DYNAMIC PATTERN FORMATION IN SWIFT-HOHENBERG EQUATIONS

BY

TUNG HOANG (*Department of Mathematics, State University of New York at Buffalo, New York*)

AND

HYUNG JU HWANG (*Department of Mathematics, Pohang University of Science and Technology,
Pohang 790-784, Korea*)

Abstract. In this paper we study nonlinear instability in the Swift-Hohenberg equation. The short-time pattern formation under the fully nonlinear setting is rigorously characterized. For unstable constant equilibria, we investigate the nonlinear dynamics of patterns which occur within the time interval of existence.

1. Introduction and main result. In this paper we study the problem of nonlinear pattern formation in the Swift-Hohenberg (SH) equation. The equation was derived in the context of thermal convection in [8]. The SH equation has been used to model patterns in a variety of fluids and biological materials [3], [2] such as Taylor-Couette flow and lasers. Related works are found in [9], [10], [11], [13]. In this article, we will rigorously establish bounds on the short-time common patterns under the fully nonlinear regime and characterize the nonlinear patterns by a finite number of the maximal growing modes over a time scale of $\ln \frac{1}{\delta}$.

We consider the generalized Swift-Hohenberg equation

$$\frac{du}{dt} = \alpha u - (1 + \Delta)^2 u - f(u) \quad (1.1)$$

on $\mathbb{T}^d = (0, L_1) \times (0, L_2) \times \dots \times (0, L_d)$. Here $\alpha > 0$ is a given constant, $f(\cdot)$ is a given function, and f satisfies $f(0) = f'(0) = 0$. We impose the following boundary conditions and initial condition

$$u = 0, \quad \frac{\partial^2 u}{\partial x_i^2} = 0, \quad \frac{\partial^4 u}{\partial x_i^4} = 0 \quad \text{for all } i = 1, 2, \dots, d \quad (1.2)$$

at $x_i = 0, L_i$ and

$$u(x, 0) = u_0(x), \quad (1.3)$$

where $u_0(x)$ is given.

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E-mail address: thoang@buffalo.edu

E-mail address: hjhwang@postech.ac.kr

We use the norm in L^2 : for every $w(x) \in L^2(\mathbb{T}^d)$.

NOTATION 1.

$$\|w\| = \|w\|_{L^2}.$$

The corresponding linearized equation then takes the form

$$\frac{du}{dt} = (\alpha - 1)u - \Delta^2 u - 2\Delta u. \quad (1.4)$$

Let $\mathbf{q} = (q_1, q_2, \dots, q_d) \in \Omega = (\mathbb{N} \cup \{0\})^d$ and let

$$e_{\mathbf{q}}(x) = \prod_{i=1}^d \sin \frac{q_i \pi x_i}{L_i}.$$

Then we have

$$\|e_{\mathbf{q}}\| = \sqrt{\frac{L_1 L_2 \dots L_d}{2^d}}$$

and $\{e_{\mathbf{q}}\}$ forms a basis of the space of functions in $\mathbb{T}^d$ which satisfy (1.2).

We will solve the linearized equation (1.4) with the boundary and initial conditions.

$$\begin{aligned} \frac{du}{dt} &= (\alpha - 1)u - \Delta^2 u - 2\Delta u, \\ u &= \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^4 u}{\partial x_i^4} = 0, \quad \text{at } x_i = 0, L_i, \quad \text{for all } i = 1, 2, \dots, d, \\ u(x, 0) &= u_0(x). \end{aligned}$$

Suppose the initial datum has the following decomposition:

$$u_0(x) = \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}} e_{\mathbf{q}}(x).$$

First we will find a solution with initial datum $u(x, 0) = e_{\mathbf{q}}(x)$. We seek a solution of the following form:

$$u(x, t) = e^{\lambda t} e_{\mathbf{q}}(x). \quad (1.5)$$

Then we have

$$\begin{cases} \frac{du}{dt} = \lambda e^{\lambda t} e_{\mathbf{q}}(x), \\ \Delta u = -q^2 e^{\lambda t} e_{\mathbf{q}}(x), \\ \Delta^2 u = q^4 e^{\lambda t} e_{\mathbf{q}}(x), \end{cases}$$

where $q = \sqrt{\sum_{i=1}^d (\frac{q_i \pi}{L_i})^2}$. Then $u(x, t)$ defined in (1.5) is a solution of (1.4) if and only if

$$\lambda = \alpha - 1 - q^4 + 2q^2 = \alpha - (q^2 - 1)^2.$$

Define $\lambda_{\mathbf{q}} = \alpha - (q^2 - 1)^2$. Then we will make the following assumption.

ASSUMPTION 1. There exists $\mathbf{q} \in \Omega$ such that $\lambda_{\mathbf{q}} > 0$.

Note that if $L_i = k\pi$, $k \in \mathbb{N}$ for some i , then for $\mathbf{q} = (q_1, q_2, \dots, q_d)$ with $q_i = k$, $q_j = 0$, for all $j \neq i$, $q^2 = \sum_{j=1}^d (\frac{q_j \pi}{L_j})^2 = 1$.

It is easy to see that there exist only finitely many values $\mathbf{q} \in \Omega$ such that $\alpha - (q^2 - 1)^2 > 0$. Define

$$\lambda_{max} = \max_{\mathbf{q} \in \Omega} \lambda_{\mathbf{q}} > 0, \quad (1.6)$$

$$\Omega_{max} = \{\mathbf{q} \in \Omega \mid \lambda_{\mathbf{q}} = \lambda_{max}\}, \quad (1.7)$$

where $\nu > 0$ is the gap between λ_{max} and the rest of $\lambda_{\mathbf{q}}$, $\mathbf{q} \in \Omega$. For $u_0 = \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}} e_{\mathbf{q}}(x)$ we define

$$L[u_0](x, t) = \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}} e^{\lambda_{\mathbf{q}} t} e_{\mathbf{q}}(x), \quad (1.8)$$

$$M[u_0](x, t) = \sum_{\mathbf{q} \in \Omega_{max}} w_{\mathbf{q}} e^{\lambda_{max} t} e_{\mathbf{q}}(x).$$

Then $L[u_0](x, t)$ is a solution of (1.4), (1.2), (1.3).

We will frequently use the following ingredients: For $d \leq 3$ by the Sobolev imbedding theorem there exists a constant $C_0 > 0$ such that for all $g \in H^2(\mathbb{T}^d)$ we have

$$\|g\|_{\infty(\mathbb{T}^d)} \leq C_0 \|g\|_{H^2(\mathbb{T}^d)}. \quad (1.9)$$

We assume that $f \in \mathbb{C}^2$ near 0 so that there exists $\eta > 0$ such that

$$C_{\eta} = \sup_{\|w\|_{\infty} \leq C_0 \eta} \|f(w)\|_{C^2} < +\infty. \quad (1.10)$$

From (1.9) and (1.10) we have

$$\sup_{\|w\|_{H^2} \leq \eta} \|f(w)\|_{C^2} \leq C_{\eta} < +\infty.$$

Then we have the following lemma.

LEMMA 1. Let $L[u_0](x, t)$ and λ_{max} be defined in (1.8) and (1.6) respectively. Then we have

$$\|L[u_0](x, t)\| \leq e^{\lambda_{max} t} \|u_0\|.$$

Proof. We have

$$L[u_0](x, t) = \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}} e^{\lambda_{\mathbf{q}} t} e_{\mathbf{q}}(x).$$

Since the $e_{\mathbf{q}}(x)$'s are orthogonal in $L^2(\mathbb{T}^d)$, we get

$$\|L[u_0](x, t)\|^2 = \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}}^2 e^{2\lambda_{\mathbf{q}} t} \|e_{\mathbf{q}}\|^2,$$

where $\lambda_{\mathbf{q}} \leq \lambda_{max}$ and so we have

$$\|L[u_0](x, t)\|^2 \leq \sum_{\mathbf{q} \in \Omega} w_{\mathbf{q}}^2 \|e_{\mathbf{q}}\|^2 e^{2\lambda_{max} t} = \|u_0\|^2 e^{2\lambda_{max} t}.$$

Hence we get

$$\|L[u_0](x, t)\| \leq e^{\lambda_{max} t} \|u_0\|. \quad \square$$

We will now state the main theorem on pattern formation under the fully nonlinear setting. We fix $\theta > 0$ to be a small constant. For any small $\delta > 0$ we define the escape time T^δ by

$$\theta = \delta \exp(\lambda_{max} T^\delta). \quad (1.11)$$

The following theorem establishes a characterization of nonlinear patterns in the SH equation by a finite combination of the maximal linear growing modes.

THEOREM 1. Let $u_0 \in H^2$, $\|u_0\| = 1$. Then there exist constants $\delta_0 > 0$, $C > 0$, $\theta > 0$ depending on α such that for all $0 < \delta \leq \delta_0$ if the initial perturbation of 0 is $u^\delta(x, 0) = \delta u_0$, then its nonlinear evolution $u^\delta(x, t)$ satisfies

$$\|u^\delta(x, t) - \delta M[u_0](x, t)\| \leq C\{e^{-\nu t} + \delta\|u_0\|_{H^2}^2 + \delta e^{\lambda_{max} t}\} \delta e^{\lambda_{max} t} \quad (1.12)$$

for $0 \leq t \leq T^\delta$. Here

$$M[u_0](x, t) = \left[\sum_{\mathbf{q} \in \Omega_{max}} w_{\mathbf{q}} e_{\mathbf{q}}(x) \right] e^{\lambda_{max} t}.$$

Notice that Theorem 1 implies instability for the Swift-Hohenberg equations (1.1), (1.2), (1.3); refer to [4, 5].

The passage from linear to nonlinear instability does not rely on systematic machinery. In the PDE system with a dissipation, characterization of pattern formation has been developed in Kirchhoff ellipses [6], reaction-diffusion systems [5], and Keller-Segel systems [4]. The methods in this paper are based on the bootstrap idea that was first introduced by Guo and Strauss.

In section 2, we will provide energy estimates to control the nonlinear terms in the SH equation and the nonlinear patterns will be established in section 3.

2. Energy estimates. In this section, we provide energy estimates to control the nonlinear terms in the SH equation (1.1).

LEMMA 2. If $u \in H^4(\mathbb{T}^d)$ satisfies (1.2), then for every derivative $|\partial| \leq 2$ we have

$$\int_{\mathbb{T}^d} (\Delta \partial u)^2 dx = \sum_{i=1}^d \int_{\mathbb{T}^d} (\partial u_{x_i x_i})^2 + 2 \sum_{1 \leq i < j \leq d} \int_{\mathbb{T}^d} (\partial u_{x_i x_j})^2.$$

Here $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$.

Proof. First of all, we note that

$$(\Delta \partial u)^2 = \left(\sum_{i=1}^d \partial u_{x_i x_i} \right)^2 = \sum_{i=1}^d (\partial u_{x_i x_i})^2 + 2 \sum_{1 \leq i < j \leq d} \partial u_{x_i x_i} \partial u_{x_j x_j}.$$

So we have

$$\int_{\mathbb{T}^d} (\Delta \partial u)^2 = \sum_{i=1}^d \int_{\mathbb{T}^d} (\partial u_{x_i x_i})^2 + 2 \sum_{1 \leq i < j \leq d} \int_{\mathbb{T}^d} \partial u_{x_i x_i} \partial u_{x_j x_j}.$$

By the boundary condition (1.2) we can integrate by parts to obtain

$$\begin{aligned}
 \int_{\mathbb{T}^d} \partial u_{x_i x_i} \partial u_{x_j x_j} &= - \int_{\mathbb{T}^d} \partial u_{x_i} \partial u_{x_j x_j x_i} \\
 &= - \int_{\mathbb{T}^d} \partial u_{x_i} \partial u_{x_i x_j x_j} \\
 &= \int_{\mathbb{T}^d} \partial u_{x_i x_j} \partial u_{x_i x_j} \\
 &= \int_{\mathbb{T}^d} \partial u_{x_i x_j}^2.
 \end{aligned}$$

Therefore we get

$$\int_{\mathbb{T}^d} (\Delta \partial u)^2 dx = \sum_{i=1}^d \int_{\mathbb{T}^d} (\partial u_{x_i x_i})^2 + 2 \sum_{1 \leq i < j \leq d} \int_{\mathbb{T}^d} (\partial u_{x_i x_j})^2. \quad \square$$

LEMMA 3. If $u(x, t)$ is a solution of (1.1), (1.2), (1.3) satisfying $\|u(x, t)\|_{H^2} \leq \eta$, then we have

$$\frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} (\partial u)^2 dx \leq C_2 \|u\|^2,$$

for

$$\begin{aligned}
 C_2 &= \frac{(\alpha - 1 + 2d + d^2 C_1)^2}{2} d^2, \\
 C_1 &= C_\eta (1 + C_0^2 \eta),
 \end{aligned}$$

where C_0 is a universal constant.

Proof. Taking second order ∂ derivatives of (1.1) we have

$$\begin{aligned}
 \frac{d}{dt} \partial u &= \alpha \partial u - (1 + \Delta)^2 \partial u - \partial f(u) \\
 &= (\alpha - 1) \partial u - \Delta^2 \partial u - 2\Delta \partial u - \partial f(u).
 \end{aligned}$$

Hence we get

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} |\partial u|^2 &= (\alpha - 1) |\partial u|^2 - (\Delta^2 \partial u) \partial u - 2(\Delta \partial u) \partial u - \partial f(u) \partial u. \\
 \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\partial u|^2 &= (\alpha - 1) \int_{\mathbb{T}^d} |\partial u|^2 - \int_{\mathbb{T}^d} (\Delta^2 \partial u) \partial u - 2 \int_{\mathbb{T}^d} (\Delta \partial u) \partial u \\
 &\quad - \int_{\mathbb{T}^d} \partial f(u) \partial u \\
 &= (\alpha - 1) \int_{\mathbb{T}^d} |\partial u|^2 - \int_{\mathbb{T}^d} |\Delta \partial u|^2 + 2 \int_{\mathbb{T}^d} |\nabla \partial u|^2 \\
 &\quad - \int_{\mathbb{T}^d} \partial f(u) \partial u,
 \end{aligned} \tag{2.1}$$

by integration by parts and using the boundary condition (1.2). We now treat the last nonlinear term $\int_{\mathbb{T}^d} \partial f(u) \partial u$.

Assume $\partial = \frac{\partial^2}{\partial x_{i_1} \partial x_{i_2}}$.

$$\partial f(u) = f'(u)\partial u + f''(u)u_{x_{i_1}}u_{x_{i_2}}.$$

Therefore we get

$$\int_{\mathbb{T}^d} \partial f(u)\partial u = \int_{\mathbb{T}^d} f'(u)(\partial u)^2 + \int_{\mathbb{T}^d} f''(u)u_{x_{i_1}}u_{x_{i_2}}\partial u.$$

By assumption (1.10) and $\|u(x, t)\|_{H^2} \leq \eta$,

$$- \int_{\mathbb{T}^d} \partial f(u)\partial u \leq C_\eta \left[\int_{\mathbb{T}^d} |\partial u|^2 + \|\nabla u\|_{L^4} \|\nabla u\|_{L^4} \|\nabla^2 u\| \right].$$

We have a constant $C_0 > 0$ such that if $g \in H^1(\mathbb{T}^d)$ and $\int_{\mathbb{T}^d} g = 0$, then

$$\|g\| + \|g\|_{L^4} \leq C_0 \|\nabla g\|. \quad (2.2)$$

Hence we get $\|\nabla u\|_{L^4} \leq C_0 \|\nabla^2 u\|$ and $\|\nabla u\|_{L^4} \leq C_0 \|u\|_{H^2} \leq C_0 \eta$. Therefore we get

$$- \int_{\mathbb{T}^d} \partial f(u)\partial u \leq C_\eta (1 + C_0^2 \eta) \|\nabla^2 u\|^2.$$

Let $C_1 = C_\eta (1 + C_0^2 \eta)$; then

$$- \int_{\mathbb{T}^d} \partial f(u)\partial u \leq C_1 \|\nabla^2 u\|^2. \quad (2.3)$$

Set $v = \partial u$:

$$\int_{\mathbb{T}^d} |\nabla \partial u|^2 = \int_{\mathbb{T}^d} |\nabla v|^2 = \sum_{i=1}^d \int_{\mathbb{T}^d} |v_{x_i}|^2 = - \sum_{i=1}^d \int_{\mathbb{T}^d} v v_{x_i x_i}.$$

Hence we get

$$\begin{aligned} \int_{\mathbb{T}^d} |\nabla \partial u|^2 &\leq \sum_{i=1}^d \left[\int_{\mathbb{T}^d} \frac{1}{4} |v_{x_i x_i}|^2 + \int_{\mathbb{T}^d} v^2 \right] \\ &\leq \frac{1}{4} \int_{\mathbb{T}^d} |\Delta \partial u|^2 + d \int_{\mathbb{T}^d} v^2, \end{aligned}$$

since $\sum_{i=1}^d \int_{\mathbb{T}^d} |v_{x_i x_i}|^2 \leq \int_{\mathbb{T}^d} |\Delta \partial u|^2$ by Lemma 2. Therefore we have

$$\begin{aligned} (\alpha - 1) \int_{\mathbb{T}^d} |\partial u|^2 - \int_{\mathbb{T}^d} |\Delta \partial u|^2 + 2 \int_{\mathbb{T}^d} |\nabla \partial u|^2 &\leq (\alpha - 1 + 2d) \int_{\mathbb{T}^d} |\partial u|^2 \\ &\quad - \frac{1}{2} \int_{\mathbb{T}^d} |\Delta \partial u|^2. \end{aligned} \quad (2.4)$$

From (2.1), (2.3), (2.4) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 &\leq (\alpha - 1 + 2d) \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 - \frac{1}{2} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\Delta \partial u|^2 \\ &\quad + d^2 C_1 \|\nabla^2 u\|^2. \end{aligned}$$

Let $A_1 = \alpha - 1 + 2d + d^2 C_1$; then

$$\frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 \leq A_1 \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 - \frac{1}{2} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\Delta \partial u|^2. \quad (2.5)$$

By Lemma 2, we have

$$\int_{\mathbb{T}^d} |\Delta \partial u|^2 = \sum_{i=1}^d \int_{\mathbb{T}^d} |\partial u_{x_i x_i}|^2 + 2 \sum_{1 \leq i < j \leq d} \int_{\mathbb{T}^d} |\partial u_{x_i x_j}|^2.$$

Hence we get

$$\begin{aligned} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\Delta \partial u|^2 &= \sum_{|\partial|=4} \int_{\mathbb{T}^d} |\partial u|^2, \\ \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 &= \sum_{i=1}^d \sum_{j=1}^d \int_{\mathbb{T}^d} u_{x_i x_j}^2 \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{\mathbb{T}^d} u u_{x_i x_i x_j x_j} \\ &\leq \frac{1}{2A_1} \sum_{|\partial|=4} \int_{\mathbb{T}^d} |\partial u|^2 + \frac{A_1}{2} d^2 \int_{\mathbb{T}^d} u^2 \\ &\leq \frac{1}{2A_1} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\Delta \partial u|^2 + \frac{A_1}{2} d^2 \int_{\mathbb{T}^d} u^2. \end{aligned} \quad (2.6)$$

From (2.5) and (2.6) we have

$$\frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 dx \leq \frac{A_1^2}{2} d^2 \int_{\mathbb{T}^d} u^2. \quad (2.7)$$

This completes the proof. $\square$

LEMMA 4. Suppose $u(x, t)$ is a solution of (1.1) such that $\|u(x, t)\|_{H^2} \leq \eta$ and

$$\|u(x, t)\| \leq 2e^{\lambda_{max} t} \|u(x, 0)\|. \quad (2.8)$$

Then we have

$$\|u(t)\|_{H^2}^2 \leq C_3 \{ \|u(0)\|_{H^2}^2 + e^{2\lambda_{max} t} \|u(0)\|^2 \},$$

where $C_3 = (2 + \sqrt{d}) \max\{\frac{4C_2}{\lambda_{max}}, 4\}$.

Proof. We first consider the second-order derivatives of $u(x, t)$. From the previous lemma and our assumptions on $\|u(t)\|_{H^2} \leq \eta$ we have

$$\frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 dx \leq C_2 \|u(t)\|^2 \leq 4C_2 e^{2\lambda_{max} t} \|u(0)\|^2. \quad (2.9)$$

Integrating (2.9) from 0 to t we have

$$\begin{aligned} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u(x, t)|^2 dx &\leq \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u(x, 0)|^2 dx + \frac{4C_2}{\lambda_{max}} e^{2\lambda_{max} t} \|u(0)\|^2 \\ &\leq \|u(0)\|_{H^2}^2 + \frac{4C_2}{\lambda_{max}} e^{2\lambda_{max} t} \|u(0)\|^2. \end{aligned} \quad (2.10)$$

For the 0^{th} and the first derivatives:

$$\|u(t)\|^2 \leq 4e^{2\lambda_{max} t} \|u(0)\|^2$$

by (2.8).

$$\begin{aligned}
\sum_{j=1}^d \int_{\mathbb{T}^d} \left(\frac{\partial u}{\partial x_j} \right)^2 &= - \sum_{j=1}^d \int_{\mathbb{T}^d} u \frac{\partial^2 u}{\partial x_j^2} \\
&\leq \frac{\sqrt{d}}{2} \sum_{j=1}^d \int_{\mathbb{T}^d} \left(\frac{\partial^2 u}{\partial x_j^2} \right)^2 + \frac{1}{2\sqrt{d}} \sum_{j=1}^d \int_{\mathbb{T}^d} u^2 \\
&\leq \frac{\sqrt{d}}{2} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial u|^2 dx + \frac{\sqrt{d}}{2} \int_{\mathbb{T}^d} u^2.
\end{aligned}$$

Thus the lemma follows. $\square$

3. Proof of the main result.

Proof. Let $u^\delta(x, t)$ be a family of solutions to the SH equation (1.1) with initial data $u^\delta(x, 0) = \delta u_0$. Define

$$\begin{aligned}
T^* &= \sup\{t > 0 \mid \|u^\delta(s) - \delta L[u_0](\cdot, s)\| \leq \frac{\delta}{2} e^{\lambda_{max}s}, \forall 0 \leq s \leq t\}, \\
T^{**} &= \sup\{t > 0 \mid \|u(s)\| \leq \eta, \forall 0 \leq s \leq t\}.
\end{aligned}$$

By the definition of T^* for $0 \leq t \leq T^*$ and by Lemma 1, we get

$$\|u^\delta(t)\| \leq \frac{3}{2} \delta e^{\lambda_{max}t}.$$

Using Lemma 4 we have

$$\|u^\delta(t)\|_{H^2} \leq \sqrt{C_3} \{\delta \|u(0)\|_{H^2} + \delta e^{\lambda_{max}t}\}. \quad (3.1)$$

We estimate the L^2 norm of $u^\delta(x, t)$ for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$. Apply Duhamel's principle to obtain

$$u^\delta(t) = \delta L[u_0](\cdot, t) + \int_0^t L[-f(u^\delta(\tau))](\cdot, t - \tau) d\tau.$$

Apply Lemma 1 to deduce

$$\|u^\delta(t) - \delta L[u_0](\cdot, t)\| \leq \int_0^t e^{\lambda_{max}(t-\tau)} \|f(u^\delta(\tau))\| d\tau. \quad (3.2)$$

Since $f(0) = f'(0) = 0$ there exists $v(x, \tau) \in \mathbb{R}$ with $|v(x, \tau)| \leq |u^\delta(x, \tau)|$ such that

$$f(u^\delta(x, \tau)) = \frac{f''(v(x, \tau))}{2} u^\delta(x, \tau)^2.$$

By $\|u^\delta(\tau)\|_{H^2} \leq \eta$ from (1.9) we have $\|u^\delta(x, \tau)\|_\infty \leq C_0 \eta$ and this implies $|v^\delta(x, \tau)| \leq C_0 \eta$. By (1.10) we have

$$|f''(v(x, \tau))| \leq C_\eta$$

so we get

$$|f(u^\delta(x, \tau))| \leq \frac{1}{2} C_\eta u^\delta(x, \tau)^2.$$

Hence we deduce

$$\|f(u^\delta(\tau))\| \leq \frac{1}{2} C_\eta \|u^\delta(\tau)\| \|u^\delta(\tau)\|_\infty.$$

By (1.9) we have

$$\|f(u^\delta(\tau))\| \leq \frac{1}{2}C_\eta \|u^\delta(\tau)\|_{H^2} C_0 \|u^\delta(\tau)\|_{H^2} = \frac{C_0 C_\eta}{2} \|u^\delta(\tau)\|_{H^2}^2. \quad (3.3)$$

From (3.2) and (3.3) we have

$$\|u^\delta(t) - \delta L[u_0](\cdot, t)\| \leq \frac{C_0 C_\eta}{2} \int_0^t e^{\lambda_{max}(t-\tau)} \|u^\delta(\tau)\|_{H^2}^2 d\tau.$$

By Lemma 4 we get

$$\|u^\delta(\tau)\|_{H^2}^2 \leq C_3 \{\delta^2 \|u(0)\|_{H^2}^2 + \delta^2 e^{2\lambda_{max}\tau}\}.$$

Therefore we obtain

$$\begin{aligned} & \|u^\delta(t) - \delta L[u_0](\cdot, t)\| \\ & \leq \frac{C_0 C_\eta C_3}{2} \int_0^t e^{\lambda_{max}(t-\tau)} \{\delta^2 \|u(0)\|_{H^2}^2 + \delta^2 e^{2\lambda_{max}\tau}\} d\tau \\ & \leq \frac{C_0 C_\eta C_3}{2} \left\{ \frac{\delta \|u(0)\|_{H^2}^2}{\lambda_{max}} + \frac{1}{\lambda_{max}} \delta e^{\lambda_{max}t} \right\} \delta e^{\lambda_{max}t}. \end{aligned} \quad (3.4)$$

We choose $\theta > 0$ and $\delta_0 > 0$ such that

$$2\sqrt{C_3}\theta < \eta, \quad (3.5)$$

$$\frac{C_0 C_\eta C_3}{2} \theta < \frac{\lambda_{max}}{4}, \quad (3.6)$$

$$\delta_0 \|u_0\|_{H^2} \leq \theta, \quad (3.7)$$

$$\frac{C_0 C_\eta C_3}{2} \delta_0 \|u_0\|_{H^2}^2 < \frac{\lambda_{max}}{4}. \quad (3.8)$$

Then we will prove by contradiction that for $0 < \delta \leq \delta_0$ we have

$$T^\delta = \min\{T^\delta, T^*, T^{**}\}$$

and our theorem follows.

Case 1) Suppose that T^{**} is minimum. We can let $t = T^{**}$ in (3.1) to get

$$\begin{aligned} \|u^\delta(T^{**})\|_{H^2} & \leq \sqrt{C_3} \{\delta \|u(0)\|_{H^2} + \delta e^{\lambda_{max}T^\delta}\} \\ & \leq \sqrt{C_3} \{\delta \|u(0)\|_{H^2} + \theta\} \\ & \leq 2\sqrt{C_3}\theta \end{aligned}$$

by our choice of δ_0 in (3.7). By the choice of θ in (3.5):

$$\|u^\delta(T^{**})\|_{H^2} < \eta.$$

This is a contradiction to the definition of T^{**} .

Case 2) Suppose that T^* is the smallest in $\{T^\delta, T^*, T^{**}\}$. We let $t = T^*$ in (3.4) to get

$$\begin{aligned} & \|u^\delta(T^*) - \delta L[u_0](\cdot, T^*)\| \\ & \leq \frac{C_0 C_\eta C_3}{2} \left\{ \frac{\delta \|u(0)\|_{H^2}^2}{\lambda_{max}} + \frac{1}{\lambda_{max}} \delta e^{\lambda_{max} T^\delta} \right\} \delta e^{\lambda_{max} T^*} \\ & \leq \frac{C_0 C_\eta C_3}{2} \left\{ \frac{\delta \|u(0)\|_{H^2}^2}{\lambda_{max}} + \frac{1}{\lambda_{max}} \theta \right\} \delta e^{\lambda_{max} T^*}. \end{aligned}$$

For $0 < \delta \leq \delta_0$ by our choice of θ, δ_0 in (3.8) and (3.6) we have

$$\|u^\delta(T^*) - \delta L[u_0](\cdot, T^*)\| < \frac{\delta}{2} e^{\lambda_{max} T^*}.$$

This contradicts the definition of T^* . Thus this completes the proof. $\square$

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