FROM A MULTIDIMENSIONAL QUANTUM HYDRODYNAMIC MODEL TO THE CLASSICAL DRIFT-DIFFUSION EQUATION

BY

YEPING LI

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

Abstract. In the paper, we discuss the combined semiclassical and relaxation-time limits of a multidimensional isentropic quantum hydrodynamical model for semiconductors with small momentum relaxation time and Planck constant. The quantum hydrodynamic equations consist of the isentropic Euler equations for the particle density and current density including the quantum Bohm potential and a momentum relaxation term. The momentum equation is highly nonlinear and contains a dispersive term with third-order derivatives. The equations are self-consistently coupled to the Poisson equation for the electrostatic potential. With the help of the Maxwell-type iteration, we prove that, as the Planck constant and the relaxation time tend to zero, periodic initial-value problems of a scaled isentropic quantum hydrodynamic model have unique smooth solutions existing in the time interval where the classical drift-diffusion models have smooth solutions.

1. Introduction. In the present paper, we consider the initial value problem of the quantum hydrodynamic model for semiconductors where an additional relaxation term is involved in the linear momentum equation to model the interaction between the electron and crystal lattice. The rescaled multidimensional quantum hydrodynamic models for semiconductors (see [5, 6]) are then given by

$$\begin{cases} \partial_t n + \frac{1}{\epsilon} \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \frac{1}{\epsilon} \nabla \cdot (nu \otimes u + p(n)) = \frac{n \nabla \phi}{\epsilon} + \frac{h^2}{4\epsilon} \nabla \cdot (n \nabla^2 (\log n)) - \frac{nu}{\epsilon^2}, \\ \lambda^2 \Delta \phi = n - b(x). \end{cases} \quad (1.1)$$

The variables are the electron density n , the mean velocity u , and the electrostatic potential ϕ . $p(n)$ is the given strictly increasing function and denotes the pressure. The function $b(x)$ stands for the prescribed density of positive charged background ions (doping profile). The parameters are the (scaled) Planck constant h , the momentum relaxation time ϵ , and the Debye length λ . The quantum hydrodynamic equations can

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E-mail address: ypleemei@yahoo.com.cn

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be interpreted as Euler equations for a charged isentropic gas, containing the nonlinear dispersive term $\frac{h^2}{4}\nabla \cdot (n\nabla^2(\log n))$ and the relaxation term $\frac{nu}{\epsilon^2}$. Moreover,

$$\frac{h^2}{4}\nabla \cdot (n\nabla^2(\log n)) = \frac{h^2}{2}n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right),$$

and $\frac{\Delta\sqrt{n}}{\sqrt{n}}$ is the quantum Bohm potential. Note that the scaling

$$t = \epsilon\tilde{t}$$

converts (1.1) back into the original quantum isentropic model in [7, 8] with $\tilde{t}$ as its time variable. The scaled-time variable t was first introduced in [22] to study the relation between the classical hydrodynamical models and corresponding drift-diffusion models.

Since we are interested in the combined semiclassical and relaxation limits, we can take $\lambda = 1, h = \epsilon$. On the other hand, the operator $\nabla\Delta^{-1}$ is a bound linear operator on $L^2(\mathbb{T}^3)$, where and in the sequel, $\mathbb{T}^3$ is the three-dimensional unit torus $(0, 1] \times (0, 1] \times (0, 1]$. It is convenient to make use of the variable transformation $n = w^2$ as in [11, 12] and the enthalpy $f(n)$ satisfying $f'(n) = \frac{p'(n)}{n}$. With the above simplifications, we can rewrite the model (1.1) for smooth solutions as

$$\begin{cases} w_t + \frac{1}{2\epsilon}w\operatorname{div}u + \frac{1}{\epsilon}\nabla w \cdot u = 0, \\ u_t + \frac{1}{\epsilon}(u \cdot \nabla)u + \frac{1}{\epsilon}\nabla f(w^2) = \frac{1}{\epsilon}\nabla\Delta^{-1}(w^2 - b(x)) + \frac{\epsilon}{2}\nabla\left(\frac{\Delta w}{w}\right) - \frac{u}{\epsilon^2}. \end{cases} \quad (1.2)$$

Recently, many efforts have been made on the quantum hydrodynamic (Euler-Poisson) system. More precisely, the existence and uniqueness of thermal equilibrium steady-state classical solutions for one and high dimensional quantum models have been studied in [3, 24]. The corresponding nonthermal equilibrium steady-state solutions have also been considered in [4, 9, 27] for general monotonic pressure functions and in [14] for general pressure functions, however, with different boundary conditions, for instance, assuming Dirichlet data for the velocity potential [9] or employing nonlinear boundary conditions [27]. For the time-dependent system, the local- and global-in-time existence of the classical solutions was obtained in a bounded domain [12, 15] (subject to boundary conditions on the density and the electrostatic potential) and on the real line [11], for ir-rotational flow in the whole space [20] assuming strictly convex pressure functions and in the multidimensional torus $\mathbb{T}^n$ [19], while paper [2] established the global existence of finite energy weak solutions to the quantum hydrodynamic system for arbitrarily large data.

Relaxation limits in the classical hydrodynamic equations have been performed extensively. In one space dimension, the relaxation limit problem for isentropic hydrodynamic models has been investigated in the compactness frameworks for nonsmooth solutions of conservation laws; see [22]. Paper [10] also discussed similar results for the one-dimensional nonisentropic hydrodynamic model for semiconductors with zero thermal conductivity. In [18, 17], the authors considered the multidimensional isentropic unipolar hydrodynamic model and the corresponding bipolar model with x in a bounded domain, assumed the existence of L^∞ -solutions in a τ -independent time interval, and justified the relaxation limit in a compactness framework for nonsmooth solutions. In [26, 21], the authors studied the diffusive relaxation of multidimensional isentropic and

nonisentropic hydrodynamical models for semiconductors by the Maxwell iteration. Finally, paper [12] first investigated the relaxation limit for the quantum hydrodynamic models, where the relaxation-time limits both in the stationary and in the transient case are proved. Similar results about classical hydrodynamic models for semiconductors have been extensively discussed in [1, 16, 28]. However, the combined semiclassical and relaxation limit for the quantum hydrodynamical model for semiconductors is not studied. Motivated by [12, 26, 21], we expect that the quantum hydrodynamic model and the classical drift-diffusion model give similar results when ϵ is small, which can be seen formally as follows. Applying the Maxwell-type iteration to the momentum equations in (1.2) gives

$$\begin{aligned} u &= -\epsilon \nabla f(w^2) + \epsilon \nabla \Delta^{-1}(w^2 - b(x)) - \epsilon(u \cdot)u + \frac{\epsilon^3}{2} \nabla \left(\frac{\Delta w}{w} \right) - \epsilon^2 \partial_t u \\ &= -\epsilon \nabla f(w^2) + \epsilon \nabla \Delta^{-1}(w^2 - b(x)) + O(\epsilon^2). \end{aligned}$$

Substituting these truncations $u = -\epsilon \nabla f(w^2) + \epsilon \nabla \Delta^{-1}(w^2 - b(x))$ into the mass equations in (1.2), we arrive at the classical drift-diffusion model

$$2w \partial_t w - \operatorname{div}(\nabla p(w^2) - w^2 \nabla \Delta^{-1}(w^2 - b(x))) = 0; \quad (1.3)$$

this is a parabolic-type system (see [13, 23]), provided that $p'(w^2) > 0$. Now we state our main results in the following.

THEOREM 1.1. Suppose $f(w^2) \in C^8(0, \infty)$, $f'(w^2) > 0$, $b(x) \in H^6(\mathbb{T}^3)$, and that the quantum drift-diffusion model (1.3) with the initial data $w(x, 0) = w_0(x) > 0$ has a solution $w \in C([0, T_*], H^8(\mathbb{T}^3)) \cap C^1([0, T_*], H^7(\mathbb{T}^3))$ with a positive lower bound.

Then, for ϵ sufficiently small, the quantum isentropic model (1.2) with periodic initial data

$$\begin{cases} w(x, 0) = w_0(x), \\ u(x, 0) = \epsilon \nabla \Delta^{-1}(w^2(x, 0) - b(x)) - \epsilon \nabla f(w_0^2) \end{cases} \quad (1.4)$$

has a unique solution $(w^\epsilon, u^\epsilon) \in C([0, T_*], H^6(\mathbb{T}^3)) \times C([0, T_*], H^5(\mathbb{T}^3))$, and there exists a constant $K > 0$, independent of ϵ but dependent on $T_*(< \infty)$, such that

$$\sup_{t \in [0, T_*]} (\|w^\epsilon - w_\epsilon\|_4 + \epsilon \|\partial_x^5(w^\epsilon - w_\epsilon)\|_1 + \|u^\epsilon - u_\epsilon\|_5) \leq K\epsilon^2, \quad (1.5)$$

where

$$w_\epsilon = w, \quad u_\epsilon = \epsilon \nabla \Delta^{-1}(w^2 - b(x)) - \epsilon \nabla f(w^2). \quad (1.6)$$

To prove this result, we shall adopt and modify the arguments in [25, 26]. More precisely, we first construct the approximate solutions, based on the existence of smooth solutions for the corresponding classical drift-diffusion model in [13, 23]. Next, we establish the continuation principle (convergence-stability lemma) of the Cauchy problem, under the assumption of the important a priori estimate (which is indeed referred to as a convergence assumption in [25, 26]). Finally, we should show the a priori convergence assumption; namely, from the energy methods and the special nonlinear structure of the quantum hydrodynamic models, we can obtain the a priori estimate. However, we have to face up to two difficulties, contrasted with [25, 26]. The first one is from the

Bohm potential, which is a third-order dispersive term. Our strategy is to reformulate the conservation law of mass in the quantum hydrodynamic equation as a fourth-order wave equation as in [11, 12]. The second difficulty is that we cannot directly make use of the previous convergence-stability lemma, which was first formulated by W.A. Yong in [25], since it only fits into the symmetrizable hyperbolic system with relaxation term. So here we have to establish the corresponding convergence-stability lemma for our quantum hydrodynamic model with the third-order dispersive term.

REMARK 1.2. Our conclusion implies that if the classical drift-diffusion model has a global smooth solution on $[0, \infty]$ with w having a positive lower bound, then there exists $\epsilon_0 > 0$ such that the quantum isentropic hydrodynamical model has a unique smooth solution up to the time ∞ when $\epsilon < \epsilon_0$, and when $T < \infty$,

$$(w^\epsilon, u^\epsilon) = (w_\epsilon, u_\epsilon) + O(\epsilon^2).$$

Moreover, we believe that our result also holds for the more general multidimensional ($d > 3$) quantum isentropic hydrodynamic model for semiconductors.

REMARK 1.3. As far as we know, this is the first time that the convergence-stability lemma of the symmetrizable hyperbolic system has been extended to that of the nonhyperbolic case, and we apply it to investigate the combined semiclassical and relaxation limits for the multi-dimensional quantum isentropic hydrodynamic model for semiconductors.

REMARK 1.4. Employing similar arguments, we can investigate the combined semiclassical and relaxation limits for the bipolar quantum hydrodynamic model for semiconductors. Moreover, on the basis of the convergence-stability lemma in this paper, we can use the matched expansion method to discuss these limits for the relaxation limit for the quantum hydrodynamic semiconductor model with more general initial data.

This paper is organized as follows. In section 2 we are going to derive the convergence-stability result and construct the formal approximation (1.6). In section 3 we prove the validity of the formal approximation and conclude the existence of the solution to (w^ϵ, u^ϵ) in the time interval where w is well-defined.

2. The convergence-stability lemmas and formal approximations. In this section, we are going to derive convergence-stability lemmas and construct formal approximate solutions. To begin with, for

$$\begin{cases} w_t + \frac{1}{2\epsilon} w \operatorname{div} u + \frac{1}{\epsilon} \nabla w \cdot u = 0, \\ u_t + \frac{1}{\epsilon} (u \cdot \nabla) u + \frac{1}{\epsilon} \nabla f(w^2) = \frac{1}{\epsilon} \nabla \Delta^{-1} (w^2 - b(x)) + \frac{\epsilon}{2} \nabla \left(\frac{\Delta w}{w} \right) - \frac{u}{\epsilon^2}, \end{cases} \quad (2.1)$$

we have

LEMMA 2.1 (see [11]). Assume that $f(w^2) \in C^8(0, +\infty)$, $f'(w^2) > 0$, $b(x) \in H^6(\mathbb{T}^3)$ and that $(w(x, 0), u(x, 0)) \in H^6(\mathbb{T}^3) \times H^5(\mathbb{T}^3)$, $w(x, 0) > 0$. Then the unique classical solution (w, u) of the system (2.1) with the initial data $(w(x, 0), u(x, 0))$ exists for $t \in [0, T]$ and satisfies $w > 0$ in $\mathbb{T}^3 \times [0, T]$ and

$$w \in C^i([0, T]; H^{6-2i}(\mathbb{T}^3)), \quad i = 0, 1, 2, \quad u \in C^j([0, T]; H^{5-2j}(\mathbb{T}^3)), \quad j = 0, 1.$$

Fixing ϵ , and according to Lemma 2.1, there is a time interval $[0, T]$ so that (2.1) has a unique $H^6 \times H^5$ solution

$$(w^\epsilon, u^\epsilon) \in C([0, T], H^6(\mathbb{T}^3)) \times C([0, T], H^5(\mathbb{T}^3)).$$

Define

$$T_\epsilon = \sup\{T > 0 : (w^\epsilon, u^\epsilon) \in C([0, T], H^6(\mathbb{T}^3)) \times C([0, T], H^5(\mathbb{T}^3))\}.$$

Namely, $[0, T_\epsilon]$ is the maximal time interval of $H^6(\mathbb{T}^3) \times H^5(\mathbb{T}^3)$ -existence. Note that T_ϵ may tend to zero as ϵ goes to a certain singular point, say 0.

In order to show that $\lim_{\epsilon \rightarrow 0} T_\epsilon > 0$, we need the convergence-stability lemma (see [25, 26]). That is,

LEMMA 2.2. Let $b(x) \in H^6(\mathbb{T}^3)$ and $b(x) > 0$, and let $[0, T_\epsilon]$ be the maximal time interval such that (1.2) with $(w, u)(x, 0) = (\bar{w}, \bar{u})(x, \epsilon) \in H^6(\mathbb{T}^3) \times H^5(\mathbb{T}^3)$ for all (x, ϵ) and $\bar{w}(x, \epsilon) > 0$ has a unique solution $(w^\epsilon, u^\epsilon) \in C([0, T_\epsilon], H^6(\mathbb{T}^3)) \times C([0, T_\epsilon], H^5(\mathbb{T}^3))$ satisfying $w^\epsilon(x, t) > 0$. Assume there exists $T_* > 0$ and $(w_\epsilon, u_\epsilon) \in H^6(\mathbb{T}^3) \times H^5(\mathbb{T}^3)$ for each ϵ , satisfying

$$w_\epsilon(x, t) > 0$$

such that it follows that for $t \in [0, \min\{T_*, T_\epsilon\})$, as ϵ tends to the singular point,

$$\begin{aligned} \sup_{x,t} |(w^\epsilon, u^\epsilon)(x, t) - (w_\epsilon, u_\epsilon)(x, t)| &= o(1), \\ \sup_t (\|w^\epsilon(x, t) - w_\epsilon(x, t)\|_6 + \|u^\epsilon(x, t) - u_\epsilon(x, t)\|_5) &= O(1). \end{aligned}$$

Then

$$T_\epsilon > T_*$$

for all ϵ in a neighborhood of the singular point.

Proof. Otherwise, there is a sequence $\{\epsilon_k\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and $T_{\epsilon_k} \leq T_*$. Thanks to the convergence assumption, there exists $w^{\epsilon_k} > 0$. On the other hand, we deduce from

$$\|w^{\epsilon_k}\|_6 + \|u^{\epsilon_k}\|_5 \leq \|w^{\epsilon_k} - w_{\epsilon_k}\|_6 + \|u^{\epsilon_k} - u_{\epsilon_k}\|_5 + \|w_{\epsilon_k}\|_6 + \|u_{\epsilon_k}\|_5$$

and the convergence assumption that $\|w^{\epsilon_k}\|_6 + \|u^{\epsilon_k}\|_5$ is bounded uniformly with respect to $t \in [0, T_{\epsilon_k}]$. Now we could apply Lemma 2.1, beginning at a time t less than T_{ϵ_k} , to continue this solution beyond T_{ϵ_k} . This contradicts the definition of T_ϵ . This completes the proof. $\square$

In the following, we prepare to construct the approximation (w_ϵ, u_ϵ) as above. Let w solve the IVP of the classical drift-diffusion model (see [13, 23]), i.e.,

$$\begin{cases} 2w\partial_t w - \operatorname{div}(\nabla p(w^2) - w^2 \nabla \Delta^{-1}(w^2 - b(x))) = 0, \\ w(x, 0) = w_0(x). \end{cases} \quad (2.2)$$

Inspired by the Maxwell-type iteration, we take

$$\begin{cases} w_\epsilon = w, \\ u_\epsilon = \epsilon \nabla \Delta^{-1}(w^2 - b(x)) - \epsilon \nabla f(w^2). \end{cases} \quad (2.3)$$

Defining

$$R = \frac{\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon / \epsilon - \frac{\epsilon}{2} \nabla \left(\frac{\Delta w}{w} \right)}{\epsilon} = \partial_t (\nabla \Delta^{-1} (w^2 - b(x)) - \nabla f(w^2)) \\ + ((\nabla \Delta^{-1} (w^2 - b(x)) - \nabla f(w^2)) \cdot \nabla) (\nabla \Delta^{-1} (w^2 - b(x)) - \nabla f(w^2)) - \frac{1}{2} \nabla \left(\frac{\Delta w}{w} \right),$$

we have

$$\begin{cases} w_{\epsilon t} + \frac{1}{2\epsilon} w_\epsilon \operatorname{div} u_\epsilon + \frac{1}{\epsilon} \nabla w_\epsilon \cdot u_\epsilon = 0, \\ u_{\epsilon t} + \frac{1}{\epsilon} (u_\epsilon \cdot \nabla) u_\epsilon + \frac{1}{\epsilon} \nabla f(w_\epsilon^2) = \frac{1}{\epsilon} \nabla \Delta^{-1} (w_\epsilon^2 - b(x)) + \frac{\epsilon}{2} \nabla \left(\frac{\Delta w_\epsilon}{w_\epsilon} \right) - \frac{u_\epsilon}{\epsilon^2} + \epsilon R. \end{cases} \quad (2.4)$$

From well-known calculus inequalities in Sobolev spaces and the regularity of w , we have the following regularity result on (w_ϵ, u_ϵ) :

LEMMA 2.3. Assume that $f(w^2) \in C^8(0, \infty)$, $f'(w^2) > 0$, and $w \in C([0, T_*], H^7(\mathbb{T}^3)) \cap C^1([0, T_*], H^6(\mathbb{T}^3))$ has positive lower bound. Moreover, if $b(x) \in H^6(\mathbb{T}^3)$ and $b(x) > 0$, then $u_\epsilon \in C([0, T_*], H^6(\mathbb{T}^3)) \cap C^1([0, T_*], H^5(\mathbb{T}^3))$ and $R \in C([0, T_*], H^4(\mathbb{T}^3))$.

3. The proof of the a priori estimate. Thanks to Lemma 2.2, it suffices to prove the error estimates in (1.5) for $t \in [0, \min\{T_*, T_\epsilon\})$ in this section. To begin with, in order to avoid the dispersive third-order term, we reformulate (1.2)₁ of the quantum hydrodynamic equation as a fourth-order wave equation. For this, differentiating (1.2)₁ with respect to time t and replacing by (1.2)₂, we have

$$w_{tt} + \frac{1}{w} w_t^2 + \frac{1}{\epsilon^2} w_t - \frac{1}{2\epsilon^2 w} \Delta p(w^2) - \frac{1}{2\epsilon^2 w} \nabla^2 \cdot (w^2 u \otimes u) \\ + \frac{1}{2\epsilon^2 w} \operatorname{div} (w^2 \nabla \Delta^{-1} (w^2 - b(x))) + \frac{1}{4} \Delta^2 w - \frac{1}{4w} (\Delta w)^2 = 0. \quad (3.1)$$

Similarly, we can derive the wave equation satisfied by w_ϵ as the following:

$$w_{\epsilon tt} + \frac{1}{w_\epsilon} w_{\epsilon t}^2 + \frac{1}{\epsilon^2} w_{\epsilon t} - \frac{1}{2\epsilon^2 w_\epsilon} \Delta p(w_\epsilon^2) - \frac{1}{2\epsilon^2 w_\epsilon} \nabla^2 \cdot (w_\epsilon^2 u_\epsilon \otimes u_\epsilon) \\ + \frac{1}{2\epsilon^2 w_\epsilon} \operatorname{div} (w_\epsilon^2 \nabla \Delta^{-1} (w_\epsilon^2 - b(x))) + \frac{1}{4} \Delta^2 w_\epsilon - \frac{1}{4w_\epsilon} (\Delta w_\epsilon)^2 = -\frac{1}{2} \operatorname{div} R. \quad (3.2)$$

Further, setting

$$(\psi, \eta)^* = (w_\epsilon - w^\epsilon, u_\epsilon - u^\epsilon)^*,$$

from the equations in (1.2), (2.4), (3.1) and (3.2), it follows that the error (ψ, η) satisfies

$$\begin{aligned} & \psi_{tt} + \frac{1}{\epsilon^2} \psi_t + \frac{1}{4} \Delta^2 \psi + \frac{1}{w_\epsilon} w_{\epsilon t}^2 - \frac{1}{w^\epsilon} w_t^{\epsilon^2} - \frac{1}{2\epsilon^2 w_\epsilon} \Delta p(w_\epsilon^2) + \frac{1}{2\epsilon^2 w^\epsilon} \Delta p(w^{\epsilon^2}) \\ & + \frac{1}{2\epsilon^2 w_\epsilon} \operatorname{div}(w_\epsilon^2 \nabla \Delta^{-1}(w_\epsilon^2 - b(x))) - \frac{1}{2\epsilon^2 w^\epsilon} \operatorname{div}(w^{\epsilon^2} \nabla \Delta^{-1}(w^{\epsilon^2} - b(x))) \\ & - \frac{1}{2\epsilon^2 w_\epsilon} \nabla^2 \cdot (w_\epsilon^2 u_\epsilon \otimes u_\epsilon) + \frac{1}{2\epsilon^2 w^\epsilon} \nabla^2 \cdot (w^{\epsilon^2} u^\epsilon \otimes u^\epsilon) - \frac{1}{4w_\epsilon} (\Delta w_\epsilon)^2 \\ & + \frac{1}{4w^\epsilon} (\Delta w^\epsilon)^2 = -\frac{1}{2} \operatorname{div} R, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \eta_t + \frac{1}{\epsilon^2} \eta + \frac{1}{\epsilon} \nabla f(w_\epsilon^2) - \frac{1}{\epsilon} \nabla f(w^{\epsilon^2}) + \frac{1}{\epsilon} ((u_\epsilon \cdot \nabla) u_\epsilon - (u^\epsilon \cdot \nabla) u^\epsilon) \\ & = \frac{1}{\epsilon} \nabla \Delta^{-1}(w^{\epsilon^2} - b(x)) - \frac{\eta}{\epsilon^2} - \frac{\epsilon}{2} \nabla \left(\frac{\Delta w_\epsilon}{w_\epsilon} \right) + \frac{\epsilon}{2} \nabla \left(\frac{\Delta w^\epsilon}{w^\epsilon} \right) + \epsilon R. \end{aligned} \quad (3.4)$$

Moreover, we also need

$$\psi_t + \frac{1}{2\epsilon} w^\epsilon \operatorname{div} \eta + \frac{1}{2\epsilon} \psi \operatorname{div} u_\epsilon + \frac{1}{\epsilon} u^\epsilon \nabla \psi + \frac{1}{\epsilon} \eta \nabla w_\epsilon = 0. \quad (3.5)$$

For the sake of clarity, we divide the following arguments into lemmas.

LEMMA 3.1. Set

$$D = D(t) = \frac{\|\psi\|_{H^4} + \epsilon \|\partial_x^5 \psi\|_{H^1} + \|\eta\|_{H^5} + \epsilon \|\psi_t\|_{H^4}}{\epsilon}.$$

Then we have

$$\begin{aligned} & |u^\epsilon|, |u_x^\epsilon|, |u_{xx}^\epsilon|, |u_{xxx}^\epsilon| \leq C\epsilon + C\epsilon D, \\ & |w_t^\epsilon|, |w_{tx}^\epsilon|, |w_{ttx}^\epsilon|, |w_{xxx}^\epsilon|, |w_{xxxx}^\epsilon| \leq C + CD, \\ & |w_x^\epsilon|, |w_{xx}^\epsilon| \leq C + C\epsilon D. \end{aligned}$$

Proof. It is obvious from Sobolev's imbedding theorem and Lemma 2.3 that

$$|u^\epsilon| \leq |u^\epsilon - u_\epsilon| + |u_\epsilon| \leq C \|u^\epsilon - u_\epsilon\|_5 + \|u_\epsilon\|_5 \leq C\epsilon + C\epsilon D.$$

Similarly, we can prove other estimates.

LEMMA 3.2. Under the assumptions of Theorem 1.1, it follows that for $0 \leq \alpha \leq 4$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\partial_x^\alpha \eta)^2 dx + \frac{15}{16\epsilon^2} \int (\partial_x^\alpha \eta)^2 dx \leq \frac{\|\partial_x^\alpha \psi_t\|^2}{2} \\ & + C\epsilon^4 + C(1 + D^{10})(\|\psi\|_{\alpha+1}^2 + \|\eta\|_\alpha^2 + \epsilon^2 \|\partial_x^\alpha \Delta \psi\|^2). \end{aligned} \quad (3.6)$$

□

Proof. Multiplying (3.4) by η and integrating the resultant equation over Ω , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \eta^2 dx + \frac{1}{\epsilon^2} \int \eta^2 dx + \frac{1}{\epsilon} \int (\nabla(f(w_\epsilon^2) - f(w^{\epsilon^2})) + (u_\epsilon \cdot \nabla) u_\epsilon - (u^\epsilon \cdot \nabla) u^\epsilon) \eta dx \\ & = \int \left(\frac{1}{\epsilon} \nabla \Delta^{-1} \psi + \frac{1}{2\epsilon} \nabla \left(\frac{\Delta w_\epsilon}{w_\epsilon} - \frac{\Delta w^\epsilon}{w^\epsilon} \right) + \epsilon R \right) \eta dx. \end{aligned} \quad (3.7)$$

From the Cauchy-Schwarz and Young inequalities, one has

$$\begin{aligned}
& \frac{1}{2\epsilon} \int ((u_\epsilon \cdot \nabla)u_\epsilon - (u^\epsilon \cdot \nabla)u^\epsilon)\eta dx + \frac{1}{\epsilon} \int \nabla \Delta^{-1}(w_\epsilon + w^\epsilon)\psi \eta dx + \int \epsilon R \eta dx \\
&= -\frac{1}{2\epsilon} \int ((u_\epsilon \cdot \nabla)\eta - (\eta \cdot \nabla)u^\epsilon)\eta dx + \frac{1}{\epsilon} \int \nabla \Delta^{-1}(w_\epsilon + w^\epsilon)\psi \eta dx + \int \epsilon R \eta dx \\
&\leq \frac{\|\eta\|^2}{16\epsilon^2} + C\epsilon^4 + C(1+D)(\|\psi\|^2 + \|\eta\|^2 + \|\nabla \eta\|^2). \tag{3.8}
\end{aligned}$$

Moreover, noting that

$$f(w_\epsilon^2) - f(w^{\epsilon^2}) = (w_\epsilon^2 - w^{\epsilon^2}) \int_0^1 f'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))d\sigma,$$

one has

$$\begin{aligned}
\|f(w_\epsilon^2) - f(w^{\epsilon^2})\| &\leq C\|\psi\| \left\| \int_0^1 f'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))d\sigma \right\|_4 \\
&\leq C\|\psi\| \int_0^1 \|f'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))\|_4 d\sigma \\
&\leq C\|\psi\| \int_0^1 (1 + \|\sigma(w_\epsilon^2 - w^{\epsilon^2})\|_4) d\sigma \\
&\leq C(1 + D^5)\|\psi\|.
\end{aligned}$$

So, we can obtain

$$\frac{1}{\epsilon} \int \nabla(f(w_\epsilon^2) - f(w^{\epsilon^2}))\eta dx \leq \frac{\|\operatorname{div} \eta\|^2}{16\epsilon^2} + C(1 + D^{10})\|\psi\|^2. \tag{3.9}$$

Using integration by parts, we can compute

$$\begin{aligned}
\frac{\epsilon}{2} \int \nabla \left(\frac{\Delta w_\epsilon}{w_\epsilon} - \frac{\Delta w^\epsilon}{w^\epsilon} \right) \eta dx &= -\frac{\epsilon}{2} \int \left(\frac{w^\epsilon \Delta \psi - \psi \Delta w^\epsilon}{w_\epsilon w^\epsilon} \right) \operatorname{div} \eta dx \\
&\leq \frac{\|\operatorname{div} \eta\|^2}{16\epsilon^2} + C(1+D)(\|\psi\|^2 + \epsilon^2 \|\Delta \psi\|^2). \tag{3.10}
\end{aligned}$$

Insertion of the above inequalities into (3.7) yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \eta^2 dx + \frac{15}{16\epsilon^2} \int \eta^2 dx \leq \frac{\|\operatorname{div} \eta\|^2}{8\epsilon^2} + C\epsilon^4 \\
&+ C(1 + D^{10})(\|(\eta, \nabla \eta)\|^2 + \|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\Delta \psi\|^2). \tag{3.11}
\end{aligned}$$

Similarly, taking ∂_x^α ($4 \geq \alpha \geq 1$) and in a completely similar way, we can obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int (\partial_x^\alpha \eta)^2 dx + \frac{15}{16\epsilon^2} \int (\partial_x^\alpha \eta)^2 dx \leq \frac{\|\partial_x^\alpha \operatorname{div} \eta\|^2}{8\epsilon^2} + C\epsilon^4 \\
&+ C(1 + D^{10})(\|\partial_x^\alpha \Delta \psi\|^2 + \|(\psi, \eta)\|_{|\alpha|+1}^2). \tag{3.12}
\end{aligned}$$

Moreover, from (3.5), we can deduce that

$$\|\operatorname{div} \eta\|^2 \leq 4\epsilon^2 \|\psi_t\| + C\epsilon^2 \|\eta\|^2 + \epsilon^2(1 + D^2) \|\nabla \psi\|^2 + C\|\eta\|^2 \tag{3.13}$$

and

$$\|\partial_x^\alpha \operatorname{div} \eta\|^2 \leq 4\epsilon^2 \|\partial_x^\alpha \psi_t\| + C\epsilon^2 \|\eta\|_{|\alpha|}^2 + \epsilon^2(1 + D^2) \|\nabla \psi\|_{|\alpha|}^2 + C\|\eta\|_{|\alpha|}^2. \tag{3.14}$$

Therefore, combining (3.12), (3.13) and (3.14), we can establish (3.6). This completes the proofs. $\square$

LEMMA 3.3. Under the assumptions of Theorem 1.1, we have for $0 \leq \alpha \leq 4$,

$$\begin{aligned}
& \frac{d}{dt} \int \left(\frac{1}{2\epsilon^2} (\partial_x^\alpha \psi)^2 + \partial_x^\alpha \psi \partial_x^\alpha \psi_t + (\partial_x^\alpha \psi_t)^2 + \frac{1}{4} (\partial_x^\alpha \Delta \psi)^2 \right) dx + \frac{1}{4} \int (\partial_x^\alpha \Delta \psi)^2 dx \\
& + \frac{d}{dt} \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} \partial_x^\alpha (\nabla \psi)^2 dx \\
& + \left(\frac{27}{16\epsilon^2} - \frac{17}{16} \right) \int (\partial_x^\alpha \psi_t)^2 dx + \frac{1}{2\epsilon^2} \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{w_\epsilon} (\partial_x^\alpha \nabla \psi)^2 dx \\
& \leq C(1 + D^3)(\|\partial_x^\alpha \Delta \psi\|^2 + \|\partial_x^\alpha \psi_t\|^2) + \frac{C}{\epsilon^2} (\|\psi\|_{1+|\alpha|}^2 + \|\eta\|_{1+|\alpha|}^2) + C\epsilon^2. \tag{3.15}
\end{aligned}$$

Proof. Multiplying (3.3) by ψ and integrating the resultant equation over Ω , one has

$$\begin{aligned}
& \frac{d}{dt} \int \left(\psi \psi_t + \frac{1}{2\epsilon^2} \psi^2 \right) dx - \int \psi_t^2 dx + \frac{1}{4} \int (\Delta \psi)^2 dx + \int \left(\frac{1}{w_\epsilon} w_{\epsilon t}^2 - \frac{1}{w^\epsilon} w_t^{\epsilon^2} \right. \\
& - \frac{1}{2\epsilon^2 w_\epsilon} \Delta(p(w_\epsilon^2) - p(w^{\epsilon^2})) + \nabla^2 \cdot (w_\epsilon^2 u_\epsilon \otimes u_\epsilon - w^{\epsilon^2} u^\epsilon \otimes u^\epsilon) \\
& + \frac{1}{2\epsilon^2 w_\epsilon} \operatorname{div}(w_\epsilon^2 \nabla \Delta^{-1}(w_\epsilon^2 - \epsilon b(x))) - \frac{1}{2\epsilon^2 w^\epsilon} \operatorname{div}(w^{\epsilon^2} \nabla \Delta^{-1}(w^{\epsilon^2} - b(x))) \\
& \left. - \frac{1}{4w_\epsilon} (\Delta w_\epsilon)^2 + \frac{1}{4w^\epsilon} (\Delta w^\epsilon)^2 \right) \psi dx = -\frac{1}{2} \int \operatorname{div} R \cdot \psi dx. \tag{3.16}
\end{aligned}$$

For the integral terms in (3.16), from Cauchy-Schwarz's inequalities, we have

$$\begin{aligned}
& - \int \left(\frac{1}{w_\epsilon} w_{\epsilon t}^2 - \frac{1}{w^\epsilon} w_t^{\epsilon^2} \right) \psi dx - \frac{1}{2} \int \operatorname{div} R \cdot \psi dx \\
& = - \int \frac{-(w_{\epsilon t})^2 \psi + w_\epsilon (w_\epsilon + w^\epsilon)_t \psi_t}{w_\epsilon w^\epsilon} \psi dx - \frac{1}{2} \int \operatorname{div} R \cdot \psi dx \\
& \leq \frac{1}{16} \|\psi_t\|^2 + \frac{C(1+D)}{\epsilon^2} \|\psi\|^2 + C\epsilon^2
\end{aligned}$$

and

$$\begin{aligned}
& - \frac{1}{4} \int \left(\frac{(\Delta w_\epsilon)^2}{w_\epsilon} - \frac{(\Delta w^\epsilon)^2}{w^\epsilon} \right) \psi dx + \frac{1}{2\epsilon^2} \int \left(\frac{\operatorname{div}(w_\epsilon^2 \nabla \Delta^{-1}(w_\epsilon^2 - b(x)))}{w_\epsilon} \right. \\
& \left. - \frac{\operatorname{div}(w^{\epsilon^2} \nabla \Delta^{-1}(w^{\epsilon^2} - b(x)))}{w^\epsilon} \right) \psi dx \\
& = - \frac{1}{4\epsilon^2} \int \frac{-(\Delta w_\epsilon)^2 \psi + w_\epsilon (\Delta w_\epsilon + \Delta w^\epsilon) \Delta \psi}{w_\epsilon w^\epsilon} \psi dx \\
& + \frac{1}{2\epsilon^2} \int \frac{2w_\epsilon \nabla w_\epsilon \nabla \Delta^{-1}(w_\epsilon^2 - b(x)) + w_\epsilon^2 (w_\epsilon^2 - b(x))}{w_\epsilon} \\
& - \frac{2w^\epsilon \nabla w^\epsilon \nabla \Delta^{-1}(w^{\epsilon^2} - b(x)) + w^{\epsilon^2} (w_\epsilon^2 - b(x))}{w^\epsilon} \psi dx \\
& \leq \frac{C(1+D^2)}{\epsilon^2} (\|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\Delta \psi\|^2).
\end{aligned}$$

Similarly, integration by parts and Young's inequalities lead to

$$\begin{aligned}
& -\frac{1}{2\epsilon^2} \int \left(\frac{\nabla^2 \cdot (w_\epsilon^2 u_\epsilon \otimes u_\epsilon)}{w_\epsilon} - \frac{\nabla^2 \cdot (w^{\epsilon^2} u^\epsilon \otimes u^\epsilon)}{w^\epsilon} \right) \psi dx \\
&= -\frac{1}{2\epsilon^2} \int \left(\frac{\nabla^2(w_\epsilon^2)u_\epsilon^2 + 2\nabla(w_\epsilon^2)\nabla \cdot (u_\epsilon^2) + w_\epsilon^2 \nabla^2 \cdot (u_\epsilon^2)}{w_\epsilon} \right. \\
&\quad \left. - \frac{\nabla^2(w^{\epsilon^2})u^{\epsilon^2} + 2\nabla(w^{\epsilon^2})\nabla \cdot (u^{\epsilon^2}) + w^{\epsilon^2} \nabla^2 \cdot (u^{\epsilon^2})}{w^\epsilon} \right) \psi dx \\
&\leq \frac{C(1+D^3)}{\epsilon^2} (\|\psi\|^2 + \|\nabla\psi\|^2 + \epsilon^2 \|\Delta\psi\|^2 + \|\eta\|^2 + \|\nabla\eta\|^2).
\end{aligned}$$

Similar to (3.9), we have

$$\begin{aligned}
& -\frac{1}{2\epsilon^2} \int \left(\frac{\Delta p(w_\epsilon^2)}{w_\epsilon} - \frac{\Delta p(w^{\epsilon^2})}{w^\epsilon} \right) \psi dx \\
&= -\frac{1}{2\epsilon^2} \int \left(\frac{1}{w^\epsilon} \Delta(p(w_\epsilon^2) - p(w^{\epsilon^2})) + \Delta p(w_\epsilon^2) \left(\frac{1}{w_\epsilon} - \frac{1}{w^\epsilon} \right) \right) \psi dx \\
&= \frac{1}{2\epsilon^2} \int \nabla \left(\frac{\psi}{w^\epsilon} \right) \nabla (p(w_\epsilon^2) - p(w^{\epsilon^2})) dx + \frac{1}{2\epsilon^2} \int \frac{\Delta p(w_\epsilon^2)}{w_\epsilon w^\epsilon} \psi^2 dx \\
&\leq \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\nabla\psi)^2 dx + \frac{C(1+D^2)}{\epsilon^2} (\|\psi\|^2 + \|\nabla\psi\|^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \frac{d}{dt} \int (\psi\psi_t + \frac{1}{2\epsilon^2} \psi^2) dx - \frac{17}{16} \int \psi_t^2 dx + \frac{1}{4} \int (\Delta\psi)^2 dx \\
&+ \frac{1}{2\epsilon^2} \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{w_\epsilon} (\nabla\psi)^2 dx \\
&\leq \frac{C(1+D^{10})}{\epsilon^2} \|(\psi, \eta)\|_1^2 + C(1+D^3) \|\Delta\psi\|^2 + C\epsilon^2.
\end{aligned} \tag{3.17}$$

Next, multiplying (3.3) by $2\psi_t$ and integrating the resultant equation over Ω , one has

$$\begin{aligned}
& \frac{d}{dt} \int (\psi_t^2 + \frac{1}{4} (\Delta\psi)^2) dx + \frac{2}{\epsilon^2} \int \psi_t^2 dx + 2 \int \left(\frac{1}{w_\epsilon} w_{\epsilon t}^2 - \frac{1}{w^\epsilon} w_t^{\epsilon^2} - \frac{1}{2\epsilon^2 w_\epsilon} \Delta p(w_\epsilon^2) \right. \\
&+ \frac{1}{2\epsilon^2 w^\epsilon} \Delta p(w^{\epsilon^2}) - \frac{1}{2\epsilon^2 w_\epsilon} \nabla^2(w_\epsilon^2 u_\epsilon \otimes u_\epsilon) + \frac{1}{2\epsilon^2 w^\epsilon} \nabla^2(w^{\epsilon^2} u^\epsilon \otimes u^\epsilon) \\
&+ \frac{1}{2\epsilon^2 w_\epsilon} \operatorname{div}(w_\epsilon^2 \nabla \Delta^{-1}(w_\epsilon^2 - b(x))) - \frac{1}{2\epsilon^2 w^\epsilon} \operatorname{div}(w^{\epsilon^2} \nabla \Delta^{-1}(w^{\epsilon^2} - b(x))) \\
&- \left. \frac{1}{4w_\epsilon} (\Delta w_\epsilon)^2 + \frac{1}{4w^\epsilon} (\Delta w^\epsilon)^2 \right) \psi_t dx = - \int \operatorname{div} R \psi_t dx.
\end{aligned}$$

In completely the same way, we have

$$\begin{aligned}
& 2 \int \left(\frac{1}{w_\epsilon} w_{\epsilon t}^2 - \frac{1}{w^\epsilon} w_t^{\epsilon 2} \right) \psi_t dx + \int \operatorname{div} R \psi_t dx - \frac{1}{2} \int \left(\frac{(\Delta w_\epsilon)^2}{w_\epsilon} - \frac{(\Delta w^\epsilon)^2}{w^\epsilon} \right) \psi_t dx \\
& + \frac{1}{\epsilon^2} \int \left(\frac{\operatorname{div}(w_\epsilon^2 \Delta^{-1}(w_\epsilon^2 - b(x)))}{w_\epsilon} - \frac{\operatorname{div}(w^{\epsilon 2} \Delta^{-1}(w^{\epsilon 2} - b(x)))}{w^\epsilon} \right) \psi_t dx \\
& = \int \frac{w^\epsilon (w_\epsilon + w^\epsilon)_t \psi_t + (w_t^\epsilon)^2 \psi}{w_\epsilon w^\epsilon} \psi_t dx + \int \operatorname{div} R \psi_t dx - \frac{1}{2\epsilon^2} \int \left(\frac{\Delta w_\epsilon^2}{w_\epsilon} - \frac{\Delta w^{\epsilon 2}}{w^\epsilon} \right) \psi_t dx \\
& + \frac{1}{\epsilon^2} \int \left(\frac{\operatorname{div}(w_\epsilon^2 \Delta^{-1}(w_\epsilon^2 - b(x)))}{w_\epsilon} - \frac{\operatorname{div}(w^{\epsilon 2} \Delta^{-1}(w^{\epsilon 2} - b(x)))}{w^\epsilon} \right) \psi_t dx \\
& \leq C(1 + D^2) \|(\psi, \psi_t)\|^2 + \frac{1}{8\epsilon^2} \|\psi_t\|^2 + \frac{C(1 + D)}{\epsilon^2} (\|\psi\|^2 + \epsilon^2 \|\Delta \psi\|^2) + C\epsilon^2.
\end{aligned}$$

Moreover, with the aid of integration by parts, we have

$$\begin{aligned}
& -\frac{1}{\epsilon^2} \int \left(\frac{\Delta p(w_\epsilon^2)}{w_\epsilon} - \frac{\Delta p(w^{\epsilon 2})}{w^\epsilon} \right) \psi_t dx \\
& = \frac{1}{\epsilon^2} \int (\nabla \left(\frac{1}{w_\epsilon} \right) \nabla (p(w_\epsilon^2) - p(w^{\epsilon 2}))_x \psi_t + \frac{1}{w_\epsilon} p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon 2}))(w_\epsilon + w^\epsilon) ((\nabla \psi)^2)_t) dx \\
& \geq \frac{d}{dt} \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon 2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\nabla \psi)^2 dx - \frac{C(1 + D^{10})}{\epsilon^2} \|\psi\|_1^2 - \frac{1}{16\epsilon^2} \|\psi_t\|^2
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{\epsilon^2} \int \left(\frac{\nabla^2(w_\epsilon^2 u_\epsilon \otimes u_\epsilon)}{w_\epsilon} - \frac{\nabla^2(w^{\epsilon 2} u^\epsilon \otimes u^\epsilon)}{w^\epsilon} \right) \psi_t dx \\
& = -\frac{1}{\epsilon^2} \int \left(\frac{\nabla^2(w_\epsilon^2) u_\epsilon^2 + 2\nabla(w_\epsilon^2) \nabla(u_\epsilon^2) + w_\epsilon^2 \nabla^2(u_\epsilon^2)}{w_\epsilon} \right. \\
& \quad \left. - \frac{\nabla^2(w^{\epsilon 2}) u^{\epsilon 2} + 2\nabla(w^{\epsilon 2}) \nabla(u^{\epsilon 2}) + w^{\epsilon 2} \nabla^2(u^{\epsilon 2})}{w^\epsilon} \right) \psi_t dx \\
& \leq -\frac{2}{\epsilon^2} \int \frac{w_\epsilon^2 w^\epsilon u_\epsilon \nabla^2 u_\epsilon - w^{\epsilon 2} w^\epsilon u^\epsilon \nabla^2 u^\epsilon}{w_\epsilon w^\epsilon} \psi_t dx + \frac{1}{16\epsilon^2} \|\psi_t\|^2 \\
& \quad + \frac{C(1 + D^3)}{\epsilon^2} (\|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\nabla^2 \psi\|^2 + \|\eta\|^2 + \|\nabla \eta\|^2) \\
& \leq -\frac{2}{\epsilon^2} \int \frac{-u_\epsilon \nabla^2 u_\epsilon w_\epsilon^2 \psi + w_\epsilon^2 w^\epsilon \nabla^2 u_\epsilon \eta + w_\epsilon \nabla^2 u_\epsilon u^\epsilon (w_\epsilon^2 - w^{\epsilon 2}) + u^\epsilon w_\epsilon w^{\epsilon 2} \nabla^2 \eta}{w_\epsilon w^\epsilon} \psi_t dx \\
& \quad + \frac{1}{16\epsilon^2} \|\psi_t\|^2 + \frac{C(1 + D^3)}{\epsilon^2} (\|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\nabla^2 \psi\|^2 + \|\eta\|^2 + \|\nabla \eta\|^2) \\
& \leq \frac{4}{\epsilon} \int u^\epsilon w^\epsilon \operatorname{div} \left(\frac{1}{w^\epsilon} (\psi_t + \frac{1}{2\epsilon} \eta u_{\epsilon x} + \frac{1}{\epsilon} u^\epsilon \nabla \psi + \frac{1}{\epsilon} \eta \nabla w_\epsilon) \right) \psi_t dx \\
& \quad + \frac{1}{16\epsilon^2} \|\psi_t\|^2 + \frac{C(1 + D^3)}{\epsilon^2} (\|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\nabla^2 \psi\|^2 + \|\eta\|^2 + \|\nabla \eta\|^2) \\
& \leq C(1 + D^3) \|\psi_t\|^2 + \frac{1}{8\epsilon^2} \|\psi_t\|^2 \\
& \quad + \frac{C(1 + D^3)}{\epsilon^2} (\|\psi\|^2 + \|\nabla \psi\|^2 + \epsilon^2 \|\nabla^2 \psi\|^2 + \|\eta\|^2 + \|\nabla \eta\|^2).
\end{aligned}$$

Further, one has

$$\begin{aligned} & \frac{d}{dt} \int (\psi_t^2 + \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\nabla \psi)^2 + \frac{1}{4} (\Delta \psi)^2) dx + \frac{27}{16\epsilon^2} \int \psi_t^2 dx \\ & \leq C(1 + D^3) \|(\psi, \psi_t)\|^2 + \frac{C(1 + D^{10})}{\epsilon^2} (\epsilon^2 \|\Delta \psi\|^2 + \|(\psi, \eta)\|_1^2) + C\epsilon^2. \end{aligned} \quad (3.18)$$

Combining (3.17) and (3.18) leads to

$$\begin{aligned} & \frac{d}{dt} \int (\frac{1}{2\epsilon^2} \psi^2 + \psi \psi_t + \psi_t^2 + \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\nabla \psi)^2 + \frac{1}{4} (\Delta \psi)^2) dx \\ & + (\frac{27}{16\epsilon^2} - \frac{17}{16}) \int \psi_t^2 dx + \int (\frac{1}{4} (\Delta \psi)^2 + \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\nabla \psi)^2) dx \\ & \leq C(1 + D^3) \|(\psi, \psi_t)\|^2 + \frac{C(1 + D^{10})}{\epsilon^2} (\epsilon^2 \|\Delta \psi\|^2 + \|(\psi, \eta)\|_1^2) + C\epsilon^2. \end{aligned} \quad (3.19)$$

In order to obtain a higher-order estimate, we differentiate with respect to x ; therefore by repeating the previous steps, we have

$$\begin{aligned} & \frac{d}{dt} \int (\frac{1}{2\epsilon^2} (\partial_x^\alpha \psi)^2 + \partial_x^\alpha \psi \partial_x^\alpha \psi_t + (\partial_x^\alpha \psi_t)^2 + \frac{1}{4\epsilon^2} (\partial_x^\alpha \Delta \psi)^2) dx \\ & + \frac{d}{dt} \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\partial_x^\alpha \nabla \psi)^2 dx \\ & + (\frac{27}{16\epsilon^2} - \frac{17}{16}) \int (\partial_x^\alpha \psi_t)^2 dx + \int \frac{p'(w_\epsilon^2 + \sigma(w_\epsilon^2 - w^{\epsilon^2}))(w_\epsilon + w^\epsilon)}{2\epsilon^2 w_\epsilon} (\partial_x^\alpha \nabla \psi)^2 dx \\ & \leq C(1 + D^3) \|(\partial_x^\alpha \psi, \partial_x^\alpha \psi_t)\|^2 + \frac{C(1 + D^{10})}{\epsilon^2} (\epsilon^2 \|\partial_x^\alpha \Delta \psi\|^2 + \|(\psi, \eta)\|_{|\alpha|+1}^2) + C\epsilon^2. \end{aligned} \quad (3.20)$$

This completes the proofs of Lemma 3.3. $\square$

Combining (3.6) and (3.20) yields

$$\begin{aligned} & \frac{d}{dt} (\|(\partial_x^\alpha \psi, \partial_x^\alpha \eta, \epsilon \partial_x^\alpha \Delta \psi)\|^2 + \epsilon^2 \|\partial_x^\alpha \psi_t\|^2) + \frac{1}{\epsilon^2} \|(\epsilon \partial_x^\alpha \Delta \psi, \epsilon \partial_x^\alpha \psi_t, \partial_x^\alpha \eta)\|^2 \\ & \leq C\epsilon^4 + C(1 + D^{10}) (\|(\psi, \eta)\|_{|\alpha|+1}^2 + \epsilon^2 \|\partial_x^\alpha \psi_t\|^2 + \epsilon \|\partial_x^\alpha \Delta \psi\|^2). \end{aligned} \quad (3.21)$$

Then we integrate (3.21) from 0 to T with $[0, T] \subset [0, \min\{T_\epsilon, T_*\})$ to obtain

$$\begin{aligned} & (\|(\partial_x^\alpha \psi, \partial_x^\alpha \eta, \epsilon \partial_x^\alpha \Delta \psi)\|^2 + \epsilon^2 \|\partial_x^\alpha \psi_t\|^2) + \int_0^T \frac{1}{\epsilon^2} \|(\epsilon \partial_x^\alpha \Delta \psi, \epsilon \partial_x^\alpha \psi_t, \partial_x^\alpha \eta)\|^2 dt \\ & \leq CT\epsilon^4 + C \int_0^T (1 + D^{10}) (\|(\psi, \eta)\|_{|\alpha|+1}^2 + \epsilon^2 \|\partial_x^\alpha \psi_t\|^2 + \epsilon^2 \|\partial_x^\alpha \Delta \psi\|^2) dt. \end{aligned}$$

Here we have used the fact that the initial data are in equilibrium. Summing up the last inequality over all α satisfying $|\alpha| \leq 2$, and noting that

$$\|\partial_x^5 \psi\| \leq C(\|\partial_x^4 \psi\| + \|\partial_x^6 \psi\|),$$

and using (3.14), we get

$$\begin{aligned} & \|\psi\|_4^2 + \epsilon^2 \|\partial_x^5 \psi\|_1^2 + \epsilon^2 \|\psi_t\|_4^2 + \|\eta\|_5^2 \\ & + \frac{1}{\epsilon^2} \int_0^T (\|\psi\|_4^2 + \epsilon^2 \|\partial_x^5 \psi\|_1^2 + \epsilon^2 \|\psi_t\|_4^2 + \|\eta\|_5^2) dt \\ & \leq CT_* \epsilon^4 + C \int_0^T (1 + D^{10}) (\|\psi\|_4^2 + \epsilon^2 \|\partial_x^5 \psi\|_1^2 + \epsilon^2 \|\psi_t\|_4^2 + \|\eta\|_5^2) dt. \end{aligned} \quad (3.22)$$

We apply Gronwall's lemma to (3.22) to get

$$\|\psi\|_4^2 + \epsilon^2 \|\partial_x^5 \psi\|_1^2 + \|\eta\|_5^2 + \epsilon^2 \|\psi_t\|_4^2 \leq CT_* \epsilon^4 \exp[C \int_0^T (1 + D^{10}) dt]. \quad (3.23)$$

From the definition of $D(t)$ and (3.23), it follows that

$$D(T)^2 \leq CT_* \epsilon^2 \exp[C \int_0^T (1 + D^{10}) dt] \equiv \Phi(T), \quad (3.24)$$

which implies

$$\Phi'(t) = C(1 + D^{10})\Phi(t) \leq C\Phi(t) + C\Phi^6(t). \quad (3.25)$$

Applying the nonlinear Gronwall-type inequality in [25] to (3.25) yields

$$\Phi(t) \leq e^{CT_*} \quad (3.26)$$

for $t \in [0, \min\{T_*, T_\epsilon\})$ if we choose ϵ so small that

$$\Phi(0) = CT_* \epsilon^2 \leq e^{-CT_*}.$$

Further, it follows from (3.24) and (3.26) that there exists a constant c , independent of ϵ , such that

$$D(T) \leq c \quad (3.27)$$

for $T \in [0, \min\{T_*, T_\epsilon\})$. Finally, the theorem is concluded from the definition of $D(t)$, (3.23) and (3.27). This completes the proof of Theorem 1.1.

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