CHAPMAN-ENSKOG $\Rightarrow$ VISCOSITY-CAPILLARITY

BY

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Dedicated to Constantine M. Dafermos on the Occasion of his 70th Birthday

Abstract. This paper reviews earlier work of A. Gorban and I. Karlin for the exact summation of the Chapman-Enskog expansion for the linearized Grad’s 13-moment equations. One consequence of their exact summation, not noted in their papers, is that the exact summation yields a nonlocal version of Korteweg’s theory of capillarity, which has proved to be useful as an admissibility criterion in gas dynamics.

Introduction. Sometime ago I proposed the application of Korteweg’s theory of capillarity [4], [5] as an admissibility criterion for selection of weak solutions of the equations of gas dynamics [6], [7], [8]. In order to provide an independent basis for this theory I was then led into a career detour: the historic road of Boltzmann’s kinetic theory of gases and the fluid mechanic derivations of Hilbert and more relevant to me, the expansion technique of Chapman-Enskog. As is well known, the Chapman-Enskog expansion when truncated after the first-order term in the Knudsen number ϵ yields the Navier-Stokes-Fourier theory, while truncation after the second-order term at the Burnett level appears to yield something resembling Korteweg’s theory but with the wrong sign on the crucial term. This discrepancy between Korteweg’s theory and Chapman-Enskog had been noted earlier by P. Rosenau in a series of insightful papers [9], [10], [11] and as Rosenau pointed out, the error is not in the expansion, the error is in the truncation of the expansion. But if we are not to truncate the implication is that the series must be summed either exactly or approximately. Fortunately my friends A. Gorban and I. Karlin have provided an exact sum for the linearized Grad’s 13-moment approximation to the Boltzmann equation [1], [2], [3]. Thus it was an appealing exercise to redo their exact summation for this article and show that the exact sum of the Chapman-Enskog expansion for the Grad system does indeed yield a nonlocal version of Korteweg’s theory of capillarity (the viscosity-capillarity equations) with the good signs. As a birthday present

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for Costas Dafermos, I am happy to present him with this small gift in commemoration of four decades of friendship: a new entropy equality with good signs!

The paper is divided into several sections as noted below:

- (1) The Boltzmann equation and the Chapman-Enskog expansion.
- (2) Grad's 13-moment equations.
- (3) The Chapman-Enskog expansion of the Grad 10-moment system.
- (4) Gorban and Karlin's exact summation of the Chapman-Enskog expansion.
- (5) Equations of hydrodynamics, dispersion relation, and the entropy equality.
- (6) References.

1. The Boltzmann equation and the Chapman-Enskog expansion. As usual the starting point of the kinetic theory of gases is found in the Boltzmann equation [12]

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = \frac{Q(f, f)}{\epsilon}. \quad (1.1)$$

Here $f(t, x, \xi)$ denotes the probability of finding a particle of gas at point $x \in \mathbb{R}^3$, at time t , moving with velocity ξ . $Q(f, f)$ is the collision operator and $\epsilon > 0$ denotes the Knudsen number. As $t \rightarrow \infty$ we expect fast decay to a slow invariant manifold M , which will be governed by the five macroscopic hydrodynamic fluid variables: ρ (density) $\in \mathbb{R}$, u (velocity) $\in \mathbb{R}^3$, T (temperature) $\in \mathbb{R}$. The Chapman-Enskog expansion is a formal method for computing f on this invariant manifold as a series in powers of ϵ , i.e.,

$$f_{CE}(M) = f^{(0)}(M) + \epsilon f^{(1)}(M) + \epsilon^2 f^{(2)}(M) + \dots$$

From (1.1) we see that

$$Q(f^{(0)}, f^{(0)}) = 0 \quad (1.2)$$

so that $f^{(0)}(M)$ is the equilibrium Maxwellian distribution. Historically truncation of the Chapman-Enskog expansion at order 1, ϵ , ϵ^2 , ϵ^3 are called the Euler, Navier-Stokes-Fourier, Burnett, and super-Burnett approximations, respectively. The apparent ability to derive the Navier-Stokes-Fourier theory from the kinetic theory of gases was of course a strong motivation to continue the expansion to higher orders of ϵ . However, as first noted by Bobylev, truncation at Burnett order yields instability of fluid equilibrium, a decidedly unphysical result. (See Bobylev's article [13] for a recent discussion.) As is now known, the problem with the Chapman-Enskog expansion is not the expansion itself but its truncation. (In this regard the work of P. Rosenau [9], [10], [11] prophesied the papers of Gorban and Karlin.)

Since the Chapman-Enskog procedure becomes more and more computationally tedious as we proceed to higher and higher orders of ϵ , a simpler set of equations is needed to recover all the terms in the series. It was the remarkable observation of Gorban and Karlin that Grad's 13-moment equations when linearized about the rest state provide such a system and that the resulting Chapman-Enskog expansion can be exactly summed.

2. Grad's 13-moment equations. The linearized Grad 13-moment equations are obtained by (i) first computing the 13-moment equations from the Boltzmann equation, (ii) invoking Grad's closure rule for the distribution function f , (iii) linearizing about the

rest state of constant density $\rho_0 > 0$, constant temperature $T_o > 0$, and velocity $u = 0$. In appropriate nondimensional form the linearized Grad 13-moment equations are:

$$\partial_t \rho = -\nabla \cdot u, \quad (2.1)$$

$$\partial_t u = -\nabla \rho - \nabla T - \nabla \cdot \sigma, \quad (2.2)$$

$$\partial_t T = -\frac{2}{3}(\nabla \cdot u + \nabla \cdot q), \quad (2.3)$$

$$\begin{aligned} \partial_t \sigma = & -((\nabla u) + (\nabla u)^T - \frac{2}{3}I\nabla \cdot u) \\ & -\frac{2}{5}((\nabla q) + (\nabla q)^T - \frac{2}{3}I\nabla \cdot q), \end{aligned} \quad (2.4)$$

$$\partial_t q = -\frac{5}{3}\nabla T - \nabla \cdot \sigma - \frac{2}{3}q, \quad (2.5)$$

where the pressure $p = \rho + T$ in linear theory, σ is the extra stress, and $q \in \mathbb{R}^3$ is the heat flux.

An even simpler set of equations is obtained by using only the first 10 moments in one space dimension. This formally amounts to using just (2.1)-(2.4) with $q = 0$, i.e.,

$$\partial_t \rho = -u_x, \quad (2.6)$$

$$\partial_t u = -p_x - \sigma_x, \quad (2.7)$$

$$\partial_t T = -\frac{2}{3}u_x, \quad (2.8)$$

$$\partial_t \sigma = -\frac{4}{5}u_x - \sigma. \quad (2.9)$$

But addition of (2.6), (2.8) (recall $p = \rho + T$) yields the system of three balance equations

$$\partial_t p = -\frac{5}{3}u_x, \quad (2.10)$$

$$\partial_t u = -p_x - \sigma_x, \quad (2.11)$$

$$\partial_t \sigma = -\frac{4}{5}u_x - \sigma. \quad (2.12)$$

If we rescale space and time, $x = \frac{x'}{\epsilon}$, $t = \frac{t'}{\epsilon}$ and then drop the primes, we introduce the Knudsen number into the system, i.e.,

$$\partial_t p = -\frac{5}{3}u_x, \quad (2.13)$$

$$\partial_t u = -p_x - \sigma_x, \quad (2.14)$$

$$\partial_t \sigma = -\frac{4}{5}u_x - \frac{\sigma}{\epsilon}. \quad (2.15)$$

Of course we could solve for σ in (2.15) to obtain

$$\sigma(x, t) = e^{-t/\epsilon} \sigma(x, 0) + \int_0^t \exp\left(-\frac{(s-t)}{\epsilon}\right) \left(-\frac{4}{3}u_x(s, x)\right) ds,$$

which when inserted into (2.13), (2.14) will give the visco-elastic dynamics, i.e., the rapid decay to the invariant slow manifold but not a computation of the slow manifold itself.

3. The Chapman-Enskog expansion of the Grad 10-moment system. Following Gorban and Karlin [1], [2], [3] let us perform the Chapman-Enskog expansion for the Grad 10-moment equations (2.13)-(2.15). First we write

$$\sigma_{CE} = \epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \dots, \quad (3.1)$$

where $\sigma^{(n)}$ depends on current values of p, u and their space derivatives. Substitute (3.1) into (2.13)-(2.15) to obtain

$$p_t = -\frac{5}{3}u_x, \quad (3.2)$$

$$u_t = -p_x - (\epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \dots)_x, \quad (3.3)$$

$$\epsilon \sigma_t^{(0)} + \epsilon^2 \sigma_t^{(1)} + \epsilon^3 \sigma_t^{(2)} + \epsilon^4 \sigma_t^{(3)} + \dots = -\frac{4}{3}u_x - (\sigma^{(0)} + \epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \epsilon^3 \sigma^{(3)} + \dots). \quad (3.4)$$

Equation (3.4) immediately yields

$$\sigma^{(0)} = -\frac{4}{3}u_x. \quad (3.5)$$

Insert relation (3.5) into (3.3), (3.4) to obtain

$$u_t = -p_x - \left(-\epsilon \frac{4}{3}u_x + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \epsilon^4 \sigma^{(3)} + \dots \right)_x, \quad (3.6)$$

$$\begin{aligned} & \left(-\frac{4}{3}\epsilon u_{xt} + \epsilon^2 \sigma_t^{(1)} + \epsilon^3 \sigma_t^{(2)} + \epsilon^4 \sigma_t^{(3)} + \dots \right) \\ &= -(\epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \epsilon^3 \sigma^{(3)} + \dots). \end{aligned} \quad (3.7)$$

Now use (3.6) to eliminate the time differentiated term u_{xt} in (3.7) to obtain

$$\begin{aligned} & -\frac{4}{3}\epsilon \left(-p_{xx} + \epsilon \frac{4}{3}u_{xxx} - \epsilon^2 \sigma_{xx}^{(1)} + \dots \right) \\ & \quad + \epsilon^2 \sigma_t^{(1)} + \epsilon^3 \sigma_t^{(2)} + \epsilon^4 \sigma_t^{(3)} + \dots \\ &= -(\epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \epsilon^3 \sigma^{(3)} + \dots). \end{aligned} \quad (3.8)$$

The balance at order ϵ yields

$$\sigma^{(1)} = -\frac{4}{3}p_{xx} \quad (3.9)$$

and the balance at order ϵ^2 gives

$$\begin{aligned} & -\frac{4}{3} \left(\frac{4}{3} \right) u_{xxx} + \sigma_t^{(1)} = -\sigma^{(2)}, \\ & -\frac{4}{3} \left(\frac{4}{3} \right) u_{xxx} - \frac{4}{3} p_{xxt} = -\sigma^{(2)}, \\ & -\frac{4}{3} \left(\frac{4}{3} u_{xxx} - \frac{5}{3} u_{xxx} \right) = -\sigma^{(2)}, \end{aligned}$$

and hence $\sigma^{(2)}$ is given by

$$\sigma^{(2)} = -\frac{4}{3} \left(\frac{1}{3} \right) u_{xxx}. \quad (3.10)$$

In summary the Chapman-Enskog expansion for the 10-moment Grad equations (2.13), (2.15) has the form

$$\sigma_{cE} = -\frac{4}{3} \left(\epsilon u_x + \epsilon^2 p_{xx} + \frac{\epsilon^3}{3} u_{xxx} + \dots \right). \quad (3.11)$$

If we truncate at orders ϵ , ϵ^2 , ϵ^3 and compute the spectrum of (2.13)-(2.15) in the Fourier frequency domain we recover the dispersion relations

$$\text{Navier-Stokes: } \omega_{\pm} = -\frac{2}{3}k^2 \pm i|k|\sqrt{4k^2 - 15},$$

$$\text{Burnett: } \omega_{\pm} = -\frac{2}{3}k^2 \pm i|k|\sqrt{8k^2 + 15},$$

$$\text{Super-Burnett: } \omega_{\pm} = \frac{2}{9}k^2(k^2 - 3) \pm \frac{1}{9}i|k|\sqrt{4k^6 - 24k^4 - 72k^2 - 135}.$$

Hence $\text{Re } \omega_{\pm}(k) \leq 0$ for the Navier-Stokes and Burnett truncation but for wave number k , $|k| > \sqrt{3}$ the super-Burnett truncation yields a Bobilev instability.

4. Gorban and Karlin's exact summation of the Chapman-Enskog expansion. Notice from the first few terms in (3.11) that σ_{cE} has the form

$$\begin{aligned} \sigma_{cE} &= \sum_{n=0}^{\infty} \epsilon^{2n+1} a_n \partial_x^{2n+1} u \\ &\quad + \sum_{n=0}^{\infty} \epsilon^{2n+2} b_n \partial_x^{2(n+1)} p. \end{aligned} \quad (4.1)$$

In fact these expansions are in the primed space variable x' . Hence if we scale x' and t' back to the original x, t variables $x = \frac{x'}{\epsilon}$, $t = \frac{t'}{\epsilon}$ we obtain

$$\sigma_{cE} = \sum_{n=0}^{\infty} a_n \partial_x^{2n+1} u + \sum_{n=0}^{\infty} b_n \partial_x^{2(n+1)} p. \quad (4.2)$$

Also when we did the Chapman-Enskog expansion in the previous section we found each $\sigma^{(n)}$ recursively in terms of the previous $\sigma^{(0)}, \dots, \sigma^{(n-1)}$. In their papers [1], [2], [3] Gorban and Karlin wrote down this recursion relation explicitly

$$\sigma^{(n)} = - \sum_{m=0}^{\infty} \partial_t^{(m)} \sigma^{(n-1-m)}, \quad (4.3)$$

where the notation “ $\partial_t^{(m)}$ ” does not mean the m^{th} derivative but is a “Chapman-Enskog” time derivative defined as follows:

$$\begin{aligned} \partial_t^{(m)} \partial_x^{\ell} u &= \begin{cases} -\partial_x^{\ell+1} p, & m = 0, \\ -\partial_x^{\ell+1} \sigma^{(m-1)}, & m \geq 1, \end{cases} \\ \partial_t^{(m)} \partial_x^{\ell} p &= \begin{cases} -\frac{5}{3} \partial_x^{\ell+1} u, & m = 0, \\ 0, & m \geq 1, \end{cases} \\ \sigma^{(0)} &= -\frac{4}{3} u_x. \end{aligned}$$

So from (4.2) we know that

$$\sigma^{(2n)} = a_n \partial_x^{2n+1} u, \quad (4.4)$$

$$\sigma^{(2n+1)} = b_n \partial_x^{2(n+1)} p, \quad (4.5)$$

which we substitute into (4.3) and use the definition of $\partial_t^{(m)}$ to find

$$\begin{aligned} b_n \partial_x^{2(n+1)} p &= \sum_{m=1}^{\infty} a_{n-m} \partial_x^{2(n-m+1)} b_{m-1} \partial_x^{2m} p \\ &\quad + a_n \partial_x^{2(n+1)} p, \\ a_n \partial_x^{2n+1} u &= \frac{5}{3} b_{n-1} \partial_x^{2n+1} u \\ &\quad + \sum_{m=0}^{n-1} a_{n-m-1} \partial_x^{2(n-m)} a_m \partial_x^{2(m+1)} u \end{aligned}$$

and hence

$$\begin{aligned} a_{n+1} &= \frac{5}{3} b_n + \sum_{m=0}^n a_m a_{n-m}, \\ b_{n+1} &= a_{n+1} + \sum_{m=0}^n a_{n-m} b_m. \end{aligned} \quad (4.6)$$

Notice that the recursion relations for a_n, b_n given in (4.6) are in convolution form, which proves to be the key property for summation of the expansion σ_{CE} .

From here on it is easier to work in the Fourier frequency domain, so we define the Fourier transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

and the inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk.$$

Then from (4.2) we find the Fourier transform of $\hat{\sigma}_{CE}$:

$$\hat{\sigma}_{CE} = \sum_{n=0}^{\infty} -ik a_n (-k^2)^n \hat{u} + \sum_{n=0}^{\infty} -k^2 b_n (-k^2)^n \hat{p} \quad (4.7)$$

and defining

$$A(k^2) = \sum_{n=0}^{\infty} a_n (-k^2)^n, \quad (4.8)$$

$$B(k^2) = \sum_{n=0}^{\infty} b_n (-k^2)^n, \quad (4.9)$$

we can write

$$\hat{\sigma}_{CE} = -ik A(k^2) \hat{u} - k^2 B(k^2) \hat{p}. \quad (4.10)$$

Hence if we know $A(k^2)$, $B(k^2)$ we know $\hat{\sigma}_{CE}$ and the Chapman-Enskog expansion has been summed. As mentioned earlier the key to the computation of A , B is the convolution nature of (4.6). Multiply both equations in (4.6) by $(-k^2)^{n+1}$:

$$\begin{aligned} a_{n+1}(-k^2)^{n+1} &= \frac{5}{3}b_n(-k^2)^n(-k^2) \\ &\quad - k^2 \sum_{m=0}^n a_n(-k^2)^m a_{n-m}(-k^2)^{n-m}, \\ b_{n+1}(-k^2)^{n+1} &= a_{n+2}(-k^2)^{n+1} \\ &\quad - k^2 \sum_{m=0}^n a_{n-m}(-k^2)^{n-m} b_m(-k^2)^m, \end{aligned}$$

then sum n from 0 to ∞ changing the order of summation in the terms on the right-hand side. This yields

$$A - a_0 = -k^2 \left\{ \frac{5}{3}B + A^2 \right\}, \quad (4.11)$$

$$B - b_0 = A - a_0 - k^2 AB, \quad (4.12)$$

where we set $a_0 = b_0 = -\frac{4}{3}$ so as to agree with the known first two terms in σ_{CE} .

Solve for A in (4.11) to obtain

$$A = \frac{B}{1 - k^2 B} \quad (4.13)$$

and substitute into (4.10) to obtain a cubic equation for B . If we set $C = k^2 B$ this cubic equation is simply

$$-\frac{5}{3}(1-C)^2 \left(C + \frac{4}{5} \right) - \frac{C}{k^2} = 0. \quad (4.14)$$

The change of variable $C = y + \frac{2}{5}$ puts the cubic in the form

$$y^3 + \frac{3}{5} \left(-\frac{7}{5} + \frac{1}{k^2} \right) y + \frac{6}{5} \left(3 + \frac{1}{5k^2} \right) = 0$$

or

$$y^3 + ay + b = 0$$

with a, b having the obvious values. A direct computation shows that

$$\frac{b^2}{4} + \frac{a^3}{27} > 0$$

and Cardano's formula tells us that the equation for y and hence the equation for C has one real root.

Further information on the real root is obtained by setting the cubic in (4.13) to be $F(C)$, i.e.,

$$F(C) = -\frac{5}{3}(1-C)^2 \left(C + \frac{4}{5} \right) - \frac{C}{k^2}.$$

Then we immediately see that $F(C) > 0$, $C < -\frac{4}{5}$, and $F(C) < 0$ for $C > 1$, so the real root must lie in $(-\frac{4}{5}, 1)$. Furthermore $F(0) = -\frac{4}{3}$ and hence the sign change of $F(C)$ in fact lies in $(-\frac{4}{5}, 0)$. Also for the root $C(k^2)$ we have

$$\frac{dC}{dk^2} \left\{ \left(-\frac{10}{5}(1-C) \left(C + \frac{4}{4} \right) \right) - \frac{5}{3}(1-C)^2 \right\} = -\frac{C}{k^4}$$

and so in the interval $(-\frac{4}{5}, 0)$, $\text{sign } \frac{dC}{dk^2} = \text{sign } \frac{C}{k^4} = \text{sign } C$. Thus $\frac{dC}{dk^2} < 0$ and $C(k^2)$ is monotone decreasing in k^2 , $C(0) = 0$, and $C(k^2) \rightarrow -\frac{4}{5}$ as $|k| \rightarrow \infty$.

We can now return to (4.10) and take inverse Fourier transforms to recover the exact sum of the Chapman-Enskog expansion

$$\sigma_{CE} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ikA(k^2)\hat{u} - k^2B(k^2)\hat{p})e^{ikx}dk, \quad (4.15)$$

where

$$A = \left(\frac{C}{1-C} \right) \frac{1}{k^2}, \quad B = \frac{C}{k^2}.$$

Thus the extra stress σ_{CE} is represented as a Fourier integral operator.

5. Equations of hydrodynamics, dispersion relation, and the entropy equality. In the previous section the exact sum of the Chapman-Enskog expansion was presented in equation (4.15). Hence in Fourier space the hydrodynamics become

$$\begin{aligned} \hat{p}_t &= \frac{5}{3}ik\hat{u}, \\ \hat{u}_t &= ik\hat{p} + ik(-ikA(k^2)\hat{u}(t, k) - k^2B(k^2)\hat{p}). \end{aligned} \quad (5.1)$$

We set $\hat{p}(t, k) = e^{\omega t}P(k)$, $\hat{u}(t, k) = e^{\omega t}U(k)$ and hence

$$\begin{bmatrix} -\omega & \frac{5}{3}ik \\ ik - ik^3B & k^2A - \omega \end{bmatrix} \begin{bmatrix} P \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (5.2)$$

We set the determinant of the coefficient matrix to zero and find the dispersion relation (spectrum in Fourier space)

$$\omega^2 - \omega k^2A + \frac{5}{3}(k^2 - k^4B) = 0. \quad (5.3)$$

Hence

$$\omega = \frac{1}{2} \left(\frac{C}{1-C} \right) \pm |k| \left(\left(\frac{C}{1-C} \right)^2 \frac{1}{k^2} - \frac{20}{3}(1-C) \right)^{1/2}$$

but recall that C satisfies the cubic

$$-\frac{5}{3}(C-1)^2 \left(C + \frac{4}{5} \right) = \frac{C}{k^2}$$

and hence $-\frac{5}{3}C \left(C + \frac{4}{5} \right) = \frac{C^2}{(1-C)^2} \frac{1}{k^2}$ and the formula for ω becomes

$$\omega = \frac{1}{2} \left(\frac{C}{C-1} \right) \pm i|k| \left(\frac{5C^2 - 16C + 20}{3} \right)^{1/2}, \quad (5.4)$$

where we note that the quadratic form $5C^2 - 16C + 20$ is always positive.

Since $C \rightarrow -\frac{4}{5}$ as $|k| \rightarrow \infty$ we see that $\operatorname{Re} \omega \rightarrow -\frac{2}{9}$ as $|k| \rightarrow \infty$. A sketch of the spectrum is given in Figure 1.

FIG. 1. A sketch of the spectrum for the hydrodynamic equations (5.1)

One issue not addressed by Gorban and Karlin in their papers was the derivation of an “entropy” equality. This computation is elementary. Multiply (5.1a) by $\frac{3}{5}\bar{\hat{p}}$, (5.1b) by $\bar{\hat{u}}$ (where the overbar denotes complex conjugation). This yields

$$\begin{aligned} & \frac{1}{2} \partial_t \left(\frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 \right) - ik(\bar{\hat{p}}\hat{u} + \hat{p}\bar{\hat{u}}) \\ & = k^2 A(k^2) |\hat{u}|^2 + ik\bar{\hat{u}}(-k^2 B(k^2) \hat{p}). \end{aligned}$$

But now use the relation (5.1a) to write the equality as

$$\begin{aligned} & \frac{1}{2} \partial_t \left(\frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 - \frac{3}{5} k^2 B(k^2) |\hat{p}|^2 \right) \\ & - ik(\bar{\hat{p}}\hat{u} + \hat{p}\bar{\hat{u}}) = k^2 A(k^2) |\hat{u}|^2. \end{aligned} \quad (5.5)$$

This is the “entropy” equality in Fourier space. Notice that the “entropy” $\frac{1}{2} \left(\frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 - \frac{3}{5} k^2 B(k^2) |\hat{p}|^2 \right)$ and the “dissipation” $k^2 A(k^2) |\hat{u}|^2$ each have the desired signs (positive and negative, respectively) since A and B are both negative for $k \neq 0$. In fact integration of (5.5) yields

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 dk \\ & + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk \\ & + \int_{-\infty}^{\infty} \left(\overline{\frac{\partial \hat{p}}{\partial x}} \right) \hat{u} + \left(\overline{\frac{\partial \hat{u}}{\partial x}} \right) \hat{p} dk \\ & = \int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk \end{aligned}$$

or with an application of Parseval's identity,

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |p|^2 + |u|^2 dx \\ & + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk \\ & = \int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk. \end{aligned} \quad (5.6)$$

The term

$$\int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk$$

represents a nonlocal version of the capillarity, whereas the term

$$\int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk$$

is a nonlocal version of the viscous dissipation, and we see that the hydrodynamics given by the exact summation for the Chapman-Enskog expansion is a nonlocal version of Korteweg's theory of capillarity [4], [5]. As noted earlier this idea had been put forward sometime ago by P. Rosenau [10], [11] but it seems quite satisfying to see that it follows directly from the equations of Grad via the remarkable contributions of Gorban and Karlin. Note that to leading order in k , $C = -C_0|k|^2$, $B = -C_0$, $A = -C_0$, $C_0 > 0$, and (5.6) becomes the classical entropy relation of Korteweg's theory

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |p|^2 + |u|^2 dx \\ & + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} C_0 (\partial_x p)^2 dx \\ & = - \int_{-\infty}^{\infty} C_0 (\partial_x u)^2 dx. \end{aligned} \quad (5.7)$$

In earlier work of mine [6], [7], Korteweg's theory of capillarity was proposed as an admissibility criterion (the viscosity-capillarity criterion) for weak solutions of the equations of gas dynamics. Here the appearance of Korteweg's theory with the good signs as the relevant coefficients as a direct result of the exact summation of the Chapman-Enskog expansion is for me a very pleasant corollary of the work of Gorban and Karlin. An additional point of interest is that the hydrodynamical equations (5.1) may be written in another form, which some may find appealing. We rewrite (5.1) as the second-order equation

$$\frac{3}{5} \hat{p}_{tt} + k^2 \hat{p} + k^2 \left(-A(k^2) \left(\frac{3}{5} \right) \hat{p}_t - k^2 B(k^2) \hat{p} \right) = 0 \quad (5.8)$$

and then attempt to write it in factored form as

$$\left(\frac{3}{5} \frac{\partial}{\partial t} + \mu_1(k) \right) \left(\frac{\partial}{\partial t} + \mu_2(k) \right) \hat{p} = -k^2 \hat{p}. \quad (5.9)$$

A comparison of (5.8) and (5.9) yields

$$\begin{aligned}\mu_1(k) + \frac{3}{5}\mu_2(k) &= -\frac{3}{5}k^2 A(k^2), \\ \mu_1(k)\mu_2(k) &= -k^4 B(k^2).\end{aligned}$$

Hence if we define $\hat{v}$ by the formula

$$\frac{\partial \hat{p}}{\partial t} + \mu_2(k)\hat{p} = ik\hat{v},$$

system (5.9) now takes the form

$$\begin{aligned}\frac{3}{5}\partial_t \hat{v} &= ik\hat{p} - \mu_1(k)\hat{v}, \\ \partial_t \hat{p} &= ik\hat{v} - \mu_2(k)\hat{p},\end{aligned}\tag{5.10}$$

or in physical space-time,

$$\begin{aligned}\frac{3}{5}\partial_t v &= -\partial_x p - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mu_1(k) \hat{v}(k, t) dk, \\ \partial_t p &= -\partial_x v - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mu_2(k) \hat{p}(k, t) dk.\end{aligned}\tag{5.11}$$

The formulas for $\mu_1(k), \mu_2(k)$ are

$$\begin{aligned}2\mu_1(k) &= -\frac{3}{5} \left(\frac{C}{1-C} \right) \\ &\quad \pm \frac{3}{5} i|k| \left(\frac{5C^2 - 16C}{3} \right)^{1/2}, \\ 2\mu_2(k) &= - \left(\frac{C}{1-C} \right) \\ &\quad \pm i|k| \left(\frac{5C^2 - 16C}{3} \right)^{1/2}.\end{aligned}\tag{5.12}$$

Since $5C^2 - 16C > 0$ (remember $C < 0$), we see that the damping coefficients $\mu_1(k), \mu_2(k)$ are in fact complex-valued, reflecting the “capillarity” effect in the system.

Finally I note that Gorban and Karlin in [3] carried their exact summation to the full 13-moment Grad system in three space dimensions. Hence the extension of the entropy equations to the full systems can be done as well but at a price of lengthier computations which were beyond the scope of this paper.

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