GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A MILDLY DEGENERATE KIRCHHOFF WAVE EQUATION WITH BOUNDARY DAMPING

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Abstract. In this article a degenerate nonlinear dissipative wave equation of Kirchhoff type with nonlinear boundary damping is considered. We prove the existence, uniqueness and regularity of the global solution of the system when the initial data are small enough and the geometry of the domain satisfies suitable assumptions. We also obtain the polynomial decay property of the global solution.

1. Introduction. Let Ω be an open bounded set in $\mathbb{R}^n$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Assume that Γ_0 has positive boundary measure and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ (this assumption excludes simply connected regions). We consider the following degenerate nonlinear wave equation of Kirchhoff type with boundary damping:

$$\begin{cases} u''(x, t) - M(\|\nabla u(x, t)\|^2)\Delta u(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u(x, t) = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ M(\|\nabla u(x, t)\|^2)\partial_\nu u(x, t) + g(u'(x, t)) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where u is transverse displacement of the wave, $\|\cdot\|$ represents the norm on $L^2(\Omega)$, ν denotes the unit outward normal vector on boundary, and $M : [0, \infty) \rightarrow [0, \infty)$ is a continuous function. System (1.1) is called a nondegenerate equation when $M(\xi) \geq \tilde{M} > 0$ for all $\xi \geq 0$, and a degenerate one when $M(\xi) \geq 0$ for all $\xi \geq 0$. In the case of $M(\xi) \equiv \hat{M} > 0$, system (1.1) is a usual linear wave equation with boundary damping.

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The nonlinear function $g(\cdot)$ is assumed in $C^1(R)$, strictly increasing and zero at the origin.

System (1.1) originated in the nonlinear vibration of an elastic string, which was introduced by G. Kirchhoff ([9]) in the 1880s:

$$u''(x, t) - (a + b\|u_x(x, t)\|^2)u_{xx}(x, t) = 0 \quad \text{in } (0, L) \times R^+,$$

where a and b are positive constants, and L is the length of the string. Later, more general models were considered by Carrier ([4]) and Lions ([13]).

Since then, the local and global solvability of nonlinear systems of Kirchhoff type has been studied under various assumptions concerned with the function M , dissipative terms and initial data (see, e.g., [1, 6, 8]). Among the literature, the latest result about local solvability of a nonlinear system of Kirchhoff type can be found in [8]. They proved the existence, uniqueness and regularity of the local solution for model (1.1) with $k = 0$ or $u|_\Gamma = 0$ when M is a degenerate differential function and $M(\xi) \neq 0$ for ξ belonging to a neighborhood of the norm of initial data.

It is clear that an extra dissipative term is necessary to obtain the existence, uniqueness and regularity of the global solution of a nonlinear system of Kirchhoff type. In the non-degenerate case, system (1.1) describes a pre-stressed structure. Brito [3], Nishihara [14] and Cavalcanti [5] proved that there exists a unique global solution with exponential decay property for the system if the initial data is small enough, damping is applied on Ω and the nondegenerate function M is C^1 -continuous. A similar result was obtained by Lasiecka and Ong [12] if the dissipative term is on the boundary, i.e., the nondegenerate system (1.1).

Degenerate nonlinear systems of Kirchhoff type were considered by Nishihara and Yamada ([15]), Ono ([16]), Ghisi ([7]) and the references therein. They proved that when $M(\xi)$ behaves like ξ^γ ($\gamma > 0$), a velocity dissipative term is applied in Ω and the initial data is small and regular enough, there exists a unique global solution of the system and the solution decays with polynomial rate. It is more difficult to handle a degenerate case than a nondegenerate one. Furthermore, the difficulty increases when the damping is applied on the boundary. The method in the references cannot be applied directly to the degenerate Kirchhoff system with boundary damping.

In this paper we study the global existence and decay properties of the solution of a degenerate Kirchhoff wave equation system with nonlinear boundary damping. The nonlinear coefficient function is assumed to be $M(\xi) = \xi^\gamma$ ($\gamma > 1$). To prove the existence, uniqueness and regularity of the global solution of the system, we first introduce estimates of higher order energies divided by $\|\nabla u\|^s$ ($s > 0$) to overcome difficulties due to the degenerateness of $M(\xi)$. Then by the idea of iterativeness of the time and assumptions on the initial data, we show that $\|\nabla u(t)\| > 0$ for all $t > 0$, i.e., the degenerateness situation never does occur because of the dissipation. Thus, the local solution of system (1.1) can be continued globally in time. We also obtain the polynomial decay of the solution of system (1.1) by the classical Gronwall lemma.

Our paper is organized as follows. In Section 2 we present the local existence of the solution of the system. In Section 3 we prove the existence, uniqueness and regularity of the global solution. The decay property of the solution is given in Section 4.

2. Preliminaries. In what follows, $H^r(\Omega)$ denotes the usual Sobolev space for any $r \in \mathbb{R}$. $H_{\Gamma_0}^1(\Omega)$ denotes the space $\{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0\}$. Let X be a Banach space. We denote by $C^m([0, T]; X)$ the space of all m times continuously differentiable functions defined on $[0, T]$ with values in X , and write $C([0, T]; X)$ for $C^0([0, T]; X)$. The scalar product and norm in $L^2(\Omega)$ and $L^2(\Gamma)$ are represented by $\|\cdot\|$, $(\cdot, \cdot)$ and $|\cdot|_\Gamma$, $\langle \cdot, \cdot \rangle_\Gamma$, respectively.

In this paper we consider the global existence and decay property of the solution of the following degenerate Kirchhoff wave equation with boundary damping:

$$\begin{cases} u''(x, t) - \|\nabla u(x, t)\|^{2\gamma} \Delta u(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u(x, t) = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \|\nabla u(x, t)\|^{2\gamma} \partial_\nu u(x, t) + g(u'(x, t)) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) & \text{in } \Omega. \end{cases} \quad (2.1)$$

We first present assumptions on the function $g(\cdot)$ and the geometry of the domain.

(G) $g(0) = 0$, $c_1 \leq g'(s) \leq c_2$, $c_3|s| \leq |g(s)| \leq c_4|s|$, $c_i > 0$, $i = 1, 2, 3, 4$.

(B1) $m(x) \cdot \nu(x) \leq 0$ on Γ_0 , where $m(x) \doteq x - x_0$, and x_0 is an arbitrary fixed point in $\mathbb{R}^n$.

The natural energy of system (2.1) is defined by

$$E(t) = \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2\gamma + 2} \|\nabla u(t)\|^{2\gamma+2}. \quad (2.2)$$

It is clear from (G) that

$$\frac{d}{dt} E(t) = -\langle g(u'(t)), u'(t) \rangle_{\Gamma_1} \leq 0, \quad (2.3)$$

i.e., $E(t)$ decreases on $[0, \infty)$.

By applying the Banach contraction mapping theorem, we get the following local existence theorem.

THEOREM 2.1. Let $\gamma > 0$, $g(\cdot)$ satisfy (G) and the initial data $u_0 \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$, $u_1 \in H_{\Gamma_0}^1(\Omega)$. Suppose $\|\nabla u_0\| > 0$. Then there exists $T = T(\|\nabla u_0\|) > 0$ such that system (2.1) has a unique local solution $u(x, t)$ with regularity

$$u \in C([0, T]; H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T]; H_{\Gamma_0}^1(\Omega)) \cap C^2([0, T]; L^2(\Omega)). \quad (2.4)$$

Proof. First, we consider the functional space

$$X_{T,L} \doteq \left\{ w(t) \in C([0, T]; H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T]; H_{\Gamma_0}^1(\Omega)) : w(0) = u_0, \right. \\ \left. w'(0) = u_1, \|\nabla w(t)\|^2, \|\nabla w'(t)\|^2, \|\Delta w(t)\|^2 \leq L \text{ on } [0, T] \right\},$$

where $T > 0$ and $L > 0$ satisfy

$$T = \frac{M_0}{4L}, \quad M_0 \doteq \|\nabla u_0\|^2. \quad (2.5)$$

Then, for any $w \in X_{T,L}$, we have by the Lagrange theorem that

$$\|\nabla w(t)\|^2 \geq \frac{M_0}{2}, \quad \forall t \in [0, T]. \quad (2.6)$$

Define a map $\mathcal{S}$ as follows. For $w \in X_{T,L}$, $u = \mathcal{S}w$ is the unique solution of the following linear problem:

$$\begin{cases} u''(x, t) - \|\nabla w(x, t)\|^{2\gamma} \Delta u(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u(x, t) = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \|\nabla w(x, t)\|^{2\gamma} \partial_\nu u(x, t) + g(u'(x, t)) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) & \text{in } \Omega. \end{cases} \quad (2.7)$$

Now we prove that $\mathcal{S}$ maps $X_{T,L}$ into itself if (2.5) holds. By multiplying the first equation of system (2.7) by $2u'$ and integrating it over Ω , we have

$$\begin{aligned} \frac{d}{dt} \left(\|u'(t)\|^2 + \|\nabla w(t)\|^{2\gamma} \|\nabla u(t)\|^2 \right) &= 2\gamma (\nabla w(t), \nabla w'(t)) \|\nabla w(t)\|^{2\gamma-2} \|\nabla u(t)\|^2 \\ &\quad - 2\langle g(u'(t)), u'(t) \rangle_{\Gamma_1}. \end{aligned} \quad (2.8)$$

It follows from (G), (2.6) and (2.8) that

$$\frac{d}{dt} \left(\|u'(t)\|^2 + \|\nabla w(t)\|^{2\gamma} \|\nabla u(t)\|^2 \right) \leq 4\gamma L M_0^{-1} \|\nabla w(t)\|^{2\gamma} \|\nabla u(t)\|^2, \quad \forall 0 \leq t \leq T. \quad (2.9)$$

Thus, we have from the Gronwall inequality that for all $0 < t \leq T$,

$$\|u'(t)\|^2 + \|\nabla w(t)\|^{2\gamma} \|\nabla u(t)\|^2 \leq (\|u_1\|^2 + \|\nabla u_0\|^{2\gamma+2}) e^{\varepsilon_1 L T}, \quad \varepsilon_1 \doteq 4\gamma M_0^{-1}. \quad (2.10)$$

Consequently,

$$\|u'(t)\|^2, \|\nabla u(t)\|^2 \leq \max \left\{ 1, 2^\gamma M_0^{-\gamma} \right\} (\|u_1\|^2 + \|\nabla u_0\|^{2\gamma+2}) e^{\varepsilon_1 L T}, \quad \forall 0 \leq t \leq T. \quad (2.11)$$

Moreover, we multiply the first equation of (2.7) by $2\Delta u'$ and integrate it over Ω ,

$$\begin{aligned} \frac{d}{dt} F(t) &= 2\gamma (\nabla w(t), \nabla w'(t)) \left[\|\nabla w\|^{2\gamma-2} \|\Delta u(t)\|^2 + \frac{2\langle u''(t), g(u'(t)) \rangle_{\Gamma_1}}{\|\nabla w(t)\|^{2\gamma+2}} \right] \\ &\quad - \frac{2\langle g'(u'(t))u''(t), u''(t) \rangle_{\Gamma_1}}{\|\nabla w(t)\|^{2\gamma}}, \end{aligned} \quad (2.12)$$

where $F(t) \doteq \|\nabla u'(t)\|^2 + \|\nabla w(t)\|^{2\gamma} \|\Delta u(t)\|^2$. Thus, by using (G) and the Cauchy-Schwarz inequality,

$$\frac{d}{dt} F(t) \leq 2\gamma (\nabla w(t), \nabla w'(t)) \|\nabla w(t)\|^{2\gamma-2} \|\Delta u(t)\|^2 + \frac{2\gamma^2 (\nabla w(t), \nabla w'(t))^2}{c_1 \|\nabla w(t)\|^{2\gamma+4}} |g(u'(t))|_{\Gamma_1}^2. \quad (2.13)$$

Note that by the trace theorem, there exists a positive constant C_Ω such that

$$|u'|_{\Gamma_1}^2 \leq C_\Omega \|\nabla u'\|^2. \quad (2.14)$$

Combining (G), (2.13) with (2.14) yields

$$\frac{d}{dt} F(t) \leq 2\gamma \|\nabla w'(t)\| \|\nabla w(t)\|^{2\gamma-1} \|\Delta u(t)\|^2 + \frac{2c_4^2 C_\Omega \gamma^2 \|\nabla w'(t)\|^2 \|\nabla u'(t)\|^2}{c_1 \|\nabla w(t)\|^{2\gamma+2}}. \quad (2.15)$$

Thus, by (2.6) and (2.15),

$$\frac{d}{dt} F(t) \leq \varepsilon_2 L F(t), \quad \forall 0 \leq t \leq T, \quad (2.16)$$

where

$$\varepsilon_2 \doteq \max \{ \varepsilon_1, 2^{\gamma+2} c_4^2 C_\Omega \gamma^2 c_1^{-1} M_0^{-(\gamma+1)} \}.$$

Therefore,

$$\|\nabla u'(t)\|^2, \|\Delta u(t)\|^2 \leq \max \{1, 2^\gamma M_0^{-\gamma}\} F(0) e^{\varepsilon_2 L T}, \quad \forall 0 \leq t \leq T. \quad (2.17)$$

Set

$$L \doteq M^* \max \{1, 2^\gamma M_0^{-\gamma}\} (\|u_1\|^2 + \|\nabla u_0\|^{2\gamma+2} + \|\nabla u_1\|^2 + \|\nabla u_0\|^{2\gamma} \|\Delta u_0\|^2), \quad (2.18)$$

where

$$M^* \geq e^{\frac{\varepsilon_2 M_0}{4}}.$$

Hence, we obtain that $\mathcal{S}$ maps $X_{T,L}$ into itself from (2.11) and (2.17) if T and L satisfy (2.5) and (2.18).

By a similar technique, we can prove the contraction property of map $\mathcal{S}$. Thus, by the contraction mapping principle, we obtain that there exists a unique local solution $u(x, t)$ of system (2.1) with smooth initial data $u_0 \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$, $u_1 \in H_{\Gamma_0}^1(\Omega)$, and $u(x, t)$ satisfies (2.4). $\square$

3. Global existence. In this section we shall consider the global existence of the solution of system (2.1). The main result of this section is as follows.

THEOREM 3.1. Assume (G) and (B1) hold, $\gamma \geq 1$. Let the initial data $u_0 \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$, $u_1 \in H_{\Gamma_0}^1(\Omega)$ satisfy

$$\|\nabla u_0\| > 0 \quad (3.1)$$

and

$$\|\nabla u_0\|^{2\gamma} \|\Delta u_0\|^2 + \|\nabla u_1\|^2 + \frac{\|\Delta u_0\|^2}{\|\nabla u_0\|^{2\gamma+2}} + \frac{\|\nabla u_1\|^2}{\|\nabla u_0\|^{4\gamma+2}} < \kappa, \quad (3.2)$$

where κ is a given positive constant. Then system (2.1) has a unique global solution $u(x, t)$ with regularity

$$u \in C([0, \infty); H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)) \cap C^1([0, \infty); H_{\Gamma_0}^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)).$$

REMARK 3.1. In Theorem 3.1, κ is a constant depending on Ω , $g(\cdot)$ and γ . More precisely, κ satisfies

$$\kappa < \frac{\beta^2}{4\lambda}, \quad (3.3)$$

where $\beta > 0$ satisfies (3.14) and λ is defined by (3.18).

To prove Theorem 3.1, we need to introduce several auxiliary functions. Since $\|\nabla u_0\| > 0$, we can define

$$T_1 \doteq \sup \{t \in [0, \infty) : \|\nabla u(s)\| > \alpha, \quad \forall 0 \leq s < t\},$$

where α is a sufficiently small positive constant. Then, for $0 \leq t \leq T_1$, we set

$$\begin{aligned} H_1(t) &\doteq \|\nabla u(x, t)\|^{2\gamma} \|\Delta u(x, t)\|^2 + \|\nabla u'(x, t)\|^2, \\ H_2(t) &\doteq \frac{\|\Delta u(x, t)\|^2}{\|\nabla u(x, t)\|^{2\gamma+2}} + \frac{\|\nabla u'(x, t)\|^2}{\|\nabla u(x, t)\|^{4\gamma+2}}, \\ \rho(t) &\doteq (\Delta u(x, t), 2m(x) \cdot \nabla u'(x, t)) + (n-1)(\Delta u(x, t), u'(x, t)), \\ \Phi_p(t) &\doteq \frac{\|\nabla u'(x, t)\|}{\|\nabla u(x, t)\|^p}, \quad p \in \mathbb{N}. \end{aligned}$$

It is clear from (2.3) that for any $0 \leq t < T_1$,

$$|\rho(t)| \leq \rho_0 H_2(t), \quad (3.4)$$

where

$$\begin{aligned} \rho_0 &\doteq e_0^{2\gamma+2} + \left[R^2 + \frac{(n-1)^2}{4} \right] e_0^{4\gamma+2}, \\ R &\doteq \max_{x \in \Omega} |m(x)|, \\ e_0 &\doteq [(2\gamma+2)E(0)]^{\frac{1}{2\gamma+2}}. \end{aligned} \quad (3.5)$$

We can get the following properties of H_1 and H_2 by direct computation.

LEMMA 3.1. Let $u(x, t)$ satisfy system (2.1) and let assumption (G) hold. Then, for any $0 \leq t < T_1$,

$$\frac{d}{dt} H_1(t) \leq -\frac{c_1 |u''(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|^{2\gamma}} + \frac{4\gamma^2}{c_1} \Phi_{\gamma+1}^2(t) |g(u'(t))|_{\Gamma_1}^2 + 2\gamma \Phi_1(t) H_1(t), \quad (3.6)$$

$$\frac{d}{dt} H_2(t) \leq -\frac{c_1 |u''(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|^{6\gamma+2}} + \frac{4\gamma^2}{c_1} \Phi_{3\gamma+2}^2(t) |g(u'(t))|_{\Gamma_1}^2 + (4\gamma+2) \Phi_1(t) H_2(t). \quad (3.7)$$

The following lemma is useful and its proof is given in the Appendix.

LEMMA 3.2. Assume (G) and (B1) hold, $\gamma \geq 1$. Let $u(x, t)$ satisfy system (2.1). Then, for any $0 \leq t < T_1$,

$$\begin{aligned} &\int_0^t H_1(s) ds \\ &\leq C_0 H_1(0) + 2\rho_0 H_2(0) + \int_0^t \left\{ \frac{R |u''(s)|_{\Gamma_1}^2}{\|\nabla u(s)\|^{2\gamma}} + \frac{2C_B |g'(u'(s)) u''(s)|_{\Gamma_1}^2}{\|\nabla u(s)\|^{4\gamma}} + 2RC_{\delta, \epsilon} |u''(s)|_{\Gamma_1}^2 \right. \\ &\quad + \frac{|u'(s)|_{\Gamma_1}^2}{4C_\Omega} + \frac{8\gamma^2}{c_1} \left[c_1 C_B \Phi_{2\gamma+1}^2(s) + \delta \Phi_{\gamma+1}^2(s) + \rho_0 \Phi_{3\gamma+2}^2(s) \right] |g(u'(s))|_{\Gamma_1}^2 \\ &\quad \left. + \left[4\gamma \delta \Phi_1(s) + 2(4\gamma+2) \rho_0 \Phi_{4\gamma+3}(s) + 8\gamma^2 RC_{\delta, \epsilon} e_0^{2\gamma} \Phi_1^2(s) \right] H_1(s) \right\} ds, \end{aligned} \quad (3.8)$$

where $0 < \delta < \frac{t}{2}$, $\epsilon, \epsilon_0 > 0$ are arbitrary, $C_{\delta, \epsilon}, C_{\delta, \epsilon_0}, C_t$ are positive constants, and

$$\begin{aligned} C_0 &= 2\delta + 2RC_{\delta, \epsilon_0} C_t (1 + e_0^{2\gamma}), \\ C_B &= 3R + 2RC_{\delta, \epsilon} + (n-1)^2 C_\Omega + 2RC_{\delta, \epsilon_0} C_t. \end{aligned}$$

Now we prove Theorem 3.1. Set

$$\mathcal{E}(t) \doteq \lambda_1 H_1(t) + \lambda_2 H_2(t),$$

where λ_i ($i = 1, 2, 3$) are positive constants satisfying

$$\begin{aligned} \lambda_1 &\geq \max \left\{ \frac{R(1 + 2C_{\delta, \epsilon} e_0^{2\gamma})}{c_1}, C_0 \right\}, \\ \lambda_2 &\geq \max \left\{ \frac{2c_2^2 C_B e_0^{2\gamma+2}}{c_1}, 2\rho_0 \right\}. \end{aligned} \quad (3.9)$$

Then, we have from (G), Lemmas 3.1-3.2 that

$$\begin{aligned} \mathcal{E}(t) + \int_0^t H_1 ds &\leq 2\mathcal{E}(0) + \int_0^t \left\{ \frac{|u'|_{\Gamma_1}^2}{4C_\Omega} + \frac{4\gamma^2 c_4^2}{c_1} \left[2c_1 C_B \Phi_{2\gamma+1}^2 + (2\delta + \lambda_1) \Phi_{\gamma+1}^2 + (2\rho_0 + \lambda_2) \Phi_{3\gamma+2}^2 \right] |u'|_{\Gamma_1}^2 \right. \\ &\quad \left. + \left[2\gamma(2\delta + \lambda_1) \Phi_1 + (4\gamma + 2)(2\rho_0 + \lambda_2) \Phi_{4\gamma+3} + 8\gamma^2 R C_{\delta, \epsilon} e_0^{2\gamma} \Phi_1^2 \right] H_1 \right\} ds. \end{aligned} \quad (3.10)$$

Moreover, for a positive constant β , define

$$T_2 \doteq \sup \{ t \in [0, \infty) : \Phi_{2\gamma+1}(s) < \beta, \quad \forall \quad 0 \leq s < t \}.$$

It is clear that $T_2 > 0$. Suppose

$$T_2 < T_1. \quad (3.11)$$

Then,

$$\Phi_{2\gamma+1}(T_2) = \beta. \quad (3.12)$$

It follows from (2.3), (2.14) and (3.10) that for any $0 \leq t \leq T_2$,

$$\mathcal{E}(t) + \int_0^t H_1(s) ds \leq 2\mathcal{E}(0) + \int_0^t \left(\frac{1}{4} + \mu_1 \beta + \mu_2 \beta^2 \right) H_1(s) ds, \quad (3.13)$$

where

$$\begin{aligned} \mu_1 &\doteq 2\gamma(2\delta + \lambda_1) e_0^{2\gamma} + \frac{(4\gamma + 2)(2\rho_0 + \lambda_2)}{\alpha^{2\gamma+2}}, \\ \mu_2 &\doteq \frac{4\gamma^2 c_4^2 C_\Omega}{c_1} \left[2c_1 C_B + (2\delta + \lambda_1) e_0^{2\gamma} + \frac{2\rho_0 + \lambda_2}{\alpha^{2\gamma+2}} \right] + 8\gamma^2 R C_{\delta, \epsilon} e_0^{6\gamma}. \end{aligned}$$

Let β satisfy

$$0 < \beta < \frac{-\mu_1 + (\mu_1^2 + \mu_2)^{\frac{1}{2}}}{2\mu_2}. \quad (3.14)$$

Then,

$$\mu_1 \beta + \mu_2 \beta^2 < \frac{1}{4}. \quad (3.15)$$

Thus, it follows from (3.13)-(3.15) that

$$\mathcal{E}(t) \leq 2\mathcal{E}(0), \quad \forall \quad 0 \leq t \leq T_2. \quad (3.16)$$

Consequently, for any $0 \leq t \leq T_2$,

$$H_1(t) + H_2(t) \leq \lambda [H_1(0) + H_2(0)], \quad (3.17)$$

where

$$\lambda \doteq \frac{2 \max \{ \lambda_1, \lambda_2 \}}{\min \{ \lambda_1, \lambda_2 \}}. \quad (3.18)$$

On the other hand, it is clear that for any $0 \leq t < T_1$,

$$\Phi_{2\gamma+1}(t) \leq H_2^{\frac{1}{2}}(t). \quad (3.19)$$

Combining (3.11), (3.17) with (3.19) yields that for any $0 \leq t \leq T_2 < T_1$,

$$\Phi_{2\gamma+1}(t) \leq [\lambda(H_1(0) + H_2(0))]^{\frac{1}{2}}. \quad (3.20)$$

Therefore, from Remark 3.1, we have from (3.20) that

$$\Phi_{2\gamma+1}(t) < \frac{\beta}{2}, \quad 0 \leq t \leq T_2. \quad (3.21)$$

We get the contradiction to (3.12). Therefore, $T_1 \leq T_2$.

$t \geq 0$. By the definition of T_1 , we have that

$$\|\nabla u(T_1)\| = 0. \quad (3.22)$$

Since $T_1 \leq T_2$, we have that estimation (3.17) holds when $0 \leq t \leq T_1$. Then we have from (3.22) that

$$\|\nabla u'(T_1)\| = 0. \quad (3.23)$$

Induce a variable $v(t) \doteq u(T_1 - t)$. Then $v(t)$ satisfies

$$\begin{cases} v'' - \|\nabla v\|^{2\gamma} \Delta v = 0 & \text{in } \Omega \times [0, T_1], \\ v = 0 & \text{on } \Gamma_0 \times [0, T_1], \\ \|\nabla v\|^{2\gamma} \partial_\nu v + g(-v') = 0 & \text{on } \Gamma_1 \times [0, T_1], \\ v(0) = 0, v'(0) = 0 & \text{in } \Omega. \end{cases} \quad (3.24)$$

It is clear that

$$\frac{d}{dt} E(v(t)) = -\langle g(-v'(t)), v'(t) \rangle_{\Gamma_1}. \quad (3.25)$$

Moreover, from (3.17),

$$\|\nabla u'(T_1 - t)\|^2 \leq \lambda \mathcal{E}(0) \|\nabla u(T_1 - t)\|^{4\gamma+2}. \quad (3.26)$$

Hence, by the trace theorem and (3.25)-(3.26), we get that

$$\frac{d}{dt} E(v(t)) \leq c_4 C_\Omega \lambda \mathcal{E}(0) e_0^{2\gamma} (2\gamma + 2) E(v(t)), \quad \forall \ 0 \leq t \leq T_1. \quad (3.27)$$

Notice that $E(v(0)) = 0$. Thus, it follows from (3.27) that $E(v(T_1)) = 0$. Consequently, $\|\nabla u(0)\| = 0$, which contradicts assumption (3.1). Therefore, $\|\nabla u(t)\| > 0$ for all $t \geq 0$, and (3.17) holds for all $t \geq 0$. The proof is completed. $\square$

4. Polynomial decay. In this section, we shall study the asymptotic behavior of the global solution of system (2.1). First, we introduce the following geometry assumption on domain Ω .

(B2) $m(x) \cdot \nu(x) \geq m_0 > 0$ on Γ_1 .

The main result of this section is as follows.

THEOREM 4.1. Assume (G), (B1) and (B2) hold, $\gamma \geq 1$. Let the initial data $u_0 \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$, $u_1 \in H_{\Gamma_0}^1(\Omega)$ satisfy $\|\nabla u_0\| > 0$ and

$$\|\nabla u_0\|^{2\gamma} \|\Delta u_0\|^2 + \|\nabla u_1\|^2 + \frac{\|\Delta u_0\|^2}{\|\nabla u_0\|^{2\gamma+2}} + \frac{\|\nabla u_1\|^2}{\|\nabla u_0\|^{4\gamma+2}} < \tilde{\kappa}, \quad (4.1)$$

where $\tilde{\kappa}$ is a given positive constant. Then there exists a positive constant C_E depending on Ω , $g(\cdot)$, γ and the initial data such that

$$E(t) \leq \frac{C_E}{(1+t)^{\frac{2\gamma+2}{\gamma}}}, \quad \forall t \geq 0. \quad (4.2)$$

REMARK 4.1. In (4.1), the constant $\tilde{\kappa}$ satisfies

$$\tilde{\kappa} \leq \min \left\{ \kappa, \frac{1}{4\eta\lambda C_\Omega(\gamma+1)} \right\}, \quad (4.3)$$

where κ , η , λ and C_Ω are defined by Remark 3.1, (4.7), (3.18) and (2.14), respectively.

In (4.2), the constant C_E is defined as

$$C_E \doteq (\tilde{\lambda} - \tilde{\rho}_0)^{\frac{(1-\theta)(2\gamma+2)}{\gamma}} C_{\mathcal{E}}^{\frac{2\gamma+2}{\gamma}},$$

where $\tilde{\rho}_0$, $\tilde{\lambda}$, θ and $C_{\mathcal{E}}$ are defined in (4.8), (4.10), (4.19) and (4.21).

Thanks to estimates (3.17) and (4.2), we have immediately the following polynomial decay results.

THEOREM 4.2. Assume that all conditions of Theorem 4.1 are satisfied. Let $u(x, t)$ be the global solution of system (2.1). Then we have for all $t \geq 0$,

$$\begin{aligned} 0 < \|\nabla u(t)\|^2 &\leq \frac{[(2\gamma+2)C_E]^{\frac{1}{\gamma+1}}}{(1+t)^{\frac{2}{\gamma}}}, \\ \|u'(t)\|^2 &\leq \frac{2C_E}{(1+t)^{\frac{2\gamma+2}{\gamma}}}, \\ \|\Delta u(t)\|^2 &\leq \frac{(2\gamma+2)\lambda\kappa C_E}{(1+t)^{\frac{2\gamma+2}{\gamma}}}, \\ \|\nabla u'(t)\|^2 &\leq \frac{\lambda\kappa[(2\gamma+2)C_E]^{\frac{2\gamma+1}{\gamma+1}}}{(1+t)^{\frac{4\gamma+2}{\gamma}}}, \end{aligned}$$

where C_E is defined in Remark 4.1, and λ and κ are defined in Remark 3.1.

Proof of Theorem 4.1. We first set

$$\tilde{\rho}(t) \doteq 2(u'(t), m \cdot \nabla u(t)) + (n-1)(u'(t), u(t)).$$

Similar to the proof of Lemma 3.2, we multiply the first equation of system (2.1) by $[2m \cdot \nabla u(t) + (n-1)u(t)](E(t))^p$ ($p > 0$),

$$\begin{aligned} & (E(t))^p (\|u'(t)\|^2 + \|\nabla u(t)\|^{2\gamma+2}) + \frac{d}{dt} [(E(t))^p \tilde{\rho}(t)] \\ &= (E(t))^p \|\nabla u(t)\|_\Omega^{2\gamma} \int_{\Gamma_0} (m \cdot \nu) |\nabla u(t)|^2 d\Gamma + (E(t))^p \int_{\Gamma_1} (m \cdot \nu) |u'(t)|^2 d\Gamma \\ &+ (E(t))^p \|\nabla u(t)\|_\Omega^{2\gamma} \int_{\Gamma_1} \left[-(m \cdot \nu) |\nabla u(t)|^2 + 2\partial_\nu u(t)(m \cdot \nabla u(t)) \right. \\ &\left. + (n-1)\partial_\nu u(t) u(t) \right] d\Gamma + \tilde{\rho}(t) \frac{d}{dt} (E(t))^p. \end{aligned} \quad (4.4)$$

Thus, by using assumptions (B1), (B2), (2.3) and the Cauchy-Schwarz inequality,

$$\begin{aligned} & 2(E(t))^{p+1} + \frac{d}{dt} [(E(t))^p \tilde{\rho}(t)] \\ & \leq (E(t))^p \left[R|u'(t)|_{\Gamma_1}^2 + \frac{R^2}{m_0} \|\nabla u(t)\|_\Omega^{2\gamma} |\partial_\nu u(t)|_{\Gamma_1}^2 \right. \\ & \left. + (n-1) \|\nabla u(t)\|_\Omega^{2\gamma} \langle \partial_\nu u(t), u(t) \rangle_{\Gamma_1} \right] - (E(t))^{p-1} \tilde{\rho}(t) \langle g(u'(t)), u'(t) \rangle_{\Gamma_1}. \end{aligned} \quad (4.5)$$

From (G), the third equation of (2.1), (2.14) and (4.5), we have

$$\begin{aligned} & \frac{3}{2} (E(t))^{p+1} + \frac{d}{dt} [(E(t))^p \tilde{\rho}(t)] \\ & \leq R(E(t))^p |u'(t)|_{\Gamma_1}^2 + c_4 (E(t))^{p-1} |\tilde{\rho}(t)| |u'(t)|_{\Gamma_1}^2 + \frac{\eta (E(t))^p |u'(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|_\Omega^{2\gamma}}, \end{aligned} \quad (4.6)$$

where

$$\eta \doteq \frac{c_4^2 R^2}{m_0} + c_4^2 C_\Omega (n-1)^2 (\gamma+1). \quad (4.7)$$

Furthermore, it is clear that

$$|\tilde{\rho}(t)| \leq \sqrt{2}(2R+n-1)(2\gamma+2)^{\frac{1}{2\gamma+2}} (E(t))^{\frac{\gamma+2}{2\gamma+2}} \doteq \tilde{\rho}_0 (E(t))^{\frac{\gamma+2}{2\gamma+2}}. \quad (4.8)$$

Therefore, by (2.3), (4.6) and (4.8),

$$\begin{aligned} & \frac{3}{2} (E(t))^{p+1} + \frac{d}{dt} [(E(t))^p \tilde{\rho}(t)] \\ & \leq \left[R(E(0))^{1-\frac{\gamma+2}{2\gamma+2}} + c_4 \tilde{\rho}_0 \right] (E(t))^{p-1+\frac{\gamma+2}{2\gamma+2}} |u'(t)|_{\Gamma_1}^2 + \frac{\eta (E(t))^p |u'(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|_\Omega^{2\gamma}}. \end{aligned} \quad (4.9)$$

Set

$$\tilde{\mathcal{E}}(t) \doteq \tilde{\lambda} (E(t))^{p+\frac{\gamma+2}{2\gamma+2}} + (E(t))^p \tilde{\rho}(t),$$

where

$$\tilde{\lambda} > 2 \max \left\{ \tilde{\rho}_0, c_3^{-1} \left(p + \frac{\gamma+2}{2\gamma+2} \right)^{-1} \left[R(E(0))^{1-\frac{\gamma+2}{2\gamma+2}} + c_4 \tilde{\rho}_0 \right] \right\}. \quad (4.10)$$

Then, from (4.8),

$$(\tilde{\lambda} - \tilde{\rho}_0) (E(t))^{p+\frac{\gamma+2}{2\gamma+2}} \leq \tilde{\mathcal{E}}(t) \leq (\tilde{\lambda} + \tilde{\rho}_0) (E(t))^{p+\frac{\gamma+2}{2\gamma+2}}. \quad (4.11)$$

Notice that by (G),

$$\frac{d}{dt}(E(t))^{p+\frac{\gamma+2}{2\gamma+2}} \leq -c_3 \left(p + \frac{\gamma+2}{2\gamma+2}\right) (E(t))^{p-1+\frac{\gamma+2}{2\gamma+2}} |u'|_{\Gamma_1}^2. \quad (4.12)$$

Therefore, we have by (4.9) and (4.12) that

$$\frac{3}{2}(E(t))^{p+1} + \frac{d}{dt} \tilde{\mathcal{E}}(t) \leq \frac{\eta(E(t))^p |u'(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|_{\Omega}^{2\gamma}}. \quad (4.13)$$

Moreover, it follows from (3.17) that

$$\frac{\|\nabla u'(t)\|^2}{\|\nabla u(t)\|^{4\gamma+2}} \leq H_2(t) \leq \lambda \mathcal{E}(0), \quad \forall t \geq 0. \quad (4.14)$$

Consequently,

$$\frac{|u'(t)|_{\Gamma_1}^2}{\|\nabla u(t)\|^{2\gamma}} \leq \lambda C_{\Omega} (2\gamma+2) \mathcal{E}(0) E(t), \quad \forall t \geq 0. \quad (4.15)$$

Replacing (4.15) into (4.13) yields that

$$\frac{3}{2}(E(t))^{p+1} + \frac{d}{dt} \tilde{\mathcal{E}}(t) \leq \eta \lambda C_{\Omega} (2\gamma+2) \mathcal{E}(0) (E(t))^{p+1}. \quad (4.16)$$

Thus, it follows from Remark 4.1 and (4.16) that

$$(E(t))^{p+1} + \frac{d}{dt} \tilde{\mathcal{E}}(t) \leq 0. \quad (4.17)$$

From (4.11) and (4.17), we obtain that

$$[(\tilde{\lambda} + \tilde{\rho}_0)^{-1} \tilde{\mathcal{E}}(t)]^{\theta} + \frac{d}{dt} \tilde{\mathcal{E}}(t) \leq 0, \quad (4.18)$$

where

$$\theta \doteq \frac{p+1}{p+\frac{\gamma+2}{2\gamma+2}} > 1. \quad (4.19)$$

By (4.18), there exists a positive constant $C_{\mathcal{E}}$ such that

$$(\tilde{\mathcal{E}}(t))^{\theta-1} \leq \frac{C_{\mathcal{E}}}{1+t}, \quad (4.20)$$

where

$$C_{\mathcal{E}} \doteq \frac{1}{\min \{(\tilde{\lambda} + \tilde{\rho}_0)^{-\theta} (\theta-1), (\tilde{\mathcal{E}}(0))^{1-\theta}\}}. \quad (4.21)$$

Thus, the desired result is reached from (4.11) and (4.20). $\square$

REMARK 4.2. In [12], Lasiecka and Ong obtained the exponential stability of the solution of the nondegenerate system when $g(\cdot)$ satisfies (G). For the degenerate case, since the auxiliary function $\tilde{\rho}(t)$ is not of the same order as $E(t)$ by (4.8), we can only reach the polynomial decay of the solution. On the other hand, we cannot estimate the tangential trace in (4.4) by Lemma 1 because of the multiplier $(E(t))^p$. Therefore, we need the geometry condition (B2) in Theorem 4.1.

5. Appendix. In this section we prove Lemma 3.2. Denote $w(x, t) = u'(x, t)$. Then $w(x, t)$ satisfies the following system:

$$\begin{cases} w'' - \|\nabla u\|^{2\gamma} \Delta w - \frac{2\gamma(\nabla u, \nabla u') w'}{\|\nabla u\|^2} = 0 & \text{in } \Omega \times [0, T_1), \\ w = 0 & \text{on } \Gamma_0 \times [0, T_1), \\ \partial_\nu w = -\frac{g'(w)w'}{\|\nabla u\|^{2\gamma}} + \frac{2\gamma(\nabla u, \nabla u') g(w)}{\|\nabla u\|^{2\gamma+2}} & \text{on } \Gamma_1 \times [0, T_1). \end{cases} \quad (5.1)$$

To prove Lemma 3.2, we need the following estimate of the tangential trace $|\partial_\tau w|_{\Gamma_1}^2$.

LEMMA 1 ([11, 12]). Let $w(x, t)$ satisfy system (5.1) and $0 < t < T_1$. Then for any $0 < \delta < \frac{t}{2}$ and $\epsilon, \epsilon_0 > 0$, there exist positive constants $C_{\delta, \epsilon}, C_{\delta, \epsilon_0}$ such that the following trace estimate holds:

$$\begin{aligned} \int_\delta^{t-\delta} |\partial_\tau w(s)|_{\Gamma_1}^2 ds &\leq C_{\delta, \epsilon} \int_0^t \left(|\partial_\nu w(s)|_{\Gamma_1}^2 + |w'(s)|_{\Gamma_1}^2 + 4\gamma^2 \Phi_1^2(s) \|w'(s)\|_{H^{-\frac{1}{2}+\epsilon}(\Omega)}^2 \right) ds \\ &\quad + C_{\delta, \epsilon_0} \|w(\cdot)\|_{H^{\frac{1}{2}+\epsilon_0}((0, t) \times \Omega)}^2. \end{aligned} \quad (5.2)$$

By using the multiplier method and Lemma 1, we obtain the following inequality.

LEMMA 2. Assume (B1), (G) hold and $\gamma \geq 1$. Let $w(x, t)$ satisfy system (5.1) and $0 < t < T_1$. Then for any $0 < \delta < \frac{t}{2}$ and $\epsilon, \epsilon_0 > 0$, there exist positive constants $C_{\delta, \epsilon}, C_{\delta, \epsilon_0}$ such that

$$\begin{aligned} \int_0^t H_1 ds &\leq 2\delta H_1(0) + 2\rho_0 H_2(0) + \int_0^t \left[\frac{R|w'|_{\Gamma_1}^2}{\|\nabla u\|^{2\gamma}} + 2RC_{\delta, \epsilon} |w'|_{\Gamma_1}^2 + \frac{|w|_{\Gamma_1}^2}{4C_\Omega} \right. \\ &\quad + (3R + 2RC_{\delta, \epsilon} + (n-1)^2 C_\Omega) |\partial_\nu w|_{\Gamma_1}^2 + \frac{8\gamma^2}{c_1} (\delta \Phi_{\gamma+1}^2 + \rho_0 \Phi_{3\gamma+2}^2) |g(w)|_{\Gamma_1}^2 \\ &\quad + 4\gamma \delta \Phi_1 H_1 + 2(4\gamma + 2) \rho_0 \Phi_1 H_2 + 8\gamma^2 RC_{\delta, \epsilon} \Phi_1^2 \|w'\|_{H^{-\frac{1}{2}+\epsilon}(\Omega)}^2 \left. \right] ds \\ &\quad + 2RC_{\delta, \epsilon_0} \|w\|_{H^{\frac{1}{2}+\epsilon_0}((0, t) \times \Omega)}^2. \end{aligned} \quad (5.3)$$

Proof. For $0 < S \leq T < T_1$, we multiply the first equation of (5.1) by $\frac{m \cdot \nabla w}{\|\nabla u\|^{2\gamma}}$ and integrate it on $\Omega \times [S, T]$ to get

$$\begin{aligned} &\frac{(w', m \cdot \nabla w)}{\|\nabla u\|^{2\gamma}} \Big|_{s=S}^T + \int_S^T \left[\frac{n}{2} \frac{\|w'\|^2}{\|\nabla u\|^{2\gamma}} + \left(1 - \frac{n}{2}\right) \|\nabla w\|^2 \right] ds \\ &= \frac{1}{2} \int_S^T \int_{\Gamma_0} (m \cdot \nu) |\nabla w|^2 d\Gamma ds + \frac{1}{2} \int_S^T \int_{\Gamma_1} \frac{(m \cdot \nu) |w'|^2}{\|\nabla u\|^{2\gamma}} d\Gamma ds \\ &\quad - \frac{1}{2} \int_S^T \int_{\Gamma_1} (m \cdot \nu) |\nabla w|^2 d\Gamma ds + \int_S^T \int_{\Gamma_1} \partial_\nu w (m \cdot \nabla w) d\Gamma ds. \end{aligned} \quad (5.4)$$

Furthermore, multiplying the first equation of (5.1) by $\frac{w}{\|\nabla u\|^{2\gamma}}$, we get that

$$\frac{(w', w)}{\|\nabla u\|^{2\gamma}} \Big|_{s=S}^T + \int_S^T \left(\|\nabla w\|^2 - \frac{\|w'\|^2}{\|\nabla u\|^{2\gamma}} \right) ds = \int_S^T \langle \partial_\nu w, w \rangle_{\Gamma_1} ds. \quad (5.5)$$

From (5.4) and (5.5), we have

$$\begin{aligned} \rho(T) - \rho(S) + \int_S^T \left(\frac{\|w'\|^2}{\|\nabla u\|^{2\gamma}} + \|\nabla w\|^2 \right) ds \\ = \int_S^T \int_{\Gamma_0} (m \cdot \nu) |\nabla w|^2 d\Gamma ds + \int_S^T \int_{\Gamma_1} \frac{(m \cdot \nu) |w'|^2}{\|\nabla u\|^{2\gamma}} d\Gamma ds - \int_S^T \int_{\Gamma_1} (m \cdot \nu) |\nabla w|^2 d\Gamma ds \\ + \int_S^T \int_{\Gamma_1} \partial_\nu w [2(m \cdot \nabla w) + (n-1)w] d\Gamma ds. \end{aligned} \quad (5.6)$$

Thus, by using assumption (B1) in (5.6),

$$\begin{aligned} \int_S^T H_1 ds \leq |\rho(T)| + |\rho(S)| + \int_S^T \left[\frac{R|w'|_{\Gamma_1}^2}{\|\nabla u\|^{2\gamma}} + \frac{|w|_{\Gamma_1}^2}{4C_\Omega} + 2R|\partial_\tau w|_{\Gamma_1}^2 \right. \\ \left. + (3R + (n-1)^2 C_\Omega) |\partial_\nu w|_{\Gamma_1}^2 \right] ds, \end{aligned} \quad (5.7)$$

where C_Ω is the constant defined in (2.14). From (5.2) and (5.7),

$$\begin{aligned} \int_\delta^{t-\delta} H_1 ds \leq |\rho(t-\delta)| + |\rho(\delta)| + \int_\delta^{t-\delta} \left[\frac{R|w'|_{\Gamma_1}^2}{\|\nabla u\|^{2\gamma}} + \frac{|w|_{\Gamma_1}^2}{4C_\Omega} + 2RC_{\delta,\epsilon} |w'|_{\Gamma_1}^2 + (3R + 2RC_{\delta,\epsilon} \right. \\ \left. + (n-1)^2 C_\Omega) |\partial_\nu w|_{\Gamma_1}^2 + 8\gamma^2 RC_{\delta,\epsilon} \Phi_1^2 \|w'\|_{H^{-\frac{1}{2}+\epsilon}(\Omega)}^2 \right] ds + 2RC_{\delta,\epsilon_0} \|w\|_{H^{\frac{1}{2}+\epsilon_0}((0,t)\times\Omega)}^2. \end{aligned} \quad (5.8)$$

Moreover, we have from (3.6) that

$$\int_0^\delta H_1 ds + \int_{t-\delta}^t H_1 ds \leq 2\delta H_1(0) + 2\delta \int_0^t \left(\frac{4\gamma^2}{c_1} \Phi_{\gamma+1}^2 |g(w)|_{\Gamma_1}^2 + 2\gamma \Phi_1 H_1 \right) ds. \quad (5.9)$$

By (3.4) and (3.7),

$$|\rho(t-\delta)| + |\rho(\delta)| \leq 2\rho_0 H_2(0) + 2\rho_0 \int_0^t \left[\frac{4\gamma^2}{c_1} \Phi_{3\gamma+2}^2 |g(w)|_{\Gamma_1}^2 + (4\gamma+2) \Phi_1 H_2 \right] ds. \quad (5.10)$$

Therefore, we get the desired result from (5.8)-(5.10). $\square$

Now we give the estimate for the last term in (5.3).

LEMMA 3. Assume $\gamma \geq 1$. Let $w(x, t)$ satisfy system (5.1) and $0 < t < T_1$. Then there exists a constant C_t such that

$$\|w\|_{H^{\frac{1}{2}+\epsilon_0}((0,t)\times\Omega)}^2 \leq C_t \left[\int_0^t |\partial_\nu w|_{\Gamma_1}^2 ds + \|\nabla w(0)\|^2 + \|w'(0)\|^2 \right]. \quad (5.11)$$

Proof. Suppose there exists a sequence $\{w_n\}$ satisfying (5.1) such that

$$\|w_n\|_{H^{\frac{1}{2}+\epsilon_0}((0,t)\times\Omega)}^2 = 1, \quad \lim_{n \rightarrow \infty} \left[\int_0^t |\partial_\nu w_n|_{\Gamma_1}^2 ds + \|\nabla w_n(0)\|^2 + \|w'_n(0)\|^2 \right] = 0. \quad (5.12)$$

Then, there exist $\tilde{w}_0 \in H^1(\Omega)$ and $\tilde{w}_1 \in L^2(\Omega)$,

$$w_n(0) \rightarrow \tilde{w}_0 \quad \text{weakly in } H^1(\Omega), \quad (5.13)$$

$$w'_n(0) \rightarrow \tilde{w}_1 \quad \text{weakly in } L^2(\Omega). \quad (5.14)$$

Denote by $\tilde{w}$ the solution of (5.1) with initial data $\tilde{w}_0, \tilde{w}_1$. Then,

$$w_n \rightarrow \tilde{w} \quad \text{weak star in } L^\infty((0, t), H^1(\Omega)), \quad (5.15)$$

$$w'_n \rightarrow \tilde{w}' \quad \text{weak star in } L^\infty((0, t), L^2(\Omega)). \quad (5.16)$$

Therefore, by [18],

$$w_n \rightarrow \tilde{w} \quad \text{strongly in } H^{\frac{1}{2}+\epsilon_0}((0, t) \times \Omega). \quad (5.17)$$

Thus, $\tilde{w}$ satisfies (5.1) and

$$\|\tilde{w}\|_{H^{\frac{1}{2}+\epsilon_0}((0, t) \times \Omega)}^2 = 1, \quad (5.18)$$

$$\int_0^t |\partial_\nu \tilde{w}|_{\Gamma_1}^2 ds = 0. \quad (5.19)$$

However, by (5.1) and (5.19), we have $\tilde{w} = 0$ in $(0, t) \times \Omega$, which contradicts (5.18). $\square$

By Lemma 3 and (2.3), we have that

$$\|w\|_{H^{\frac{1}{2}+\epsilon_0}((0, t) \times \Omega)}^2 \leq C_t \int_0^t |\partial_\nu w|_{\Gamma_1}^2 ds + C_t(1 + e_0^{2\gamma})H_1(0). \quad (5.20)$$

Therefore, Lemma 3.2 is proved by combining Lemma 2 with (5.20).

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