullback $\mathcal{D}$ -attractor, one can refer to [14, 19, 51]. Also, the above defined global pullback attractor possesses the weak attracting property of kernel sections; see [24] for details.

For the existence of pullback attractors, the following result was proved in [11, 51].

THEOREM 2.2 ([51]). *Let $\{V(t, \tau)\}_{t \geq \tau}$ be a continuous process on E . If there exists a family of compact pullback absorbing sets $\{\mathcal{B}(t)\}_{t \in \mathbb{R}} \subset E$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, and $\mathcal{A}(t) \subset \mathcal{B}(t)$ for all $t \in \mathbb{R}$. Moreover*

$$\mathcal{A}(t) = \overline{\bigcup_{\mathcal{B} \in \mathcal{B}(E)} \omega_{\mathcal{B}}(t)}, \quad \text{where} \quad \omega_{\mathcal{B}}(t) = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{s \geq n} V(t, t-s)\mathcal{B}}.$$

REMARK 2.3. Caraballo and Real [19] pointed out that if the family of sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is uniformly (with respect to $t \in \mathbb{R}$) bounded or the family of compact absorbing sets is uniformly bounded, then the pullback attractor is unique. But the uniqueness of the pullback attractor defined above does not hold in general.

We conclude this section with the following results.

LEMMA 2.2 ([24]). *Let E_0, E_1, E be three Banach spaces satisfying $E_1 \hookrightarrow E \subset E_0$, where the embedding $E_1 \hookrightarrow E$ is compact. Assume $p_1 \geq 1$ and $p_0 > 1$. Set*

$$W_{p_1, p_0}(0, T; E_1, E_0) = \{\psi(t), t \in [0, T] : \psi(t) \in L^{p_1}(0, T; E_1), \psi'(t) \in L^{p_0}(0, T; E_0)\}.$$

Then the embedding is compact: $W_{p_1, p_0}(0, T; E_1, E_0) \hookrightarrow L^{p_1}(0, T; E), \forall T > 0$.

LEMMA 2.3 (Uniform Gronwall Lemma [53]). *Let $\Upsilon(t), \Phi(t), \Psi(t)$ be three positive locally integrable functions on $[\tau, +\infty)$ such that Φ' is locally integrable on $[\tau, +\infty)$ and*

$$\frac{d\Phi(t)}{dt} \leq \Upsilon(t)\Phi(t) + \Psi(t) \quad \text{for } t \geq \tau,$$

$$\int_t^{t+r} \Upsilon(s)ds \leq a_1, \quad \int_t^{t+r} \Psi(s)ds \leq a_2, \quad \int_t^{t+r} \Phi(s)ds \leq a_3, \quad \text{for } t \geq \tau,$$

where r, a_1, a_2 and a_3 are positive constants. Then

$$\Phi(t+r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \quad \forall t \geq \tau.$$

3. Existence of pullback attractors with L^2 -regularity. In this section, we will prove the existence of pullback attractors $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$ in the spaces $\mathcal{C}_H$ and E_H^2 , respectively. We see that both $\mathcal{C}_H$ and E_H^2 possess L^2 -regularity. As mentioned previously in the introduction, the idea and arguments of this section originate from Caraballo and Real [19].

LEMMA 3.1. *Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then for any given initial time $\tau \in \mathbb{R}$, the following hold.*

(I) Let $(u_0, \psi), (v_0, \varphi) \in E_H^2$ be two couples of initial data, and let $u(\cdot) = u(\cdot; \tau, (u_0, \psi))$ and $v(\cdot) = v(\cdot; \tau, (v_0, \varphi))$ be the corresponding solutions to (2.14). Then there exists a constant $c > 0$ (independent of the initial time and data) such that

$$\begin{aligned} \|u(t) - v(t)\|^2 &\leq \left(\frac{L_f^2}{c_1 \mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \|u_0 - v_0\|^2 \right) \\ &\quad \cdot \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1} \right) d\xi \right), \quad \forall t \geq \tau. \end{aligned}$$

(II) For $\forall t \geq \tau$, denote by $u_t(\cdot) = u_t(\cdot; \tau, (u_0, \psi))$ and $v_t(\cdot) = v_t(\cdot; \tau, (v_0, \varphi))$ the corresponding solutions to (2.14) with initial conditions $(u_0, \psi), (v_0, \varphi) \in E_H^2$. Then

$$\|u_t - v_t\|_{L_H^2}^2 \leq \left(\left(\frac{L_f^2 \nu}{c_1 \mu_1} + 1 \right) \|\psi - \varphi\|_{L_H^2}^2 + \nu \|u_0 - v_0\|^2 \right) \exp\left(\int_{\tau}^t \frac{L_f^2 + c}{c_1 \mu_1} \|u(\xi)\|_w^2 d\xi \right).$$

Moreover, if $t \geq \tau + \nu$, then

$$\|u_t - v_t\|_{c_H}^2 \leq \left(\frac{L_f^2}{c_1 \mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \|u_0 - v_0\|^2 \right) \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1} \right) d\xi \right).$$

(III) If $\psi, \varphi \in \mathcal{C}_H$ and $u_0 = \psi(0), v_0 = \varphi(0)$, then for any $t \geq \tau$,

$$\|u(t) - v(t)\|_{c_H}^2 \leq \left(\frac{\nu L_f^2}{c_1 \mu_1} + 1 \right) \|\psi - \varphi\|_{c_H}^2 \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1} \right) d\xi \right).$$

Proof. (I) Set $w = u - v$. Then w is a solution of the following problem:

$$\begin{aligned} \frac{\partial w(t)}{\partial t} + 2\mu_1 A w(t) + B(u(t)) - B(v(t)) + N(u(t)) - N(v(t)) \\ = f(t, u_t) - f(t, v_t), \quad t > \tau, \end{aligned} \quad (3.23)$$

$$w(\tau) = u_0 - v_0, \quad w(t) = \psi(t - \tau) - \varphi(t - \tau), \quad t \in [\tau - \nu, \tau]. \quad (3.24)$$

Multiplying (3.23) by $w(t)$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 + 2\mu_1 a(w(t), w(t)) + \langle B(u(t)) - B(v(t)), w \rangle \\ + \langle N(u(t)) - N(v(t)), w(t) \rangle = (f(t, u_t) - f(t, v_t), w(t)), \quad t > \tau. \end{aligned} \quad (3.25)$$

Integrating by parts, we have

$$\langle B(u) - B(v), w \rangle = b(u, u, w) - b(v, v, w) = b(w, u, w). \quad (3.26)$$

Then using the Hölder and Gagliardo-Nirenberg inequalities, we get

$$|\langle B(u) - B(v), w \rangle| = |b(w, u, w)| \leq c_1 \mu_1 \|w\|_w^2 + \frac{c^2}{4c_1 \mu_1} \|w\|^2 \|u\|_w^2; \quad (3.27)$$

hereafter c is a positive constant depending on the Gagliardo-Nirenberg inequality and thus essentially on Ω . Since $\alpha \in (0, 1)$, we have (see [41])

$$\langle N(u) - N(v), w \rangle = \sum_{i,j=1}^2 \int_{\Omega} [\mu(u) e_{ij}(u) - \mu(v) e_{ij}(v)] e_{ij}(w) dx \geq 0. \quad (3.28)$$

Also, clearly,

$$(f(t, u_t) - f(t, v_t), w(t)) \leq \frac{1}{2c_1\mu_1} \|f(t, u_t) - f(t, v_t)\|^2 + \frac{c_1\mu_1}{2} \|w(t)\|_w^2. \quad (3.29)$$

It then follows from (3.25)–(3.29) that

$$\frac{d}{dt} \|w(t)\|^2 + c_1\mu_1 \|w(t)\|_w^2 \leq \frac{c^2}{2c_1\mu_1} \|w(t)\|^2 \|u(t)\|_w^2 + \frac{1}{c_1\mu_1} \|f(t, u_t) - f(t, v_t)\|^2 \quad (3.30)$$

and thus

$$\frac{d}{dt} \|w(t)\|^2 \leq \frac{c^2}{2c_1\mu_1} \|w(t)\|^2 \|u(t)\|_w^2 + \frac{1}{c_1\mu_1} \|f(t, u_t) - f(t, v_t)\|^2. \quad (3.31)$$

Integrating (3.31) over $[\tau, t]$ and using (2.13), we obtain

$$\begin{aligned} & \|w(t)\|^2 - \|w(\tau)\|^2 \\ & \leq \frac{c^2}{2c_1\mu_1} \int_{\tau}^t \|w(\xi)\|^2 \|u(\xi)\|_w^2 d\xi + \frac{1}{c_1\mu_1} \int_{\tau}^t \|f(\xi, u_{\xi}) - f(\xi, v_{\xi})\|^2 d\xi \\ & \leq \frac{c^2}{2c_1\mu_1} \int_{\tau}^t \|w(\xi)\|^2 \|u(\xi)\|_w^2 d\xi + \frac{L_f^2}{c_1\mu_1} \int_{\tau-\nu}^t \|u(\xi) - v(\xi)\|^2 d\xi \\ & \leq \frac{L_f^2}{c_1\mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \int_{\tau}^t \left(\frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1\mu_1} \right) \|w(\xi)\|^2 d\xi, \end{aligned} \quad (3.32)$$

whence

$$\|w(t)\|^2 \leq \|u_0 - v_0\|^2 + \frac{L_f^2}{c_1\mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \int_{\tau}^t \left(\frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1\mu_1} \right) \|w(\xi)\|^2 d\xi, \quad \forall t \geq \tau,$$

and from which it follows by using the Gronwall Lemma that

$$\|w(t)\|^2 \leq (\|u_0 - v_0\|^2 + \frac{L_f^2}{c_1\mu_1} \|\psi - \varphi\|_{L_H^2}^2) \cdot \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1\mu_1} \right) d\xi\right), \quad \forall t \geq \tau.$$

The proof of **(I)** is complete.

(II) If $t \geq \tau + \nu$, then $t + \theta \geq \tau$ for any $\theta \in [-\nu, 0]$ and by **(I)**, we have

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &= \int_{-\nu}^0 \|u(t + \theta) - v(t + \theta)\|^2 d\theta \\ &\leq \int_{-\nu}^0 \sup_{\theta \in [-\nu, 0]} \|u(t + \theta) - v(t + \theta)\|^2 d\theta \\ &\leq \nu \left(\frac{L_f^2}{c_1\mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \|u_0 - v_0\|^2 \right) \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1\mu_1} \right) d\xi\right). \end{aligned}$$

If $\tau \leq t < \tau + \nu$, we clearly have

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &= \int_{-\nu}^0 \|u(t + \theta) - v(t + \theta)\|^2 d\theta \\ &\leq \left(\left(\frac{\nu L_f^2}{c_1\mu_1} + 1 \right) \|\psi - \varphi\|_{L_H^2}^2 + \nu \|u_0 - v_0\|^2 \right) \\ &\quad \times \exp\left(\int_{\tau}^t \left(\frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1\mu_1} \right) d\xi\right). \end{aligned}$$

Thus, for any $t \geq \tau$, there holds

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &\leq \left(\left(\frac{\nu L_f^2}{c_1 \mu_1} + 1\right) \|\psi - \varphi\|_{L_H^2}^2 + \nu \|u_0 - v_0\|^2\right) \\ &\quad \times \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right). \end{aligned}$$

Now, if $t \geq \tau + \nu$, then $t + \theta \geq \tau$ for any $\theta \in [-\nu, 0]$. By Theorem 2.1, we get

$$\begin{aligned} \|u_t - v_t\|_{\mathcal{C}_H}^2 &= \max_{\theta \in [-\nu, 0]} \|u(t + \theta) - v(t + \theta)\|^2 \\ &\leq \left(\frac{L_f^2}{c_1 \mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \|u_0 - v_0\|^2\right) \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right). \end{aligned}$$

Hence (II) holds.

(III) If $\psi, \varphi \in \mathcal{C}_H$ and $u_0 = \psi(0), v_0 = \varphi(0)$, then by (I), for any $t \geq \tau$, we have

$$\begin{aligned} &\|u(t) - v(t)\|^2 \\ &\leq \left(\frac{L_f^2}{c_1 \mu_1} \|\psi - \varphi\|_{L_H^2}^2 + \|u_0 - v_0\|^2\right) \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right) \\ &\leq \left(\nu \frac{L_f^2}{c_1 \mu_1} \|\psi - \varphi\|_{\mathcal{C}_H}^2 + \|\psi - \varphi\|_{\mathcal{C}_H}^2\right) \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right). \end{aligned}$$

Noticing that $u(t) - v(t) = \psi(t - \tau) - \varphi(t - \tau)$ when $\tau - \nu \leq t < \tau$, we get

$$\|u(t) - v(t)\|^2 \leq \left(\frac{\nu L_f^2}{c_1 \mu_1} + 1\right) \|\psi - \varphi\|_{\mathcal{C}_H}^2 \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right),$$

for $\forall t \geq \tau - \nu$. Therefore,

$$\begin{aligned} \|u_t - v_t\|_{\mathcal{C}_H}^2 &= \max_{\theta \in [-\nu, 0]} \|u(t + \theta) - v(t + \theta)\|^2 \\ &\leq \left(\frac{\nu L_f^2}{c_1 \mu_1} + 1\right) \|\psi - \varphi\|_{\mathcal{C}_H}^2 \cdot \exp\left(\int_\tau^t \left(\frac{c^2}{2c_1 \mu_1} \|u(\xi)\|_w^2 + \frac{L_f^2}{c_1 \mu_1}\right) d\xi\right), \quad \forall t \geq \tau. \end{aligned}$$

The proof of Lemma 3.1 is now complete. $\square$

Note that from Lemma 3.1 one can directly deduce the uniqueness of solutions to problem (2.14). Thus, for each initial time $\tau \in \mathbb{R}$, Theorem 2.1 ensures that problem (2.14) admits a unique solution $u(\cdot) = u(\cdot; \tau, (u_0, \psi)) \in L^2(\tau - \nu, T; H) \cap L^2(\tau, T; W) \cap \mathcal{C}([\tau, T]; H)$ for all $T > \tau$. We now can proceed as [19] did, with two different ways to construct the process associated to problem (2.14). Firstly, we can define a continuous process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau} : \mathcal{C}_H \mapsto \mathcal{C}_H$ as

$$\mathcal{U}(t, \tau)\psi = u_t(\cdot; \tau, (\psi(0), \psi)), \quad \forall \psi \in \mathcal{C}_H \text{ and } t \geq \tau. \quad (3.33)$$

Secondly, we can also define a continuous process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau} : E_H^2 \mapsto E_H^2$ via

$$\mathcal{S}(t, \tau)(u_0, \psi) = (u(t; \tau, (u_0, \psi)), u_t(\cdot; \tau, (u_0, \psi))), \quad \forall (u_0, \psi) \in E_H^2 \text{ and } t \geq \tau. \quad (3.34)$$

We next prove that $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ and $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possess a family of uniformly bounded pullback absorbing sets in $\mathcal{C}_H$ and E_H^2 , respectively. To this end, we introduce a family of mappings $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau} : E_H^2 \mapsto L_H^2$ defined by

$$\tilde{\mathcal{U}}(t, \tau)(u_0, \psi) = u_t(\cdot; \tau, (u_0, \psi)), \quad \forall (u_0, \psi) \in E_H^2 \text{ and } t \geq \tau. \quad (3.35)$$

Then we have

$$\mathcal{U}(t, \tau)\psi = \tilde{\mathcal{U}}(t, \tau)(\psi(0), \psi), \quad \forall \psi \in \mathcal{C}_H \text{ and } t \geq \tau, \quad (3.36)$$

and the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ can be represented as

$$\begin{aligned} \mathcal{S}(t, \tau)(u_0, \psi) &= (u(t; \tau, (u_0, \psi)), u_t(\cdot; \tau, (u_0, \psi))) \\ &= (u(t; \tau, (u_0, \psi)), \tilde{\mathcal{U}}(t, \tau)(u_0, \psi)), \quad \forall (u_0, \psi) \in E_H^2, t \geq \tau. \end{aligned} \quad (3.37)$$

Also, we define a linear mapping J as

$$J : \mathcal{C}_H \ni \psi \longmapsto J(\psi) = (\psi(0), \psi) \in H \times \mathcal{C}_H. \quad (3.38)$$

Then J is clearly continuous from $\mathcal{C}_H$ into $H \times \mathcal{C}_H$ and also into E_H^2 . Since for all $(u_0, \psi) \in E_H^2$ it follows that $\mathcal{U}(t, \tau)(u_0, \psi) = u_t(\cdot; \tau, (u_0, \psi)) \in \mathcal{C}_H$ provided $t \geq \tau + \nu$ (see Lemma 3.1 (II)), we then can rewrite $\mathcal{S}(t, \tau)(u_0, \psi)$ as

$$\mathcal{S}(t, \tau)(u_0, \psi) = J(\tilde{\mathcal{U}}(t, \tau)(u_0, \psi)), \quad \forall (u_0, \psi) \in E_H^2, t \geq \tau + \nu. \quad (3.39)$$

As will be shown below, the above facts will help us to establish the estimates for the processes $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ and $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in a direct way by using the obtained estimates for the mappings $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$.

The following definition and lemma were presented in [19].

DEFINITION 3.1. The family of bounded sets $\{\mathcal{B}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$ is said to be pullback absorbing sets for $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$ in E_H^2 if for any given bounded set $\tilde{D} \subset E_H^2$ and each $t \in \mathbb{R}$, there exists $\tilde{T}_{\tilde{D}}(t) > 0$ such that for any $s \geq \tilde{T}_{\tilde{D}}(t)$, it follows that $\tilde{\mathcal{U}}(t, t-s)\tilde{D} \subset \mathcal{B}(t)$.

Note that $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$ is not a process and the word “absorbing” in the above definition should be understood in a generalized sense.

LEMMA 3.2 ([19]). *Let $\{\mathcal{B}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$ be the family of bounded absorbing sets for $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$. Then*

- (i) $\{\mathcal{B}(t)\}_{t \in \mathbb{R}}$ is the family of pullback absorbing sets for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$;
- (ii) $\{J(\mathcal{B}(t))\}_{t \in \mathbb{R}}$ is the family of pullback absorbing sets for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in $H \times \mathcal{C}_H$.

LEMMA 3.3. Assume **(A₁)**–**(A₅)** hold and $g \in H$. Then the family of mappings $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$ possesses a family of bounded pullback absorbing sets $\{\mathcal{B}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$. Moreover, there exists a bounded set $\mathcal{B}_1 \subset \mathcal{C}_H$ such that $\mathcal{B}(t) = \mathcal{B}_1$ for all $t \in \mathbb{R}$.

Proof. Let $\tilde{D} \subset E_H^2$ be bounded. Then there exists an $\tilde{r} > 0$ such that

$$\|u_0\|^2 + \|\psi\|_{L_H^2}^2 \leq \tilde{r}, \quad \forall (u_0, \psi) \in \tilde{D}.$$

Take any $(u_0, \psi) \in \tilde{D}$ and $\tau \in \mathbb{R}$, and denote the corresponding solution as usual: $u(\cdot) = u(\cdot; \tau, (u_0, \psi))$. Multiplying the equation in (2.14) by $u(t)$ and using Lemma 2.1, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + 2c_1\mu_1 \|u(t)\|_W^2 + \langle B(u(t)), u(t) \rangle + \langle N(u(t)), u(t) \rangle \\ &= (g, u(t)) + (f(t, u_t), u(t)) \\ &\leq \frac{1}{2c_1\mu_1} \|g\|^2 + \frac{3c_1\mu_1}{2} \|u(t)\|^2 + \frac{1}{4c_1\mu_1} \|f(t, u_t)\|^2. \end{aligned} \quad (3.40)$$

Since $\langle B(u(t)), u(t) \rangle = 0$, by the non-negativity of the term $\langle N(u(t)), u(t) \rangle$ and the fact that $\|\phi\|_W \geq \|\phi\|$ for all $\phi \in W$, we deduce from (3.40) that

$$\frac{d}{dt} \|u(t)\|^2 + c_1 \mu_1 \|u(t)\|^2 \leq \frac{1}{c_1 \mu_1} \|g\|^2 + \frac{1}{2c_1 \mu_1} \|f(t, u_t)\|^2. \quad (3.41)$$

Now by $(\mathbf{A}_5)$, we can choose a $\beta \in (0, \beta_0)$ such that

$$\frac{L_f^2}{c_1 \mu_1} + \beta < c_1 \mu_1. \quad (3.42)$$

For the above β , we get by using (3.41) that

$$\begin{aligned} \frac{d}{dt} (e^{\beta t} \|u(t)\|^2) &= \beta e^{\beta t} \|u(t)\|^2 + e^{\beta t} \frac{d}{dt} \|u(t)\|^2 \\ &\leq \frac{1}{c_1 \mu_1} \|g\|^2 e^{\beta t} + \frac{\|f(t, u_t)\|^2}{2c_1 \mu_1} e^{\beta t} + (\beta - c_1 \mu_1) \|u(t)\|^2 e^{\beta t}. \end{aligned} \quad (3.43)$$

Integrating (3.43) over $[\tau, t]$, we can get

$$\begin{aligned} e^{\beta t} \|u(t)\|^2 - e^{\beta \tau} \|u_0\|^2 &\leq \frac{1}{c_1 \mu_1} \int_{\tau}^t \|g\|^2 e^{\beta \xi} d\xi + \int_{\tau}^t \frac{\|f(\xi, u_{\xi})\|^2}{2c_1 \mu_1} e^{\beta \xi} d\xi \\ &\quad + (\beta - c_1 \mu_1) \int_{\tau}^t \|u(\xi)\|^2 e^{\beta \xi} d\xi \\ &= \frac{\|g\|^2}{c_1 \mu_1 \beta} e^{\beta t} + \frac{L_f^2}{2c_1 \mu_1} \int_{\tau-\nu}^{\tau} \|\psi(\xi - \tau)\|^2 e^{\beta \xi} d\xi \\ &\quad + \left(\frac{L_f^2}{2c_1 \mu_1} + \beta - c_1 \mu_1 \right) \int_{\tau}^t \|u(\xi)\|^2 e^{\beta \xi} d\xi \\ &\leq \frac{\|g\|^2}{c_1 \mu_1 \beta} e^{\beta t} + \frac{L_f^2 e^{\beta \tau}}{2c_1 \mu_1} \int_{-\nu}^0 \|\psi(\theta)\|^2 d\theta, \quad \forall t \geq \tau. \end{aligned}$$

Hence,

$$e^{\beta t} \|u(t)\|^2 \leq \frac{\|g\|^2}{c_1 \mu_1 \beta} e^{\beta t} + e^{\beta \tau} \tilde{r} \left(1 + \frac{L_f^2}{2c_1 \mu_1} \right), \quad \forall t \geq \tau,$$

and

$$\|u(t)\|^2 \leq \frac{\|g\|^2}{c_1 \mu_1 \beta} + \tilde{r} \left(1 + \frac{L_f^2}{2c_1 \mu_1} \right) e^{-\beta(t-\tau)}, \quad \forall t \geq \tau. \quad (3.44)$$

If $t \geq \tau + \nu$, then for any $\theta \in [-\nu, 0]$, we have $t + \theta \geq \tau$ and

$$\|u(t + \theta)\|^2 \leq \frac{\|g\|^2}{c_1 \mu_1 \beta} + \tilde{r} \left(1 + \frac{L_f^2}{2c_1 \mu_1} \right) e^{\beta \nu} e^{-\beta(t-\tau)}, \quad \forall \theta \in [-\nu, 0].$$

Consequently,

$$\|u_t\|_{c_H}^2 = \max_{\theta \in [-\nu, 0]} \|u(t + \theta)\|^2 \leq \frac{\|g\|^2}{c_1 \mu_1 \beta} + \tilde{r} \left(1 + \frac{L_f^2}{2c_1 \mu_1} \right) e^{\beta \nu} e^{-\beta(t-\tau)}, \quad \forall t \geq \tau + \nu. \quad (3.45)$$

We now consider the initial time $t - s$ instead of τ , and then (3.45) implies

$$\begin{aligned} \|u_t\|_{\mathcal{C}_H}^2 &= \|\tilde{\mathcal{U}}(t, t-s)(u_0, \psi)\|_{\mathcal{C}_H}^2 \\ &\leq \frac{\|g\|^2}{c_1\mu_1\beta} + \tilde{r}\left(1 + \frac{L_f^2}{2c_1\mu_1}\right)e^{\beta\nu}e^{-\beta s}, \quad \forall t \in \mathbb{R}, s \geq \nu. \end{aligned} \quad (3.46)$$

Setting

$$R_H^2 = \frac{2\|g\|^2}{c_1\mu_1\beta}, \quad (3.47)$$

we deduce from (3.46) that there exists $\tilde{T}_{\tilde{D}}(t) = \tilde{T}_{\tilde{D}}$ (independent of t) such that for any $s \geq \tilde{T}_{\tilde{D}}$ and any $(u_0, \psi) \in \tilde{D} \subset E_H^2$,

$$\|\tilde{\mathcal{U}}(t, t-s)(u_0, \psi)\|_{\mathcal{C}_H}^2 \leq R_H^2,$$

that is to say,

$$\{\mathcal{B}(t)\}_{t \in \mathbb{R}} = \mathcal{B}_{\mathcal{C}_H}(0, R_H) \doteq \mathcal{B}_1, \quad (3.48)$$

where $\mathcal{B}_{\mathcal{C}_H}(0, R_H)$ denotes the ball in $\mathcal{C}_H$ centered at zero with radius R_H , is the family of uniformly bounded pullback absorbing sets for the mappings $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$. The proof is complete. $\square$

As a corollary of Lemma 3.2 and Lemma 3.3, we have the following result.

LEMMA 3.4. Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a family of uniformly bounded pullback absorbing sets $\{\mathcal{B}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}} \subset \mathcal{C}_H$ and $\mathcal{B}_{\mathcal{C}_H}(t) = \mathcal{B}_{\mathcal{C}_H}(0, R_H)$ for all $t \in \mathbb{R}$. Also, the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a family of uniformly bounded pullback absorbing sets $\{\mathcal{B}_{E_H^2}(t)\}_{t \in \mathbb{R}} \subset E_H^2$ and $\mathcal{B}_{E_H^2}(t) = \mathcal{B}_H(0, R_H) \times \mathcal{B}_{L_H^2}(0, \nu^{1/2}R_H)$ for all $t \in \mathbb{R}$.

Similar to Lemma 3.3, we next prove that the family of mappings $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$ possesses a family of uniformly bounded pullback absorbing sets in $\mathcal{C}_W$.

LEMMA 3.5. Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then there exist positive constants $\tilde{R}_W, \tilde{\eta}_1$ and $\tilde{\eta}_2$ such that for any bounded set $\tilde{D} \subset E_H^2$,

$$\|\tilde{\mathcal{U}}(t, t-s)(u_0, \psi)\|_{\mathcal{C}_W}^2 = \max_{\theta \in [-\nu, 0]} \|u(t+\theta; t-s, (u_0, \psi))\|_W^2 \leq \tilde{R}_W^2, \quad (3.49)$$

$$\int_{t+\theta_1}^{t+\theta_2} \|Au(\xi; t-s, (u_0, \psi))\|^2 d\xi \leq \tilde{\eta}_1 |\theta_1 - \theta_2| + \tilde{\eta}_2, \quad (3.50)$$

for all $s \geq \tilde{T}_{\tilde{D}} + \nu + 1$, $t \in \mathbb{R}$, $(u_0, \psi) \in E_H^2$, and $\theta_1, \theta_2 \in [-\nu, 0]$.

Proof. Let $\tilde{D} \subset E_H^2$ be bounded. Then there exists $\tilde{r} > 0$ such that

$$\|u_0\|^2 + \|\psi\|_{L_H^2}^2 \leq \tilde{r}, \quad \forall (u_0, \psi) \in \tilde{D}.$$

Take a fixed $t_0 \in \mathbb{R}$ and $s \geq \tilde{T}_{\tilde{D}}$ (the absorbing time corresponding to the set $\mathcal{B}_{\mathcal{C}_H}(0, R_H)$ in Lemma 3.3), and consider the solution $u(\cdot) = u(\cdot; t_0 - s, (u_0, \psi))$ for $(u_0, \psi) \in \tilde{D}$.

Similar to (3.41), we have

$$\frac{d}{dt}\|u(t)\|^2 + c_1\mu_1\|u(t)\|_w^2 \leq \frac{\|g\|^2}{c_1\mu_1} + \frac{1}{2c_1\mu_1}\|f(t, u_t)\|^2. \quad (3.51)$$

For $t \geq t_0$, we integrate (3.51) over $[t, t+1]$ to obtain

$$\begin{aligned} & \|u(t+1)\|^2 - \|u(t)\|^2 + c_1\mu_1 \int_t^{t+1} \|u(\xi)\|_w^2 d\xi \\ & \leq \frac{\|g\|^2}{c_1\mu_1} + \frac{L_f^2}{2c_1\mu_1} \int_{t-\nu}^t \|u(\xi)\|^2 d\xi + \frac{L_f^2}{2c_1\mu_1} \int_t^{t+1} \|u(\xi)\|_w^2 d\xi, \end{aligned}$$

and consequently, we have

$$\begin{aligned} \left(c_1\mu_1 - \frac{L_f^2}{2c_1\mu_1}\right) \int_t^{t+1} \|u(\xi)\|_w^2 d\xi & \leq \frac{\|g\|^2}{c_1\mu_1} + \frac{L_f^2}{2c_1\mu_1} \int_{t-\nu}^t \|u(\xi)\|^2 d\xi + \|u(t)\|^2 \\ & \leq \frac{\|g\|^2}{c_1\mu_1} + \frac{\nu\tilde{r}L_f^2}{2c_1\mu_1} + R_H^2, \quad t \geq t_0, \end{aligned}$$

and

$$\int_t^{t+1} \|u(\xi)\|_w^2 d\xi \leq \tilde{R}_H^2, \quad t \geq t_0, \quad (3.52)$$

where

$$\tilde{R}_H^2 = \frac{1}{c_1\mu_1 - \frac{L_f^2}{2c_1\mu_1}} \left(\frac{\|g\|^2}{c_1\mu_1} + \frac{\nu\tilde{r}L_f^2}{2c_1\mu_1} + R_H^2 \right).$$

We now multiply the equation in (2.14) by Au and get for $\xi \geq t_0$ that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \xi} a(u(\xi), u(\xi)) + 2\mu_1 \|Au(\xi)\|^2 + \langle B(u(\xi)), Au(\xi) \rangle + \langle N(u(\xi)), Au(\xi) \rangle \\ = (g, Au(\xi)) + (f(\xi, u_\xi), Au(\xi)). \end{aligned} \quad (3.53)$$

We next evaluate the terms in (3.53). Firstly,

$$\begin{aligned} |(g, Au(\xi))| + |(f(\xi, u_\xi), Au(\xi))| & \leq \|Au(\xi)\|(\|g\| + \|f(\xi, u_\xi)\|) \\ & \leq \frac{1}{\mu_1}(\|g\|^2 + \|f(\xi, u_\xi)\|^2) + \frac{\mu_1}{2}\|Au(\xi)\|^2. \end{aligned} \quad (3.54)$$

Secondly, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned} |\langle B(u(\xi)), Au(\xi) \rangle| & \leq c\|u(\xi)\|^{1/2}\|u(\xi)\|_{H^1(\Omega)}\|u(\xi)\|_w^{1/2}\|Au(\xi)\| \\ & \leq \frac{\mu_1}{2}\|Au(\xi)\|^2 + \frac{c^4}{2c_1^2\mu_1}\|u(\xi)\|^2\|u(\xi)\|_{H^1(\Omega)}^4. \end{aligned} \quad (3.55)$$

Thirdly, to estimate the term $\langle N(u(\xi)), Au(\xi) \rangle$, we set

$$F(y) = 2\mu_0(\varepsilon + |y|^2)^{-\alpha/2}y,$$

where

$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{R}^4 \quad \text{and} \quad |y|^2 = \sum_{i=1}^4 y_i^2, \quad y_i \in \mathbb{R}, \quad i = 1, 2, 3, 4.$$

Then the first-order Fréchet derivative of $F(y)$ is

$$DF(y) = 2\mu_0 (\varepsilon + |y|^2)^{-\alpha/2} \begin{pmatrix} 1 - \frac{\alpha y_1^2}{\varepsilon + |y|^2} & -\frac{\alpha y_1 y_2}{\varepsilon + |y|^2} & -\frac{\alpha y_1 y_3}{\varepsilon + |y|^2} & -\frac{\alpha y_1 y_4}{\varepsilon + |y|^2} \\ -\frac{\alpha y_1 y_2}{\varepsilon + |y|^2} & 1 - \frac{\alpha y_2^2}{\varepsilon + |y|^2} & -\frac{\alpha y_2 y_3}{\varepsilon + |y|^2} & -\frac{\alpha y_2 y_4}{\varepsilon + |y|^2} \\ -\frac{\alpha y_1 y_3}{\varepsilon + |y|^2} & -\frac{\alpha y_2 y_3}{\varepsilon + |y|^2} & 1 - \frac{\alpha y_3^2}{\varepsilon + |y|^2} & -\frac{\alpha y_3 y_4}{\varepsilon + |y|^2} \\ -\frac{\alpha y_1 y_4}{\varepsilon + |y|^2} & -\frac{\alpha y_2 y_4}{\varepsilon + |y|^2} & -\frac{\alpha y_3 y_4}{\varepsilon + |y|^2} & 1 - \frac{\alpha y_4^2}{\varepsilon + |y|^2} \end{pmatrix}.$$

Since $0 < \alpha < 1$, we have

$$\left| -\frac{\alpha y_i y_j}{\varepsilon + |y|^2} \right| < \left| \frac{y_i y_j}{\varepsilon + |y|^2} \right| < \frac{1}{\varepsilon}, \quad i, j = 1, 2, 3, 4,$$

and

$$0 < 1 - \frac{\alpha y_i^2}{\varepsilon + |y|^2} < 1, \quad i = 1, 2, 3, 4.$$

Consequently,

$$\|DF(y)\| \leq 2\mu_0 (\varepsilon + |y|^2)^{-\alpha/2} \sqrt{4 + \frac{12}{\varepsilon^2}} \leq 2\mu_0 \varepsilon^{-\alpha/2} \sqrt{4 + \frac{12}{\varepsilon^2}}, \quad \forall y \in \mathbb{R}^4.$$

Similarly, the second-order Fréchet derivative of $F(y)$ is a three-dimensional matrix

$$D^2 F(y) = \left(\frac{\partial^2 F_i(y)}{\partial y_j \partial y_k} \right), \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4, \quad k = 1, 2, 3, 4,$$

where $F_i(y) = 2\mu_0 (\varepsilon + |y|^2)^{-\alpha/2} y_i$. By some computations we see that the first-order and second-order Fréchet derivatives of $F(y)$ satisfy

$$\|DF(y)\| + \|D^2 F(y)\| \leq c_3(\mu_0, \varepsilon, \alpha) \doteq c_3, \quad \forall y \in \mathbb{R}^4, \quad (3.56)$$

where c_3 is a positive constant depending only on μ_0, ε and α . For any $y, z \in \mathbb{R}^4$,

$$F(y) - F(z) = \int_0^1 DF(y + \tau(z - y))(z - y) d\tau.$$

Taking $y = e(u) = (e_{ij}(u))$, $z = e(0) = (e_{ij}(0))$, applying the integration by parts first and then the above inequality about $F(y)$, we have

$$\begin{aligned} |\langle N(u(\xi)), Au(\xi) \rangle| &= \left| - \int_{\Omega} \{ \nabla \cdot [F(e(u(\xi))) - F(e(0))] \} \cdot Au(\xi) dx \right| \\ &\leq c_3 (\|u(\xi)\|_{H^1(\Omega)} + \|u(\xi)\|_w) \|Au(\xi)\| \\ &\leq \frac{\mu_1}{2} \|Au(\xi)\|^2 + \frac{c_3^2}{2\mu_1} \|u(\xi)\|_w^2. \end{aligned} \quad (3.57)$$

It then follows from (3.53)–(3.57) and Lemma 2.1 that

$$\begin{aligned} &\frac{1}{2} \frac{\partial}{\partial \xi} a(u(\xi), u(\xi)) + \frac{\mu_1}{2} \|Au(\xi)\|^2 \\ &\leq \frac{1}{\mu_1} (\|g\|^2 + \|f(\xi, u_\xi)\|^2) + \frac{c^4}{2c_1^2 \mu_1} \|u(\xi)\|^2 \|u(\xi)\|_{H^1(\Omega)}^4 + \frac{c_3^2}{2\mu_1} \|u(\xi)\|_w^2, \end{aligned} \quad (3.58)$$

whence, by Lemma 2.1 and the fact that $\|\phi\|_{H^1(\Omega)} \leq \|\phi\|_W$ for any $\phi \in W$, we get

$$\begin{aligned} & \frac{\partial}{\partial \xi} a(u(\xi), u(\xi)) \\ & \leq \frac{2\|g\|^2}{\mu_1} + \frac{2\|f(\xi, u_\xi)\|^2}{\mu_1} + \frac{c^4}{c_1^2 \mu_1} \|u(\xi)\|^2 \|u(\xi)\|_W^4 + \frac{c_3^2}{\mu_1} \|u(\xi)\|_W^2 \\ & \leq \frac{2\|g\|^2}{\mu_1} + \frac{2\|f(\xi, u_\xi)\|^2}{\mu_1} + \left(\frac{c^4 \|u(\xi)\|^2}{c_1^2 \mu_1} \|u(\xi)\|_W^2 + \frac{c_3^2}{\mu_1} \right) \frac{a(u(\xi), u(\xi))}{c_1}. \end{aligned} \quad (3.59)$$

Noticing that $\xi \geq t_0$, we get by using $(\mathbf{A}_3)$ that

$$\frac{2\|g\|^2}{\mu_1} + \frac{2\|f(\xi, u_\xi)\|^2}{\mu_1} \leq \frac{2\|g\|^2}{\mu_1} + \frac{2C_f^2}{\mu_1} \|u(\xi)\|_{c_H}^2 \leq \frac{2\|g\|^2}{\mu_1} + \frac{2C_f^2}{\mu_1} R_H^2, \quad (3.60)$$

$$\frac{c^4 \|u(\xi)\|^2}{c_1^2 \mu_1} \|u(\xi)\|_W^2 + \frac{c_3^2}{\mu_1} \leq \frac{c^4 R_H^2}{c_1^2 \mu_1} \|u(\xi)\|_W^2 + \frac{c_3^2}{\mu_1}. \quad (3.61)$$

Thus

$$\int_t^{t+1} \left(\frac{2\|g\|^2}{\mu_1} + \frac{2\|f(\xi, u_\xi)\|^2}{\mu_1} \right) d\xi \leq \frac{2\|g\|^2}{\mu_1} + \frac{2C_f^2}{\mu_1} R_H^2 \doteq a_1, \quad \forall t \geq t_0, \quad (3.62)$$

and by (3.52), (3.61) and Lemma 2.1, we get

$$\begin{aligned} \frac{1}{c_1} \int_t^{t+1} \left(\frac{c^4 \|u(\xi)\|^2}{c_1^2 \mu_1} \|u(\xi)\|_W^2 + \frac{c_3^2}{\mu_1} \right) d\xi & \leq \frac{c^4 R_H^2}{c_1^3 \mu_1} \int_t^{t+1} \|u(\xi)\|_W^2 d\xi + \frac{c_3^2}{c_1 \mu_1} \\ & \leq \frac{c^4 R_H^2 \tilde{R}_H^2}{c_1^3 \mu_1} + \frac{c_3^2}{c_1 \mu_1} \\ & \doteq a_2, \quad \forall t \geq t_0, \end{aligned} \quad (3.63)$$

$$\begin{aligned} \int_t^{t+1} a(u(\xi), u(\xi)) d\xi & \leq \int_t^{t+1} c_2 \|u(\xi)\|_W^2 d\xi \\ & \leq c_2 \tilde{R}_H^2 \doteq a_3, \quad \forall t \geq t_0. \end{aligned} \quad (3.64)$$

Using the Uniform Gronwall Lemma (see Lemma 2.3) for $s \geq \tilde{T}_{\tilde{D}}$, we deduce from (3.59) and (3.62)–(3.64) that

$$a(u(\xi), u(\xi)) \leq (a_3 + a_1)e^{a_2}, \quad \forall \xi \geq t_0 + 1, \quad \forall s \geq \tilde{T}_{\tilde{D}},$$

and again by Lemma 2.1,

$$\|u(\xi)\|_W^2 \leq \frac{1}{c_1} a(u(\xi), u(\xi)) \leq \frac{1}{c_1} (a_3 + a_1)e^{a_2}, \quad \forall \xi \geq t_0 + 1, \quad \forall s \geq \tilde{T}_{\tilde{D}}. \quad (3.65)$$

Now if $s \geq \tilde{T}_{\tilde{D}} + \nu + 1$, then (3.65) implies

$$\sup_{\theta \in [-\nu, 0]} \|u(t_0 + \theta)\|_W^2 \leq \frac{1}{c_1} (a_3 + a_1)e^{a_2} \doteq \tilde{R}_W^2, \quad (3.66)$$

where the constants a_1, a_2, a_3 and c_1 appearing in (3.66) are independent of the fixed time $t_0 \in \mathbb{R}$. Thus (3.66) holds true for any $t_0 \in \mathbb{R}$. We now denote $u_t(\cdot) = u_t(\cdot; t-s, (u_0, \psi))$. Theorem 2.1 ensures that $u_t(\cdot) \in \mathcal{C}_W$ for $\forall t \in \mathbb{R}$ provided $s > \nu$. Hence

$$\|u_t\|_{\mathcal{C}_W}^2 \leq \tilde{R}_W^2, \quad \forall t \in \mathbb{R}, \quad \forall s \geq \tilde{T}_{\tilde{D}} + \nu + 1,$$

and (3.49) holds. We next prove (3.50). Indeed, we deduce from (3.58) that

$$\begin{aligned} \|Au(\xi)\|^2 &\leq \frac{2\|g\|^2}{\mu_1^2} + \frac{2\|f(\xi, u_\xi)\|^2}{\mu_1^2} + \frac{c^4}{c_1^2\mu_1^2} \|u(\xi)\|^2 \|u(\xi)\|_{H^1(\Omega)}^4 \\ &\quad + \frac{c_3^2}{2\mu_1^2} \|u(\xi)\|_v^2 - \frac{1}{\mu_1} \frac{\partial}{\partial \xi} a(u(\xi), u(\xi)). \end{aligned} \quad (3.67)$$

Choose $s \geq \tilde{T}_D + \nu + 1$ and $\theta_1, \theta_2 \in [-\nu, 0]$ (let $\theta_2 > \theta_1$ without loss of generality). Then we get

$$\begin{aligned} &\int_{t+\theta_1}^{t+\theta_2} \|Au(\xi)\|^2 d\xi \\ &\leq \frac{2\|g\|^2}{\mu_1^2} |\theta_2 - \theta_1| + \frac{2}{\mu_1^2} \int_{t+\theta_1}^{t+\theta_2} \|f(\xi, u_\xi)\|^2 d\xi + \frac{c^4}{c_1^2\mu_1^2} \int_{t+\theta_1}^{t+\theta_2} \|u(\xi)\|^2 \|u(\xi)\|_{H^1(\Omega)}^4 d\xi \\ &\quad + \frac{c_3^2}{2\mu_1^2} \int_{t+\theta_1}^{t+\theta_2} \|u(\xi)\|_w^2 d\xi + \frac{1}{\mu_1} a(u(t+\theta_1), u(t+\theta_1)) \\ &\leq \frac{2\|g\|^2}{\mu_1^2} |\theta_2 - \theta_1| + \frac{2C_f^2}{\mu_1^2} \int_{t+\theta_1}^{t+\theta_2} \|u_\xi\|_{C_H}^2 d\xi + \frac{c^4 R_H^2}{c_1^2\mu_1^2} \int_{t+\theta_1}^{t+\theta_2} \|u(\xi)\|_{H^1(\Omega)}^4 d\xi \\ &\quad + \frac{c_3^2 \tilde{R}_w^2}{2\mu_1^2} |\theta_2 - \theta_1| + \frac{c_2}{\mu_1} \|u(t+\theta_1)\|_w^2 \\ &\leq \left(\frac{2\|g\|^2}{\mu_1^2} + \frac{2C_f^2 R_H^2}{\mu_1^2} + \frac{c^4 R_H^2 \tilde{R}_w^4}{c_1^2\mu_1^2} + \frac{c_3^2 \tilde{R}_w^2}{2\mu_1^2} \right) |\theta_2 - \theta_1| + \frac{c_2}{\mu_1} \tilde{R}_w^2. \end{aligned}$$

The proof of Lemma 3.5 is now complete. $\square$

LEMMA 3.6. Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then there exist positive constants R_w, η_1 and η_2 such that for any bounded set $\mathcal{B} \subset C_H$ and for $T_{\mathcal{B}} = \tilde{T}_{J(\mathcal{B})}$ (the absorbing time corresponding to the set $\mathcal{B}_{C_H}(0, R_H)$ in Lemma 3.3), the following holds:

$$\begin{aligned} \|\mathcal{U}(t, t-s)\psi\|_{C_w}^2 &= \|u_t(\cdot; t-s, J(\psi))\|_{C_w}^2 \\ &= \max_{\theta \in [-\nu, 0]} \|u(t+\theta; t-s, J(\psi))\|_w^2 \leq R_w^2, \end{aligned} \quad (3.68)$$

$$\int_{t+\theta_1}^{t+\theta_2} \|Au(\xi; t-s, J(\psi))\|^2 d\xi \leq \eta_1 |\theta_1 - \theta_2| + \eta_2, \quad (3.69)$$

for all $s \geq T_{\mathcal{B}} + \nu + 1$, $t \in \mathbb{R}$, $\psi \in \mathcal{B}$, and $\theta_1, \theta_2 \in [-\nu, 0]$. Moreover, the two families of sets $\{\mathcal{B}_{C_w}(t)\}_{t \in \mathbb{R}} \subset C_w$, $\{\mathcal{B}_{E_w^2}(t)\}_{t \in \mathbb{R}} \subset E_w^2$, where $\mathcal{B}_{C_w}(t) = \mathcal{B}_{C_w}(0, R_w)$, $\mathcal{B}_{E_w^2}(t) = \mathcal{B}_w(0, R_w) \times \mathcal{B}_{L_w^2}(0, \nu^{1/2} R_w)$ for all $t \in \mathbb{R}$, are uniformly bounded pullback absorbing sets of the processes $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in C_w and $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_w^2 , respectively.

Proof. First, (3.68) and (3.69) can be deduced directly from (3.49) and (3.50). Then (3.68) implies that the family of sets $\{\mathcal{B}_{C_w}(t)\}_{t \in \mathbb{R}} \subset C_w$, where $\mathcal{B}_{C_w}(t) = \mathcal{B}_{C_w}(0, R_w)$ (the ball in C_w centered at zero with radius R_w), $t \in \mathbb{R}$, are pullback absorbing for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in C_w . Finally, note that $\{J(\mathcal{B}_{C_w}(t))\}_{t \in \mathbb{R}}$ is a family of uniformly bounded pullback absorbing sets for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_w^2 . At the same time, $\|\psi\|_{L_w^2}^2 \leq \nu \|\psi\|_{C_w}$ for any $\psi \in C_w$, and $J(\mathcal{B}_{C_w}(t)) = \{(\psi(0), \psi) : \psi \in \mathcal{B}_{C_w}(0, R_w)\}$. It then follows

that $J(\mathcal{B}_{C_W}(t)) \subset \mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W) \doteq \mathcal{B}_{E_W^2}(t)$, $t \in \mathbb{R}$, where $\mathcal{B}_W(0, R_W)$ denotes the ball in W centered at zero with radius R_W and $\mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W)$ denotes the ball in L_W^2 centered at zero with radius $\nu^{1/2}R_W$. $\square$

At this stage, we can state and prove the main result of this section, which reads as

THEOREM 3.1. *Let $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a unique uniformly bounded pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}} \subset \mathcal{B}_{C_W}(0, R_W) \subset \mathcal{C}_H$ and the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a unique uniformly bounded pullback attractor $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}} \subset \mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W) \subset E_H^2$. Furthermore, $\mathcal{A}_{E_H^2}(t) \subset H \times \mathcal{C}_H$ for all $t \in \mathbb{R}$, and the two attractors are related via*

$$\mathcal{A}_{E_H^2}(t) = J(\mathcal{A}_{C_H}(t)), \quad \forall t \in \mathbb{R}. \quad (3.70)$$

Proof. Let $\{\mathcal{B}_{C_W}(t)\}_{t \in \mathbb{R}}$ be defined as in Lemma 3.6. Then $\{\mathcal{B}_{C_W}(t)\}_{t \in \mathbb{R}}$ is a family of uniformly bounded pullback absorbing sets for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_W$. For simplicity, write $\mathcal{B}_2 = \mathcal{B}_{C_W}(0, R_W)$ and set $\tilde{\mathcal{B}}_2 = J(\mathcal{B}_2)$. Then Lemma 3.5 shows that there exists $T_{\mathcal{B}_2} = \tilde{T}_{\tilde{\mathcal{B}}_2} + \nu + 1$ such that

$$\tilde{\mathcal{U}}(t, t-s)\tilde{\mathcal{B}}_2 \subset \mathcal{B}_2, \quad \forall t \in \mathbb{R} \text{ and } s \geq T_{\mathcal{B}_2}.$$

Writing

$$\mathcal{B}_3(t) = \bigcup_{s \geq T_{\mathcal{B}_2}} \tilde{\mathcal{U}}(t, t-s)\tilde{\mathcal{B}}_2 \subset \mathcal{B}_2 \subset \mathcal{C}_W, \quad t \in \mathbb{R},$$

we see that $\{\mathcal{B}_3(t)\}_{t \in \mathbb{R}}$ is a family of uniformly bounded pullback absorbing sets of $\{\tilde{\mathcal{U}}(t, \tau)\}_{t \geq \tau}$. We next use the Ascoli-Arzelà theorem to prove that for each $t \in \mathbb{R}$, $\mathcal{B}_3(t)$ is relatively compact in $\mathcal{C}_H$. To this end, we need to check the following two points.

(1) For each $t \in \mathbb{R}$, $\mathcal{B}_3(t)$ is equi-continuous in $\mathcal{C}_H$; that is, given any $\epsilon > 0$, $\exists \delta > 0$ such that if $|\theta_2 - \theta_1| \leq \delta$, then $\|\tilde{\mathcal{U}}(t, t-s)J(\psi)(\theta_2) - \tilde{\mathcal{U}}(t, t-s)J(\psi)(\theta_1)\| \leq \epsilon$, for $\forall s \geq \tilde{T}_{\mathcal{B}_2}$ and $\forall \psi \in \mathcal{B}_2$.

(2) For each $t \in \mathbb{R}$ and $\forall \theta \in [-\nu, 0]$,

$$\bigcup_{s \geq \tilde{T}} \bigcup_{\psi \in \mathcal{B}_2} \tilde{\mathcal{U}}(t, t-s)(J(\psi))(\theta) \text{ is pre-compact in } H.$$

Indeed, on the one hand, we have

$$\begin{aligned} & \|\tilde{\mathcal{U}}(t, t-s)(J(\psi))(\theta_2) - \tilde{\mathcal{U}}(t, t-s)(J(\psi))(\theta_1)\| \\ &= \|u(t + \theta_2, t-s, (J(\psi))) - u(t + \theta_1, t-s, (J(\psi)))\| \end{aligned}$$

for $\forall t \in \mathbb{R}$, $s \geq T_{\mathcal{B}_2}$ and $\forall \psi \in \mathcal{B}_2$, $\theta_1, \theta_2 \in [-\nu, 0]$. Denoting $u(\cdot) = u(\cdot; t-s, J(\psi))$ and assuming (without loss of generality) $\theta_2 > \theta_1$, we obtain

$$\begin{aligned} & \|u(t + \theta_2) - u(t + \theta_1)\| \\ &= \left\| \int_{t+\theta_1}^{t+\theta_2} \frac{\partial}{\partial \xi} u(\xi) d\xi \right\| \leq \int_{t+\theta_1}^{t+\theta_2} \left\| \frac{\partial}{\partial \xi} u(\xi) \right\| d\xi \\ &\leq \int_{t+\theta_1}^{t+\theta_2} (\|2\mu_1 A u(\xi)\| + \|B(u(\xi))\| + \|N(u(\xi))\| + \|g\| + \|f(\xi, u_\xi)\|) d\xi. \quad (3.71) \end{aligned}$$

By the Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} |\langle B(u(\xi)), \phi \rangle| &\leq \sum_{i,j=1}^2 \left| \int_{\Omega} u_i \frac{\partial u_i}{\partial x_j} \phi_j dx \right| \leq \|u\|_{L^4(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|\phi\| \\ &\leq c \|u\| \|\Delta u\| \|\phi\|, \quad \forall \phi \in H, \end{aligned}$$

and thus by using (2.8), we obtain

$$\begin{aligned} \int_{t+\theta_1}^{t+\theta_2} \|B(u(\xi))\| d\xi &\leq c R_H \int_{t+\theta_1}^{t+\theta_2} \|\Delta u(\xi)\| d\xi \leq c R_H \int_{t+\theta_1}^{t+\theta_2} \|u(\xi)\|_W d\xi \\ &\leq \frac{c R_H}{c_1} \int_{t+\theta_1}^{t+\theta_2} \|Au(\xi)\| d\xi, \quad s \geq T_{\mathcal{B}_2}. \end{aligned} \quad (3.72)$$

Analogously, applying the same derivations to (3.57), we have

$$\int_{t+\theta_1}^{t+\theta_2} \|N(u(\xi))\| d\xi \leq c_3 \int_{t+\theta_1}^{t+\theta_2} \|\Delta u(\xi)\| d\xi \leq \frac{c_3}{c_1} \int_{t+\theta_1}^{t+\theta_2} \|Au(\xi)\| d\xi, \quad s \geq T_{\mathcal{B}_2}. \quad (3.73)$$

It then follows from (3.69) and (3.71)–(3.73) that

$$\begin{aligned} &\|u(t+\theta_2) - u(t+\theta_1)\| \\ &\leq \|g\| |\theta_1 - \theta_2| + \int_{t+\theta_1}^{t+\theta_2} \|f(\xi, u_\xi)\| d\xi + (2\mu_1 + \frac{c R_H}{c_1} + \frac{c_3}{c_1}) \int_{t+\theta_1}^{t+\theta_2} \|Au(\xi)\| d\xi \\ &\leq (\|g\| + C_f R_H) |\theta_1 - \theta_2| \\ &\quad + (2\mu_1 + \frac{c R_H}{c_1} + \frac{c_3}{c_1}) |\theta_1 - \theta_2|^{1/2} \left(\int_{t+\theta_1}^{t+\theta_2} \|Au(\xi)\|^2 d\xi \right)^{1/2} \\ &\leq (\|g\| + C_f^2 R_H) |\theta_1 - \theta_2| + (2\mu_1 + \frac{c R_H}{c_1} + \frac{c_3}{c_1}) |\theta_1 - \theta_2|^{1/2} (\eta_1 |\theta_1 - \theta_2| + \eta_2)^{1/2}, \end{aligned}$$

which gives the desired equi-continuity, i.e. (1) holds. On the other hand, for any fixed $\theta \in [-\nu, 0]$ and $t \in \mathbb{R}$, the set

$$\{u(t+\theta; t-s, J(\psi)) : s \geq T_{\mathcal{B}_2}, \quad \psi \in \mathcal{B}_2\}$$

is pre-compact in H because it is bounded in W (see (3.49)) and the embedding $W \hookrightarrow H$ is compact. Thus (2) holds. Therefore, $\{\mathcal{B}_3(t)\}_{t \in \mathbb{R}}$ is a family of compact (uniformly bounded) pullback absorbing sets for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$. Then Lemma 3.2 ensures that $\{\mathcal{B}_3(t)\}_{t \in \mathbb{R}}$ and $\{J(\mathcal{B}_3(t))\}_{t \in \mathbb{R}}$ are respectively the families of compact (uniformly bounded) pullback absorbing sets for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$ and for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_H^2 , respectively. Thus the existence of the uniformly bounded pullback attractors $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$ and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_H^2 follows, respectively.

We next prove (3.70). By (3.36), (3.39) and the invariance property of the pullback attractor, we have for $\forall t \geq \tau$ that

$$\begin{aligned} \mathcal{S}(t, \tau) J(\mathcal{A}_{\mathcal{C}_H}(\tau)) &= J(\tilde{\mathcal{U}}(t, \tau) J(\mathcal{A}_{\mathcal{C}_H}(\tau))) = J(\mathcal{U}(t, \tau) \mathcal{A}_{\mathcal{C}_H}(\tau)) \\ &= J(\mathcal{A}_{\mathcal{C}_H}(t)). \end{aligned} \quad (3.74)$$

Since $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}} \subset \mathcal{B}_2$ are uniformly (with respect to $t \in \mathbb{R}$) bounded in $\mathcal{C}_H$, $\{J(\mathcal{A}_{\mathcal{C}_H}(t))\}_{t \in \mathbb{R}}$ are uniformly (with respect to $t \in \mathbb{R}$) bounded in E_H^2 . Then by the pullback attracting property of the pullback attractor and (3.74), we get

$$\begin{aligned} & \text{Dist}_{E_H^2} \left(J(\mathcal{A}_{\mathcal{C}_H}(t)), \mathcal{A}_{E_H^2}(t) \right) \\ &= \text{Dist}_{E_H^2} \left(\mathcal{S}(t, t-s) J(\mathcal{A}_{\mathcal{C}_H}(t-s)), \mathcal{A}_{E_H^2}(t) \right) \quad (\forall s > 0) \\ &= \lim_{s \rightarrow +\infty} \text{Dist}_{E_H^2} \left(\mathcal{S}(t, t-s) J(\mathcal{A}_{\mathcal{C}_H}(t-s)), \mathcal{A}_{E_H^2}(t) \right) \\ &= 0, \end{aligned}$$

and thus

$$J(\mathcal{A}_{\mathcal{C}_H}(t)) \subset \mathcal{A}_{E_H^2}(t), \quad \forall t \in \mathbb{R}. \quad (3.75)$$

Noticing that $\{J(\mathcal{A}_{\mathcal{C}_H}(t))\}_{t \in \mathbb{R}}$ is the family of uniformly pullback attracting sets for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_H^2 and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$ are uniformly (with respect to $t \in \mathbb{R}$) bounded in E_H^2 , we have, also using the invariance property and pullback attracting property of the pullback attractor,

$$\begin{aligned} & \text{Dist}_{E_H^2} \left(\mathcal{A}_{E_H^2}(t), J(\mathcal{A}_{\mathcal{C}_H}(t)) \right) \\ &= \text{Dist}_{E_H^2} \left(\mathcal{S}(t, t-s) \mathcal{A}_{E_H^2}(t-s), J(\mathcal{A}_{\mathcal{C}_H}(t)) \right) \quad (\forall s > 0) \\ &= \lim_{s \rightarrow +\infty} \text{Dist}_{E_H^2} \left(\mathcal{S}(t, t-s) \mathcal{A}_{E_H^2}(t-s), J(\mathcal{A}_{\mathcal{C}_H}(t)) \right) \\ &= 0, \end{aligned}$$

and hence $\mathcal{A}_{E_H^2}(t) \subset J(\mathcal{A}_{\mathcal{C}_H}(t))$, $\forall t \in \mathbb{R}$, by which and (3.75) we get (3.70). The proof of Theorem 3.1 is complete. $\square$

4. Existence of a pullback attractor in E_w^2 . In this section, we establish that the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a uniform bounded pullback attractor $\{\mathcal{A}_{E_w^2}(t)\}_{t \in \mathbb{R}} \subset E_w^2$. The main step is to prove that the family of uniformly bounded absorbing sets $\{\mathcal{B}_{E_w^2}(t)\}_{t \in \mathbb{R}}$ obtained in Lemma 3.6 is compact in E_w^2 . The argument used here is the energy equality method, which is due to J. M. Ball (see, e.g., [5]). The energy equality method has been proved to be very useful for the investigation of the asymptotic compactness of dynamical systems, especially for the case that the spatial domain is unbounded (see, e.g., [31, 49, 56]).

By Theorem 2.1 (2), we can also define a continuous process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau} : E_w^2 \mapsto E_w^2$ via

$$\mathcal{S}(t, \tau)(u_0, \psi) = (u(t; \tau, (u_0, \psi)), u_t(\cdot; \tau, (u_0, \psi))), \quad \forall (u_0, \psi) \in E_w^2 \text{ and } t \geq \tau.$$

DEFINITION 4.1. The process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ is said to be pullback asymptotically compact in E_w^2 if for each $t \in \mathbb{R}$, any bounded sequence $\{(u_0^{(n)}, \psi^{(n)})\}$ of E_w^2 , and $\{s_n\} \subset \mathbb{R}_+$ with $s_n \rightarrow +\infty$ as $n \rightarrow \infty$, the sequence $\{\mathcal{S}(t, t-s_n)(u_0^{(n)}, \psi^{(n)})\}$ has a convergent subsequence.

LEMMA 4.1. Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ is pullback asymptotically compact in E_w^2 .

Proof. By Lemma 3.6, we only need to prove

$$\left\{ \begin{array}{l} \text{For each } t \in \mathbb{R}, \text{ any } \{s_n\} \subset \mathbb{R}_+ \text{ with } s_n \rightarrow +\infty \text{ as } n \rightarrow \infty \text{ and} \\ \text{any } \{(u_0^{(n)}, \psi^{(n)})\} \subset \mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W), \text{ the sequence} \\ \{\mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)})\} \text{ has a convergent subsequence.} \end{array} \right. \quad (4.76)$$

For brevity, we let $\{s_n\} \subset [T^*, +\infty)$, where

$$T^* = T_{\mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W)} \quad (4.77)$$

denotes the corresponding absorbing time for $\mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W)$ in Lemma 3.6. Then $\{\mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)})\} \subset \mathcal{B}_W(0, R_W) \times \mathcal{B}_{L_W^2}(0, \nu^{1/2}R_W) \subset W \times L_W^2$. Note that $W \times L_W^2$ is a reflexive Banach space. So there exists a subsequence (still denoted by $\{\mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)})\}$) such that

$$\mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)}) \rightharpoonup (w_0, \psi) \text{ weakly in } W \times L_W^2 \text{ as } n \rightarrow \infty \quad (4.78)$$

for some $(w_0, \psi) \in W \times L_W^2$. Similarly, for each $T \in \mathbb{N}$ ($T < s_n$ for n being large enough), we have

$$\mathcal{S}(t - T, t - s_n)(u_0^{(n)}, \psi^{(n)}) \rightharpoonup (w_T, \psi_T) \text{ weakly in } W \times L_W^2 \text{ as } n \rightarrow \infty \quad (4.79)$$

for some $(w_T, \psi_T) \in W \times L_W^2$. By the property of the process, we have

$$\begin{aligned} & \mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)}) \\ &= \mathcal{S}(t, t - T)\mathcal{S}(t - T, t - s_n)(u_0^{(n)}, \psi^{(n)}) \\ &\rightharpoonup \mathcal{S}(t, t - T)(w_T, \psi_T) \text{ weakly in } W \times L_W^2 \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.80)$$

By the uniqueness of the limit, we get $(w_0, \psi) = \mathcal{S}(t, t - T)(w_T, \psi_T)$ for each $T \in \mathbb{N}$. Setting

$$\mathcal{S}(t - T, t - s_n)(u_0^{(n)}, \psi^{(n)}) = (w_T^{(n)}, \psi_T^{(n)}),$$

we have

$$\begin{aligned} \mathcal{S}(t, t - s_n)(u_0^{(n)}, \psi^{(n)}) &= \mathcal{S}(t, t - T)(w_T^{(n)}, \psi_T^{(n)}) \\ &\rightharpoonup \mathcal{S}(t, t - T)(w_T, \psi_T) \\ &= (w_0, \psi) \text{ weakly in } W \times L_W^2 \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.81)$$

Denote by $u(t) = u(t; \tau, (u_0, \psi))$ the solution of problem (2.14) corresponding to the initial data (u_0, ψ) and the initial time $\tau \in \mathbb{R}$. Write

$$u^{(n)}(t) = u(t; t - s_n, (u_0^{(n)}, \psi^{(n)})) = u(t; t - T, (w_T^{(n)}, \psi_T^{(n)})). \quad (4.82)$$

Then (4.78) implies that

$$u^{(n)}(t) = u(t; t - T, (w_T^{(n)}, \psi_T^{(n)})) \rightharpoonup w_0 \text{ weakly in } W \text{ as } n \rightarrow +\infty. \quad (4.83)$$

Lemma 2.1 shows that the inner product $a(\cdot, \cdot)$ can induce a norm which is equivalent to $\|\cdot\|_W$. Thus (4.83) gives us

$$a(w_0, w_0) \leq \liminf_n a(u(t; t - T, (w_T^{(n)}, \psi_T^{(n)})), u(t; t - T, (w_T^{(n)}, \psi_T^{(n)}))). \quad (4.84)$$

We next prove

$$a(w_0, w_0) \geq \limsup_n a(u(t; t - T, (w_T^{(n)}, \psi_T^{(n)})), u(t; t - T, (w_T^{(n)}, \psi_T^{(n)}))). \quad (4.85)$$

Define $[\![\cdot, \cdot]\!] : D(A) \times D(A) \mapsto \mathbb{R}$ as

$$[\![u, v]\!] = 2\mu_1(Au, Av) - \frac{c_1^2}{c_2}\mu_1 a(u, v), \quad \forall u, v \in D(A) \cap W.$$

Then we have, using (2.7) and (2.8),

$$\begin{aligned} [\![u]\!]^2 = [\![u, u]\!] &\geq 2\mu_1 \|Au\|^2 - c_1^2 \mu_1 \|u\|_W^2 \\ &\geq 2\mu_1 \|Au\|^2 - \mu_1 \|Au\|^2 = \mu_1 \|Au\|^2 \end{aligned}$$

and

$$\mu_1 \|Au\|^2 \leq [\![u]\!]^2 \leq 2\mu_1 \|Au\|^2. \quad (4.86)$$

Let $k = \frac{c_1^2}{c_2}$. Multiplying the equation in (2.14) by $Au(t)$, we obtain

$$\begin{aligned} &\frac{d}{dt} a(u(t), u(t)) + k\mu_1 a(u(t), u(t)) \\ = &(g, Au(t)) + (f(t, u_t), Au(t)) \\ &- \langle B(u(t)), Au(t) \rangle - \langle N(u(t)), Au(t) \rangle - [\![u(t)]\!]^2. \end{aligned} \quad (4.87)$$

Using the formula of constant variation, we get

$$a(u(t), u(t)) = a(u_0, u_0) e^{-k\mu_1(t-\tau)} + \int_\tau^t e^{-k\mu_1(t-\xi)} \mathcal{K}(g, f(\xi, u_\xi), u(\xi)) d\xi, \quad (4.88)$$

where

$$\begin{aligned} \mathcal{K}(g, f(\xi, u_\xi), u(\xi)) &= (g, Au(\xi)) + (f(\xi, u_\xi), Au(\xi)) - \langle B(u(\xi)), Au(\xi) \rangle \\ &\quad - \langle N(u(\xi)), Au(\xi) \rangle - [\![u(\xi)]\!]^2. \end{aligned}$$

Now by (4.82) and (4.88), we have

$$\begin{aligned}
& a(u^{(n)}(t), u^{(n)}(t)) \\
&= a(w_T^{(n)}, \psi_T^{(n)})e^{-k\mu_1 T} \\
&+ \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(g, Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \\
&+ \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \\
&- \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle B(u(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right\rangle d\xi \\
&- \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle N(u(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right\rangle d\xi \\
&- \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|u(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)}))\|^2 d\xi. \tag{4.89}
\end{aligned}$$

We next evaluate the terms in the right-hand side of (4.89). Firstly, by Lemma 3.6, we have

$$a(w_T^{(n)}, \psi_T^{(n)})e^{-k\mu_1 T} \leq c_2^2 \|w_T^{(n)}\|_W^2 e^{-k\mu_1 T} \leq c_2^2 R_W^2 e^{-k\mu_1 T}, \tag{4.90}$$

when n is large enough such that $s_n > T + T^*$ (see (4.77)). Secondly, the following three limiting relations can be proved essentially by the same way as that in [56]:

$$\begin{aligned}
& \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|u(t; t-T, (w_T^{(n)}, \psi_T^{(n)}))\|^2 d\xi \\
& \leq \liminf_n \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|u(t; t-T, (w_T, \psi_T))\|^2 d\xi, \tag{4.91}
\end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle B(u(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right\rangle d\xi \\
&= \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle B(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \right\rangle d\xi, \tag{4.92}
\end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle N(u(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right\rangle d\xi \\
&= \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left\langle N(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \right\rangle d\xi. \tag{4.93}
\end{aligned}$$

Finally, we prove

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t-T, (w_T^{(n)}, \psi_T^{(n)}))), Au(\xi; t-T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \\
&= \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \right) d\xi. \tag{4.94}
\end{aligned}$$

On the one hand, the embedding $W \times L_W^2 \hookrightarrow H \times L_H^2$ is compact. Thus, by (4.79), we have

$$(w_T^{(n)}, \psi_T^{(n)}) \longrightarrow (w_T, \psi_T) \text{ strongly in } H \times L_H^2 \text{ as } n \rightarrow \infty. \tag{4.95}$$

Then by Lemma 3.1 (II),

$$\lim_{n \rightarrow \infty} \|u_\xi(\cdot; t - T, (w_T^{(n)}, \psi_T^{(n)})) - u_\xi(\cdot; t - T, (w_T, \psi_T))\|_{C_H} = 0, \quad (4.96)$$

and by (A₃),

$$\lim_{n \rightarrow \infty} \|f(\xi, u_\xi(\cdot; t - T, (w_T^{(n)}, \psi_T^{(n)}))) - f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T)))\| = 0. \quad (4.97)$$

At the same time, from Theorem 2.1 (2), we see that for each $n \in \mathbb{N}$,

$$\int_{t-T}^t \|Au_\xi(\cdot; t - T, (w_T^{(n)}, \psi_T^{(n)}))\|^2 d\xi < +\infty. \quad (4.98)$$

It then follows from (4.97) and (4.98) that

$$\begin{aligned} & \left| \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t - T, (w_T^{(n)}, \psi_T^{(n)}))) , Au(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \right. \\ & \quad \left. - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T))) , Au(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \right| \\ & \leq \left\{ \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|f(\xi, u_\xi(\cdot; t - T, (w_T^{(n)}, \psi_T^{(n)}))) \right. \\ & \quad \left. - f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T)))\|^2 d\xi \right\}^{1/2} \\ & \quad \cdot \left(\int_{t-T}^t e^{-k\mu_1(t-\xi)} \|Au(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)}))\|^2 d\xi \right)^{1/2} \\ & \longrightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.99)$$

On the other hand, since $(w_T^{(n)}, \psi_T^{(n)}) \rightharpoonup (w_T, \psi_T)$ weakly in $W \times L_W^2$ as $n \rightarrow \infty$, we can use the similar argument as that in [56] to prove

$$u(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)})) \rightharpoonup u(\xi; t - T, (w_T, \psi_T)) \text{ weakly in } L^2(t - T, t; D(A)) \text{ as } n \rightarrow \infty.$$

Thus,

$$\begin{aligned} & Au(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)})) \rightharpoonup Au(\xi; t - T, (w_T, \psi_T)) \\ & \text{weakly in } L^2(t - T, t; H) \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.100)$$

Obviously, $e^{-k\mu_1(t-\xi)} f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T))) \in L^2(t - T, t; H)$ and then by (4.100),

$$\begin{aligned} & \left| \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T))) , Au(\xi; t - T, (w_T^{(n)}, \psi_T^{(n)})) \right) d\xi \right. \\ & \quad \left. - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \left(f(\xi, u_\xi(\cdot; t - T, (w_T, \psi_T))) , Au(\xi; t - T, (w_T, \psi_T)) \right) d\xi \right| \\ & \longrightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.101)$$

(4.99) and (4.101) imply (4.94). Now taking (4.90)–(4.94) into account, we get

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} a(u^{(n)}(t), u^{(n)}(t)) \\
& \leq c_2^2 R_w^2 e^{-k\mu_1 T} + \int_{t-T}^t e^{-k\mu_1(t-\xi)} (g, Au(\xi; t-T, (w_T, \psi_T))) d\xi \\
& \quad + \int_{t-T}^t e^{-k\mu_1(t-\xi)} (f(\xi, u_\xi(\cdot; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T))) d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \langle B(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \rangle d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \langle N(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \rangle d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|u(\xi; t-T, (w_T, \psi_T))\|^2 d\xi. \tag{4.102}
\end{aligned}$$

At the same time, applying (4.88) to $w_0 = u(t; t-T, (w_T, \psi_T))$, we obtain

$$\begin{aligned}
a(w_0, w_0) &= a(w_T, w_T) e^{-k\mu_1 T} + \int_{t-T}^t e^{-k\mu_1(t-\tau)} (g, Au(\xi; t-T, (w_T, \psi_T))) d\xi \\
& \quad + \int_{t-T}^t e^{-k\mu_1(t-\xi)} (f(\xi, u_\xi(\cdot; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T))) d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \langle B(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \rangle d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \langle N(u(\xi; t-T, (w_T, \psi_T))), Au(\xi; t-T, (w_T, \psi_T)) \rangle d\xi \\
& \quad - \int_{t-T}^t e^{-k\mu_1(t-\xi)} \|u(\xi; t-T, (w_T, \psi_T))\|^2 d\xi, \quad \text{for each } T \in \mathbb{N}. \tag{4.103}
\end{aligned}$$

From (4.102) and (4.103) we get that for each $T \in \mathbb{N}$,

$$\limsup_{n \rightarrow \infty} a(u^{(n)}(t), u^{(n)}(t)) \leq (c_2^2 R_w^2 - a(w_T, w_T)) e^{-k\mu_1 T} + a(w_0, w_0). \tag{4.104}$$

Recalling that $w_T \in \mathcal{B}_W(0, R_W)$, and letting $T \rightarrow +\infty$ in (4.104), we get

$$\limsup_{n \rightarrow \infty} a(u^{(n)}(t), u^{(n)}(t)) \leq a(w_0, w_0), \tag{4.105}$$

that is, (4.85) holds. It follows from (4.84) and (4.85) that

$$\lim_{n \rightarrow \infty} a(u^{(n)}(t), u^{(n)}(t)) = \lim_{n \rightarrow \infty} a(u(t; t-s_n, (u_0^{(n)}, \psi^{(n)}))) = a(w_0, w_0),$$

and then Lemma 2.1 shows that

$$\lim_{n \rightarrow \infty} \|u^{(n)}(t)\|_W = \|w_0\|_W. \tag{4.106}$$

Since W is a Hilbert space, combining (4.78) and (4.106), we have

$$\lim_{n \rightarrow \infty} \|u^{(n)}(t) - w_0\|_W = 0,$$

i.e.

$$u^{(n)}(t) \longrightarrow w_0 \quad \text{strongly in } W \text{ as } n \rightarrow \infty. \tag{4.107}$$

Now we denote by $u_t(\theta; t - s_n, (u_0, \psi)) = u(t + \theta; t - s_n, (u_0, \psi)) \in L_w^2$ the solution of (2.14) with initial data $(u_0, \psi) \in L_w^2$ at initial time $t - s_n$. Then for each $\theta_0 \in [-\nu, 0]$, we get

$$u_t(\theta_0; t - s_n, (u_0^{(n)}, \psi^{(n)})) \doteq u_t^{(n)}(\theta_0) = u(t + \theta_0; t - s_n, (u_0^{(n)}, \psi^{(n)})) \longrightarrow \psi(\theta_0)$$

strongly in W as $n \rightarrow \infty$. Therefore, we have

$$\|u_t^{(n)}(\theta)\|_w^2 \longrightarrow \|\psi(\theta)\|_w^2 \text{ everywhere for } \theta \in [-\nu, 0] \text{ as } n \rightarrow \infty.$$

Note that $u_t^{(n)}(\theta_0) \in \mathcal{B}_w(0, R_w)$ for n being large enough and $\psi(\theta) \in L_w^2$. By Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} \int_{-\nu}^0 \|u_t^{(n)}(\theta)\|_w^2 d\theta = \int_{-\nu}^0 \|\psi(\theta)\|_w^2 d\theta. \quad (4.108)$$

By (4.78),

$$u_t^{(n)}(\theta) \rightharpoonup \psi(\theta) \text{ weakly in } L_w^2 \text{ as } n \rightarrow \infty. \quad (4.109)$$

Also, L_w^2 is a Hilbert space, and (4.108) and (4.109) imply

$$u_t^{(n)}(\theta) \longrightarrow \psi(\theta) \text{ strongly in } L_w^2 \text{ as } n \rightarrow +\infty. \quad (4.110)$$

So (4.107) and (4.110) tell us that

$$\begin{aligned} S(t, t - s_n)(u_0^{(n)}, \psi^{(n)}) &= (u(t; t - s_n, (u_0^{(n)}, \psi^{(n)})), u_t(\theta; t - s_n, (u_0^{(n)}, \psi^{(n)}))) \\ &\longrightarrow (w_0, \psi(\theta)) \text{ strongly in } E_w^2 \text{ as } n \rightarrow \infty. \end{aligned}$$

The proof of Lemma 4.1 is now complete. $\square$

The main result of this section is

THEOREM 4.1. *Let $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a unique uniformly bounded pullback attractor $\{\mathcal{A}_{E_w^2}(t)\}_{t \in \mathbb{R}} \subset \mathcal{B}_w(0, R_w) \times \mathcal{B}_{L_w^2}(0, \nu^{1/2} R_w) \subset E_w^2$. Furthermore, $\mathcal{A}_{E_w^2}(t) \subset W \times \mathcal{C}_w$ for all $t \in \mathbb{R}$.*

Proof. Lemma 3.6 shows that $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a family of uniformly bounded pullback absorbing sets in E_w^2 , while Lemma 4.1 shows that $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ is pullback asymptotically compact in E_w^2 . At the same time, the continuity of $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_w^2 is obvious. Thus, we can prove, similar to the semigroup case (see, e.g., [53]) that $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ possesses a unique uniformly bounded pullback attractor $\{\mathcal{A}_{E_w^2}(t)\}_{t \in \mathbb{R}} \subset \mathcal{B}_w(0, R_w) \times \mathcal{B}_{L_w^2}(0, \nu^{1/2} R_w) \subset E_w^2$. The fact that $\mathcal{A}_{E_w^2}(t) \subset W \times \mathcal{C}_w$ for all $t \in \mathbb{R}$ follows from Theorem 2.1 (2) and Lemma 3.6. The proof is complete. $\square$

REMARK 4.1. Let $\mathcal{H}$ be a Hilbert space or a uniform convex Banach space. If $h_n \rightharpoonup h$ weakly in $\mathcal{H}$ and $\|h_n\|_{\mathcal{H}} \longrightarrow \|h\|_{\mathcal{H}}$ as $n \rightarrow \infty$, then $h_n \longrightarrow h$ strongly in $\mathcal{H}$. This property of a Hilbert space plays a very important role when we prove Lemma 4.1 via the energy equality method. Since $\mathcal{C}_w$ is not a Hilbert space, the energy equality method seems difficult to be applicable for us to prove that the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a pullback attractor $\{\mathcal{A}_{\mathcal{C}_w}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_w$. But we can prove that the pullback attractor $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$ is indeed a family of compact sets of $\mathcal{C}_w$ (see Lemma 5.2 below).

5. Regularity of pullback attractors. In this section we first prove that the pullback attractor $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}}$ coincides with $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$. Then we establish that the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a uniformly bounded pullback attractor $\{\mathcal{A}_{C_W}(t)\}_{t \in \mathbb{R}}$ in C_W , and moreover

$$\mathcal{A}_{C_W}(t) = \mathcal{A}_{C_H}(t), \quad \forall t \in \mathbb{R}.$$

We next use the Uniform Gronwall Lemma to prove that the solution $\mathcal{S}(t, t-s)(u_0, \psi)$ with initial condition (u_0, ψ) in any bounded sets of E_H^2 will enter a bounded set of E_W^2 provided s is large enough.

LEMMA 5.1. *Assume (A₁)–(A₅) hold and $g \in H$. Let $\mathcal{B}_{E_H^2}$ be a bounded set of E_H^2 . Then for any $(u_0, \psi) \in \mathcal{B}_{E_H^2}$ and any $t \in \mathbb{R}$, there exist a time $s_0^* = s_0^*(\mathcal{B}_{E_H^2})$ and a positive constant M such that*

$$\begin{aligned} \|\mathcal{S}(t, t-s)(u_0, \psi)\|_{E_W^2} &= \|(u(t; t-s, (u_0, \psi)), u_t(\cdot; t-s, (u_0, \psi)))\|_{E_W^2} \\ &\leq M, \quad \forall s \geq s_0^*. \end{aligned} \quad (5.111)$$

Proof. For simplicity, we denote $u' = \frac{\partial u}{\partial t}$ and $u = u(t) = u(t; t-s, (u_0, \psi))$. Multiplying the equation in (2.14) by u' and then integrating the resulting equality over Ω , we obtain

$$\|u'\|^2 + 2\mu_1 a(u, u') + \langle B(u), u' \rangle + \langle N(u), u' \rangle = (g, u') + (f(t, u_t), u'). \quad (5.112)$$

Set

$$\Gamma(|e(u)|^2) = \int_0^{|e(u)|^2} \mu_0(\varepsilon + \xi)^{-\alpha/2} d\xi.$$

Then

$$\frac{d\Gamma}{dt} = \sum_{i,j=1}^2 \mu(u) e_{ij}(u) \frac{\partial e_{ij}(u)}{\partial t} = \sum_{i,j=1}^2 \mu(u) e_{ij}(u) e_{ij}(u').$$

Thus,

$$\langle N(u), u' \rangle = \sum_{i,j=1}^2 \int_{\Omega} \mu(u) e_{ij}(u) e_{ij}(u') dx = \frac{d}{dt} \left(\int_{\Omega} \Gamma(|e(u)|^2) dx \right). \quad (5.113)$$

Substituting (5.113) into (5.112), we obtain

$$\begin{aligned} &\|u'\|^2 + \frac{d}{dt} \left(\mu_1 a(u, u) + \int_{\Omega} \Gamma(|e(u)|^2) dx \right) \\ &= -\langle B(u), u' \rangle + (g, u') + (f(t, u_t), u') \\ &\leq \left| \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial u_j}{\partial x_i} u'_j dx \right| + \|g\|^2 + \|f(t, u_t)\|^2 + \frac{1}{2} \|u'\|^2 \\ &\leq \|u\|_{L^4(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|u'\| + \|g\|^2 + \|f(t, u_t)\|^2 + \frac{1}{2} \|u'\|^2. \end{aligned} \quad (5.114)$$

By the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^4(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|u'\| \leq c \|\Delta u\|^2 \|u'\| \leq \frac{c^2}{2} \|u\|_W^4 + \frac{1}{2} \|u'\|^2. \quad (5.115)$$

Inserting (5.115) into (5.114), we obtain by using Lemma 2.1 that

$$\begin{aligned} \frac{d}{dt} \left(\mu_1 a(u, u) + \int_{\Omega} \Gamma(|e(u)|^2) dx \right) &\leq \frac{c^2}{2} \|u\|_w^4 + \|g\|^2 + \|f(t, u_t)\|^2 \\ &\leq \frac{c^2}{2} \|u\|_w^2 \cdot \frac{\mu_1 a(u, u)}{c_1 \mu_1} + \|g\|^2 + \|f(t, u_t)\|^2. \end{aligned} \quad (5.116)$$

So we have

$$\frac{d\Phi(t)}{dt} \leq \Upsilon(t)\Phi(t) + \Psi(t), \quad (5.117)$$

where

$$\begin{aligned} \Phi(t) &= \mu_1 a(u(t), u(t)) + \int_{\Omega} \Gamma(|e(u(t))|^2) dx, \\ \Upsilon(t) &= \frac{c^2}{2c_1\mu_1} \|u(t)\|_w^2, \quad \Psi(t) = \|g\|^2 + \|f(t, u_t)\|^2. \end{aligned}$$

Now taking the inner product $(\cdot, \cdot)$ of the equation in (2.14) with u and integrating the resulting equality over $[t, t+1]$, we obtain

$$\begin{aligned} &\int_t^{t+1} 2\mu_1 a(u(\xi), u(\xi)) d\xi \\ &\leq \int_t^{t+1} \|u(\xi)\| (\|g(\xi)\| + \|f(\xi, u_\xi)\|) d\xi + \frac{1}{2} \|u(t)\|^2 \\ &\leq \int_t^{t+1} (\|u(\xi)\|^2 + 2\|g(\xi)\|^2 + 2\|f(\xi, u_\xi)\|^2) d\xi + \frac{1}{2} \|u(t)\|^2. \end{aligned} \quad (5.118)$$

Denote by $s_0(\mathcal{B}_{E_H^2})$ the absorbing time corresponding to $\mathcal{B}_H(0, R_H) \times \mathcal{B}_{L_H^2}(0, \nu^{1/2} R_H)$ in Lemma 3.4 and let $s \geq s_0$. Then from (5.118) it follows that

$$\begin{aligned} 2c_1\mu_1 \int_t^{t+1} \|u(\xi)\|_w^2 d\xi &\leq \int_t^{t+1} 2\mu_1 a(u(\xi), u(\xi)) d\xi \\ &\leq R_H^2 + 2\|g\|^2 + 2 \int_t^{t+1} \|f(\xi, u_\xi)\|^2 d\xi + \frac{1}{2} R_H^2 \\ &\leq \frac{3}{2} R_H^2 + 2\|g\|^2 + 2C_f^2 \int_t^{t+1} \|u_\xi\|_{\mathcal{C}_H}^2 d\xi \\ &\leq \frac{3}{2} R_H^2 + 2\|g\|^2 + 2C_f^2 R_H^2, \quad \forall s \geq s_0. \end{aligned} \quad (5.119)$$

Hence, by Lemma 2.1,

$$\begin{aligned} \int_t^{t+1} \Upsilon(\xi) d\xi &= \int_t^{t+1} \frac{c^2}{2c_1\mu_1} \|u(\xi)\|_w^2 d\xi \leq \frac{c^2}{4c_1\mu_1^2} \left(\frac{3}{2} R_H^2 + 2\|g\|^2 + 2C_f^2 R_H^2 \right) \\ &\doteq a_1, \quad \forall s \geq s_0. \end{aligned} \quad (5.120)$$

Clearly, we have

$$\int_t^{t+1} \Psi(\xi) d\xi = \int_t^{t+1} (\|g\|^2 + \|f(\xi, u_\xi)\|^2) d\xi \leq \|g\|^2 + C_f^2 R_H^2 \doteq a_2. \quad \forall s \geq s_0, \quad (5.121)$$

where a_1, a_2 are positive constants. We next show that there exists a positive constant a_3 such that

$$\int_t^{t+1} \Phi(\xi) d\xi \leq a_3, \quad \forall s \geq s_0.$$

From (5.119) and Lemma 2.1 we obtain

$$\int_t^{t+1} \mu_1 a(u(\xi), u(\xi)) d\xi \leq \frac{3}{4} R_H^2 + \|g\|^2 + C_f^2 R_H^2, \quad \forall s \geq s_0. \quad (5.122)$$

At the same time, we have $0 < (\varepsilon + \xi)^{-\alpha/2} \leq \varepsilon^{-\alpha/2}$ for all $\xi \geq 0$ and $0 < \alpha < 1$. Thus

$$\Gamma(|e(u)|^2) = \int_0^{|e(u)|^2} \mu_0 (\varepsilon + \xi)^{-\alpha/2} d\xi \leq \mu_0 \varepsilon^{-\alpha/2} |e(u)|^2,$$

from which and (5.122) it follows that

$$\begin{aligned} \int_t^{t+1} \int_{\Omega} \Gamma(|e(u(\xi))|^2) dx d\xi &\leq \mu_0 \varepsilon^{-\alpha/2} \int_t^{t+1} \int_{\Omega} |e(u(\xi))|^2 dx d\xi \\ &\leq 4\mu_0 \varepsilon^{-\alpha/2} \int_t^{t+1} \|u(\xi)\|_W^2 d\xi \\ &\leq 4\mu_0 \varepsilon^{-\alpha/2} \int_t^{t+1} \frac{\mu_1 a(u(\xi), u(\xi))}{c_1 \mu_1} d\xi \\ &\leq \frac{4\mu_0 \varepsilon^{-\alpha/2}}{c_1 \mu_1} \cdot \left(\frac{3}{4} R_H^2 + \|g\|^2 + C_f^2 R_H^2 \right), \quad \forall s \geq s_0, \end{aligned} \quad (5.123)$$

where we have used the fact that $\int_{\Omega} |e(u)|^2 dx \leq 4\|u\|_W^2$. It follows from (5.122) and (5.123) that

$$\int_t^{t+1} \Phi(\xi) d\xi \leq \left(1 + \frac{4\mu_0 \varepsilon^{-\alpha/2}}{c_1 \mu_1} \right) \left(\frac{3}{4} R_H^2 + \|g\|^2 + C_f^2 R_H^2 \right) \doteq a_3, \quad \forall s \geq s_0. \quad (5.124)$$

Taking Lemma 2.3, (5.117), (5.120), (5.121) and (5.124) into account, we obtain

$$\Phi(t) \leq (a_3 + a_2) e^{a_1}, \quad \forall s \geq s_0 + 1, \quad \forall t \in \mathbb{R}.$$

Therefore,

$$\|u(t)\|_W^2 \leq \frac{1}{c_1} a(u(t), u(t)) \leq \frac{\Phi(t)}{c_1 \mu_1} \leq \frac{(a_3 + a_2) e^{a_1}}{c_1 \mu_1} \doteq M_3, \quad s \geq s_0 + 1, \quad \forall t \in \mathbb{R},$$

and if $s \geq s_0^* = s_0 + \nu + 1$, then

$$\begin{aligned} \|u_t(\cdot; t-s, (u_0, \psi))\|_{L_W^2}^2 &= \int_{-\nu}^0 \|u(t+\theta; t-s, (u_0, \psi))\|_W^2 d\theta \\ &\leq \int_{-\nu}^0 \sup_{\theta \in [-\nu, 0]} \|u(t+\theta; t-s, (u_0, \psi))\|_W^2 d\theta \\ &\leq \nu M_3, \quad \forall t \in \mathbb{R}, \quad \forall s \geq s_0^*. \end{aligned}$$

The proof of Lemma 5.1 is now complete. $\square$

THEOREM 5.1. *Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then*

$$J(\mathcal{A}_{\mathcal{C}_H}(t)) = \mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R}. \quad (5.125)$$

Proof. By (3.70), we only need to prove

$$\mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R}. \quad (5.126)$$

On the one hand, $\mathcal{A}_{E_W^2}(t)$ is bounded in $W \times L_W^2$ for any $t \in \mathbb{R}$. Thus $\mathcal{A}_{E_W^2}(t)$ is bounded in $H \times L_H^2$ for any $t \in \mathbb{R}$. By the invariance property and pullback attracting property of the pullback attractor, we have

$$\begin{aligned} \text{Dist}_{E_H^2}(\mathcal{A}_{E_W^2}(t), \mathcal{A}_{E_H^2}(t)) &= \text{Dist}_{E_H^2}(\mathcal{S}(t, t-s)\mathcal{A}_{E_W^2}(t-s), \mathcal{A}_{E_H^2}(t)) \quad (\forall s \in \mathbb{R}_+) \\ &= \lim_{s \rightarrow +\infty} \text{Dist}_{E_H^2}(\mathcal{S}(t, t-s)\mathcal{A}_{E_W^2}(t-s), \mathcal{A}_{E_H^2}(t)) \\ &= 0, \quad \forall t \in \mathbb{R}, \end{aligned}$$

which implies

$$\mathcal{A}_{E_W^2}(t) \subseteq \mathcal{A}_{E_H^2}(t), \quad \forall t \in \mathbb{R}. \quad (5.127)$$

On the other hand, Theorem 4.1 and Lemma 5.1 show that $\mathcal{A}_{E_H^2}(t)$ is bounded in $W \times L_W^2$ for any $t \in \mathbb{R}$. Then also by the invariance property and pullback attracting property of the pullback attractor, we obtain

$$\begin{aligned} \text{Dist}_{E_H^2}(\mathcal{A}_{E_H^2}(t), \mathcal{A}_{E_W^2}(t)) &\leq \text{Dist}_{E_W^2}(\mathcal{A}_{E_H^2}(t), \mathcal{A}_{E_W^2}(t)) \\ &= \text{Dist}_{E_W^2}(\mathcal{S}(t, t-s)\mathcal{A}_{E_H^2}(t-s), \mathcal{A}_{E_W^2}(t)) \quad (\forall s \in \mathbb{R}_+) \\ &= \lim_{s \rightarrow +\infty} \text{Dist}_{E_W^2}(\mathcal{S}(t, t-s)\mathcal{A}_{E_H^2}(t-s), \mathcal{A}_{E_W^2}(t)) \\ &= 0, \quad \forall t \in \mathbb{R}, \end{aligned}$$

which gives

$$\mathcal{A}_{E_H^2}(t) \subseteq \mathcal{A}_{E_W^2}(t), \quad \forall t \in \mathbb{R}. \quad (5.128)$$

We readily get (5.126) from (5.127) and (5.128). The proof is complete. $\square$

We next use (5.125) to prove that the pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}}$ in C_H obtained in Theorem 3.1 is indeed a family of compact sets of C_W .

LEMMA 5.2. Assume **(A₁)**–**(A₅)** hold and $g \in H$. Then the uniformly bounded pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in C_H is a family of compact sets of C_W .

Proof. Given $t \in \mathbb{R}$, let $\{\psi_n(\theta)\}$ be a bounded sequence of $\mathcal{A}_{C_H}(t)$. We only need to prove that $\{\psi_n(\theta)\}$ possesses a convergent subsequence in C_W . Indeed, by (5.125) and Theorem 4.1 we see that $\mathcal{A}_{C_H}(t) \subset C_W$ and is compact in $C_H \cap L_W^2$. Thus, there exist a subsequence (still denoted by $\{\psi_n(\theta)\}$) and a $\psi(\theta) \in C_H \cap L_W^2$ such that

$$\psi_n(\theta) \longrightarrow \psi(\theta) \text{ strongly in } L_W^2 \text{ as } n \rightarrow \infty, \quad (5.129)$$

$$\psi_n(\theta) \longrightarrow \psi(\theta) \text{ strongly in } C_H \text{ as } n \rightarrow \infty. \quad (5.130)$$

From (5.129), we get

$$\left\{ \begin{array}{l} \text{For any } \epsilon > 0, \text{ there exists an } N_0 \in \mathbb{N}, \text{ such that} \\ \int_{-\nu}^0 \|\psi_n(\theta) - \psi_m(\theta)\|_W^2 d\theta < \epsilon \text{ provided } n, m > N_0. \end{array} \right. \quad (5.131)$$

Note that the strong convergence of the sequence $\{\psi_n(\theta)\}$ in $\mathcal{C}_W$ is equivalent to its uniform convergence. Therefore, if $\{\psi_n(\theta)\}$ is not uniformly convergent in $\mathcal{C}_W$, then by the Cauchy Criteria on the convergence of a sequence of functions, there exists an $\epsilon_0 > 0$ such that for any $N \in \mathbb{N}$, there exist $n_0, m_0 > N$ and $\theta_0 \in [-\nu, 0]$, yielding

$$\|\psi_{n_0}(\theta_0) - \psi_{m_0}(\theta_0)\|_W \geq \epsilon_0. \quad (5.132)$$

Since $\psi_{n_0}(\theta), \psi_{m_0}(\theta) \in \mathcal{A}_{\mathcal{C}_H}(t) \subset \mathcal{C}_W$, there exists a neighborhood of θ_0 (denoted by $(\theta_0 - \delta, \theta_0 + \delta)$ for some $\delta > 0$) such that

$$\|\psi_{n_0}(\theta) - \psi_{m_0}(\theta)\|_W \geq \frac{\epsilon_0}{2} \text{ for all } \theta \in (\theta_0 - \delta, \theta_0 + \delta) \subset [-\nu, 0],$$

and thus we have

$$\int_{-\nu}^0 \|\psi_{n_0}(\theta) - \psi_{m_0}(\theta)\|_W^2 d\theta \geq \int_{\theta_0 - \delta}^{\theta_0 + \delta} \|\psi_{n_0}(\theta) - \psi_{m_0}(\theta)\|_W^2 d\theta \geq \delta \epsilon_0,$$

which contradicts with (5.131). Thus, $\{\psi_n(\theta)\}$ is uniformly convergent in $\mathcal{C}_W$. By (5.130) and the uniqueness of the limit, we get

$$\psi_n(\theta) \longrightarrow \psi(\theta) \text{ strongly in } \mathcal{C}_W.$$

The proof is complete. $\square$

THEOREM 5.2. *Assume $(\mathbf{A}_1)$ – $(\mathbf{A}_5)$ hold and $g \in H$. Then the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a uniformly bounded pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_W$. Moreover,*

$$\mathcal{A}_{\mathcal{C}_W}(t) = \mathcal{A}_{\mathcal{C}_H}(t) \text{ and } J(\mathcal{A}_{\mathcal{C}_W}(t)) = J(\mathcal{A}_{\mathcal{C}_H}(t)) = \mathcal{A}_{E_H^2}(t) = \mathcal{A}_{E_W^2}(t), \forall t \in \mathbb{R}. \quad (5.133)$$

Proof. We first prove that the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a uniformly bounded pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_W$. To this end, we establish that the family of uniformly bounded absorbing sets $\{\mathcal{B}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ of $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ (see Lemma 3.6) are compact sets of $\mathcal{C}_W$. Indeed, the embedding $\mathcal{C}_W \hookrightarrow \mathcal{C}_H$ is continuous and for all $t \in \mathbb{R}$, $\mathcal{B}_{\mathcal{C}_W}(t) = \mathcal{B}_{\mathcal{C}_W}(0, R_W)$ is a bounded set of $\mathcal{C}_H$. Obviously, there exists a time $s^* = s(\mathcal{B}_{\mathcal{C}_W}(0, R_W)) > 0$ such that

$$\mathcal{U}(t, t-s)\mathcal{B}_{\mathcal{C}_W}(0, R_W) \subset \mathcal{B}_{\mathcal{C}_W}(0, R_W), \quad \forall t \in \mathbb{R}, \quad \forall s > s^*.$$

Thus,

$$\begin{aligned} \text{Dist}_{\mathcal{C}_H}(\mathcal{B}_{\mathcal{C}_W}(t), \mathcal{A}_{\mathcal{C}_H}(t)) &= \text{Dist}_{\mathcal{C}_H}(\mathcal{B}_{\mathcal{C}_W}(0, R_W), \mathcal{A}_{\mathcal{C}_H}(t)) \\ &\leq \lim_{s \rightarrow +\infty} \text{Dist}_{\mathcal{C}_H}(\mathcal{U}(t, t-s)\mathcal{B}_{\mathcal{C}_W}(0, R_W), \mathcal{A}_{\mathcal{C}_H}(t)) = 0, \end{aligned}$$

and $\mathcal{B}_{\mathcal{C}_W}(t) \subseteq \mathcal{A}_{\mathcal{C}_H}(t)$ for each $t \in \mathbb{R}$, while Lemma 5.2 shows that $\mathcal{A}_{\mathcal{C}_H}(t)$ is compact in $\mathcal{C}_W$. Hence $\mathcal{B}_{\mathcal{C}_W}(t)$ is compact in $\mathcal{C}_W$ and the process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a uniformly bounded pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ in $\mathcal{C}_W$. Recall that the embedding $\mathcal{C}_W \hookrightarrow \mathcal{C}_H$ is continuous, so $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ is also a uniformly bounded pullback attractor for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$. By the uniqueness of the (uniformly bounded) pullback attractor (see Remark 2.3), we get $\mathcal{A}_{\mathcal{C}_W}(t) = \mathcal{A}_{\mathcal{C}_H}(t)$ for each $t \in \mathbb{R}$, and then (5.133) is clear. The proof is complete. $\square$

6. Conclusions and final remarks. We first use the idea of Caraballo and Real [19] to prove the existence of pullback attractors $\{\mathcal{A}_{\mathcal{C}_H}(t)\}_{t \in \mathbb{R}}$ and $\{\mathcal{A}_{E_H^2}(t)\}_{t \in \mathbb{R}}$ for the processes $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ and $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_H$ and E_H^2 , respectively. Then we apply the energy equality method of Ball to establish that the process $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ is pullback asymptotically compact in E_W^2 and consequently, we obtain the existence of the uniformly bounded pullback attractor $\{\mathcal{A}_{E_W^2}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{S}(t, \tau)\}_{t \geq \tau}$ in E_W^2 . Meanwhile, we utilize the Uniform Gronwall Lemma to verify the regularity of the pullback attractor by showing $\mathcal{A}_{E_W^2}(t) = \mathcal{A}_{E_H^2}(t)$ for all $t \in \mathbb{R}$, which in turn allows us to prove the existence of a uniformly bounded pullback attractor $\{\mathcal{A}_{\mathcal{C}_W}(t)\}_{t \in \mathbb{R}}$ for $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ in $\mathcal{C}_W$.

The regularity

$$\mathcal{A}_{E_W^2}(t) = \mathcal{A}_{E_H^2}(t), \quad \mathcal{A}_{\mathcal{C}_W}(t) = \mathcal{A}_{\mathcal{C}_H}(t), \quad \text{for all } t \in \mathbb{R},$$

shows that the pullback attractors associated to (1.1)–(1.2) do not depend on the energy space chosen for the mathematical studying. Furthermore, it reveals the pullback asymptotic smoothing effect of the fluid with delays in the sense that the solutions become eventually more regular (possessing H^2 -regularity) than the initial data (possessing L^2 -regularity). This effect is caused essentially by the special extra stress tensor in the addressed equations of the non-Newtonian fluid. Also, this effect implies, in the case of delays, that the regularity of the fluid in its history state does not play an important role on the regularity of its eventual state.

We next give some remarks on some possible extensions.

REMARK 6.1. Caraballo and Real [19] gave some applications with forcing terms containing variable delays and distributed delays to the Navier-Stokes models. We find that these examples are also valid to the non-Newtonian fluid after some slight modifications. Similar to [19] for the Navier-Stokes models, some results for the autonomous version of the non-Newtonian fluid can be deduced as particular cases of the non-autonomous case discussed in this paper.

REMARK 6.2. If the spatial domain is unbounded, the embedding $W \hookrightarrow H$ is no longer compact and the argument used in the proof of Theorem 3.1 is inadequate. Caraballo, Łukaszewicz and Real [14] obtained the existence of pullback attractors for 2D non-autonomous Navier-Stokes equations in 2D unbounded domains in which the Poincaré inequality holds. The technique used in [14] is asymptotic compactness based on the energy equations of the associated systems. As mentioned previously in the introduction, this method was first formulated by Ball (see, e.g., [5]) for autonomous systems, and by Łukaszewicz and Sadowski [40] to extend to the non-autonomous systems in the autonomous framework (see Ju [31] and Rosa [49]). This approach is also valid for the non-Newtonian fluid with delays when the spatial domain is unbounded but in which the Poincaré inequality holds.

REMARK 6.3. Zhao and Zhou [59] proved the existence and regularity of the pullback attractors (in the version of cocycle attractor) for the non-Newtonian fluid without delays. Schmalfuss [51] gave conditions for the existence of a pullback attractors for cocycles based on the so-called pullback convergence and proved the unique existence of pullback (or cocycle) attractor for the generalized non-autonomous Navier-Stokes equations. One can find that the conditions with respect to the t -regularity for the right-hand side in the

addressed Navier-Stokes equations is weaker than that of [59]. It seems that we can also prove, using the argument of [51], the existence and regularity of cocycle attractors for the non-Newtonian fluid with delays under weaker assumptions (similar to that in [51]) with respect to the t -regularity.

REMARK 6.4. Caraballo, Łukaszewicz and Real [13] proved the existence of a pullback attractor for a non-autonomous 2D Navier-Stokes model with a quite general non-autonomous term, a case in which the theory of uniform attractors does not work. We guess that the method in [13] could be extended to treat the Navier-Stokes model with delays and is also applicable to the non-Newtonian fluid with delays.

REMARK 6.5. Rosa [49] proved the existence of an L^2 -compact global attractor for the Navier-Stokes equations on two-dimensional unbounded domains in which the Poincaré inequality holds. Later Ju [31] extended the results of [49] by proving that the global attractor obtained in [49] is compact not only in the L^2 -norm but also in the H^1 -norm and first pointed out the regularity of the global attractor for the Navier-Stokes equations without delays. We note that the pullback attractors of the retarded Navier-Stokes equations also bear the similar regularity.

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