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# ELECTRO-MAGNETO-ENCEPHALOGRAPHY AND FUNDAMENTAL SOLUTIONS

By

G. DASSIOS (*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom*)

AND

A. S. FOKAS (*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom*)

**Abstract.** Electroencephalography (EEG) and Magnetoencephalography (MEG) are two important methods for the functional imaging of the brain. For the case of the spherical homogeneous model, we elucidate the mathematical relations of these two methods. In particular, we derive and analyse three different representations for the electric and magnetic potentials, as well as the corresponding electric and magnetic induction fields, namely: integral representations involving the Green and the Neumann kernels, representations in terms of eigenfunction expansions, and closed form expressions. We show that the parts of the EEG and MEG fields in the interior of the brain that are due to the induction current are related via Kelvin's inversion transformation. We also derive closed form expressions for the interior and exterior vector potentials of the corresponding magnetic induction fields.

**1. Introduction.** Electrochemically generated neuronal currents in the brain give rise to a magnetic field, which in turn excites an induction current within the conductive brain tissue. This electromagnetic activity of the brain is recorded by measuring the electric potential on the scalp (EEG) and the magnetic induction field at distances 4-6 cm from the head (MEG). There exists an extensive literature on how to utilise the EEG and MEG recordings [13], [17] in order to determine the electric potential and the magnetic induction field outside the head. From the mathematical point of view, the two basic problems for EEG and MEG are the *forward problem*, namely find the interior and exterior fields in terms of the neuronal current and the *inverse problem,* namely find the neuronal current in terms of the electric potential or in terms of the magnetic field. The fact that neither of these inverse problems has a unique solution was known to Helmholtz in 1853. Nevertheless, complete quantitative results on the non-uniqueness of the inverse MEG problem were obtained only recently, in [9], [10] for the spherical model

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The first author is on leave from the University of Patras and ICEHT/FORTH, Greece.

and in [8] for a star-shaped conductor (see also [1]). To our knowledge, no corresponding results were known for EEG until recently [7]. It was shown in [7] that at least for the spherical brain-model, EEG and MEG provide strictly complementary information about the current, namely the components of the current obtained from EEG and from MEG live in two orthogonal complements of the vector space used for the representation of the neuronal current. In fact, it is shown in [7], [8] that EEG and MEG depend on the scalar and the vector invariants, respectively, of a certain dyadic field, which provides the appropriate tensor field characterising the unique solution of the relative inverse problem. Simple analytic algorithms for the solution of the inverse problem for both EEG and MEG corresponding to a single dipole, a finite set of dipoles, and a current localised within a small sphere, are presented in [5]. For the more realistic ellipsoidal brain-model, the direct problems are discussed in [3], [4].

In this paper we concentrate on the spherical model and in particular we discuss these different mathematical formulations for EEG and MEG. One of these formulations involves integrals of the Green and Neumann kernels. We show that the complementarity of the EEG and MEG [7] is due to the fact that the electric and magnetic potentials are generated mathematically through the action of a directional derivative on two *orthogonal* directions: the direction of the dipole's moment for EEG and the direction perpendicular to the moment and to the position vector of the dipole for MEG. For the case of the electric potential, the above directional derivative acts on an integral representation involving the product of the Neumann kernel with respect to the observation point times the normal derivative of the Green kernel with respect to the source point. For the case of the magnetic potential, the relative directional derivative acts on a similar integral representation. In both cases, the passage of the information from the interior source to the exterior observation point occurs via the surface integral over the boundary of the conductive sphere involving the two kernels. In addition to integral representations, we also present closed form solutions, as well as eigenfunction expansions.

As was mentioned earlier, the electric and magnetic potentials and fields are generated by the neuronal activity of the brain and by the induction current excited within the brain tissue as a result of this activity. We show here that the parts of the interior and exterior fields that are due to this secondary inductive activity are images of each other under the harmonicity-preserving Kelvin transformation. Therefore, expressing the internal field in terms of the external one and vice versa is mathematically straightforward. Similar results hold for the vector potentials of the magnetic induction fields. In fact, the Kelvin connection obliterates the possibility to obtain extra information from the interior field.

The paper is organised as follows. Section 2 summarises the mathematical formulation of EEG and MEG. Electroencephalography is discussed in Section 3, and magnetoencephalography is the topic of Section 4.

**2. Mathematical formulation.** Mathematically, the electromagnetic activity of the human brain is governed by the quasi-static theory of Maxwell's equations [16], [19]:

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}^p + \sigma \boldsymbol{E}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

where $\sigma$ is the conductivity, $\mu_0$ is the magnetic permeability, $\boldsymbol{J}^p$ is the neuronal current, $\boldsymbol{B}$ is the magnetic induction field, and $\boldsymbol{E}$ is the electric field. Equation (1) implies the existence of the electric potential $u$ such that

$$\boldsymbol{E} = -\nabla u. \tag{4}$$

For the single dipole excitation at the point $\boldsymbol{\tau}$ with moment $\boldsymbol{Q}$, the current is defined by

$$\boldsymbol{J}^p(\boldsymbol{r}) = \boldsymbol{Q}\delta(\boldsymbol{r}-\boldsymbol{\tau}), \tag{5}$$

where $\delta$ denotes the Dirac measure. For this current inside a homogeneous conductor $\Omega$, the electric potential and the magnetic induction field satisfy the Geselowitz [11], [12] formulae

$$4\pi\sigma u(\boldsymbol{r}) = \boldsymbol{Q}\cdot\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} - \sigma\oint_{\partial\Omega} u(\boldsymbol{r}')\hat{\boldsymbol{n}}(\boldsymbol{r}')\cdot\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}ds(\boldsymbol{r}'), \quad \boldsymbol{r}\notin\partial\Omega \tag{6}$$

and

$$\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{Q}\times\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} - \sigma\oint_{\partial\Omega} u(\boldsymbol{r}')\hat{\boldsymbol{n}}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}ds(\boldsymbol{r}'), \quad \boldsymbol{r}\notin\partial\Omega, \tag{7}$$

respectively, where $\partial\Omega$ denotes the boundary of the conductive medium $\Omega$. The interior electric potential $u$ satisfies the Neumann problem

$$\sigma\Delta u(\boldsymbol{r}) = \nabla\cdot\boldsymbol{J}^p(\boldsymbol{r}), \quad \boldsymbol{r}\in\Omega, \tag{8}$$

$$\partial_n u(\boldsymbol{r}) = 0, \quad \boldsymbol{r}\in\partial\Omega. \tag{9}$$

Equations (6) and (7) are the scalar and the vector invariants [8] of the dyadic equation

$$\widetilde{\boldsymbol{D}}(\boldsymbol{r}) = \boldsymbol{Q}\otimes\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} - \sigma\oint_{\partial\Omega} u(\boldsymbol{r}')\widehat{\boldsymbol{n}}(\boldsymbol{r}')\otimes\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}|^3}ds(\boldsymbol{r}'), \tag{10}$$

which completely characterises the electric and magnetic activity of the brain [7].

In the particular case that the conductor is a sphere of radius $\alpha$ with conductivity $\sigma$, the dipolar current (5) generates the electric potentials $u^{\pm}$, the electric fields $\boldsymbol{E}^{\pm}$, the scalar magnetic potentials $U^{+}$, and the magnetic induction fields $\boldsymbol{B}^{\pm}$, where the superscripts $+$ and $-$ denote exterior and interior fields, respectively.

The potentials $u^{\pm}$ solve the boundary value problems

$$\sigma\Delta u^-(\boldsymbol{r}) = \boldsymbol{Q}\cdot\nabla\delta(\boldsymbol{r}-\boldsymbol{\tau}), \quad r<\alpha, \tag{11}$$

$$\partial_n u^-(\boldsymbol{r}) = 0, \quad r=\alpha \tag{12}$$

and

$$\Delta u^+(\boldsymbol{r}) = 0, \quad r>\alpha, \tag{13}$$

$$u^+(\boldsymbol{r}) = u^-(\boldsymbol{r}), \quad r=\alpha, \tag{14}$$

$$u^+(\boldsymbol{r}) = O(1/r^2), \quad r\to\infty, \tag{15}$$

respectively.

In the region $\Omega^c$ exterior to the conductor, the field $\boldsymbol{B}^+$ is both irrotational and solenoidal, and therefore it can be represented as the gradient of a harmonic function

$$\boldsymbol{B}^+(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\nabla U^+(\boldsymbol{r}), \tag{16}$$

where $U^+$ is called the scalar magnetic potential. The magnetic potential satisfies the exterior problem

$$\begin{aligned} \Delta U^+(\boldsymbol{r}) &= 0, \quad r > \alpha, & (17)\\ \partial_r U^+(\boldsymbol{r}) &= -\frac{\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3}, \quad r = \alpha, & (18)\\ U^+(\boldsymbol{r}) &= O(1/r^2), \quad r \to \infty, & (19) \end{aligned}$$

where the notation $\widehat{}$ on the top of a vector denotes unit magnitude, and the Neumann condition (18) is obtained from formula (7). Equation (7) also provides the interior magnetic field $\boldsymbol{B}^-$.

## 3. Electroencephalography.

A particular solution of (11) is given by

$$u_p(\boldsymbol{r}) = \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} = \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|}. \tag{20}$$

Then

$$u^-(\boldsymbol{r}) = u_p(\boldsymbol{r}) + w(\boldsymbol{r}), \tag{21}$$

where the harmonic function $w$ satisfies the Neumann condition

$$\partial_r w(\boldsymbol{r}) = -\frac{1}{4\pi\sigma}\partial_r\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|}, \quad r = \alpha. \tag{22}$$

It was shown in [6] that an interior harmonic function satisfying (22) can be represented in the form

$$\begin{aligned} w(\boldsymbol{r}) &= \frac{1}{4\pi\alpha}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})\partial_r w(\alpha\hat{\boldsymbol{r}}')\,ds(\boldsymbol{r}')\\ &= -\frac{1}{(4\pi)^2\alpha\sigma}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})\partial_{r'}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}'-\boldsymbol{\tau}|}\,ds(\boldsymbol{r}'), \end{aligned} \tag{23}$$

where the Neumann kernel $N$ is given by

$$N(\boldsymbol{r}',\boldsymbol{r}) = \frac{2r'}{|\boldsymbol{r}'-\boldsymbol{r}|} - \ln[|\boldsymbol{r}'-\boldsymbol{r}| + \hat{\boldsymbol{r}}'\cdot(\boldsymbol{r}'-\boldsymbol{r})]. \tag{24}$$

Let $G$ denote the fundamental solution of the Laplace equation

$$G(\boldsymbol{r}',\boldsymbol{\tau}) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}'-\boldsymbol{\tau}|} \tag{25}$$

and define the integral

$$D(\boldsymbol{r},\boldsymbol{\tau}) = \frac{1}{4\pi\alpha}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})\partial_{r'}G(\alpha\hat{\boldsymbol{r}}',\boldsymbol{\tau})\,ds(\boldsymbol{r}'), \tag{26}$$

where the observation point $\boldsymbol{r}$ appears in the Neumann function and the source position $\boldsymbol{\tau}$ appears in the double layer kernel. Then, equations (21)-(26) imply

$$\sigma u^-(\boldsymbol{r}) = \boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}[D(\boldsymbol{r},\boldsymbol{\tau}) - G(\boldsymbol{r},\boldsymbol{\tau})]. \tag{27}$$

Using the values $u^{-}(\alpha\hat{\boldsymbol{r}})$ in (14) and Poisson's integral formula for the sphere [6], we obtain the following representation for the exterior electric potential:

$$u^{+}(\boldsymbol{r}) = -\frac{(\alpha^2 - r^2)}{4\pi\alpha}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\oint_{r'=\alpha}\frac{D(\boldsymbol{r}',\boldsymbol{\tau}) - G(\boldsymbol{r}',\boldsymbol{\tau})}{|\boldsymbol{r}' - \boldsymbol{r}|^3}\,ds(\boldsymbol{r}'). \tag{28}$$

Straightforward calculations, using either eigenfunction expansions or images [4], lead to the following solutions:

$$\begin{aligned} u^{-}(\boldsymbol{r}) &= \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} + \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\frac{\tau^n r^n}{\alpha^{2n+1}}P_n(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{\tau}})\right] \\ &= \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\left[\frac{1}{P} + \frac{\alpha}{r}\frac{1}{R} - \frac{1}{\alpha}\ln\frac{rR + \boldsymbol{r}\cdot\boldsymbol{R}}{2\alpha^2}\right] \\ &= \frac{\boldsymbol{Q}}{4\pi\sigma}\cdot\left[\frac{\boldsymbol{P}}{P^3} + \frac{\alpha}{r}\frac{\boldsymbol{R}}{R^3} + \frac{1}{\alpha R}\frac{R\boldsymbol{r} + r\boldsymbol{R}}{Rr + \boldsymbol{r}\cdot\boldsymbol{R}}\right], \quad r| < \alpha, \end{aligned} \tag{29}$$

where

$$\boldsymbol{P} = \boldsymbol{r} - \boldsymbol{\tau}, \quad P = |\boldsymbol{P}|, \quad \boldsymbol{R} = \tfrac{\alpha^2}{r^2}\boldsymbol{r} - \boldsymbol{\tau}, \quad R = |\boldsymbol{R}| \tag{30}$$

and

$$\begin{aligned} u^{+}(\boldsymbol{r}) &= \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\sum_{n=1}^{\infty}\frac{2n+1}{n}\frac{\tau^n}{r^{n+1}}P_n(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\tau}}) \\ &= \frac{1}{4\pi\sigma}\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\left[\frac{2}{P} - \frac{1}{r}\ln\frac{rP + \boldsymbol{r}\cdot\boldsymbol{P}}{2r^2}\right] \\ &= \frac{\boldsymbol{Q}}{4\pi\sigma}\cdot\left[2\frac{\boldsymbol{P}}{P^3} + \frac{1}{rP}\frac{P\boldsymbol{r} + r\boldsymbol{P}}{Pr + \boldsymbol{r}\cdot\boldsymbol{P}}\right], \quad r > \alpha. \end{aligned} \tag{31}$$

The integral representation for $u^{+}$also follows from the standard Poisson integral formula for the sphere [6].

In summing the series in (29) and (31) we have used the fact that the function

$$f_\theta(\rho) = \sum_{n=1}^{\infty}\frac{\rho^n}{n}P_n(\cos\theta), \quad \rho < 1 \tag{32}$$

solves the initial value problem

$$\begin{aligned} \rho f_\theta'(\rho) &= (1 - 2\rho\cos\theta + \rho^2)^{-1/2} - 1, \\ f_\theta(0) &= 0 \end{aligned} \tag{33}$$

and therefore

$$f_\theta(\rho) = -\ln\frac{1 - \rho\cos\theta + \sqrt{1 - 2\rho\cos\theta + \rho^2}}{2}. \tag{34}$$

The contributions of the induction current to the electric fields $u^-$, $u^+$ are given by

$$f^-(\boldsymbol{r}) = \frac{\boldsymbol{Q}}{4\pi\sigma}\cdot\frac{\alpha}{r}\left[\frac{\boldsymbol{R}}{R^3}+\frac{r}{\alpha^2 R}\frac{R\boldsymbol{r}+r\boldsymbol{R}}{Rr+\boldsymbol{r}\cdot\boldsymbol{R}}\right], \quad r<\alpha, \tag{35}$$

$$f^+(\boldsymbol{r}) = \frac{\boldsymbol{Q}}{4\pi\sigma}\cdot\left[\frac{\boldsymbol{P}}{P^3}+\frac{1}{rP}\frac{P\boldsymbol{r}+r\boldsymbol{P}}{Pr+\boldsymbol{r}\cdot\boldsymbol{P}}\right], \quad r>\alpha, \tag{36}$$

respectively. In view of the definitions (30) and Kelvin's theorem

$$\Delta f(\boldsymbol{r}) = \left(\frac{r}{\alpha}\right)^5 \Delta\frac{\alpha}{r} f\left(\frac{\alpha^2}{r^2}\boldsymbol{r}\right) \tag{37}$$

it follows that $f^-$, $f^+$ are images of each other under the harmonicity-preserving Kelvin transformation $\boldsymbol{r}\mapsto(\alpha/r)^2\boldsymbol{r}$ [15].

The electric fields $\boldsymbol{E}^-$ and $\boldsymbol{E}^+$ can be determined using equation (4) and any of the expressions in (29) and (31), respectively.

**4. Magnetoencephalography.** Equations (7) or (10) yield

$$\frac{4\pi}{\mu_0}\hat{\boldsymbol{r}}\cdot\boldsymbol{B}(\boldsymbol{r}) = \partial_r U(\boldsymbol{r}) = -\frac{\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} = -\frac{1}{r}\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|}, \quad r>\alpha. \tag{38}$$

In analogy with (23) we obtain

$$\begin{aligned} U(\boldsymbol{r}) &= -\frac{1}{4\pi\alpha}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})\partial_r U(\alpha\hat{\boldsymbol{r}}')\,ds(\boldsymbol{r}') \\ &= \frac{1}{4\pi\alpha^2}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}'-\boldsymbol{\tau}|}\,ds(\boldsymbol{r}'), \end{aligned} \tag{39}$$

where the minus sign accounts for the exterior problem and $N$ is defined by (24). Define the integral

$$S(\boldsymbol{r},\boldsymbol{\tau}) = \frac{1}{\alpha^2}\oint_{r'=\alpha} N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r})G(\alpha\hat{\boldsymbol{r}}',\boldsymbol{\tau})\,ds(\boldsymbol{r}'). \tag{40}$$

Then, the magnetic potential is represented by

$$U(\boldsymbol{r}) = -(\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}})S(\boldsymbol{r},\boldsymbol{\tau}), \quad r>\alpha. \tag{41}$$

Note that $N$ depends on the observation point $\boldsymbol{r}$, $G$ depends on the source point $\boldsymbol{\tau}$ and the directional differentiation depends solely on the variables of the dipole.

The function $U$ defined by (41) is harmonic; using the identity

$$\partial_r N(\alpha\hat{\boldsymbol{r}}',\boldsymbol{r}) = \frac{\alpha}{r}\frac{\alpha^2-r^2}{|\alpha\hat{\boldsymbol{r}}'-\boldsymbol{r}|^3} \tag{42}$$

as well as Poisson's integral formula for the sphere [6], we recover the Neumann data (18):

$$\begin{aligned}\partial_r U(\boldsymbol{r}) &= -\frac{1}{r}\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\left[\frac{\alpha^2-r^2}{\alpha}\oint_{r'=\alpha}\frac{G(\alpha\hat{\boldsymbol{r}}',\boldsymbol{\tau})}{|\alpha\hat{\boldsymbol{r}}'-\boldsymbol{r}|^3}ds(\boldsymbol{r}')\right]\\ &= -\frac{1}{r}\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|}\\ &= -\frac{\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3}.\end{aligned} \tag{43}$$

Utilising the classical expansion

$$\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}|} = \sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'), \quad r'<r \tag{44}$$

and the identity

$$\partial_{r'}\ln[|\boldsymbol{r}'-\boldsymbol{r}|+\hat{\boldsymbol{r}}'\cdot(\boldsymbol{r}'-\boldsymbol{r})] = \frac{1}{|\boldsymbol{r}'-\boldsymbol{r}|}, \tag{45}$$

we obtain the expansion

$$N(\boldsymbol{r}',\boldsymbol{r}) = \sum_{n=0}^{\infty}\frac{2n+1}{n+1}\left(\frac{r'}{r}\right)^{n+1} P_n(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'), \quad r'<r. \tag{46}$$

By substituting (44) (with $\boldsymbol{r}$ replaced by $\boldsymbol{\tau}$) and (46) into (39), applying the addition theorem [18],

$$\frac{4\pi}{2n+1}\sum_{m=-n}^{n}Y_n^{m^*}(\widehat{\boldsymbol{r}})Y_n^m(\widehat{\boldsymbol{r}}') = P_n(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{r}}'), \tag{47}$$

where $Y_n^m$ denotes the orthogonalised complex form of spherical harmonics, and using orthogonality, we obtain the following expansion:

$$U(\boldsymbol{r}) = \boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{\tau^n}{r^{n+1}}P_n(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\tau}}), \quad r>\alpha. \tag{48}$$

Using (32)-(34) to sum the series in (48) and applying the relevant directional differentiation we obtain

$$U(\boldsymbol{r}) = \frac{\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\boldsymbol{r}}{F(\boldsymbol{r},\boldsymbol{\tau})}, \tag{49}$$

where

$$F(\boldsymbol{r},\boldsymbol{r}_0) = |\boldsymbol{r}-\boldsymbol{\tau}|\left[|\boldsymbol{r}||\boldsymbol{r}-\boldsymbol{\tau}|+\boldsymbol{r}\cdot(\boldsymbol{r}-\boldsymbol{\tau})\right]. \tag{50}$$

The expression (49) for the scalar magnetic potential coincides with the one obtained in [2], [20] via integration.

For the evaluation of the exterior magnetic field we can use formula (16) and either (48) or (49). Alternatively, in order to demonstrate how the two fields are connected via the Kelvin transformation, we prefer to follow the interior-exterior expansion approach [14].

From equations (31) and (47) we obtain

$$\sigma u(\alpha\hat{\boldsymbol{r}}) = \boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{1}{n}\frac{\tau^n}{\alpha^{n+1}}Y_n^m(\widehat{\boldsymbol{r}})Y_n^{m*}(\widehat{\boldsymbol{\tau}}), \tag{51}$$

where $u$ is either the interior or the exterior potential. Substituting (51) into (7) and using the identity

$$-\boldsymbol{r}'\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \boldsymbol{r}\times\nabla_{\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{52}$$

we arrive at

$$\begin{aligned}\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) &= \boldsymbol{Q}\times\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3}\\ &\quad+(\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}})(\boldsymbol{r}\times\nabla_{\boldsymbol{r}})\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{1}{n}\left(\frac{\tau}{\alpha}\right)^n Y_n^{m*}(\widehat{\boldsymbol{\tau}})I_n^{m\pm}(\boldsymbol{r}), \quad r\neq\alpha,\end{aligned} \tag{53}$$

where

$$I_n^{m\pm}(\boldsymbol{r}) = \oint_{|\hat{\boldsymbol{\rho}}|=1}\frac{Y_n^m(\widehat{\boldsymbol{\rho}})}{|\boldsymbol{r}-\alpha\hat{\boldsymbol{\rho}}|}\,ds(\hat{\boldsymbol{\rho}}) \tag{54}$$

and the $+$ and $-$ signs indicate the regions exterior and interior to the sphere, respectively. Appropriate use of equation (44) and orthogonality leads to the expressions

$$I_n^{m-}(\boldsymbol{r}) = \frac{4\pi}{2n+1}\frac{r^n}{\alpha^{n+1}}Y_n^m(\widehat{\boldsymbol{r}}), \quad \text{for } r<\alpha \tag{55}$$

and

$$I_n^{m+}(\boldsymbol{r}) = \frac{4\pi}{2n+1}\frac{\alpha^n}{r^{n+1}}Y_n^m(\widehat{\boldsymbol{r}}), \quad \text{for } r>\alpha. \tag{56}$$

Using equations (32)-(34) and (47) to sum the series, we obtain the following closed form expression:

$$\frac{4\pi}{\mu_0}\boldsymbol{B}^{\pm}(\boldsymbol{r}) = \boldsymbol{Q}\times\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3}-(\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}})(\boldsymbol{r}\times\nabla_{\boldsymbol{r}})L^{\pm}(\boldsymbol{r},\boldsymbol{\tau}) \tag{57}$$

with

$$L^-(\boldsymbol{r},\boldsymbol{\tau}) = \frac{1}{\alpha}\ln\frac{F(\bar{\boldsymbol{r}},\boldsymbol{\tau})}{2\bar{r}^2|\bar{\boldsymbol{r}}-\boldsymbol{\tau}|}, \quad \text{for } r<\alpha, \tag{58}$$

and

$$L^+(\boldsymbol{r},\boldsymbol{\tau}) = \frac{1}{r}\ln\frac{F(\boldsymbol{r},\boldsymbol{\tau})}{2r^2|\boldsymbol{r}-\boldsymbol{\tau}|}, \quad \text{for } r>\alpha, \tag{59}$$

where the function $F$ is given by (50), and $\bar{\boldsymbol{r}}=(\alpha^2/r^2)\boldsymbol{r}$ is the Kelvin image of $\boldsymbol{r}$ with respect to the conductive sphere. The two functions $L^-$ and $L^+$ are connected via Kelvin's theorem (37), and since the operator $\boldsymbol{r}\times\nabla_{\boldsymbol{r}}$ remains invariant under the Kelvin transformation (since it involves only tangential operations), it follows that the parts of the fields $\boldsymbol{B}^-$ and $\boldsymbol{B}^+$ that are due to the induction current are connected via Kelvin's theorem. Hence, as far as the effect of the conductive brain tissue is concerned, the MEG fields behave exactly the same way as the EEG fields.

The fields $\boldsymbol{B}^{\pm}$ are solenoidal (see (3)) and therefore they can be represented in terms of the vector potentials $\boldsymbol{A}^{\pm}$ by

$$\frac{4\pi}{\mu_0}\boldsymbol{B}^{\pm}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}^{\pm}(\boldsymbol{r}). \tag{60}$$

Hence, equation (57) implies that

$$\boldsymbol{A}^{\pm}(\boldsymbol{r}) = \frac{\boldsymbol{Q}}{|\boldsymbol{r}-\boldsymbol{\tau}|} + \boldsymbol{r}(\boldsymbol{Q}\cdot\nabla_{\boldsymbol{\tau}})\ L^{\pm}(\boldsymbol{r},\boldsymbol{\tau})\ , \tag{61}$$

where $L^{\pm}$ are given by (58), (59).

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