ASYMPTOTIC STABILITY FOR THREE-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH TIME-VARYING COEFFICIENTS

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Abstract. This paper is concerned with the asymptotic stability of the zero solution of three-dimensional linear differential systems with variable coefficients. The coefficients are not assumed to be positive. A concept innovated by László Hatvani plays a vital role in our results. Sufficient conditions are also given for the zero solution to be uniformly stable. Some suitable examples are included to illustrate our results. Finally, certain changes of variable are used to broaden the application of our results.

1. Introduction. In this paper, we present sufficient conditions for the zero solution of time-varying three-dimensional linear differential systems to be asymptotically stable.

As for time-invariant linear differential systems, the study of stability can date back to the nineteenth century. It goes without saying that the steam engine invented by J. Watt played a leading role in the Industrial Revolution. The steam engine continued being improved by the development of the Industrial Revolution. The velocity of a machine is measured and regulated by a centrifugal speed regulator called a governor. A governor plays the role which keeps the operation speed of a steam engine constant. In the process of the design change of the steam engine, most governors stopped acting well and the operation speed started changing continuously. Destabilization of the operation speed is called the hunting phenomenon, and it was a great blow to the industrial world. Many scientists and engineers did their best to escape from this social predicament. J. C. Maxwell, who is famous for the Maxwell equations, ascertained this cause first. He described the structure of governors in 1868 by a third-order linear differential equation

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(see [8]) and succeeded in the analysis of the hunting phenomenon of governors. Maxwell's result was generalized by E. J. Routh who was a classmate of Maxwell and a stability criterion for the n -th order linear differential equation was given (see [12]). About 20 years later, without knowing Routh's method, A. Hurwitz gave another stability criterion expressed simply by determinants (see [6]). It was shown later that Routh's method and Hurwitz's method are equivalent although the forms are different from each other. At present, their methods are called the Routh-Hurwitz criterion. This stability criterion is still used as an effective basic theorem in control engineering.

Applying the Routh-Hurwitz stability criterion to the time-invariant linear differential equation

$$x''' + ax'' + bx' + cx = 0, \quad (1.1)$$

we can easily see that $a > 0$ and $ab > c > 0$ are necessary and sufficient conditions for the equilibrium $(x, x', x'') = (0, 0, 0)$ of (1.1) to be asymptotically stable (for the definition, see Section 2). Judging from this fact, it might be inferred that the equilibrium of the equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0 \quad (1.2)$$

is asymptotically stable if

$$a(t) > 0 \quad \text{and} \quad a(t)b(t) > c(t) > 0 \quad \text{for } t \geq 0. \quad (1.3)$$

However, this assertion is not true. In fact, consider the equation

$$x''' + \frac{3}{1+t}x'' + (2+t)x' + \frac{1}{1+t}x = 0. \quad (1.4)$$

Then, assumption (1.3) is satisfied with $a(t) = 3/(1+t)$, $b(t) = 2+t$ and $c(t) = 1/(1+t)$, but equation (1.4) has a solution $x(t) = \mu(2+t)/(1+t)$, where μ is an arbitrary number. Since the solution does not tend to zero unless $\mu = 0$, the equilibrium of (1.4) is not asymptotically stable.

There are a number of transformations into a system equivalent to (1.1). For example, let $y = x'$ and $z = x''$ be new variables. Then equation (1.1) becomes the system

$$x' = y, \quad y' = z, \quad z' = -cx - by - az, \quad (1.5)$$

which is called the companion form. Let $y = x'$ and $z = x'' + cx/a$. Then equation (1.1) becomes the system

$$x' = y, \quad y' = -\frac{c}{a}z, \quad z' = -\left(b - \frac{c}{a}\right)y - az. \quad (1.6)$$

System (1.6) is usually called Schwarz's form, which is convenient for verifying the Routh-Hurwitz stability criterion. The equilibrium of (1.1) corresponds to the zero solution $(x(t), y(t), z(t)) = (0, 0, 0)$ of (1.5) or (1.6).

Systems (1.5) and (1.6) naturally have constant coefficients. As shown by the fact that assumption (1.3) is insufficient for the equilibrium of (1.2) to be asymptotically stable, if we change from constant coefficients to time-varying coefficients, then it is very hard to analyze the asymptotic stability of the zero solution of a system of the companion form or Schwarz's form.

To avoid this difficulty, we would like to propose a new form. We will direct our attention to an equivalent system of the form

$$x' = \sqrt{\frac{c}{a}}y, \quad y' = -\sqrt{\frac{c}{a}}x, \quad z' = -\left(b - \frac{c}{a}\right)y - az. \quad (1.7)$$

In equation (1.2), we assume that $a(t)$ and $c(t)$ are continuously differentiable, and $b(t)$ is continuous for $t \geq 0$. Let

$$e^2(t) = \exp(-2A(t)/3) \left\{ d + \frac{2}{3} \int_0^t c(s) \exp(2A(s)/3) ds \right\},$$

where $A(t) = \int_0^t a(s)ds$ and d is an arbitrary positive number. Then, letting

$$y = \frac{1}{e(t)}x' \quad \text{and} \quad z = \frac{1}{e(t)}x'' - \frac{e'(t)}{e^2(t)}x' + e(t)x,$$

we have the following system equivalent to (1.2):

$$x' = e(t)y, \quad y' = -e(t)x + f(t)z, \quad z' = -g(t)y - h(t)z, \quad (1.8)$$

where $f(t) = 1$,

$$g(t) = b(t) - e^2(t) - \frac{1}{3}a'(t) - \frac{2}{9}a^2(t) + \frac{a(t)c(t)}{3e^2(t)} + \frac{c'(t)}{3e^2(t)} - \frac{c^2(t)}{9e^4(t)}$$

and

$$h(t) = \frac{1}{3}a(t) + \frac{2c(t)}{3e^2(t)}$$

for $t \geq 0$ (for details, see Section 4). System (1.8) has the same form as (1.7). In the case that $a(t) = a > 0$, $b(t) = b > 0$ and $c(t) = c > 0$ for $t \geq 0$, system (1.8) coincides with (1.7) (we may choose $c/a > 0$ as d).

As for system (1.8), what kind of condition on time-varying coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ will guarantee that the zero solution is asymptotically stable? To assure the asymptotic stability of the equilibrium of (1.2), in addition to assumption (1.3), what kind of condition will be required? The purpose of this paper is to answer these questions.

In Section 2, we state our main result on the asymptotic stability of the zero solution of (1.8). To this end, we make several assumptions on the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$. Also, we give sufficient conditions for the zero solution of (1.8) to be uniformly stable. In Section 3, we prove our main result. For illustration of our main theorem, we take some concrete examples and those figures of a positive orbit in Section 4. Finally, we present some corollaries.

2. Assumptions. Consider a system of differential equations of the form

$$x' = e(t)y, \quad y' = -e(t)x + f(t)z, \quad z' = -g(t)y - h(t)z, \quad (E)$$

where the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are continuous and $g(t)/f(t)$ is differentiable for $t \geq 0$.

Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ and $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, and let $\|\cdot\|$ be the Euclidean norm. We denote the solution of (E) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is clear that system (E) has the zero solution $\mathbf{x}(t) \equiv \mathbf{0}$.

The zero solution is said to be *stable*, if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly stable* if it is stable and δ can be chosen to be independent of t_0 . The zero solution is said to be *attractive* if for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \rightarrow 0$ as $t \rightarrow \infty$. For homogeneous linear differential systems such as (E), if the zero solution is attractive, then it is stable. The zero solution of (E) is said to be *asymptotically stable* if it is attractive. The asymptotic stability and the attractivity are completely different concepts in general systems.

We also assume that $f(t)$ and $g(t)$ are bounded for $t \geq 0$ and that

(i) $f(t)g(t) > 0$ for $t \geq 0$ and $\liminf_{t \rightarrow \infty} f(t)g(t) > 0$.

Then there exist positive numbers k and K such that

$$k \leq \frac{f(x)}{g(x)} \leq K \quad \text{for } t \geq 0. \quad (2.1)$$

In fact, since $f(t)$ and $g(t)$ are bounded for $t \geq 0$, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$|f(t)| \leq c_1 \quad \text{and} \quad |g(t)| \leq c_2 \quad (2.2)$$

for $t \geq 0$. Because of assumption (i), there exists a $c_3 > 0$ with

$$f(t)g(t) \geq c_3 \quad \text{for } t \geq 0. \quad (2.3)$$

Hence, it is clear that

$$\frac{f(t)}{g(t)} = \frac{f(t)g(t)}{g^2(t)} \geq \frac{c_3}{c_2^2} \quad \text{for } t \geq 0.$$

We can find a number $c_4 > 0$ satisfying

$$|g(t)| \geq c_4 \quad \text{for } t \geq 0.$$

If the assertion is false, then there exists a sequence $\{t_n\}$ tending to ∞ such that

$$|g(t_n)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It then follows from (2.3) that

$$|f(t_n)| \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

This contradicts (2.2). We therefore conclude that

$$\frac{f(t)}{g(t)} = \frac{|f(t)|}{|g(t)|} \leq \frac{c_1}{c_4} \quad \text{for } t \geq 0.$$

Let $k = c_3/c_2^2$ and $K = c_1/c_4$. Then we obtain the inequality (2.1).

Since $g(t)/f(t)$ is differentiable for $t \geq 0$, we may define

$$\psi(t) = 2h(t) + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)} \right)'.$$

For the sake of simplicity, we denote

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\}.$$

Then we have the following result.

LEMMA 2.1. In addition to assumption (i), if

$$(ii) \int_0^\infty \psi_-(s) ds < \infty,$$

then the zero solution of (E) is uniformly stable.

Proof. Let $\mathbf{x} = (x, y, z)$ and define

$$V(t, \mathbf{x}) = \frac{1}{2} \left(x^2 + y^2 + \frac{f(t)}{g(t)} z^2 \right) \quad \text{and} \quad W(t, \mathbf{x}) = V(t, \mathbf{x}) \exp(-\Psi(t))$$

on $[0, \infty) \times \mathbb{R}^3$, where

$$\Psi(t) = \int_0^t \psi_-(s) ds.$$

Because of assumption (ii), there exists an $L > 0$ such that $\Psi(t) < L$ for $t \geq 0$. Hence, together with the inequality (2.1), we have

$$\begin{aligned} \frac{1}{2} (x^2 + y^2 + kz^2) e^{-L} &\leq V(t, \mathbf{x}) e^{-L} \leq W(t, \mathbf{x}) \\ &\leq V(t, \mathbf{x}) \leq \frac{1}{2} (x^2 + y^2 + Kz^2). \end{aligned}$$

Differentiate $V(t, \mathbf{x})$ along any solution of (E) obtaining

$$\dot{V}_{(E)}(t, \mathbf{x}) = -\frac{1}{2} \psi(t) \frac{f(t)}{g(t)} z^2 \leq \frac{1}{2} \psi_-(t) \frac{f(t)}{g(t)} z^2 \leq \psi_-(t) V(t, \mathbf{x}).$$

Hence, we get

$$\dot{W}_{(E)}(t, \mathbf{x}) = \left\{ \dot{V}_{(E)}(t, \mathbf{x}) - \psi_-(t) V(t, \mathbf{x}) \right\} \exp(-\Psi(t)) \leq 0$$

for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^3$. Thus, $W(t, \mathbf{x})$ is positive definite and decrescent, and $\dot{W}_{(E)}(t, \mathbf{x})$ is nonpositive. We therefore conclude that the zero solution of (E) is uniformly stable by using a Liapunov-type theorem due to Persidskiĭ [10] (as to the direct method of Liapunov, for example, see the books [1, 2, 7, 9, 11, 14]). $\square$

For the sake of brevity, let $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and $v(t) = V(t, \mathbf{x}(t))$. Then, as shown in the proof of Lemma 2.1, we have

$$v'(t) \leq \psi_-(t) v(t) \quad \text{for } t \geq t_0. \quad (2.4)$$

The inequality (2.4) guarantees the convergence of $v(t)$.

LEMMA 2.2. Under the assumption (ii), if (2.4) holds, then $v'(t)$ is absolutely integrable, and therefore, $v(t)$ has a nonnegative limiting value.

Proof. By (2.4), we have

$$v(t) \leq v(t_0) \exp(\Psi(t) - \Psi(t_0)) \leq v(t_0) e^L$$

for $t \geq t_0$, where L is the number given in the proof of Lemma 2.1. Hence, using (2.4) again, we obtain

$$v'(t) \leq v(t_0) e^L \psi_-(t) \quad \text{for } t \geq t_0.$$

Since the right-hand side of this inequality is nonnegative for $t \geq t_0$, we get

$$v'_+(t) \leq v(t_0) e^L \psi_-(t).$$

Integrate both sides from t_0 to ∞ to obtain

$$\int_{t_0}^{\infty} v'_+(s) ds \leq v(t_0) L e^L < \infty.$$

On the other hand, since $v(t) \geq 0$ for $t \geq t_0$, we have

$$\int_{t_0}^{\infty} v'_-(s) ds \leq v(t_0) + \int_{t_0}^{\infty} v'_+(s) ds < \infty.$$

We therefore conclude that

$$\int_{t_0}^{\infty} |v'(s)| ds = \int_{t_0}^{\infty} (v'_+(s) + v'_-(s)) ds < \infty,$$

as required. This completes the proof of Lemma 2.2. $\square$

We have already assumed the continuity of $e(t)$ in the opening paragraph of this section. Let us make an additional assumption on $e(t)$. To this end, we define the union of intervals as follows: for any triplex (α, β, γ) with $0 < \alpha \leq \beta$ and $\gamma > 0$, let

$$I_t(\alpha, \beta, \gamma) \stackrel{\text{def}}{=} \bigcup \{[s, s + \alpha] \subset [t, t + \beta] : |e(\tau)| \geq \gamma \text{ for } \tau \in [s, s + \alpha]\}.$$

To take some examples, if $e(t) = 1$ for $t \geq 0$, then $I_t(\alpha, \beta, \gamma) \equiv [t, t + \beta]$ for any (α, β, γ) with $0 < \alpha \leq \beta$ and $0 < \gamma \leq 1$; if $e(t) = \sin t$ for $t \geq 0$, then

$$I_t(2\pi/3, 2\pi, 1/2) = \begin{cases} \left[\frac{1}{6}\pi, \frac{5}{6}\pi\right] \cup \left[\frac{7}{6}\pi, \frac{11}{6}\pi\right] & \text{if } 0 \leq t \leq \frac{1}{6}\pi, \\ \left[\left(n + \frac{1}{6}\right)\pi, \left(n + \frac{5}{6}\right)\pi\right] \cup \left[\left(n + \frac{7}{6}\right)\pi, \left(n + \frac{11}{6}\right)\pi\right] & \text{if } \left(n - \frac{1}{6}\right)\pi \leq t \leq \left(n + \frac{1}{6}\right)\pi, \\ \left[\left(n + \frac{1}{6}\right)\pi, \left(n + \frac{5}{6}\right)\pi\right] & \text{if } \left(n - \frac{5}{6}\right)\pi < t < \left(n - \frac{1}{6}\right)\pi. \end{cases}$$

Similarly, if $e(t)$ satisfies that $\liminf_{t \rightarrow \infty} |e(t)| > 0$ or if $e(t)$ is a nontrivial periodic function, then $I_t(\alpha, \beta, \gamma)$ is not empty for all $t \geq 0$. Needless to say, there are cases where $I_t(\alpha, \beta, \gamma)$ is an empty set for any (α, β, γ) . For example, consider the case in which $e(t) = \sin(t^2)$ for $t \geq 0$. Since $e(t)$ is over-oscillation, we cannot choose a suitable constant $\alpha > 0$. Hence, $I_t(\alpha, \beta, \gamma)$ is empty for t sufficiently large. To show another case that $I_t(\alpha, \beta, \gamma)$ is empty, we define a sequence $\{d_n\}$ by $d_n = n(n-1)/2$ for $n \in \mathbb{N}$ and consider the C^1 -function

$$e(t) = \begin{cases} \sin^2 t & \text{for } t \in [d_n\pi, (d_n + 1)\pi], \\ 0 & \text{for } t \in [(d_n + 1)\pi, d_{n+1}\pi]. \end{cases} \quad (2.5)$$

In this case, the part where $e(t) > 0$ is the sum of the open intervals $(d_n\pi, (d_n + 1)\pi)$. The distance between each interval and the next expands as n increases. Hence, for any $\alpha > 0$ and $\gamma > 0$, we cannot find β with $\alpha \leq \beta$ such that $I_t(\alpha, \beta, \gamma)$ is nonempty.

We here assume that

- (iii) there exists a triplex (α, β, γ) with $0 < \alpha \leq \beta$ and $\gamma > 0$ such that $I_t(\alpha, \beta, \gamma)$ is nonempty for t sufficiently large.

Assumption (iii) is satisfied in the case in which $\liminf_{t \rightarrow \infty} |e(t)| > 0$ or $e(t)$ is a nontrivial periodic function, including an almost periodic function, but it is not satisfied in the case in which $e(t) = \sin(t^2)$ or $e(t)$ is given by (2.5).

REMARK 2.1. If $e(t)$ satisfies assumption (iii) for a triplex $(\alpha_0, \beta_0, \gamma_0)$, then it also satisfies assumption (iii) for (α, β, γ) with $0 < \alpha < \alpha_0$, $\beta > \beta_0$ and $0 < \gamma < \gamma_0$.

To state our main result, we have to introduce a concept that plays an important role in this paper. A nonnegative function ϕ is said to be *integrally positive* if

$$\int_J \phi(s) ds = \infty$$

for every set $J = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n < \sigma_n < \tau_{n+1}$ and $\sigma_n - \tau_n \geq \lambda > 0$. If, in addition, the set J satisfies $\tau_{n+1} \leq \sigma_n + \Lambda$ for some $\Lambda > 0$, then we say that the function ϕ is *weakly integrally positive*. We can cite $1/(1+t)$ or $\sin^2 t/(1+t)$ as a function that is weakly integrally positive, but not integrally positive (refer to [3, 4, 5, 13]). If ϕ is (weakly) integrally positive, then it naturally satisfies

$$\lim_{t \rightarrow \infty} \int^t \phi(s) ds = \infty.$$

REMARK 2.2. Suppose that ϕ is weakly integrally positive. If

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} \phi(s) ds < \infty$$

for a number $\Lambda > 0$ and two sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $\tau_n < \sigma_n < \tau_{n+1} \leq \sigma_n + \Lambda$, then $\liminf_{n \rightarrow \infty} (\sigma_n - \tau_n) = 0$.

We are now in a position to present the main result. We will give its proof in the next section.

THEOREM 2.3. Let $e(t)$, $f(t)$, $g(t)$ and $h(t)$ be bounded for $t \geq 0$. Suppose that assumptions (i)–(iii) are satisfied. If

- (iv) $\psi_+(t)$ is weakly integrally positive,

then the zero solution of (E) is asymptotically stable.

3. Proof of the main result. Before proving Theorem 2.3, it is helpful to transform system (E) into an equivalent system. In cylindrical coordinates $(x, y, z) \rightarrow (r, \theta, z)$ by $x = r \cos \theta$ and $y = r \sin \theta$, system (E) takes the form

$$r' = f(t)z \sin \theta, \quad \theta' = \frac{f(t)z}{r} \cos \theta - e(t), \quad z' = -g(t)r \sin \theta - h(t)z. \quad (\tilde{E})$$

Let $(r(t), \theta(t), z(t))$ be the solution of $(\tilde{E})$ corresponding to $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$.

Proof of Theorem 2.3. By means of Lemma 2.1, the zero solution of (E) is uniformly stable, that is, given any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for } t \geq t_0. \quad (3.1)$$

In order to prove that the zero solution of (E) is asymptotically stable, it is enough to show that it is attractive, namely, $\mathbf{x}(t)$ converges to $\mathbf{0}$ as t increases.

For convenience of notation, set

$$u(t) = \frac{f(t)}{2g(t)} z^2(t).$$

Then, by (2.1) we have

$$v(t) = V(t, \mathbf{x}(t)) = \frac{1}{2}(x^2(t) + y^2(t)) + u(t) \geq \frac{1}{2}(x^2(t) + y^2(t) + kz^2(t)) \quad (3.2)$$

for $t \geq t_0$. Recall that $v(t)$ satisfies

$$v'(t) = -\psi(t)u(t) \leq \psi_-(t)v(t) \quad (3.3)$$

for $t \geq t_0$ (see the calculation of $\dot{V}_{(E)}(t, \mathbf{x})$ in the proof of Lemma 2.1). Hence, from Lemma 2.2 we see that $v(t)$ has a limiting value $v_0 \geq 0$. If $v_0 = 0$, then by (3.2), the solution $\mathbf{x}(t)$ tends to $\mathbf{0}$ as $t \rightarrow \infty$. This completes the proof. Hereafter, we consider only the case in which $v_0 > 0$. We will demonstrate that this case does not occur.

From (2.1) and (3.1), we see that $u(t)$ is bounded. Hence, $u(t)$ has the inferior limit and the superior limit. First, we will show that the inferior limit of $u(t)$ is zero, and we will then show that the superior limit of $u(t)$ is also zero.

Suppose that $\liminf_{t \rightarrow \infty} u(t) > 0$. Then there exist an $\varepsilon_1 > 0$ and a $T_1 \geq t_0$ such that $u(t) > \varepsilon_1$ for $t \geq T_1$. From (3.3) and Lemma 2.2 it follows that

$$\infty > \int_{t_0}^{\infty} |v'(s)| ds = \int_{t_0}^{\infty} |\psi(s)| u(s) ds \geq \int_{t_0}^{\infty} \psi_+(s) u(s) ds > \varepsilon_1 \int_{T_1}^{\infty} \psi_+(s) ds.$$

This contradicts assumption (iii). Thus, we see that $\liminf_{t \rightarrow \infty} u(t) = 0$.

By way of a contradiction, we suppose that $\nu \stackrel{\text{def}}{=} \limsup_{t \rightarrow \infty} u(t) > 0$. Since $e(t)$, $f(t)$ and $h(t)$ are bounded for $t \geq 0$, there exist numbers $\bar{e} > 0$, $\bar{f} > 0$ and $\bar{h} > 0$ such that

$$|e(t)| \leq \bar{e}, \quad |f(t)| \leq \bar{f} \quad \text{and} \quad |h(t)| \leq \bar{h} \quad (3.4)$$

for $t \geq 0$. As mentioned in Section 2, we can choose a $\underline{g} > 0$ such that

$$|g(t)| \geq \underline{g} \quad \text{for } t \geq 0. \quad (3.5)$$

Since $v(t)$ tends to a positive value v_0 as $t \rightarrow \infty$, there exists a $T_2 \geq t_0$ such that

$$0 < \frac{v_0}{2} < v(t) < \frac{3v_0}{2} \quad \text{for } t \geq T_2. \quad (3.6)$$

Let $\varepsilon_2 > 0$ be so small that $\varepsilon_2 < \nu/2$ and

$$\frac{\sqrt{2\varepsilon_2/k}}{\sqrt{v_0 - 2\varepsilon_2}} < \min \left\{ \frac{\gamma}{6\bar{f}}, \frac{\alpha\gamma\underline{g}\sin(\alpha\gamma/4)}{2(\gamma + 6\bar{e}) + \alpha\gamma\bar{h}} \right\}, \quad (3.7)$$

where α and γ are the numbers given in assumption (iii). Since $\liminf_{t \rightarrow \infty} u(t) = 0 < \nu = \limsup_{t \rightarrow \infty} u(t)$, we can find two divergent sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < \sigma_n < \tau_{n+1}$ such that

$$\begin{aligned} u(t) &\geq \varepsilon_2 \quad \text{for } \tau_n < t < \sigma_n, \\ 0 \leq u(t) &\leq \varepsilon_2 \quad \text{for } \sigma_n < t < \tau_{n+1} \end{aligned} \quad (3.8)$$

and $u(\tau_n) = u(\sigma_n) = \varepsilon_2$. Since $u(t) \geq kz^2(t)/2$ for $t \geq t_0$, we have

$$|z(t)| \leq \sqrt{\frac{2}{k}u(t)} < \sqrt{\frac{2\varepsilon_2}{k}} \quad \text{for } \sigma_n \leq t \leq \tau_{n+1}. \quad (3.9)$$

Using (3.1), (3.2), (3.6) and (3.8), we also obtain

$$\sqrt{v_0 - 2\varepsilon_2} < \sqrt{2(v(t) - u(t))} = \sqrt{x^2(t) + y^2(t)} < \varepsilon$$

for $\sigma_n \leq t \leq \tau_{n+1}$, namely,

$$\sqrt{v_0 - 2\varepsilon_2} < r(t) < \varepsilon \quad \text{for } \sigma_n \leq t \leq \tau_{n+1}. \quad (3.10)$$

From (3.9) and (3.10), we see that the solution $(r(t), \theta(t), z(t))$ of $(\tilde{E})$ stays in the thin disc

$$D = \left\{ (r, \theta, z) : \sqrt{v_0 - 2\varepsilon_2} < r < \varepsilon, -\pi < \theta \leq \pi \text{ and } |z| \leq \sqrt{2\varepsilon_2/k} \right\}$$

for $\sigma_n \leq t \leq \tau_{n+1}$.

We will show that $\tau_{n+1} - \sigma_n \leq \beta$ for $n \in \mathbb{N}$ sufficiently large, where β is the number given in assumption (iii). If the assertion is not true, then there exists a sufficiently large $n_0 \in \mathbb{N}$ with $\tau_{n_0+1} - \sigma_{n_0} > \beta$. Because of assumption (iii), there exist two numbers ω_1 and ω_2 with $\sigma_{n_0} \leq \omega_1 < \omega_2 \leq \sigma_{n_0} + \beta$ and $\omega_2 - \omega_1 = \alpha$ such that

$$|e(t)| \geq \gamma > 0 \quad \text{for } \omega_1 \leq t \leq \omega_2.$$

Taking Remark 2.1 into account, we may regard the product $\alpha\gamma$ as being small, say, $0 < \alpha\gamma < 2\pi$. Since $e(t)$ is continuous for $t \geq 0$, there are two cases to consider: (a) $e(t) \geq \gamma$ for $t \in [\omega_1, \omega_2]$ and (b) $e(t) \leq -\gamma$ for $t \in [\omega_1, \omega_2]$. We consider only the former because the latter can be carried out in the same manner. By (3.4), (3.9) and (3.10), we have

$$\begin{aligned} -\frac{\bar{f}\sqrt{2\varepsilon_2/k}}{\sqrt{v_0 - 2\varepsilon_2}} - \bar{e} &< -\frac{|f(t)||z(t)|}{r(t)} - |e(t)| \\ &\leq \theta'(t) \leq \frac{|f(t)||z(t)|}{r(t)} - e(t) < \frac{\bar{f}\sqrt{2\varepsilon_2/k}}{\sqrt{v_0 - 2\varepsilon_2}} - \gamma \end{aligned}$$

for $\omega_1 \leq t \leq \omega_2$. From (3.7), we see that

$$-\frac{1}{6}\gamma - \bar{e} < \theta'(t) < -\frac{5}{6}\gamma \quad \text{for } \omega_1 \leq t \leq \omega_2,$$

and therefore, we obtain

$$\frac{5}{6}\alpha\gamma < \theta(\omega_1) - \theta(\omega_2) < \left(\frac{1}{6}\gamma + \bar{e}\right)\alpha. \quad (3.11)$$

We now define a pair of regions Ω_+ and Ω_- by

$$\begin{aligned} \Omega_+ &= \{(r, \theta) : \sqrt{v_0 - 2\varepsilon_2} < r < \varepsilon \text{ and } \alpha\gamma/4 < \theta < \pi - \alpha\gamma/4\} \\ \text{and} \quad \Omega_- &= \{(r, \theta) : \sqrt{v_0 - 2\varepsilon_2} < r < \varepsilon \text{ and } -\pi + \alpha\gamma/4 < \theta < -\alpha\gamma/4\}. \end{aligned}$$

The regions Ω_+ and Ω_- are nonempty because $0 < \alpha\gamma < 2\pi$. Since the union of Ω_+ and Ω_- is symmetric with respect to the horizontal line $\theta = 0$ and the vertical line $r = 0$, we

may regard $\theta(\omega_1)$ as being in the first quadrant. Consider the movement of $(r(t), \theta(t))$. As already shown, $(r(t), \theta(t))$ stays in the annulus

$$A = \{(r, \theta) : \sqrt{v_0 - 2\varepsilon_2} < r < \varepsilon \text{ and } -\pi < \theta \leq \pi\}$$

for $\omega_1 \leq t \leq \omega_2$ and it moves clockwise. If $\theta(\omega_1) \in \Omega_+$, then there are three possibilities:

- $(r(t), \theta(t))$ remains in Ω_+ for $\omega_1 \leq t \leq \omega_2$;
- $(r(t), \theta(t))$ leaves Ω_+ and enters into the region $A \setminus (\Omega_+ \cup \Omega_-)$; in other words, there exists an ω_3 with $\omega_1 < \omega_3 \leq \omega_2$ such that $(r(t), \theta(t)) \in \Omega_+$ for $\omega_1 \leq t < \omega_3$ and $(r(t), \theta(t)) \in A \setminus (\Omega_+ \cup \Omega_-)$ for $\omega_3 \leq t < \omega_2$;
- $(r(t), \theta(t))$ passes through the region $A \setminus (\Omega_+ \cup \Omega_-)$ and reaches Ω_- ; in other words, there exist two numbers ω_3 and ω_4 with $\omega_1 < \omega_3 < \omega_4 < \omega_2$ such that $(r(t), \theta(t)) \in \Omega_+$ for $\omega_1 \leq t < \omega_3$, $(r(t), \theta(t)) \in A \setminus (\Omega_+ \cup \Omega_-)$ for $\omega_3 \leq t \leq \omega_4$ and $(r(t), \theta(t)) \in \Omega_-$ for $\omega_4 < t \leq \omega_2$.

If $\theta(\omega_1) \notin \Omega_+$, then $(r(t), \theta(t))$ passes through the region $A \setminus (\Omega_+ \cup \Omega_-)$ and enters into Ω_- . In this case, it follows from (3.11) that $(r(t), \theta(t))$ does not remain in the region $A \setminus (\Omega_+ \cup \Omega_-)$ for $\omega_1 \leq t \leq \omega_2$. Thus, we can choose an ω_4 with $\omega_1 < \omega_4 < \omega_2$ such that $(r(t), \theta(t)) \in A \setminus (\Omega_+ \cup \Omega_-)$ for $\omega_1 \leq t \leq \omega_4$ and $(r(t), \theta(t)) \in \Omega_-$ for $\omega_4 < t \leq \omega_2$. To sum up, in all cases, we can find two numbers ω_3 and ω_4 with $\omega_1 \leq \omega_3 \leq \omega_4 \leq \omega_2$ such that $\theta(\omega_3) - \theta(\omega_4) \leq \alpha\gamma/2$ and

$$(r(t), \theta(t)) \in \begin{cases} \Omega_+ & \text{for } \omega_1 < t < \omega_3, \\ \Omega_- & \text{for } \omega_4 < t < \omega_2 \end{cases}$$

(in the first case, we should consider that $\omega_3 = \omega_4 = \omega_2$; in the second case, we should consider that $\omega_4 = \omega_2$; in the fourth case, we should consider that $\omega_3 = \omega_1$). Of course, if $\omega_1 = \omega_3$ (resp., $\omega_4 = \omega_2$), then we regard the interval (ω_1, ω_3) (resp., (ω_4, ω_2)) as an empty set. Using (3.11) again, we have

$$\begin{aligned} \frac{5}{6}\alpha\gamma &< \theta(\omega_1) - \theta(\omega_2) \\ &= (\theta(\omega_1) - \theta(\omega_3)) + (\theta(\omega_3) - \theta(\omega_4)) + (\theta(\omega_4) - \theta(\omega_2)) \\ &\leq (\theta(\omega_1) - \theta(\omega_3)) + \frac{1}{2}\alpha\gamma + (\theta(\omega_4) - \theta(\omega_2)). \end{aligned}$$

Hence, we conclude that

$$\max\{\theta(\omega_1) - \theta(\omega_3), \theta(\omega_4) - \theta(\omega_2)\} > \frac{1}{6}\alpha\gamma.$$

We may assume without loss of generality that $\theta(\omega_1) - \theta(\omega_3) \geq \theta(\omega_4) - \theta(\omega_2)$. Then it is clear that $\alpha\gamma/4 \leq \theta(\omega_3) \leq \theta(t) \leq \theta(\omega_1) \leq \pi/2$ for $\omega_3 \leq t \leq \omega_1$. Hence, it follows that

$$|\sin \theta(t)| = \sin |\theta(t)| \geq \sin \frac{1}{4}\alpha\gamma \quad \text{for } \omega_1 \leq t \leq \omega_3. \quad (3.12)$$

We next estimate the difference between ω_1 and ω_3 . Since $\theta'(t) > -\gamma/6 - \bar{\varepsilon}$, we have

$$\theta(\omega_1) - \theta(\omega_3) < \left(\frac{1}{6}\gamma + \bar{\varepsilon}\right)(\omega_3 - \omega_1).$$

Hence, we obtain

$$\omega_3 - \omega_1 > \frac{\theta(\omega_1) - \theta(\omega_3)}{\gamma/6 + \bar{e}} > \frac{\alpha\gamma}{\gamma + 6\bar{e}}. \quad (3.13)$$

Using (3.4), (3.5), (3.7), (3.9), (3.10), (3.12) and the third equation of $(\tilde{E})$, we get

$$\begin{aligned} |z'(t)| &\geq |g(t)||r(t)|\sin\theta(t) - |h(t)||z(t)| \\ &> \underline{g}\sqrt{v_0 - 2\varepsilon_2} \sin\frac{1}{4}\alpha\gamma - \bar{h}\sqrt{\frac{2\varepsilon_2}{k}} > 0 \end{aligned}$$

for $\omega_1 \leq t \leq \omega_3$. Since $z'(t)$ is continuous for $t \geq t_0$, we see that

$$\left| \int_{\omega_1}^{\omega_3} z'(s) ds \right| = \int_{\omega_1}^{\omega_3} |z'(s)| ds.$$

Hence, by (3.7), (3.9) and (3.13), we have

$$\begin{aligned} 2\sqrt{\frac{2\varepsilon_2}{k}} &\geq |z(\omega_1)| + |z(\omega_3)| \geq \int_{\omega_1}^{\omega_3} |z'(s)| ds \\ &> \left(\underline{g}\sqrt{v_0 - 2\varepsilon_2} \sin\frac{1}{4}\alpha\gamma - \bar{h}\sqrt{\frac{2\varepsilon_2}{k}} \right) (\omega_3 - \omega_1) \\ &> \frac{2(\gamma + 6\bar{e})}{\alpha\gamma} \sqrt{\frac{2\varepsilon_2}{k}} (\omega_3 - \omega_1) > 2\sqrt{\frac{2\varepsilon_2}{k}}, \end{aligned}$$

which is a contradiction. Thus, we have verified that $\tau_{n+1} - \sigma_n \leq b$ for $n \in \mathbb{N}$ sufficiently large.

Let $J = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. From Lemma 2.2 and (3.8), we see that

$$\infty > \int_{t_0}^{\infty} |v'(s)| ds \geq \int_{t_0}^{\infty} \psi_+(s) u(s) ds > \varepsilon_2 \int_J \psi_+(s) ds.$$

Hence, taking into account Remark 2.2 with $\Lambda = \beta$, we have

$$\liminf_{t \rightarrow \infty} (\sigma_n - \tau_n) = \infty. \quad (3.14)$$

Since $\liminf_{t \rightarrow \infty} u(t) = 0$ and $\limsup_{t \rightarrow \infty} u(t) = \nu > 0$, we can choose two sequences $\{t_n\}$ and $\{s_n\}$ with $T_2 < t_n < s_n < t_{n+1}$ such that $u(t_n) = \nu/2$, $u(s_n) = 3\nu/4$ and

$$\frac{\nu}{2} < u(t) < \frac{3\nu}{4} \quad \text{for } t_n < t < s_n.$$

Since $\varepsilon_2 < \nu/2$, we may consider that $[t_n, s_n] \subset [\tau_n, \sigma_n]$ for $n \in \mathbb{N}$ (if necessary, we can change $\{\tau_n\}$ and $\{\sigma_n\}$ into suitable subsequences of $\{\tau_n\}$ and $\{\sigma_n\}$). Hence, by (3.14), we have

$$\liminf_{n \rightarrow \infty} (s_n - t_n) = 0. \quad (3.15)$$

From (3.1), (3.2) and (3.4), we see that

$$u'(t) = v'(t) - x(t)x'(t) - y(t)y'(t) \leq |v'(t)| + |f(t)||y(t)||z(t)| \leq |v'(t)| + \bar{f}\varepsilon^2$$

for $t \geq t_0$. Integrating this inequality from t_n to s_n , we get

$$\frac{\nu}{4} = |u(s_n)| - |u(t_n)| \leq \int_{t_n}^{s_n} |v'(s)| ds + \bar{f}\varepsilon^2 (s_n - t_n)$$

for each $n \in \mathbb{N}$. By (3.15), the right-hand side of the above inequality tends to 0 as $n \rightarrow \infty$. This is a contradiction. We therefore conclude that $\limsup_{t \rightarrow \infty} u(t) = 0$.

As proven above, $u(t)$ tends to zero as $t \rightarrow \infty$. Hence, there exists a $T_3 \geq t_0$ such that

$$0 \leq u(t) < \varepsilon_2 \quad \text{for } t \geq T_3.$$

Using this estimation instead of (3.8) and repeating the same process once more, we see that the solution $(r(t), \theta(t), z(t))$ of $(\tilde{E})$ stays in the disc D for $t \geq T_3$, that is, $(r(t), \theta(t))$ remains in the annulus A for $t \geq T_3$. For this reason, we can choose two numbers ω_5 and ω_6 with $T_3 < \omega_5 < \omega_6$ such that $(r(t), \theta(t)) \in \Omega_+$ for $\omega_5 < t < \omega_6$. We can estimate that

$$\begin{aligned} \omega_6 - \omega_5 &> \frac{\theta(\omega_5) - \theta(\omega_6)}{\gamma/6 + \bar{e}} > \frac{\alpha\gamma}{\gamma + 6\bar{e}}, \\ |z'(t)| &> \underline{g}\sqrt{v_0 - 2\varepsilon_2} \sin \frac{1}{4}\alpha\gamma - \bar{h}\sqrt{\frac{2\varepsilon_2}{k}} > 0 \end{aligned}$$

for $\omega_5 < t < \omega_6$. Using these estimations, we have

$$2\sqrt{\frac{2\varepsilon_2}{k}} \geq \int_{\omega_5}^{\omega_6} |z'(s)| ds > \left(\underline{g}\sqrt{v_0 - 2\varepsilon_2} \sin \frac{1}{4}\alpha\gamma - \bar{h}\sqrt{\frac{2\varepsilon_2}{k}} \right) (\omega_6 - \omega_5) > 2\sqrt{\frac{2\varepsilon_2}{k}},$$

which is a contradiction. Thus, the case of $v_0 > 0$ does not happen. We therefore conclude that the zero solution of (E) is asymptotically stable.

The proof of Theorem 2.1 is thus complete. $\square$

4. Discussion and simulation. As a special case of (E) , we consider the linear differential system with constant coefficients,

$$x = ey, \quad y' = -ex + fz, \quad z' = -gy - hz. \quad (4.1)$$

System (4.1) is equivalent to the third-order linear differential equation

$$x''' + hx'' + (e^2 + fg)x' + e^2hx = 0.$$

By using the Routh-Hurwitz criterion, it can be shown that the zero solution of (4.1) is asymptotically stable if and only if the determinants $\Delta_1 = h$,

$$\Delta_2 = \det \begin{pmatrix} h & e^2h \\ 1 & e^2 + fg \end{pmatrix} = fgh \quad \text{and} \quad \Delta_3 = \det \begin{pmatrix} h & e^2h & 0 \\ 1 & e^2 + fg & 0 \\ 0 & h & e^2h \end{pmatrix} = e^2fgh^2$$

are positive, namely, $e \neq 0$, $fg > 0$ and $h > 0$. We will confirm that assumptions (i)–(iv) in Theorem 2.1 are satisfied. If f and g have like signs, then assumption (i) is clearly satisfied. Since $\psi(t) = 2h$ for system (4.1), $\psi_-(t) = 0$ and $\psi_+(t) = 2h$. Hence, assumptions (ii) and (iv) are satisfied for $h > 0$. If $e \neq 0$, then $I_t(\alpha, \beta, \gamma)$ exists for any $0 < \alpha < \beta$ and $0 < \gamma \leq |e|$. Hence, assumption (iii) is also satisfied. Thus, by Theorem 2.1, we conclude that if $e \neq 0$, $fg > 0$ and $h > 0$, then the zero solution of (4.1) is asymptotically stable. For the reason above, it is safe to say that our main theorem is considerably sharp.

To illustrate our main theorem, we give a simple example.

FIG. 1. A positive orbit of (E) with (4.2)

EXAMPLE 4.1. Consider system (E) with

$$e(t) = \sin 2t + \cos 3t \quad \text{and} \quad f(t) = g(t) = h(t) = 1. \quad (4.2)$$

Then the zero solution of (E) is asymptotically stable.

Assumptions (i)–(iii) are clearly satisfied. Since $e(t)$ is periodic, assumption (iv) also holds. Hence, by Theorem 2.1, we see that the zero solution of (E) is asymptotically stable. In Figure 1, we draw a positive orbit of (E) with (4.1) and its projections onto the (x, y) -plane, the (y, z) -plane and the (x, z) -plane. The initial time $t_0 = 0$ and the starting point $(x_0, y_0, z_0) = (0.8, 0.8, 1)$. The positive orbit approaches the origin $(0, 0, 0)$ by a regular but somewhat complicated motion.

The form of system (E) is by no means restrictive. We will show that the main result can be applied to the third-order linear differential equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \quad (4.3)$$

where $a(t)$ and $c(t)$ are continuously differentiable, and $b(t)$ is continuous for $t \geq 0$. If $a(t)$, $b(t)$ and $c(t)$ are the constants a , b and c , respectively, then $a > 0$ and $ab > c > 0$ are necessary and sufficient constants for the equilibrium $(x, x', x'') = (0, 0, 0)$ to be asymptotically stable. Hence, it may safely be assumed that

$$a(t) > 0 \quad \text{and} \quad a(t)b(t) > c(t) > 0 \quad \text{for } t \geq 0.$$

Denote $A(t) = \int_0^t a(s)ds$ and let

$$e^2(t) = \exp(-2A(t)/3) \left\{ d + \frac{2}{3} \int_0^t c(s) \exp(2A(s)/3) ds \right\}, \quad (4.4)$$

where d is an arbitrary positive number. Then $e^2(0) = d$ and $e^2(t)$ is a positive C^2 -function for $t \geq 0$. Hence, $e(t) \neq 0$ for $t \geq 0$. As new variables, let

$$y = \frac{1}{e(t)}x' \quad \text{and} \quad z = \frac{1}{e(t)}x'' - \frac{e'(t)}{e^2(t)}x' + e(t)x.$$

Then it is clear that $x' = e(t)y$ and $y' = -e(t)x + f(t)z$, where $f(t) \equiv 1$. From (4.4), we see that

$$\frac{e'(t)}{e(t)} = -\frac{1}{3}a(t) + \frac{1}{3}\frac{c(t)}{e^2(t)}.$$

Hence, we have

$$z = \frac{1}{e(t)}x'' + \frac{1}{3}\left(\frac{a(t)}{e(t)} - \frac{c(t)}{e^3(t)}\right)x' + e(t)x.$$

By (4.3) and a straightforward computation, we obtain

$$\begin{aligned} z' &= \frac{1}{e(t)}x''' - \frac{e'(t)}{e^2(t)}x'' + \frac{1}{3}\left(\frac{a(t)}{e(t)} - \frac{c(t)}{e^3(t)}\right)x'' \\ &\quad + \frac{1}{3}\left(\frac{a'(t)}{e(t)} - \frac{a(t)e'(t)}{e^2(t)} - \frac{c'(t)}{e^3(t)} + \frac{3c(t)e'(t)}{e^4(t)}\right)x' + e(t)x' + e'(t)x \\ &= -\frac{a(t)}{e(t)}x'' - \frac{b(t)}{e(t)}x' - \frac{c(t)}{e(t)}x + \frac{2}{3}\left(\frac{a(t)}{e(t)} - \frac{c(t)}{e^3(t)}\right)x'' \\ &\quad + \frac{1}{9}\left(\frac{3a'(t)}{e(t)} + \frac{a^2(t)}{e(t)} - \frac{4a(t)c(t)}{e^3(t)} - \frac{3c'(t)}{e^3(t)} + \frac{3c^2(t)}{e^5(t)}\right)x' + e(t)x' + e'(t)x \\ &= -\frac{1}{3}\left(\frac{a(t)}{e(t)} + \frac{2c(t)}{e^3(t)}\right)x'' - \frac{b(t)}{e(t)}x' \\ &\quad + \frac{1}{9}\left(\frac{3a'(t)}{e(t)} + \frac{a^2(t)}{e(t)} - \frac{4a(t)c(t)}{e^3(t)} - \frac{3c'(t)}{e^3(t)} + \frac{3c^2(t)}{e^5(t)}\right)x' + e(t)x' \\ &\quad - \frac{1}{3}\left(a(t)e(t) + \frac{2c(t)}{e(t)}\right)x \\ &= -\left(b(t) - e^2(t) - \frac{1}{3}a'(t) - \frac{2}{9}a^2(t) + \frac{a(t)c(t)}{3e^2(t)} + \frac{c'(t)}{3e^2(t)} - \frac{c^2(t)}{9e^4(t)}\right)\frac{1}{e(t)}x' \\ &\quad - \frac{1}{3}\left(\frac{a(t)}{e(t)} + \frac{2c(t)}{e^3(t)}\right)x'' - \frac{1}{9}\left(\frac{a^2(t)}{e(t)} + \frac{a(t)c(t)}{e^3(t)} - \frac{2c^2(t)}{e^5(t)}\right)x' \\ &\quad - \frac{1}{3}\left(a(t)e(t) + \frac{2c(t)}{e(t)}\right)x \end{aligned}$$

$$\begin{aligned}
&= -\left(b(t) - e^2(t) - \frac{1}{3}a'(t) - \frac{2}{9}a^2(t) + \frac{a(t)c(t)}{3e^2(t)} + \frac{c'(t)}{3e^2(t)} - \frac{c^2(t)}{9e^4(t)}\right)\frac{1}{e(t)}x' \\
&\quad - \frac{1}{3}\left(a(t) + \frac{2c(t)}{e^2(t)}\right)\left(\frac{1}{e(t)}x'' + \frac{1}{3}\left(\frac{a(t)}{e(t)} - \frac{c(t)}{e^3(t)}\right)x' + e(t)x\right) \\
&= -\left(b(t) - e^2(t) - \frac{1}{3}a'(t) - \frac{2}{9}a^2(t) + \frac{a(t)c(t)}{3e^2(t)} + \frac{c'(t)}{3e^2(t)} - \frac{c^2(t)}{9e^4(t)}\right)y \\
&\quad - \frac{1}{3}\left(a(t) + \frac{2c(t)}{e^2(t)}\right)z \\
&= -g(t)y - h(t)z,
\end{aligned}$$

where

$$g(t) = b(t) - e^2(t) - \frac{1}{3}a'(t) - \frac{2}{9}a^2(t) + \frac{a(t)c(t)}{3e^2(t)} + \frac{c'(t)}{3e^2(t)} - \frac{c^2(t)}{9e^4(t)} \quad (4.5)$$

and

$$h(t) = \frac{1}{3}a(t) + \frac{2c(t)}{3e^2(t)}. \quad (4.6)$$

Thus, equation (4.3) is transformed into a system of type (E).

As a simple example, let us consider the case in which $a(t) = a$, $b(t) = b$ and $c(t) = c$ for $t \geq 0$, where $a > 0$ and $ab > c > 0$. Then, $A(t) = at$ and

$$e^2(t) = \exp(-2at/3) \left\{ d + \frac{c}{a}(\exp(2at/3) - 1) \right\}$$

for $t \geq 0$. Since d is arbitrary and positive, we may take $c/a > 0$ as d . Hence, it follows that $e(t) = \pm\sqrt{c/a}$. We also have

$$g(t) = b - \frac{c}{a} - \frac{2}{9}a^2 + \frac{1}{3}a^2 - \frac{1}{9}a^2 = b - \frac{c}{a} > 0$$

and

$$h(t) = \frac{1}{3}a + \frac{2}{3}a = a > 0$$

for $t \geq 0$. Since $f(t) = 1$, we get

$$\psi(t) = 2h(t) + \frac{g'(t)}{g(t)} = 2a > 0$$

for $t \geq 0$. Hence, assumptions (i)–(iv) in Theorem 2.1 are satisfied.

If $a(t)$ and $b(t)$ are twice differentiable, and $b(t)$ is differentiable for $t \geq 0$, then $g(t)$ is also differentiable for $t \geq 0$, and hence, $\psi(t)$ is well defined. Thus, we have the following result.

COROLLARY 4.1. Let $f(t) = 1$ for $t \geq 0$ and let $e(t)$, $g(t)$ and $h(t)$ be given by (4.4), (4.5) and (4.6), respectively. Suppose that assumptions (i)–(iv) are satisfied. Then the equilibrium $(x, x', x'') = (0, 0, 0)$ of (4.3) is asymptotically stable.

In our main theorem, Theorem 2.1, we assumed the boundedness of all coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ of (E). If

$$|e(t)| > 0 \quad \text{for } t \geq 0, \quad (4.7)$$

then we can relax the boundedness. In order to explain this, we define

$$\tau = E(t) = \int_0^t |e(s)| ds$$

and assume that

$$E(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (4.8)$$

From (4.7) and (4.8) it turns out that $\tau \rightarrow \infty$ if and only if $t \rightarrow \infty$. Also, we have

$$\frac{dx}{d\tau} = \frac{e(t)}{|e(t)|} y, \quad \frac{dy}{d\tau} = -\frac{e(t)}{|e(t)|} x + \frac{f(t)}{|e(t)|} z, \quad \frac{dz}{d\tau} = -\frac{g(t)}{|e(t)|} y - \frac{h(t)}{|e(t)|} z, \quad (4.9)$$

where $t = E^{-1}(\tau)$. Let $\tilde{e}(t) = e(t)/|e(t)|$, $\tilde{f}(t) = f(t)/|e(t)|$, $\tilde{g}(t) = g(t)/|e(t)|$ and $\tilde{h}(t) = h(t)/|e(t)|$ for $t \geq 0$. Then we see that system (4.9) has the form of (E). Note that $\tilde{e}(t) = 1$ or $\tilde{e}(t) = -1$ for $t \geq 0$. Since $g(t)/f(t)$ is differentiable for $t \geq 0$, then we have

$$\tilde{\psi}(t) \stackrel{\text{def}}{=} 2\tilde{h}(t) + \frac{\tilde{f}(t)}{\tilde{g}(t)} \left(\frac{\tilde{g}(t)}{\tilde{f}(t)} \right)' = \frac{2h(t)}{|e(t)|} + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)} \right)'$$

for $t \geq 0$. Assumption (iii) is always satisfied, because $\tilde{e}(t) = 1$ or $\tilde{e}(t) = -1$ for $t \geq 0$. Thus, we have the following results.

THEOREM 4.2. Let (4.7) and (4.8) hold. Suppose that $\tilde{f}(t)$, $\tilde{g}(t)$ and $\tilde{h}(t)$ are bounded for $t \geq 0$. If

$$(v) \quad \tilde{f}(t)\tilde{g}(t) > 0 \text{ for } t \geq 0 \text{ and } \liminf_{t \rightarrow \infty} \tilde{f}(t)\tilde{g}(t) > 0,$$

$$(vi) \quad \int_0^\infty \tilde{\psi}_-(s) ds < \infty,$$

$$(vii) \quad \tilde{\psi}_+(t) \text{ is weakly integrally positive,}$$

then the zero solution of (E) is asymptotically stable.

COROLLARY 4.3. Let (4.7) and (4.8) hold and let $e(t)$, $g(t)$ and $h(t)$ be given by (4.4), (4.5) and (4.6), respectively. Suppose that $1/|e(t)|$, $g(t)/|e(t)|$ and $h(t)/|e(t)|$ are bounded for $t \geq 0$. If assumptions (v)–(vii) are satisfied, then the equilibrium $(x, x', x'') = (0, 0, 0)$ of (4.3) is asymptotically stable.

EXAMPLE 4.2. Consider system (E) with

$$e(t) = f(t) = g(t) = 1 + t \quad \text{and} \quad h(t) = \sin^2 t. \quad (4.10)$$

Then the zero solution of (E) is asymptotically stable.

Since $e(t) > 1$ for $t \geq 0$ and $E(t)$ tends to ∞ as $t \rightarrow \infty$, conditions (4.7) and (4.8) hold. In this example, $\tilde{f}(t) = \tilde{g}(t) = 1$ and $\tilde{h}(t) = \sin^2 t/(1+t)$ for $t \geq 0$. Hence, these are bounded for $t \geq 0$ and assumption (v) is satisfied. Since

$$\tilde{\psi}(t) = \frac{2h(t)}{|e(t)|} + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)} \right)' = \frac{2\sin^2 t}{1+t} > 0$$

for $t \geq 0$, we see that $\tilde{\psi}_+(t) = 2\sin^2 t/(1+t)$ and $\tilde{\psi}_-(t) = 0$ for $t \geq 0$, and therefore, assumptions (vi) and (vii) are also satisfied. Thus, by Theorem 4.2, we conclude that the zero solution of (E) is asymptotically stable. However, since $e(t)$, $f(t)$ and $g(t)$ are unbounded, Theorem 2.1 is of no use, although assumptions (i)–(iv) are satisfied.

Figure 2 below indicates the positive orbit starting from the point $(0.8, 0.8, 1)$ at the initial time $t_0 = 0$ and its projections onto the (x, y) -plane, the (y, z) -plane and the (x, z) -plane. The positive orbit rotates about the origin infinitely many times and approaches the origin.

We take another example to illustrate Theorem 4.2.

EXAMPLE 4.3. Consider system (E) with

$$e(t) = f(t) = g(t) = \frac{1}{1+t} \quad \text{and} \quad h(t) = \frac{1}{(1+t)^2}.$$

Then the zero solution of (E) is asymptotically stable.

Since $e(t) = 1/(1+t)$ and $E(t) = \log(1+t)$ for $t \geq 0$, conditions (4.7) and (4.8) hold. It is clear that $\tilde{f}(t) = \tilde{g}(t) = 1$, $\tilde{h}(t) = 1/(1+t)$ and $\tilde{\psi}(t) = 2/(1+t)$ for $t \geq 0$. Hence, we can easily confirm that $\tilde{f}(t)$, $\tilde{g}(t)$ and $\tilde{h}(t)$ are bounded for $t \geq 0$ and assumptions (v)–(vii) are satisfied. Thus, by means of Theorem 4.2, we see that the zero solution of (E) is asymptotically stable.

In Example 4.3, $f(t)g(t) = 1/(1+t)^2$ and $\psi_+(t) = 2/(1+t)^2$ for $t \geq 0$. Hence, $f(t)g(t)$ tends to zero as $t \rightarrow \infty$ and $\psi_+(t)$ is not weakly integrally positive. Also, $e(t)$ tends to zero as $t \rightarrow \infty$, and there exists no triplex (α, β, γ) with $0 < \alpha \leq \beta$ and $\gamma > 0$ such that $I_t(\alpha, \beta, \gamma)$ is nonempty for t sufficiently large. Thus, assumptions (i), (iii) and (iv) are not satisfied, and therefore, Theorem 2.1 is inapplicable to this example (we can verify assumption (ii) and the boundedness of $e(t)$, $f(t)$, $g(t)$ and $h(t)$).

FIG. 2. A positive orbit of (E) with (4.10)

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