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THE MATHEMATICS OF THE THOMSON EFFECT

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Abstract. The paper deals with the elliptic system of the thermistor problem in 3 dimensions taking into account the Thomson effect. Existence and uniqueness results are presented. The proofs are based on a reduction to a two-point problem for an ordinary differential equation.

1. Introduction. The phenomenological theory of thermoelectric effects is summarized (see [7]) in the constitutive equations

$$\mathbf{J} = \sigma(\mathbf{E} - \alpha\nabla u),\ \mathbf{q} = -\kappa\nabla u + u\alpha\mathbf{J},\ \mathbf{E} = -\nabla\varphi, \tag{1}$$

where $\mathbf{J}$ is the current density, $\mathbf{q}$ the heat flux, $\mathbf{E}$ the electric field, u the absolute temperature, φ the electric potential, σ and κ are the thermal and electric conductivity which in this paper are assumed to be positive functions of the temperature. α is also a function of u but with no definite sign. In a stationary state, equations (1), together with the balance equations

$$\nabla\cdot\mathbf{J} = 0,\ \nabla\cdot\mathbf{q} = \mathbf{E}\cdot\mathbf{J},$$

give the system of partial differential equations

$$\nabla\cdot(\sigma(u)\nabla v) = 0, \tag{2}$$

$$\nabla\cdot(\kappa(u)\nabla u) + \sigma(u)\beta(u)\nabla u\cdot\nabla v + \sigma(u)|\nabla v|^2 = 0, \tag{3}$$

where $\beta(u) = u\alpha'(u)$ and $v = \varphi + \int_{u_1}^{u}\alpha(t)dt$ is the effective potential so that

$$\mathbf{J} = -\sigma(u)\nabla v. \tag{4}$$

Let Ω be an open and bounded subset of $\mathbf{R}^3$ representing a body conductor of heat and electricity with a C^1 boundary consisting of two disjoint two-dimensional surfaces S_1, S_2 which act as electrodes to which a given difference of potential $V > 0$ is applied. The goal of this paper is to study the following problem Pb_V:

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To find $v(\mathbf{x}) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, $u(\mathbf{x}) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that equations (2) and (3) are satisfied with the boundary conditions

$$v = 0 \text{ on } S_1,\ v = V \text{ on } S_2, \tag{5}$$

$$u = u_1 \text{ on } S_1,\ u = u_2 \text{ on } S_2, \tag{6}$$

where u_1 and u_2 ($u_1 < u_2$) are given positive constants. We suppose

$$\sigma(u) \in C^0(\mathbf{R}^1),\ \kappa(u) \in C^0(\mathbf{R}^1),\ \beta(u) \in C^1(\mathbf{R}^1),\ \sigma(u) > 0,\ \kappa(u) > 0. \tag{7}$$

Thus no assumption of uniform ellipticity is made. In Section 2 we prove that a functional relation $u = \hat{u}(v)$ between the temperature and potential exists and permits the reduction of problem Pb_V to the Dirichlet problem for the Laplacian. $\hat{u}(v)$ is a solution of the following ordinary differential equation:

$$\frac{\kappa(u)}{\sigma(u)}\frac{du}{dv} = \gamma - v - \int_{u_1}^{u} \beta(t)dt \tag{8}$$

satisfying the conditions

$$u(0) = u_1,\ u(V) = u_2. \tag{9}$$

The method is not new (see W. Voigt [8] and H. Diesselhorst [5]). However, the question of existence and uniqueness for problem (8), (9), and in turn for problem Pb_V, is not treated in [5] and [8]. The special case of the metallic conduction is discussed in Section 3. Finally, Section 4 deals with a related problem in which a multiplicity of solutions may exist. Problem Pb_V is known as the "thermistor problem" when $\beta = 0$. It has been extensively studied with arbitrary boundary conditions by many authors (see, among others, [6], [1], [2] and the references therein).

2. The main theorem. If $(v(\mathbf{x}),\ u(\mathbf{x}))$ is a regular solution of problem Pb_V, we have from the maximum principle the "a priori" estimates

$$V \geq v(\mathbf{x}) \geq 0 \text{ in } \bar{\Omega}, \tag{10}$$

$$u(\mathbf{x}) \geq u_1 \text{ in } \bar{\Omega}. \tag{11}$$

THEOREM 1. If (7) holds and

$$\int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)}dt = \infty, \tag{12}$$

$$\left|\frac{\beta(u)\sigma(u)}{\kappa(u)}\right| \leq C, \text{ for all } u \geq u_1, \tag{13}$$

then

(i) there exists at least one regular solution to problem Pb_V.

(ii) The problem can be reduced to the following Dirichlet problem for the Laplacian:

$$\Delta\psi = 0 \text{ in } \Omega,\ \psi = \psi_1 \text{ on } S_1,\ \psi = \psi_2 \text{ on } S_2, \tag{14}$$

where ψ_1 and ψ_2 are constants which can be expressed in terms of the data.

Proof. If $v(\mathbf{x})$ is a solution to problem Pb_V we have, by (2),

$$\nabla \cdot (v\sigma(u)\nabla v) = \sigma(u)|\nabla v|^2,$$
$$\nabla \cdot \Big[\sigma(u) \int_{u_1}^{u} \beta(t)dt \nabla v\Big] = \sigma(u)\beta(u)\nabla u \cdot \nabla v.$$

Therefore, equation (3) can be rewritten in divergence form as

$$\nabla \cdot \Big\{\sigma(u)\Big[v\nabla v + \frac{\kappa(u)}{\sigma(u)}\nabla u + \int_{u_1}^{u} \beta(t)dt \ \nabla v\Big]\Big\} = 0. \tag{15}$$

Let $u = \hat{u}(v)$ be the sought for functional relation. Define

$$\theta = \frac{1}{2}v^2 + \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)}dt + \int_0^v \Big[\int_{u_1}^{\hat{u}(\xi)} \beta(t)dt\Big]d\xi. \tag{16}$$

Then (3) becomes

$$\nabla \cdot (\sigma(u)\nabla\theta) = 0. \tag{17}$$

Moreover, $\theta(\mathbf{x})$ satisfies the boundary conditions:

$$\theta = 0 \text{ on } S_1, \ \theta = \gamma V^{-1} \text{ on } S_2, \tag{18}$$

where γ is an unknown constant. To prove that $\theta(\mathbf{x})$ and $v(\mathbf{x})$ are related by the functional relation

$$\theta = \gamma v \tag{19}$$

we consider

$$\nabla \cdot (\sigma(\hat{u}(v))\nabla\theta) = 0 \text{ in } \Omega, \ \theta = 0 \text{ on } S_1, \ \theta = \gamma V^{-1} \text{ on } S_2, \tag{20}$$
$$\nabla \cdot (\sigma(\hat{u}(v))\nabla v) = 0 \text{ in } \Omega, \ v = 0 \text{ on } S_1, \ v = V \text{ on } S_2. \tag{21}$$

This system is uncoupled, and it is easy to see that the solution $v(\mathbf{x})$ of (21) is unique. Since $\theta(\mathbf{x}) = \gamma v(\mathbf{x})$ solves (20), we obtain (19). From (16) we have

$$\gamma v = \frac{1}{2}v^2 + \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)}dt + \int_0^v \Big[\int_{u_1}^{\hat{u}(\xi)} \beta(t)dt\Big]d\xi. \tag{22}$$

Taking the derivative of (22) with respect to v we obtain the ordinary differential equation

$$\frac{\kappa(u)}{\sigma(u)}\frac{du}{dv} = \gamma - v - B(u), \tag{23}$$

where

$$B(u) = \int_{u_1}^{u} \beta(t)dt, \tag{24}$$

which must be supplemented by (5) with the conditions:

$$u(0) = u_1, \tag{25}$$
$$u(V) = u_2. \tag{26}$$

The two-point boundary value problem (23), (25), (26) determines $\hat{u}(v)$. We prove now that (23), (25), (26) has one and only one solution for arbitrary data u_1, u_2 and V. Let us define the function

$$F : [u_1, \infty) \to [0, \infty),\ w = F(u),\ F(u) = \int_{u_1}^{u} \frac{k(t)}{\sigma(t)} dt. \tag{27}$$

F applies one-to-one $[u_1, \infty)$ onto $[0, \infty)$ by (7) and (12). Under the transformation (27), problem (23), (25), (26) becomes

$$\frac{dw}{dv} = \gamma - v - \mathfrak{B}(w), \tag{28}$$

$$w(0) = 0, \tag{29}$$

$$w(V) = F(u_2), \tag{30}$$

where

$$\mathfrak{B}(w) = B(F^{-1}(w)).$$

Since

$$\mathfrak{B}'(w) = \beta(F^{-1}(w)) \frac{\sigma(F^{-1}(w))}{\kappa(F^{-1}(w))}, \tag{31}$$

the solution of the Cauchy problem (28), (29) is defined in $[0, V]$ by (13). We claim that equation (30), i.e.

$$w(V; \gamma) = F(u_2), \tag{32}$$

is solvable with respect to γ. By (13) and (31), we have

$$\mathfrak{B}(0) - \tau w \le \mathfrak{B}(w) \le \mathfrak{B}(0) + \tau w. \tag{33}$$

Hence $w(V; \gamma)$ can be estimated from above and from below in terms of the solutions of the problems:

$$\frac{dy^+}{dv} = \gamma - v + \tau y^+,\ y^+(0) = 0,\ \frac{dy^-}{dv} = \gamma - v - \tau y^-,\ y^-(0) = 0. \tag{34}$$

We find:

$$y^-(V; \gamma) \le w(V; \gamma) \le y^+(V; \gamma). \tag{35}$$

On the other hand we have, by direct computation,

$$\lim_{\gamma \to \infty} y^-(V; \gamma) = \infty,\ \lim_{\gamma \to -\infty} y^+(V; \gamma) = -\infty. \tag{36}$$

By the continuous and differentiable dependence of $w(V; \gamma)$ on the parameter γ, $w_\gamma(v; \gamma)$ satisfies the Cauchy problem:

$$\frac{dw_\gamma}{dv} = 1 - \mathfrak{B}'(w) w_\gamma,\ w_\gamma(0) = 0. \tag{37}$$

With an easy calculation we find from (37) that

$$w_\gamma(V; \gamma) \ge \frac{e^{-CV}}{C} \Big(1 - e^{-CV}\Big) > 0. \tag{38}$$

From (38) and (36) we conclude that (32), and therefore (26), has one and only one solution $\tilde{\gamma}$. Let $\tilde{u}(v) = u(v, \tilde{\gamma})$ and consider the problem:

$$\nabla \cdot (\sigma(\tilde{u}(v))\nabla v) = 0 \text{ in } \Omega, \tag{39}$$

$$v = 0 \text{ on } S_1, \; v = V \text{ on } S_2. \tag{40}$$

Let

$$\psi = G(v), \; G(v) = \int_0^v \sigma(\tilde{u}(\xi))d\xi, \; \psi_2 = G(V). \tag{41}$$

G applies one-to-one $[0, V]$ onto $[0, \psi_2]$. If $\psi(\mathbf{x})$ solves

$$\Delta\psi = 0 \text{ in } \Omega, \; \psi = 0 \text{ on } S_1, \; \psi = \psi_2 \text{ on } S_2, \tag{42}$$

then by the maximum principle we have

$$0 \leq \psi(\mathbf{x}) \leq \psi_2 \text{ in } \bar{\Omega}. \tag{43}$$

If we define

$$v(\mathbf{x}) = G^{-1}(\psi(\mathbf{x})) \tag{44}$$

and

$$u(\mathbf{x}) = \tilde{u}(v(\mathbf{x})) \tag{45}$$

we obtain a solution to problem Pb_V. □

3. Two special cases. When equation (23) has a first integral it is possible to prove that problem Pb_V not only has a solution, but also that the solution is unique. This is the case of the metallic conduction. For metals the thermal and electric conductivity are related by the Wiedemann-Franz law [7], which reads:

$$\kappa(u) = K\sigma(u)u, \tag{46}$$

where K is a positive constant. If we assume, as in [5],

$$\beta(u) = Cu \tag{47}$$

(a linear dependence which is verified if u varies in a small interval), we have from (46) and (47),

$$\beta(u) = \tau\frac{\kappa(u)}{\sigma(u)}, \; \tau = \frac{C}{K}. \tag{48}$$

In this case it is possible to study problem Pb_V completely, in particular to prove the uniqueness of the solution. The proof is based on the following simple observation.

Lemma 1. Let $A(\theta, \psi) \in C^0(\mathbf{R}^2)$,

$$A(\theta, \psi) > 0. \tag{49}$$

If θ_i, ψ_i, $i = 1, 2$ are given constants, the problem

$$\nabla \cdot (A(\theta, \psi)\nabla\theta) = 0 \text{ in } \Omega, \; \theta = \theta_1 \text{ on } S_1, \; \theta = \theta_2 \text{ on } S_2, \tag{50}$$

$$\nabla \cdot (A(\theta, \psi)\nabla\psi) = 0 \text{ in } \Omega, \; \psi = \psi_1 \text{ on } S_1, \; \psi = \psi_2 \text{ on } S_2 \tag{51}$$

has one and only one solution $(\theta(\mathbf{x}), \psi(\mathbf{x}))$, which can be represented in terms of the solution of a Dirichlet problem for the Laplacian.

Proof. Let $(\theta(\mathbf{x}), \psi(\mathbf{x}))$ be a solution to (50), (51). Then

$$\psi(\mathbf{x}) = a\theta(\mathbf{x}) + b, \tag{52}$$

where $a = (\psi_2 - \psi_1)(\theta_2 - \theta_1)^{-1}$ and $b = \psi_1 - (\psi_2 - \psi_1)(\theta_2 - \theta_1)^{-1}$. If $w(\mathbf{x}) = \psi(\mathbf{x}) - a\theta(\mathbf{x}) - b$ we have $w = 0$ on $S_1 \cup S_2$ and $\nabla \cdot (A(\theta, \psi)\nabla w) = 0$ in Ω. Hence $w(\mathbf{x}) = 0$ in $\bar{\Omega}$ and

$$\psi(\mathbf{x}) = a\theta(\mathbf{x}) + b. \tag{53}$$

Define

$$z = L(\theta),\ \ L(\theta) = \int_{\theta_1}^{\theta} A(\xi, a\xi + b) d\xi. \tag{54}$$

If $z(\mathbf{x}) = L(\theta(\mathbf{x}))$, we get:

$$\Delta z = 0 \text{ in } \Omega,\ \ z = 0 \text{ on } S_1,\ \ z = L(\theta_2) \text{ on } S_2. \tag{55}$$

Now, if $(\tilde{\theta}, \tilde{\psi})$ is a second solution to (50), (51) and $\tilde{z}(\mathbf{x})$ is obtained as before, we have

$$\Delta \tilde{z} = 0 \text{ in } \Omega,\ \ \tilde{z} = 0 \text{ on } S_1,\ \ \tilde{z} = L(\theta_2) \text{ on } S_2. \tag{56}$$

Thus, $\tilde{z}(\mathbf{x}) = z(\mathbf{x})$ and $\theta(\mathbf{x}) = L^{-1}(\tilde{z}(\mathbf{x})) = L^{-1}(z(\mathbf{x})) = \tilde{\theta}(\mathbf{x})$, and by (52), $\tilde{\psi}(\mathbf{x}) = \psi(\mathbf{x})$. □

In the next theorem we assume for definiteness $\tau > 0$ and $u_2 = u_1 = \bar{u} > 0$. The general case can be treated similarly.

THEOREM 2. Let us suppose (48) to hold and define

$$G(v, V, \tau) = \frac{V}{\tau} \frac{1 - e^{-\tau v}}{1 - e^{-\tau V}} - \frac{v}{\tau}. \tag{57}$$

If

$$\int_{\bar{u}}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \infty, \tag{58}$$

then there exists one and only one solution of problem Pb_V, which can be constructed from the functional relation

$$u = F^{-1}(G(v, V, \tau)),\ \text{where } F(u) = \int_{\bar{u}}^{u} \frac{\kappa(t)}{\sigma(t)} dt. \tag{59}$$

Let

$$\mu = \int_{\bar{u}}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt < \infty. \tag{60}$$

If

$$G(v^*, V, \tau) \geq \mu,\ \text{where } v^* = -\frac{1}{\tau} \log \frac{1 - e^{-\tau V}}{\tau V}, \tag{61}$$

then problem Pb_V has no solution. When

$$G(v^*, V, \tau) < \mu, \tag{62}$$

there exists one and only one solution, which is again obtained from (59).

Proof. By (48), equation (23) has the integrating factor $e^{\tau v}$, which gives the first integral

$$F(u) + \frac{1}{\tau}v - \frac{1}{\tau^2} - \frac{\gamma}{\tau} = C\mathbf{e}^{-\tau v}. \tag{63}$$

Equation (63) permits us to solve problem (23), (25), (26). After computing C and γ we find the functional relation between u and v, i.e.

$$F(u) = G(v, V, \tau), \tag{64}$$

where $G(v, V, \tau)$ is given by (57). $G(v, V, \tau)$ is easily studied. We find $G(0, V, \tau) = G(V, V, \tau) = 0$; moreover, $G'(v) > 0$ if $v \in (0, v^*)$ and $G'(v) < 0$ if $v \in (v^*, V)$, where $v^* \in (0, \frac{V}{2})$ is given by (61). Hence, if (58) holds, (64) is uniquely solvable with respect to u. When (60) holds, (64) is solvable only if (62) is satisfied. Once the functional relation $u = \hat{u}(v)$ is obtained, problem Pb_V is reducible to a Dirichlet problem for the Laplacian as in Theorem 1. To prove that the solution obtained in this way is unique we use the transformation

$$\theta = e^{\tau v}\Big[\tau F(u) + v - \frac{1}{\tau} + 1\Big], \tag{65}$$

$$\psi = e^{\tau v}\Big[\tau F(u) + v - \frac{1}{\tau} - 1\Big]. \tag{66}$$

Under (65), (66) equations (2) and (3) become

$$\nabla \cdot (e^{-\tau v}\sigma(u)\nabla\theta) = 0,$$
$$\nabla \cdot (e^{-\tau v}\sigma(u)\nabla\psi) = 0,$$

where

$$v = \frac{1}{\tau}\log\frac{\theta - \psi}{2},$$
$$u = F^{-1}\Big[\frac{1}{\tau}\Big(\frac{\theta + \psi}{\theta - \psi} - \frac{1}{\tau}\log\frac{\theta - \psi}{2} + \frac{1}{\tau}\Big)\Big].$$

Since θ and ψ are constants on S_1 and S_2 all assumptions of Lemma 1 are satisfied and we can conclude that the solution is unique □

REMARK 1. The present method permits a precise estimate of the maximum of the temperature. More precisely we have:

$$\max_{\mathbf{x}\in\Omega} u(\mathbf{x}) = F^{-1}(G(v^*, V, \tau))$$

as an examination of the graphs of $F(u)$ and $G(v)$ immediately shows. Moreover, this maximum is assumed in an interior point of $\bar{\Omega}$.

When $\beta(u) = 0$ (complete absence of Thomson effect), problem (23), (25), (26) simplifies and we have:

$$\frac{\kappa(u)}{\sigma(u)}\frac{du}{dv} = \gamma - v, \tag{67}$$

$$u(0) = u_1,\ u(V) = u_2. \tag{68}$$

In (67) variables separate and, solving with condition (68), we obtain

$$F(u) = H(v), \tag{69}$$

where

$$F(u) = \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)} dt, \ H(v) = \frac{v}{V}\Big(F(u_2) + \frac{V^2}{2}\Big) - \frac{v^2}{2}.$$

We have

$$H'(v^*) = 0, \text{ with } v^* = \frac{V}{2} + \frac{F(u_2)}{V}.$$

If

$$\int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \infty,$$

problem (67), (68), and therefore problem Pb_V, has one and only one solution. In particular, when $v^* < V$, we have

$$u(\mathbf{x}_M) = \max_{\mathbf{x}\in\Omega} u(\mathbf{x}) = F^{-1}(H(v^*)), \ \mathbf{x}_M \in \Omega, \tag{70}$$

whereas, if $V^* \geq V$,

$$u(\mathbf{x}_M) = \max_{\mathbf{x}\in\Omega} u(\mathbf{x}) = u_2, \ \mathbf{x}_M \in S_2. \tag{71}$$

Assume now

$$\infty > \int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \mu. \tag{72}$$

If $v^* < V$, equation (67) is solvable with respect to u if and only if $H(v^*) < \mu$. Hence, in this case, problem Pb_V has one and only one solution. On the contrary, when $v^* \geq V$, equation (69), and therefore problem Pb_V, is always solvable, since $H(V) = F(u_2) < \mu$. The maximum of the temperature is again given by (70) and (71). To prove uniqueness, we proceed as in Theorem 2 using this time the transformation

$$\theta = \frac{1}{2}v^2 + \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)} dt,$$
$$\psi = v,$$

which reduces (2) and (3) to a form which permits the use of Lemma 1.

4. Multiplicity of solutions. In this last section we assume that the potential V is not applied directly to the electrodes S_1 and S_2, but via a one-dimensional resistor of R ohms, a natural situation, since in practice R is always greater than zero. In this model the potential Γ on S_2 is an unknown constant. By (4) the total electric current I flowing in the resistor is given by

$$I = \sigma(u_2) \int_{S_2} \frac{\partial v}{\partial n} dS,$$

where $\frac{\partial}{\partial n}$ denotes the outer normal derivative to S_2. We have problem Pb_2:

To find $v(\mathbf{x})$, $u(\mathbf{x})$ and $\Gamma \in \mathbf{R}^1$ such that

$$\nabla \cdot (\sigma(u)\nabla v) = 0 \text{ in } \Omega, \tag{73}$$

$$v = 0 \text{ on } S_1,\ v = \Gamma \text{ on } S_2, \tag{74}$$

$$\nabla \cdot (\kappa(u)\nabla u) + \sigma(u)\beta(u)\nabla u \cdot \nabla v + \sigma(u)|\nabla v|^2 = 0 \text{ in } \Omega, \tag{75}$$

$$u = u_1 \text{ on } S_1,\ u = u_2 \text{ on } S_2, \tag{76}$$

$$V - \Gamma = R\sigma(u_2)\int_{S_2} \frac{\partial v}{\partial n} dS. \tag{77}$$

LEMMA 2. If $(v(\mathbf{x}), u(\mathbf{x}), \Gamma)$ is a solution to problem Pb_2, then

$$0 < \Gamma < V. \tag{78}$$

Proof. By contradiction assume $\Gamma \leq 0$. If $\Gamma = 0$, then $v(\mathbf{x}) = 0$ and from (77) we have $\Gamma = V$. If $\Gamma < 0$, by the maximum principle in Hopf's form, we have $\frac{\partial v}{\partial n} < 0$ on S_2 by (73). Hence $V - \Gamma = R\sigma(u_2)\int_{S_2} \frac{\partial v}{\partial n} dS < 0$. Therefore $\Gamma > 0$, and as a consequence, $\frac{\partial v}{\partial n} > 0$ on S_2. Thus (78) holds. □

For every fixed $\Gamma \in (0, V)$ we can reduce problem Pb_Γ, reasoning as in Theorem 1, to the following two-point problem for an ordinary differential equation:

$$\frac{\kappa(u)}{\sigma(u)}\frac{du}{dv} = \gamma - v - B(u), \tag{79}$$

$$u(0) = u_1,\ u(\Gamma) = u_2, \tag{80}$$

which has, under the assumption (13), one and only one solution

$$u = \hat{u}(v, \Gamma).$$

Define

$$\psi = L(v, \Gamma),\ L(v, \Gamma) = \int_0^v \sigma(\hat{u}(\xi, \Gamma))d\xi,\ v \in [0, V]$$

and solve

$$\Delta\psi = 0 \text{ in } \Omega,$$

$$\psi = 0 \text{ on } S_1,\ \psi = L(\Gamma, \Gamma) \text{ on } S_2.$$

If $w(\mathbf{x})$ is given by

$$\Delta w = 0 \text{ in } \Omega,\ w = 0 \text{ on } S_1,\ w = 1 \text{ on } S_2$$

we have

$$\psi(\mathbf{x}) = L(\Gamma, \Gamma)w(\mathbf{x}), \tag{81}$$

whence $v(\mathbf{x}) = L^{-1}(\psi(\mathbf{x}), \Gamma)$ is a solution to

$$\nabla \cdot (\sigma(\hat{u}(v, \Gamma))\nabla v) = 0 \text{ in } \Omega,\ v = 0 \text{ on } S_1,\ v = \Gamma \text{ on } S_2.$$

Let

$$K = \int_{S_2} \frac{\partial w}{\partial n} dS.$$

By (81), equation (77) reads

$$V - \Gamma = RKL(\Gamma, \Gamma). \tag{82}$$

If $g(\Gamma) = V - \Gamma - RKL(\Gamma, \Gamma)$, we have $g(0) = V > 0$ and $g(V) = -RKL(\Gamma, \Gamma) < 0$. Therefore equation (82) has in $(0, V)$ at least one solution. However, uniqueness is not to be expected as can already be seen when $\beta = 0$ (see [4]). Typically there exists one or three solutions and these solutions are found in practical devices modelled after problem Pb_2. Finally, we note that the function $g(\Gamma)$ can be written in terms of the data and that the domain Ω enters only with the constant K.

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