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INITIAL BOUNDARY VALUE PROBLEM FOR SEMILINEAR HYPERBOLIC EQUATIONS AND PARABOLIC EQUATIONS WITH CRITICAL INITIAL DATA

By

XU RUNZHANG

College of Science, Harbin Engineering University, 150001, People's Republic of China

Abstract. We study the initial boundary value problem of semilinear hyperbolic equations $u_{tt} - \Delta u = f(u)$ and semilinear parabolic equations $u_t - \Delta u = f(u)$ with critical initial data $E(0) = d$ (or $J(u_0) = d$), $I(u_0) < 0$, and prove that there exist non-global solutions under classical conditions on f.

1. Introduction. It is well known that the semilinear hyperbolic equations and semilinear parabolic equations are the most important nonlinear evolution equations in the area of mathematical physics (see the previous works [1], [2], [11], [7], [13], [15]). The following are examples of important problems that are considered in this paper:

$$\begin{aligned} &u_{tt} - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \\ &u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \quad x \in \Omega, \\ &u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &u_t - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \Omega, \\ &u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0. \end{aligned} \tag{1.2}$$

A powerful technique for treating the above problems is the so-called "potential well method", which was established by Payne and Sattinger [11]. The technique for proving the global nonexistence of solutions of abstract problems that include (1.1) and (1.2) was developed first in [2]. In [1] a stronger result for (1.1) and (1.2) was established, namely pointwise blow-up in finite time. In [3] and [14], the case for which $E(0) > 0$ was first considered. A "blow-up" (global nonexistence result) was proved in these two papers

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E-mail address: xurunzh@yahoo.com.cn

by relying on the condition that $(u_0, u_1) \geq 0$. The upper bound for $E(0)$ required was smaller than the depth of the potential well.

The potential well theory has ignited considerable interest among researchers to treat various parabolic and hyperbolic equations. Because it is not possible to cite all of the more than one hundred papers on the subject, we refer the reader to [4]–[6], [8]–[10], [16]–[22] and the references therein. In [11] the authors assumed that $f(u)$ satisfies the following conditions:

$$(H)\begin{cases} (i) & f \in C^1, f(0) = f'(0) = 0; \\ (ii) & (a)\ f(u) \text{ is monotone and convex for } u > 0, \text{ concave for } u < 0 \text{ or} \\ & (b)\ f(u) \text{ is convex for } -\infty < u < \infty; \\ (iii) & \text{There exist } p \text{ and } \gamma \text{ satisfying } 2 < p+1 \leq \gamma < \frac{n+2}{n-2} \\ & \text{such that} (p+1)F(u) \leq uf(u) \text{ and } |uf(u)| \leq \gamma |F(u)| \\ & \text{where } F(u) = \int_0^u f(s)\mathrm{d}s. \end{cases}$$

And Payne and Sattinger defined

$$W = \left\{u \in H_0^1(\Omega) | I(u) > 0, J(u) < d\right\} \cup \{0\},$$
$$V = \left\{u \in H_0^1(\Omega) | I(u) < 0, J(u) < d\right\},$$

where

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x,$$
$$I(u) = \|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x,$$
$$d = \inf_{u \in \mathcal{N}} J(u),\ \mathcal{N} = \{u \in H_0^1(\Omega) | I(u) = 0, \|\nabla u\| \neq 0\},$$
$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x = \frac{1}{2}\|u_t\|^2 + J(u).$$

In the above and the following discussions we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$, $\|\cdot\|_2$ by $\|\cdot\|$ and $(u, v) = \int_\Omega uv\mathrm{d}x$. In [11], by introducing the above sets W and V, as well as other functionals, Payne and Sattinger gave a series of properties of W and V. Then, by using V, they proved the global nonexistence of solution for problem (1.1) and (1.2) in the case $E(0) < d$ (or $J(u_0) < d$ for problem (1.2)), $I(u_0) < 0$. In early 1977, in [12], Levine was informed by the authors of [11] that both proofs of Theorem 2.3 (about uniqueness) and Lemma 2.7 (in the case f is convex) in [11] are incorrect. Recently, some other incorrect proofs regarding both invariance of the set V and global non-existence (see Lemma 4.2, Theorem 4.3 and Theorem 6.3 in [11]) were pointed out and corrected in [21] by introducing a family of potential wells. In the correction (ii) on page 2667 of [21], the authors indicate that $M'(t)$ could always be negative. Of course that is correct, but on page 294 of [11], near the top of the page, Payne and Sattinger rule out this possibility. The remark in [21] may leave the impression that they did not consider this possibility.

In [21] they required the nonlinear term $f(u)$ to satisfy

$$(H_a)\begin{cases}(i) & f\in C^1, f(0)=f'(0)=0;\\ (ii) & f(u) \text{ is monotone and convex for } u>0, \text{ concave for } u<0;\\ (iii) & \text{There exist a } p \text{ and } \gamma \text{ satisfying } 2<p+1\le\gamma<\frac{n+2}{n-2}\\ & \text{such that } (p+1)F(u)\le uf(u) \text{ and } |uf(u)|\le\gamma|F(u)|,\\ & \text{where } F(u)=\int_0^u f(s)\mathrm{d}s,\end{cases}$$

a hypothesis that will also be in force here. Moreover, in [21], some new results were achieved on the invariant sets, specifically global existence of solution for the critical initial data $E(0)=d$ (or $J(u_0)=d$), $I(u_0)\ge 0$ for problems (1.1) and (1.2). But in [21] the authors did not consider all the cases related to the critical initial data. Let us explain this in detail. The main results obtained in [21] are the following theorems:

Theorem 1.1. Let $f(u)$ satisfy (H_a), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. Assume that $E(0)<d$. Then when $I(u_0)>0$ or $\|\nabla u_0\|=0$, problem (1.1) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ and $u(t)\in W$ for $0\le t<\infty$; and when $I(u_0)<0$, the problem does not admit any global weak solution.

Theorem 1.2. Let $f(u)$ satisfy (H_a), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. Assume that $E(0)=d$, $I(u_0)\ge 0$. Then problem (1.1) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ and $u(t)\in\bar W=W\cup\partial W$ for $0\le t<\infty$.

Theorem 1.3. Let $f(u)$ satisfy (H_a), $u_0(x)\in H_0^1(\Omega)$. Assume that $J(u_0)<d$. Then when $I(u_0)>0$ or $\|\nabla u_0\|=0$, problem (1.2) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^2(0,\infty;L^2(\Omega))$ and $u(t)\in W$ for $0\le t<\infty$; and when $I(u_0)<0$, the problem does not admit any global weak solution.

Theorem 1.4. Let $f(u)$ satisfy (H_a), $u_0(x)\in H_0^1(\Omega)$. Assume that $J(u_0)=d$, $I(u_0)\ge 0$. Then problem (1.2) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^2(0,\infty;L^2(\Omega))$ and $u(t)\in\bar W$ for $0\le t<\infty$.

However for the global existence of solution for either problem (1.1) with $E(0)=d$, $I(u_0)<0$ or problem (1.2) with $J(u_0)=d$, $I(u_0)<0$, there are no results in [21], [11] or any other literature. In [19] Enzo Vitillaro considered the more general case $\|u_0\|_p>\lambda_1$, $E(0)\le E_1$ for a class of abstract evolution equations. But the critical case $E(0)=E_1$ depends on the presence of the damping term. Hence these problems are still open. One of the reasons is that one cannot easily obtain the non-global existence of solution solely by the method used for $E(0)<d$ or $J(u_0)<d$. For instance, in order to prove the non-global existence of solution to problem (1.1) with critical case $E(0)=d$, $I(u_0)<0$, one must ensure the invariance of set $V'=\{u\in H_0^1(\Omega)|I(u_0)<0\}$ under the flow of (1.1) in the case of $E(0)=d$. It is natural to think whether we can use the method for case $E(0)<d$ to solve the problem with critical case $E(0)=d$. In the process of treating the case $E(0)<d$, by a contradiction method, we can suppose the invariance of the set V' does not hold; then there exists a $t_0\in(0,T)$ such that $I(u(t_0))=0$ and $I(u)<0$ for $0<t<t_0$, where T is the existence time of u. Hence we have $\|\nabla u(t_0)\|\ge r(1)$ and

$J(u(t_0)) \geq d$. Unfortunately, this result does not contradict

$$\frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0) = d, \quad 0 \leq t < T.$$

Therefore this situation yields some difficulties, so we have to work out some new ideas in order to solve this problem.

The main purpose of this paper is to resolve the open problems mentioned above by using the potential well method. We prove that if $f(u)$ satisfies (H_a), $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$ for problem (1.1); or $J(u_0) = d$, $I(u_0) < 0$ for problem (1.2), then either problem (1.1) or problem (1.2) does not admit any global weak solution. So, we seek to give positive answers to the unsolved problems existing in [21] and [11].

2. Definitions and preliminary lemmas. In this section we give some definitions and recall some preliminary lemmas.

For (1.1) and (1.2) we define $J(u)$, $I(u)$, $E(t)$, d, W and V as above. In addition, we define the functional related to the Nehari flow as

$$I_\delta(u) = \delta\|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x, \delta > 0,$$

the depth of the family of potential wells as

$$d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u),$$

$$\mathcal{N}_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) = 0, \|\nabla u\| \neq 0\}, \delta > 0;$$

and the family of potential wells as

$$W_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) > 0,\ J(u) < d(\delta)\} \cup \{0\}, \quad 0 < \delta < b;$$
$$V_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) < 0,\ J(u) < d(\delta)\}, \quad 0 < \delta < b.$$

Then some lemmas can be recalled.

Lemma 2.1 ([21], [11]). Let $f(u)$ satisfy (H_a). Then
(i) $|F(u)| \leq A|u|^\gamma$ for some $A > 0$ and $\forall u \in \mathbb{R}$.
(ii) $|uf(u)| \leq \gamma A|u|^\gamma$, $\forall u \in \mathbb{R}$.

Lemma 2.2 ([21]). Let $f(u)$ satisfy (H_a). Assume that $u \in H_0^1(\Omega)$ and $I_\delta(u) < 0$. Then $\|\nabla u\| > r(\delta)$; in particular, if $I(u) < 0$, then $\|\nabla u\| > r(1)$, where

$$r(\delta) = \left(\frac{\delta}{aC_*^\gamma}\right)^{\frac{1}{\gamma-2}}, \quad a = \sup_{u\in\mathbb{R},u\neq 0} \frac{uf(u)}{|u|^\gamma}, \ \ C_* = \sup_{u\in H_0^1(\Omega),u\neq 0} \frac{\|u\|_\gamma}{\|\nabla u\|}.$$

Lemma 2.3 ([21]). Let $f(u)$ satisfy (H_a). For $d(\delta)$ we have
(i) $d(\delta) \geq a(\delta)r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$.
(iii) $\lim_{\delta\to 0} d(\delta) = 0$ and there exists a constant b satisfying $\frac{p+1}{2} \leq b \leq \frac{\gamma}{2}$ such that $d(b) = 0$ and $d(\delta) > 0$ for $0 < \delta < b$.
(iii) $d(\delta)$ is increasing on $0 < \delta \leq 1$, decreasing on $1 \leq \delta \leq b$ and takes the maximum $d = d(1)$ at $\delta = 1$.

Throughout this paper we employ the notion of weak solution defined below.

DEFINITION 2.4. $u = u(x,t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T)$ if $u \in L^\infty\left(0,T;H_0^1(\Omega)\right)$ with $u_t \in L^\infty\left(0,T;L^2(\Omega)\right)$ satisfying

(i) $(u_t,v) + \int_0^t (\nabla u, \nabla v) d\tau = \int_0^t (f(u),v) d\tau + (u_1,v), \ \forall v \in H_0^1(\Omega), t \in (0,T)$.

(ii) $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$.

(iii) $E(t) \le E(0), \quad 0 \le t < T.$ (2.1)

DEFINITION 2.5. $u = u(x,t)$ is called a weak solution of problem (1.2) on $\Omega \times [0,T)$ if $u \in L^\infty\left(0,T;H_0^1(\Omega)\right)$ with $u_t \in L^2\left(0,T;L^2(\Omega)\right)$ satisfying

(i) $(u_t,v) + (\nabla u, \nabla v) = (f(u),v), \quad \forall v \in H_0^1(\Omega), t \in (0,T)$.

(ii) $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$.

(iii) $\int_0^t \|u_\tau\|^2 d\tau + J(u) \le J(u_0), \quad 0 \le t < T.$ (2.2)

There is a difference between problem (1.1) and problem (1.2). For problem (1.2) we have the following Lemma 2.6 to guarantee the invariance of the set V_δ. Although this invariance is not for the critical data $J(u_0) = d$, it will be shown to be sufficient to derive the nonexistence of global solution of (1.2). But for problem (1.1) this method does not appear to work.

LEMMA 2.6 ([21]). Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$. Assume that $0 < e < d$, (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > e$ for $\delta \in (\delta_1, \delta_2)$. Then all weak solutions of problem (1.2) with $0 < J(u_0) \le e$ belong to V_δ for $\delta \in (\delta_1, \delta_2)$, provided $I(u_0) < 0$.

As mentioned above, we need to give the invariance of set V_δ for problem (1.1) as follows.

LEMMA 2.7. Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $(u_0,u_1) \ge 0$. Then the following set

$$V' = \{u \in H_0^1(\Omega) | I(u) < 0\}$$

is invariant under the flow of (1.1).

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $E(0) = d$, $I(u_0) < 0$ and $(u_0,u_1) \ge 0$, T being the existence time of $u(t)$. Let us prove $I(u) < 0$ for $0 < t < T$. If it is false, then there exists a $t_0 \in (0,T)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \le t < t_0$. Hence we have $\|\nabla u\| > r(1)$ for $0 \le t < t_0$ and $\|\nabla u(t_0)\| \ge r(1) > 0$. By the definition of d we get $J(u(t_0)) \ge d$. From this and

$$\frac{1}{2}\|u_t(t_0)\|^2 + J(u(t_0)) \le E(0) = d,$$

we get $J(u(t_0)) = d$ and $\|u_t(t_0)\|^2 = 0$. Let $M(t) = \|u\|^2$. Then we have

$$\dot{M} = 2(u,u_t)$$

with

$$\dot{M}(0) = 2(u_0,u_1) \ge 0,$$
$$\ddot{M}(t) = 2\|u_t\|^2 - 2I(u) > 0, 0 \le t < t_0.$$

Hence $\dot{M}(t)$ is strictly increasing with respect to $t \in [0, t_0]$. This together with $\dot{M}(0) = 2(u_0, u_1) \geq 0$ gives $\dot{M}(t_0) = 2(u(t_0), u_t(t_0)) > 0$. This contradicts $\|u_t(t_0)\| = 0$. So this completes this proof. □

3. Main results and proof. In this section we state the main results and prove them.

THEOREM 3.1. Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$. Then the existence time of a weak solution for problem (1.1) is finite.

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$, T being the existence time of $u(t)$. Let us prove $T < \infty$. Arguing by contradiction, let us assume $T = +\infty$. Again, let

$$M(t) = \|u\|^2.$$

Then we have

$$\dot{M} = 2(u, u_t)$$

with

$$\begin{aligned} \dot{M}(0) &= 2(u_0, u_1) \geq 0, \\ \ddot{M}(t) &= 2\|u_t\|^2 - 2I(u), 0 \leq t < \infty. \end{aligned} \tag{3.1}$$

From (2.4) and $(p+1)F(u) \leq uf(u)$ we arrive at

$$\frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0) = d.$$

Hence we have

$$\begin{aligned} \ddot{M}(t) &\geq (p+3)\|u_t\|^2 + (p-1)\|\nabla u\|^2 - 2(p+1)d \\ &\geq (p+3)\|u_t\|^2 + (p-1)\lambda_1 M(t) - 2(p+1)d, \ 0 \leq t < \infty, \end{aligned} \tag{3.2}$$

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla u\|^2}{\|u\|^2}.$$

Eq. (3.1) and Lemma 2.7 yield $\ddot{M}(t) > 0$ for $0 \leq t < \infty$ and $\dot{M}(t)$ is strictly increasing for $0 \leq t < \infty$. Hence for any $t_0 > 0$ we have $\dot{M}(t) \geq \dot{M}(t_0) > 0$ for $t \geq t_0$, and

$$M(t) \geq \dot{M}(t_0)(t - t_0) + M(t_0) \geq \dot{M}(t_0)(t - t_0), \ t \geq t_0.$$

Therefore for sufficiently large t, we have $(p-1)\lambda_1 M(t) > 2(p+1)d$ and

$$\ddot{M}(t) > (p+3)\|u_t\|^2.$$

Hence

$$\begin{aligned} M\ddot{M} - \frac{p+3}{4}\left(\dot{M}\right)^2 &\geq (p+3)\left(\|u\|^2\|u_t\|^2 - (u, u_t)^2\right) \geq 0, \\ \left(M^{-\alpha}\right)'' = \frac{-\alpha}{M^{\alpha+2}}\left(M\ddot{M} - (\alpha+1)\left(\dot{M}\right)^2\right) &\leq 0, \quad \alpha = \frac{p-1}{4}. \end{aligned} \tag{3.3}$$

From (3.3) it follows that there exists a $T_1 > 0$ such that

$$\lim_{t \to T_1} M^{-\alpha}(t) = 0,$$

and

$$\lim_{t\to T_1} M(t) = +\infty,$$

which contradict $T = +\infty$. The proof is completed. $\square$

From Theorem 1.2 and Theorem 3.1 we can get a sharp condition for global existence of solution for problem (1.1) with $E(0) = d$ as follows:

THEOREM 3.2. Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$, $(u_0, u_1) \geq 0$. Then when $I(u_0) \geq 0$, problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in \bar{W} = W \cup \partial W$ for $0 \leq t < \infty$; and when $I(u_0) < 0$, the problem does not admit any global weak solution.

THEOREM 3.3. Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$. Assume that $J(u_0) = d$, $I(u_0) < 0$. Then the existence time of weak solution for problem (1.2) is finite.

Proof. Let $u(t)$ be any weak solution of problem (1.2) with $J(u_0) = d$, $I(u_0) < 0$, T being the existence time of $u(t)$. Let us prove $T < \infty$. Arguing by contradiction, we suppose $T = +\infty$. Let

$$M_1(t) = \int_0^t \|u\|^2 \mathrm{d}\tau.$$

Then

$$\ddot{M}_1(t) = 2(u_t, u) = -2I(u), \quad 0 \leq t < \infty. \tag{3.4}$$

From (2.5) and

$$\int_\Omega u f(u)\mathrm{d}x \geq (p+1)\int_\Omega F(u)\mathrm{d}x$$

we get

$$\begin{aligned} &\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1} I(u) \\ \leq &\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + J(u) \leq J(u_0) = d. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we have

$$\begin{aligned} \ddot{M}_1(t) &\geq 2(p+1)\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + (p-1)\|\nabla u\|^2 - 2(p+1)d \\ &\geq 2(p+1)\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + (p-1)\lambda_1 \dot{M}_1(t) - 2(p+1)d. \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} \left(\int_0^t (u_\tau, u)\mathrm{d}\tau\right)^2 &= \left(\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\|u\|^2 \mathrm{d}\tau\right)^2 \\ &= \frac{1}{4}\left(\|u\|^4 - 2\|u_0\|^2\|u\|^2 + \|u_0\|^4\right) \\ &= \frac{1}{4}\left(\dot{M}_1^2(t) - 2\|u_0\|^2 \dot{M}_1(t) + \|u_0\|^4\right). \end{aligned}$$

Hence we have

$$M_1\ddot{M}_1 - \frac{p+1}{2}(\dot{M}_1)^2 \geq 2(p+1)\left[\int_0^t \|u\|^2 \mathrm{d}\tau \int_0^t \|u_\tau\|^2 \mathrm{d}\tau - \left(\int_0^t (u, u_\tau)\mathrm{d}\tau\right)^2\right] + (p-1)\lambda_1 M_1(t)\dot{M}_1(t) - (p+1)\|u_0\|^2\dot{M}_1(t) - 2(p+1)dM_1(t) + \frac{p+1}{2}\|u_0\|^4.$$

From this and the Schwartz inequality we get

$$\begin{aligned} &M_1\ddot{M}_1 - \frac{p+1}{2}(\dot{M}_1)^2 \\ \geq &(p-1)\lambda_1 M_1(t)\dot{M}_1(t) - (p+1)\|u_0\|^2\dot{M}_1(t) - 2(p+1)dM_1(t) \\ = &\left(\frac{1}{2}(p-1)\lambda_1 M_1(t) - (p+1)\|u_0\|^2\right)\dot{M}_1(t) \\ &+ \left(\frac{1}{2}(p-1)\lambda_1\dot{M}_1(t) - 2(p+1)d\right)M_1(t). \end{aligned} \tag{3.7}$$

On the other hand, from $J(u_0) = d > 0$, $I(u_0) < 0$ and the continuity of $J(u)$ and $I(u)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1)) > 0$ and $I(u) < 0$ for $0 \leq t \leq t_1$. Hence $(u_t, u) = -I(u) > 0$ and $\|u_t\| > 0$ for $0 \leq t \leq t_1$. From this and the continuity of $\int_0^t \|u_\tau\|^2 \mathrm{d}\tau$ it follows that we can choose a t_1 such that

$$0 < d_1 = d - \int_0^{t_1} \|u_t\|^2 \mathrm{d}t < d.$$

And by (2.5) we have

$$0 < J(u(t_1)) \leq d - \int_0^{t_1} \|u_t\|^2 \mathrm{d}t = d_1 < d.$$

Thus if in Lemma 2.6 we take $t = t_1$ as the initial time, then we have $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t_1 \leq t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Hence we have $I_\delta(u) < 0$ and $\|\nabla u\| > r(\delta)$ for $\delta \in (1, \delta_2)$, $t_1 \leq t < \infty$, and $I_{\delta_2}(u) \leq 0$, $\|\nabla u\| \geq r(\delta_2)$ for $t_1 \leq t < \infty$. Thus from (3.4) we get

$$\begin{aligned} \ddot{M}_1(t) &= -2I(u) = 2(\delta_2 - 1)\|\nabla u\|^2 - 2I_{\delta_2}(u) \\ &\geq 2(\delta_2 - 1)\|\nabla u\|^2 \geq 2(\delta_2 - 1)r^2(\delta_2) \equiv C(\delta_2), \quad t_1 \leq t < \infty, \end{aligned} \tag{3.8}$$

$$\dot{M}_1(t) \geq C(\delta_2)(t - t_1) + \dot{M}_1(t_1) \geq C(\delta_2)(t - t_1), \quad t_1 \leq t < \infty, \tag{3.9}$$

$$M_1(t) \geq \frac{1}{2}C(\delta_2)(t - t_1)^2 + M_1(t_1) > \frac{1}{2}C(\delta_2)(t - t_1)^2, \quad t_1 \leq t < \infty. \tag{3.10}$$

From (3.9) and (3.10) it follows that for sufficiently large t we have

$$\frac{1}{2}(p-1)\lambda_1 M_1(t) > (p+1)\|u_0\|^2$$

and

$$\frac{1}{2}(p-1)\lambda_1\dot{M}_1(t) > 2(p+1)d.$$

Thus (3.7) yields

$$M_1(t)\ddot{M}_1(t) - \frac{p+1}{2}\left(\dot{M}_1(t)\right)^2 > 0,$$

which gives

$$\left(M_1^{-\alpha}\right)'' = \frac{-\alpha}{M_1^{\alpha+2}}\left(M_1\ddot{M}_1 - (\alpha+1)\left(\dot{M}_1\right)^2\right) \le 0, \quad \alpha = \frac{p-1}{2}. \tag{3.11}$$

From this it follows that there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} M_1^{-\alpha}(t) = 0$$

and

$$\lim_{t\to T_1} M_1(t) = +\infty,$$

which contradicts $T = +\infty$. □

From Theorem 1.4 and Theorem 3.3 we can obtain a sharp condition for global existence of solution for problem (1.2) with $J(u_0) = d$ as follows.

Theorem 3.4. *Let $f(u)$ satisfy (H_a), $u_0(x) \in H_0^1(\Omega)$. Assume that $J(u_0) = d$. Then when $I(u_0) \ge 0$, problem (1.2) admits a global weak solution $u(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_t(t) \in L^2(0,\infty; L^2(\Omega))$ and $u(t) \in \bar{W} = W \cup \partial W$ for $0 \le t < \infty$; and when $I(u_0) < 0$, the problem does not admit any global weak solution.*

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