REMARKS ON THE THEORY OF THE DIVERGENCE-MEASURE FIELDS

BY

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Dedicated to Costas Dafermos on his 70th birthday

Abstract. We review the theory of the (extended) divergence-measure fields providing an up-to-date account of its basic results established by Chen and Frid (1999, 2002), as well as the more recent important contributions by Silhavý (2008, 2009). We include a discussion on some pairings that are important in connection with the definition of normal trace for divergence-measure fields. We also review its application to the uniqueness of Riemann solutions to the Euler equations in gas dynamics, as given by Chen and Frid (2002). While reviewing the theory, we simplify a number of proofs allowing an almost self-contained exposition.

1. Introduction. The purpose of this paper is to review the theory of the (extended) divergence-measure fields providing an up-to-date account of its basic results established by Chen and Frid [1, 2, 3], as well as the more recent important contributions by Silhavý [11, 12]. We include a discussion on some pairings that are important in connection with the definition of normal trace for divergence-measure fields. We also review its application to the uniqueness of Riemann solutions to the Euler equations in gas dynamics, as given by Chen and Frid [3]. While reviewing the theory, we simplify a number of proofs allowing an almost self-contained exposition. We refer to [6] and the papers already mentioned for a more complete bibliography on the theory of divergence-measure fields, as well as many others of its possible applications.

2. Divergence-measure fields. We begin by recalling the definition of the divergence-measure fields.

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DEFINITION 2.1. Let $U \subset \mathbb{R}^N$ be open. For $F \in L^p(U; \mathbb{R}^N)$, $1 \leq p \leq \infty$, or $F \in \mathcal{M}(U; \mathbb{R}^N)$, set

$$|\operatorname{div} F|(U) := \sup \left\{ \int_U \nabla \varphi \cdot F : \varphi \in C_0^1(U), |\varphi(x)| \leq 1, x \in U \right\}. \quad (2.1)$$

For $1 \leq p \leq \infty$, we say that F is an L^p -divergence-measure field over U , i.e., $F \in \mathcal{DM}^p(U)$, if $F \in L^p(U; \mathbb{R}^N)$ and

$$\|F\|_{\mathcal{DM}^p(U)} := \|F\|_{L^p(U; \mathbb{R}^N)} + |\operatorname{div} F|(U) < \infty. \quad (2.2)$$

We say that F is an extended divergence-measure field over D , i.e., $F \in \mathcal{DM}^{ext}(U)$, if $F \in \mathcal{M}(U; \mathbb{R}^N)$ and

$$\|F\|_{\mathcal{DM}^{ext}(U)} := |F|(U) + |\operatorname{div} F|(U) < \infty. \quad (2.3)$$

If $F \in \mathcal{DM}^*(U)$ for any open set $U \Subset \mathbb{R}^N$, then we say that $F \in \mathcal{DM}_{loc}^*(\mathbb{R}^N)$.

We recall the following product rule proved in [1].

THEOREM 2.1 (Chen and Frid [1]). Given $F \in \mathcal{DM}^\infty(U)$ and $g \in BV(U) \cap L^\infty(U)$, then $gF \in \mathcal{DM}^\infty(U)$ and

$$\operatorname{div}(gF) = \bar{g} \operatorname{div} F + \overline{F \cdot \nabla g}, \quad (2.4)$$

in the sense of Radon measures in U , where $\bar{g}$ equals g a.e. is the limit of a mollified sequence for g through a symmetric mollifier, and $\overline{F \cdot \nabla g}$ is a Radon measure absolutely continuous with respect to $|\nabla g|$, whose absolutely continuous part with respect to the Lebesgue measure in U satisfies

$$(\overline{F \cdot \nabla g})_{ac} = F \cdot (\nabla g)_{ac}, \quad \text{a.e. in } U. \quad (2.5)$$

Moreover, $|\overline{F \cdot \nabla g}|(U) \leq \|F\|_\infty |\nabla g|(U)$.

Proof. Let $g_\delta = \omega_\delta * g$, where $\omega_\delta(x) = \delta^{-N} \eta(\frac{x}{\delta})$ with a positive symmetric mollifier ω . One easily deduces that

$$\operatorname{div}(g_\delta F) = g_\delta \operatorname{div} F + F \cdot \nabla g_\delta. \quad (2.6)$$

Now, it is well known that g_δ converges to a Borel function $\bar{g}$, $\mathcal{H}^{N-1}$ -a.e. in U (this function equals g a.e. in U).

We claim that, for a Borel set $A \subset U$, $\mathcal{H}^{N-1}(A) = 0$ implies that $|\operatorname{div} F|(A) = 0$. Indeed, since $|\operatorname{div} F|$ is a Radon measure, we may assume that A is compact. Also, we may assume that $\operatorname{div} F(A) = |\operatorname{div} F|(A)$. Hence, given $\varepsilon > 0$, we may cover A with a finite number of balls $B_i = B(x_i; r_i)$, $i = 1, \dots, J$,

$$A \subset A_\varepsilon := \bigcup_{i=1}^J B_i, \text{ such that } \sum_{i=1}^J r_i^{N-1} \leq \varepsilon. \quad (2.7)$$

We may assume also that $|\operatorname{div} F|(\partial B_i) = 0$, $i = 1, \dots, J$, since otherwise we can take r_i slightly different satisfying this property and (2.7). By using an approximation of the identity sequence, we obtain a sequence $F_\delta \in C^\infty(U; \mathbb{R}^N)$ such that $F_\delta \rightarrow F$ a.e. in U , and $|\operatorname{div} F_\delta| \rightarrow |\operatorname{div} F|$ in $\mathcal{M}(U)$. Again, we may assume that $F_\delta \rightarrow F$ a.e. in

∂B_i , $i = 1, \dots, J$. Now, by the usual Gauss-Green formula for smooth vector fields and domains with Lipschitz boundaries, we have

$$\int_{A_\varepsilon} \operatorname{div} F_\delta \, dx = \int_{\partial A_\varepsilon} F_\delta \cdot \nu \, d\mathcal{H}^{N-1},$$

so that, passing to the limit when $\delta \rightarrow 0$, we obtain

$$\int_{A_\varepsilon} \operatorname{div} F = \int_{\partial A_\varepsilon} F \cdot \nu \, d\mathcal{H}^{N-1} \leq c \|F\|_\infty \sum_{i=1}^J r_i^{N-1} \leq c \|F\|_\infty \varepsilon.$$

Since A is compact, $\chi_{A_\varepsilon} \rightarrow \chi_A$ everywhere in U , and by dominated convergence applied to the measure $|\operatorname{div} F|$, we get $|\operatorname{div} F|(A) = \operatorname{div} F(A) = 0$, which proves the claim.

Then, using the claim we just proved, we get

$$g_\delta \operatorname{div} F \rightharpoonup \bar{g} \operatorname{div} F, \quad \text{in } \mathcal{M}(U),$$

as a consequence of dominated convergence applied to the measure $\operatorname{div} F$.

On the other hand, we claim that $\{\operatorname{div}(g_\delta F)\}$ is uniformly bounded in $\mathcal{M}(U)$. Indeed, this follows from

$$\begin{aligned} \langle \operatorname{div}(g_\delta F), \phi \rangle &= - \int_U g_\delta F \cdot \nabla \phi \, dx = - \int_U F \cdot \nabla(g_\delta \phi) \, dx + \int_U \phi F \cdot \nabla g_\delta \, dx \\ &\leq \|g\|_\infty |\operatorname{div} F|(U) + \|F\|_\infty |\nabla g|(U), \end{aligned}$$

for all $\phi \in C_c^\infty(U)$, with $\|\phi\|_\infty = 1$.

Now, $\operatorname{div}(g_\delta F)$ converges to $\operatorname{div}(gF)$, in the sense of distributions over U . Then, $\operatorname{div}(g_\delta F) \rightharpoonup \operatorname{div}(gF)$ in $\mathcal{M}(U)$. Hence,

$$F \cdot \nabla g_\delta \rightharpoonup \overline{F \cdot \nabla g} := \operatorname{div}(gF) - \bar{g} \operatorname{div} F.$$

Now we prove that $\overline{F \cdot \nabla g}$ is absolutely continuous w.r.t. $|\nabla g|$. Let $A \subset D$ be such that $|\nabla g|(A) = 0$. We are going to prove that $|\overline{F \cdot \nabla g}|(A) = 0$. It suffices to consider any compact set A with $|\nabla g|(A) = 0$. Given $\varepsilon > 0$, we can cover A by a finite number, J , of balls so that

$$A \subset \bigcup_{i=1}^J B(x_i; r_i), \quad r_i < \varepsilon; \quad |\nabla g|(\bigcup_{i=1}^J B(x_i; r_i)) < \varepsilon.$$

We may assume that $|\nabla g|(\partial B(x_i; r_i)) = 0$, $i = 1, \dots, J$. Let $\phi \in C_0(\bigcup_{i=1}^J B(x_i; r_i))$. Thus

$$\begin{aligned} \langle \overline{F \cdot \nabla g}, \phi \rangle &= \lim_{\delta \rightarrow 0} \int \phi(x) F(x) \cdot \nabla g_\delta(x) \, dx \\ &= \|\phi\|_\infty \|F\|_\infty |\nabla g|(\bigcup_{i=1}^J B(x_i; r_i)) \leq \varepsilon \|\phi\|_\infty \|F\|_\infty, \end{aligned}$$

from the fact that $|\nabla g_\delta|(B) \rightarrow |\nabla g|(B)$, for all open sets $B \subset D$ with $|\nabla g|(\partial B) = 0$. Hence, we obtain

$$|\overline{F \cdot \nabla g}|(A) \leq |\overline{F \cdot \nabla g}|(\bigcup_{i=1}^J B(x_i; r_i)) \leq \varepsilon \|F\|_\infty.$$

The proof of (2.5) is a little more technical and, for that, we simply refer to [1] since it escapes our purposes here. $\square$

We now recall a result of Silhavý in [11] that is in some sense a dual formulation for the previous result, in the sense that it compensates a relaxation on the regularity of the vector field F , which now may be just a vector measure, by imposing more regularity on the function g , which now is assumed to be in $W^{1,\infty}(U)$. As we will see, its proof follows exactly the same lines as that of Theorem 2.1 just recalled.

THEOREM 2.2 (Silhavý [11]). Given $F \in \mathcal{DM}^{ext}(U)$ and $g \in W^{1,\infty}(U)$, then $gF \in \mathcal{DM}^{ext}(U)$ and

$$\operatorname{div}(gF) = g \operatorname{div} F + \overline{\nabla g \cdot F}, \quad (2.8)$$

in the sense of Radon measures in U , where $\overline{\nabla g \cdot F}$ is a Radon measure absolutely continuous with respect to $|F|$. Moreover:

- (i) $|\overline{\nabla g \cdot F}|(U) \leq \|\nabla g\|_\infty |F|(U)$.
- (ii) If $h \in W^{1,\infty}(U)$, $\overline{\nabla(gh) \cdot F} = h \overline{\nabla g \cdot F} + g \overline{\nabla h \cdot F} = \overline{\nabla g \cdot hF} + \overline{\nabla h \cdot gF}$.
- (iii) If $V \subset U$ is an open set, then $(\overline{\nabla g \cdot F} \llcorner V)_V = \overline{\nabla g \cdot F}_U \llcorner V$.
- (iv) $(\overline{\nabla g \cdot F})_{ac} = \nabla g \cdot (F)_{ac}$.

Proof. We again define g_δ as above and obtain (2.6). We have that g_δ converges locally uniformly to g so that the first term on the right-hand side of (2.6) converges to $g \operatorname{div} F$, in the sense of the Radon measures. It is also easy to see that $\nabla g_\delta \cdot F$ is uniformly bounded in $\mathcal{M}(U)$. Therefore, the left-hand side of (2.6) is also compact in $\mathcal{M}(U)$, in the weak star topology, and since it converges to $\operatorname{div}(gF)$ in the sense of distributions, it follows that $\operatorname{div}(gF)$ is indeed a Radon measure and the whole sequence $\operatorname{div}(g_\delta F)$ converges to $\operatorname{div}(gF)$. Hence, the whole sequence $\nabla g_\delta \cdot F$ converges to the Radon measure

$$\overline{\nabla g \cdot F} := \operatorname{div}(gF) - g \operatorname{div} F.$$

The assertions (i)–(ii) are proved in the standard way. Assertion (iii) is called the localization property in [11]; it follows trivially from the definitions. Finally, the proof of (iv) is entirely similar to that of the analogous assertion in Theorem 2.1. $\square$

Actually, Silhavý in [12], using machinery from Geometric Measure Theory (flat chains, normal currents, etc.) defines pairings more general than the one established by (2.8). Namely, by [12], we have the following more general fact, for which we provide here a proof that is much simpler than the one in [12].

PROPOSITION 2.1 (Silhavý [12]). If $z \in L^\infty(U; \mathbb{R}^N)$ is such that $\operatorname{curl} z \in L^\infty(U; \mathbb{R}^{N^2})$, and $F \in \mathcal{DM}^{ext}(U)$, $z_\delta = w_\delta * z$, then $z_\delta \cdot F$ converges in the sense of distributions to a well-defined Radon measure $\overline{z \cdot F}$, which is absolutely continuous with respect to $|F|$ and we have

$$|\overline{z \cdot F}|(U) \leq \|z\|_\infty |F|(U). \quad (2.9)$$

Proof. We borrow some ideas in the proof of the Div-Curl Lemma given in [9]. Let us write

$$z = \nabla g + w, \quad (2.10)$$

where g is given by

$$\begin{aligned}\Delta g &= \operatorname{div} z, & \text{in } U, \\ g &= 0, & \text{in } \partial U,\end{aligned}\tag{2.11}$$

and so $\operatorname{div} w = 0$ in the sense of distributions. Now, let us define u by

$$\begin{aligned}\Delta u &= w, & \text{in } U, \\ u &= 0, & \text{in } \partial U.\end{aligned}\tag{2.12}$$

We then have that $\zeta = -\operatorname{div} u$ is a harmonic function on U . If we set $y := w - \nabla \zeta$, we easily check that

$$y^i = \sum_{j=1}^N (u_{x_i}^j - u_{x_j}^i)_{x_j}.$$

Now, the hypothesis that $\operatorname{curl} z \in L^\infty(U; \mathbb{R}^{N^2})$ implies, by (2.12) and (2.10), that $\operatorname{curl} u \in W_{\operatorname{loc}}^{2,p}(U; \mathbb{R}^{N^2})$, for any $p \in [1, \infty)$; hence $y \in C(U)$, which in turn implies that $w \in C(U)$. Therefore, for any $V \Subset U$, we have that, in (2.10), $g \in W^{1,\infty}(V)$, and so we have

$$\lim_{\delta \rightarrow 0} z_\delta \cdot F = (\overline{\nabla g \cdot F})_V + w \cdot F, \quad \text{in the sense of the distributions in } V.$$

Then, we may define $\overline{z \cdot F} \in \mathcal{M}(U)$ by setting

$$\overline{z \cdot F}|_V := (\overline{\nabla g \cdot F})_V + w \cdot F, \quad \text{in the sense of the distributions in } V,$$

for all $V \Subset U$.

The fact that

$$\overline{z \cdot F} = \lim_{\delta \rightarrow 0} z_\delta \cdot F, \quad \text{in the sense of the distributions in } U,$$

trivially implies (2.9). $\square$

To complete our discussion concerning the pairings $\overline{z \cdot F}$, between $z \in L^\infty(U; \mathbb{R}^N)$, with $\operatorname{curl} z \in L^\infty(U; \mathbb{R}^{N^2})$, and $F \in \mathcal{DM}^{ext}(U)$, given by Proposition 2.1, we state the following weak continuity property of the Div-Curl type due to Silhavy [12].

LEMMA 2.1 (Div-Curl type weak continuity [12]). Let $(z^n) \subset L^\infty(U; \mathbb{R}^N)$ be a weak star converging sequence to $z \in L^\infty(U; \mathbb{R}^N)$, such that $(\operatorname{curl} z^n)$ is a uniformly bounded sequence in $L^\infty(U; \mathbb{R}^{N^2})$, and let (F^n) be a uniformly bounded sequence in $\mathcal{DM}^{ext}(U)$, weak star converging to $F \in \mathcal{DM}^{ext}(U)$. Then, $\overline{z^n \cdot F^n} \rightharpoonup \overline{z \cdot F}$ in the weak topology of $\mathcal{M}(U)$.

Proof. We write, as in the proof of Proposition 2.1,

$$z^n = \nabla g^n + w^n,\tag{2.13}$$

and, by what we have seen in that proof, we deduce that w^n is locally uniformly converging in $C(U)$, and so g^n is locally uniformly covering to a function $g \in W_{\operatorname{loc}}^{1,\infty}(U)$. But,

$$\overline{\nabla g^n \cdot F^n} = \operatorname{div}(g^n F^n) - g^n \operatorname{div} F^n,\tag{2.14}$$

and by what we have just said, we have that $\operatorname{div}(g^n F^n) \rightharpoonup \operatorname{div}(gF)$ in the sense of distributions in U , and $g^n \operatorname{div} F^n \rightharpoonup g \operatorname{div} F$ in the weak star topology of $\mathcal{M}(U)$. Therefore, we have $\overline{\nabla g^n \cdot F^n} \rightharpoonup \overline{\nabla g \cdot F}$ in the weak star topology of $\mathcal{M}(U)$. Hence, (2.13) and the

local uniform convergence of w^n imply that $\overline{z^n \cdot F^n} \rightharpoonup \overline{z \cdot F}$ in the weak star topology of $\mathcal{M}(U)$. $\square$

Concerning the pairing $\overline{F \cdot \nabla g}$, with $F \in \mathcal{DM}^\infty(U)$ and $g \in BV(U) \cap L^\infty(U)$, given by Theorem 2.1, we have the following weak continuity property.

LEMMA 2.2. Let $(g^n) \subset BV(U) \cap L^\infty(U)$ be such that $g^n \rightarrow g \in BV(U) \cap L^\infty(U)$ in $L^1(U)$, with g^n bounded in $L^\infty(U)$ and ∇g^n bounded in $\mathcal{M}(U)$. Let $(F^n) \subset \mathcal{DM}^\infty(U)$ be such that $F^n \rightharpoonup F \in \mathcal{DM}(U)$ in the weak star topology of $L^\infty(U; \mathbb{R}^N)$, and $|\operatorname{div} F^n - \varphi \, dx| \rightarrow 0$ in the weak star topology of $\mathcal{M}(U)$, for some $\varphi \in L^1(U)$. Then $\overline{F^n \cdot \nabla g^n} \rightharpoonup \overline{F \cdot \nabla g}$ in the weak star topology of $\mathcal{M}(U)$.

Proof. Again, we have

$$\overline{F^n \cdot \nabla g^n} = \operatorname{div}(g^n F^n) - \bar{g}^n \operatorname{div} F^n, \quad (2.15)$$

where the first term on the right-hand side is a bounded sequence in $\mathcal{M}(U)$, converging to $\operatorname{div}(gF)$ in the sense of distributions, and we claim that the second term, on the same side of (2.15), converges to $g\varphi \, dx$, in the weak star topology of $\mathcal{M}(U)$. Indeed, $|\bar{g}^n(\operatorname{div} F^n - \varphi \, dx)| \leq C|\operatorname{div} F^n - \varphi \, dx| \rightarrow 0$, in the sense of Radon measures on U . Also, (g^n) is bounded in $L^\infty(U)$ and converges a.e. to g in U , by passing to a subsequence if necessary, so that $g^n \varphi \, dx \rightharpoonup g\varphi \, dx$, weak star in $\mathcal{M}(U)$. Moreover, clearly, $\varphi = \operatorname{div} F$, and so $\bar{g} \operatorname{div} F = g\varphi \, dx$ in $\mathcal{M}(U)$. Hence, by definition, we get that the right-hand side of (2.15) converges to $\overline{F \cdot \nabla g}$ in the weak star topology of $\mathcal{M}(U)$, concluding the proof. $\square$

We now come back to the main subject of this paper and recall the Gauss-Green formula for general divergence-measure fields, first proved in [2, 3] and extended by Silhavy in [11].

THEOREM 2.3 (Chen & Frid [2, 3], Silhavy [11]). If $F \in \mathcal{DM}^{ext}(U)$, then there exists a linear functional $F \cdot \nu : \operatorname{Lip}(\partial U) \rightarrow \mathbb{R}$ such that

$$F \cdot \nu(g|\partial U) = \int_U \overline{\nabla g \cdot F} + \int_U g \operatorname{div} F, \quad (2.16)$$

for every $g \in \operatorname{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover,

$$|F \cdot \nu(h)| \leq |F|_{\mathcal{DM}(U)} |h|_{\operatorname{Lip}(\partial U)}, \quad (2.17)$$

for all $h \in \operatorname{Lip}(\partial U)$, where we use the notation

$$|g|_{\operatorname{Lip}(C)} := \sup_{x \in C} |g(x)| + \operatorname{Lip}_C(g).$$

Proof. A major step in the proof of this result is to prove that the right-hand side of (2.16) depends only on the values of g restricted to ∂U , that is, that for $g \in \operatorname{Lip}(\mathbb{R}^N)$, with $g(x) = 0$, for $x \in \partial U$, then

$$\int_U \overline{\nabla g \cdot F} + \int_U g \operatorname{div} F = 0. \quad (2.18)$$

Clearly, we may as well assume $g(x) = 0$, for $x \in \mathbb{R}^N \setminus U$ (cf. Lemma 3.2 in [11]). We first prove (2.18) in the case where $\operatorname{supp} g$ is a compact subset of U . In this case, for

$\delta > 0$ sufficiently small we have $\text{supp } g_\delta \subset U$, where, as above, $g_\delta = g * \omega_\delta$. Then, by the definition of the divergence of the (vector-valued) distribution F , we have

$$\int_U \nabla g_\delta \cdot F + \int_U g_\delta \operatorname{div} F = 0. \quad (2.19)$$

Hence, taking the limit when $\delta \rightarrow 0$ in (2.19), using the definition of $\overline{\nabla g \cdot F}$, we obtain (2.18) in this case. We now consider the case where $g \in \operatorname{Lip}(\mathbb{R}^N)$ and $g(x) = 0$, for $x \in \mathbb{R}^N \setminus U$. Let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\zeta(t) := \begin{cases} 0, & \text{if } t < 1/2, \\ 2(t - 1/2), & \text{if } 1/2 \leq t \leq 1, \\ 1, & \text{if } t > 1, \end{cases}$$

and for each $\varepsilon > 0$ let $h_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}$ be defined by

$$h_\varepsilon(x) := \begin{cases} \zeta(\varepsilon^{-1} \operatorname{dist}(x, \partial U)), & x \in U, \\ 0, & x \in \mathbb{R}^N \setminus U. \end{cases}$$

Observe that h_ε is a Lipschitz function satisfying $h_\varepsilon(x) = 1$ if $x \in U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) \geq \varepsilon\}$. Then the function $h_\varepsilon g$ is a Lipschitz function which coincides with g on U_ε and

$$\overline{\nabla(h_\varepsilon g) \cdot F} = h_\varepsilon \overline{\nabla g \cdot F} + g \overline{\nabla h_\varepsilon \cdot F}.$$

By what we have already proved, we have

$$\int_U h_\varepsilon \overline{\nabla g \cdot F} + \int_U g \overline{\nabla h_\varepsilon \cdot F} + \int_U h_\varepsilon g \operatorname{div} F = 0. \quad (2.20)$$

Now, we have

$$\int_U g \overline{\nabla h_\varepsilon \cdot F} = \int_{U \setminus U_{2\varepsilon}} g \overline{\nabla h_\varepsilon \cdot F},$$

since $\nabla h_\varepsilon \equiv 0$ in U_ε . Also, $|\nabla h_\varepsilon| \leq 2\varepsilon^{-1}$, and $|g(x)| \leq 2\operatorname{Lip}(g)\varepsilon$, for $x \in U \setminus U_{2\varepsilon}$. Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_U g \overline{\nabla h_\varepsilon \cdot F} = \lim_{\varepsilon \rightarrow 0} \int_{U \setminus U_{2\varepsilon}} g \overline{\nabla h_\varepsilon \cdot F} = 0,$$

by dominated convergence. Hence, letting $\varepsilon \rightarrow 0$ in (2.20), since $h_\varepsilon \rightarrow 1$, as $\varepsilon \rightarrow 0$, everywhere in U , we finally get (2.18).

The assertion just proved shows that the right-hand side of (2.16) depends only on $g|_{\partial U}$. Also, the inequality (2.17) is clear from (2.16), in the case where $h = H|_{\partial U}$, where $H \in \operatorname{Lip}(\mathbb{R}^N)$, and

$$|H|_{\operatorname{Lip}(\mathbb{R}^N)} = |h|_{\operatorname{Lip}(\partial U)}.$$

Now, Kirszbraun's Theorem (see, e.g., [10, 8]) guarantees, for any $h \in \operatorname{Lip}(\partial U)$, the existence of $H \in \operatorname{Lip}(\mathbb{R}^N)$ such that $H|_{\partial U} = h$ and $\operatorname{Lip}_{\mathbb{R}^N}(H) = \operatorname{Lip}_{\partial U}(h)$. Moreover, a trivial cutoff procedure ensures that $\|H\|_{L^\infty(\mathbb{R}^N)} = \|h\|_{L^\infty(\partial U)}$; this completes the proof. $\square$

We now discuss a direct way of defining the normal trace functional $F \cdot \nu : \text{Lip}(\partial U) \rightarrow \mathbb{R}$. The formula was first obtained in [2, 3], under regularity restrictions on the boundary, and in [11], for general boundaries. Before stating the corresponding result, we recall the following lemma, which is a slight modification of Lemma 3.3 of [11].

LEMMA 2.3 (Silhavy [11]). If $F \in \mathcal{DM}^{ext}(U)$, $m \in \text{Lip}(U)$, $t \in \mathbb{R}$ and if $T \subset m^{-1}(t)$ is a compact subset of U , then the restriction $\overline{\nabla m \cdot F}|_T$, of $\overline{\nabla m \cdot F}$ to T , satisfies

$$\overline{\nabla m \cdot F}|_T = 0. \quad (2.21)$$

Proof. Clearly, we can take $t = 0$. Also, multiplying m by a suitable function in $C_0^\infty(U)$, if necessary, we can assume that m has compact support in U . Therefore, we can assume that m is a Lipschitz function vanishing on $\mathbb{R}^N \setminus W$, with $W = U \setminus T$, and, in particular, also on $\mathbb{R}^N \setminus U$. Therefore, for any $\eta \in C_0^\infty(U)$, we have

$$\int_W \overline{\nabla(\eta m) \cdot F} + \int_W \eta m \operatorname{div} F = 0, \quad (2.22)$$

$$\int_U \overline{\nabla(\eta m) \cdot F} + \int_U \eta m \operatorname{div} F = 0. \quad (2.23)$$

Subtracting (2.22) from (2.23), since ηm vanishes on T , we get

$$0 = \int_T \overline{\nabla(\eta m) \cdot F} = \int_T \eta \overline{\nabla m \cdot F} + \int_T m \overline{\nabla \eta \cdot F} = \int_T \eta \overline{\nabla m \cdot F},$$

and so, since η is arbitrary, we arrive at (2.21). $\square$

The following result gives a simple formula to compute the normal trace of $\mathcal{DM}$ -fields. This formula, displayed in (i) of the statement below, was first obtained in [2, 3] under some regularity restrictions on the boundary, and later was extended to general domains in [11]. The item (ii) gives a useful necessary condition for the normal trace to be a measure over ∂U established by Silhavy [11].

THEOREM 2.4. Let $F \in \mathcal{DM}^{ext}(U)$ and $m : \mathbb{R}^N \rightarrow \mathbb{R}$ be a nonnegative Lipschitz function with $\operatorname{supp} m \subset \bar{U}$ which is strictly positive on U , and for each $\varepsilon > 0$, let $L_\varepsilon = \{x \in U : 0 < m(x) < \varepsilon\}$. Then:

(i) (cf. [2, 3] and [11]) If $g \in \text{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have

$$F \cdot \nu(g|\partial U) = - \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{L_\varepsilon} g d(\overline{\nabla m \cdot F}); \quad (2.24)$$

(ii) (cf. [11]) If

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-1} |\overline{\nabla m \cdot F}|(L_\varepsilon) < \infty, \quad (2.25)$$

then $F \cdot \nu$ is a measure over ∂U .

Proof. We repeat the proof given in [11].

(i) For each $\varepsilon > 0$ we define $m_\varepsilon(x) = \varepsilon^{-1} \min\{m(x), \varepsilon\}$. We see that m_ε is a Lipschitz function vanishing on ∂U . We have that $gm_\varepsilon \in \text{Lip}(\mathbb{R}^N)$ and

$$\overline{\nabla(gm_\varepsilon) \cdot F} = m_\varepsilon \overline{\nabla g \cdot F} + g \overline{\nabla m_\varepsilon \cdot F},$$

by the properties of the pairing $\overline{\nabla g \cdot F}$. Since gm_ε vanishes on ∂U , we have

$$\int_U m_\varepsilon d(\overline{\nabla g \cdot F}) + \int_U g d(\overline{\nabla m_\varepsilon \cdot F}) + \int_U gm_\varepsilon \operatorname{div} F = 0. \quad (2.26)$$

Now, $m_\varepsilon(x) \rightarrow 1$ everywhere in U , so that dominated convergence implies that

$$\int_U m_\varepsilon d(\overline{\nabla g \cdot F}) \rightarrow \int_U d(\overline{\nabla g \cdot F})$$

and $\int_U gm_\varepsilon \operatorname{div} F \rightarrow \int_U g \operatorname{div} F$. On the other hand, we have $m_\varepsilon = \varepsilon^{-1}m$ in L_ε , so $\nabla m_\varepsilon = \varepsilon^{-1}\nabla m$, a.e. in L_ε , which gives $\overline{\nabla m_\varepsilon \cdot F} = \varepsilon^{-1}\overline{\nabla m \cdot F}$, over L_ε . Moreover, since $U \setminus L_\varepsilon = m_\varepsilon^{-1}(1)$, by Lemma 2.3, we have $\overline{\nabla m_\varepsilon \cdot F} = 0$ on $U \setminus L_\varepsilon$. Hence, we obtain (2.24) from (2.26) when $\varepsilon \rightarrow 0$, by the definition of $F \cdot \nu|_{\partial U}$ in (2.16).

(ii) By (2.25), we have $|\overline{\nabla m \cdot F}|(L_\varepsilon) \leq C\varepsilon$, for some $C > 0$ independent of ε , so that

$$\left| \varepsilon^{-1} \int_{L_\varepsilon} g d(\overline{\nabla m \cdot F}) \right| \leq C \|g\|_{L^\infty(\mathbb{R}^N)},$$

for each $g \in \operatorname{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Therefore, given $g \in \operatorname{Lip}(\partial U)$, we may extend g to a Lipschitz function in $\mathbb{R}^N$ so that $\|g|_{\partial U}\|_{L^\infty(\partial U)} = \|g\|_{L^\infty(\mathbb{R}^N)}$, and so, by (2.24), we deduce that $|F \cdot \nu(g)| \leq C \|g\|_{L^\infty(\partial U)}$, which implies, by the Riesz representation theorem, that $F \cdot \nu$ is a measure on ∂U , as asserted. $\square$

REMARK 2.1. A typical example of m in the statement of Theorem 2.4 is provided by $m(x) = \operatorname{dist}(x, \partial U)$, for $x \in U$, and $m(x) = 0$, for $x \in \mathbb{R}^N \setminus U$.

REMARK 2.2. The following interesting example from [11] shows cases where $F \cdot \nu$ is a measure over ∂U and cases where $F \cdot \nu$ fails to be a measure. Namely, for $1 \leq \alpha < 3$, let $F : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2$ be defined by

$$F(x) = \frac{1}{|x|^\alpha} (x_2, -x_1),$$

and let $U = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1, x_2 < 0\}$. Clearly, $\operatorname{div} F = 0$, in $\mathbb{R}^2 \setminus \{0\}$, and we easily verify that

$$F \in \mathcal{DM}^p(U; \mathbb{R}^2) \quad \text{with } 1 \leq p < 2/(\alpha - 1), \text{ for } 1 < \alpha < 3, \text{ and } p = \infty, \text{ for } \alpha = 1.$$

Now, if $g \in \operatorname{Lip}(\partial U)$ and $\operatorname{supp} g \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0, |x_1| < 1\}$, we may use (2.25) with m satisfying $m(x) = -x_2$, for $-\varepsilon_0 < x_2 \leq 0$, and $|x_1| < 1 - \varepsilon_0$, with $\varepsilon_0 > 0$ small enough so that $g(x) = 0$ if $|x_1| \geq 1 - \varepsilon_0$. Also, we may consider an extension of g to $\mathbb{R}^2$ such that $g(x_1, x_2) = g(x_1, 0)$, for $|x_2| < \varepsilon_0$. Applying (2.25) with m and the extension of g so defined, we get

$$F \cdot \nu(g) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{-1}^1 g(t, s) \frac{t}{(t^2 + s^2)^{\frac{\alpha}{2}}} dt ds,$$

which gives

$$F \cdot \nu(g) = \int_{-1}^1 g(t, 0) \operatorname{sgn}(t) |t|^{1-\alpha} dt, \quad \text{for } 1 \leq \alpha < 2,$$

and

$$F \cdot \nu(g) := \lim_{\varepsilon \rightarrow 0} \int_{|t| > \varepsilon} g(t, 0) \operatorname{sgn}(t) |t|^{1-\alpha} dt, \quad \text{for } 2 \leq \alpha < 3.$$

This shows that, for $1 \leq \alpha < 2$, $F \cdot \nu$ is a measure, while, for $2 \leq \alpha < 3$, $F \cdot \nu$ is not a measure on ∂U .

REMARK 2.3. For $\varepsilon_0 > 0$ sufficiently small and $0 < s < \varepsilon_0$, we may consider the open set $U_s := \{x \in U : m(x) > s\}$, for m as in Theorem 2.4. By Theorem 2.4, for the normal trace $F \cdot \nu|_{\partial U_s}$, we have the following formula similar to (2.24),

$$F \cdot \nu(g|_{\partial U_s}) = - \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\{s < m(x) < s+\varepsilon\}} g d(\overline{\nabla m \cdot F}), \quad (2.27)$$

and, again, we have that the condition

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-1} |\overline{\nabla m \cdot F}|(\{s < m(x) < s + \varepsilon\}) < \infty \quad (2.28)$$

implies that $F \cdot \nu|_{\partial U_s}$ is a measure on ∂U_s . If we consider the monotone function $V(s) = |\overline{\nabla m \cdot F}|(\{0 < m(x) < s\})$, for $s \in (0, \varepsilon_0)$, we see that the left-hand side of (2.28) is the right-derivative of V at s , which we know exists for a.e. $s \in (0, \varepsilon_0)$. Therefore, $F \cdot \nu|_{\partial U_s}$ is a measure for a.e. $s \in (0, \varepsilon_0)$, and in this sense we may assert that “for almost all boundaries” ∂U the normal trace $F \cdot \nu|_{\partial U}$ is a measure.

3. Application to the Euler equations in gas dynamics. In this section we review the application of Theorem 2.3 to gas dynamics given in [3].

So, we consider the system of Euler equations in gas dynamics in Lagrangian coordinates (we adopt the notation of [6])

$$u_t - v_x = 0, \quad (3.1)$$

$$v_t + p(u, S)_x = 0, \quad (3.2)$$

$$(\mathcal{E} + \frac{v^2}{2})_t + (pv)_x = 0, \quad (3.3)$$

where u is the specific volume, v is the velocity, p is the pressure, S is the entropy, and $\mathcal{E}$ is the internal energy. For ideal polytropic gases, we have

$$\mathcal{E} = c_u \theta, \quad (3.4)$$

where θ is the temperature, c_u is a positive constant, and the following state equation holds:

$$pu = R\theta, \quad (3.5)$$

which, together with the Gibbs relation

$$d\mathcal{E} = \theta dS - p du, \quad (3.6)$$

closes system (3.1)-(3.3), giving

$$p(u, S) = \kappa u^{-\gamma} e^{S/c_u}, \quad (3.7)$$

where R and κ are positive constants, and $\gamma = 1 + \frac{R}{c_u}$.

If we denote by ρ the gas density, then we have $u = \rho^{-1}$. Of special interest here is the possibility of the occurrence of a vacuum, that is, a region of the space where $\rho = 0$. To deal with this possibility we must allow u to be a measure, instead of an ordinary function, as can be easily seen from the simple case of the solution of a Riemann problem

for the isentropic gas dynamics 2×2 system where a vacuum occurs. Namely, let us consider the system

$$u_t - v_x = 0, \quad (3.8)$$

$$v_t + p(u)_x = 0, \quad (3.9)$$

where $p(u) = \kappa u^{-\gamma}$, with $\gamma > 1$ and we take $\kappa = (\gamma - 1)^2/(4\gamma)$ for simplicity. We consider, for instance, an initial data of the form

$$(u(x, 0), v(x, 0)) = \begin{cases} (u_0, -M), & x < 0, \\ (u_0, M), & x > 0, \end{cases} \quad (3.10)$$

with $u_0 > 0, M > 0$. If $M - u_0^{-(\gamma-1)/2} =: v_0 > 0$, the solution of the above Riemann problem is given by

$$(u(x, t), v(x, t)) = \begin{cases} (u_0, -M), & \xi := x/t < -\sqrt{-p'(u_0)}, \\ (u_\xi, -v_0 - u_\xi^{-(\gamma-1)/2}), & -\sqrt{-p'(u_0)} \leq \xi < 0, \\ (2v_0 t \delta_0(x) \otimes dt, 0), & x = 0, t > 0, \\ (u_\xi, v_0 + u_\xi^{-(\gamma-1)/2}), & 0 < \xi < \sqrt{-p'(u_0)}, \\ (u_0, M), & \xi > \sqrt{-p'(u_0)}, \end{cases} \quad (3.11)$$

where u_ξ is given by $\sqrt{-p'(u_\xi)} = |\xi|$, and $\delta_0(x) \otimes dt$ is the product measure obtained from the product of the Dirac measure concentrated at 0 on $\mathbb{R}$ by the unidimensional Lebesgue measure on $(0, \infty)$. Observe that the self-similarity is broken on the line $x = 0$, where the singular part of u is concentrated. We also remark that, since u is a measure, we should understand $p(u)$ as the L^1_{loc} function resulting from the composition of p with the absolutely continuous part of the measure u , with respect to the Lebesgue measure over $\mathbb{R} \times (0, \infty)$, $u_{\text{abs}}(x, t)$, which is coherent with the fact that $p(u) \rightarrow 0$ as $u \rightarrow +\infty$.

In [3], the stability of the solution to the Riemann problem for the system (3.1)-(3.3), with p given by (3.7), is studied for a shock-free Riemann solution, which may or may not contain a vacuum. So, let us come back to system (3.1)-(3.3), and, for simplicity, let us write

$$p(u, S) = A(S)u^{-\gamma}, \quad B(S) = \frac{2}{\gamma-1}A(S)^{1/2}, \quad W = (u, v, S).$$

We then consider the solution of the Riemann problem for (3.1)-(3.3), formed by prescribing initial data of the type

$$W(x, 0) = \bar{W}_0(x) = (\bar{u}_0(x), \bar{v}_0(x), \bar{S}_0(x)) := \begin{cases} W_L = (u_L, v_L, S_L), & x < 0, \\ W_R = (u_R, v_R, S_R), & x > 0. \end{cases} \quad (3.12)$$

Let

$$c_L = v_L + B(S_L)u_L^{-\frac{\gamma-1}{2}}, \quad c_R = v_R - B(S_R)u_R^{-\frac{\gamma-1}{2}}.$$

We assume that

$$c_L - B(S_L) \left(\frac{A(S_R)}{A(S_L)} \right)^{\frac{\gamma-1}{2\gamma}} u^{-\frac{\gamma-1}{2}} < c_R + B(S_R) u^{-\frac{\gamma-1}{2}}, \quad (3.13)$$

$$\text{for } u \in \{u_R, \left(\frac{A(S_R)}{A(S_L)} \right)^{\frac{1}{\gamma}} u_L\}.$$

In the (u, v) -plane, we consider the curves:

$$\begin{aligned} \mathcal{L}_L &:= \{(u, v) : v + B(S_L) u^{-\frac{\gamma-1}{2}} = c_L\}, \\ \mathcal{L}_R &:= \{(u, v) : v - B(S_R) u^{-\frac{\gamma-1}{2}} = c_R\}, \\ \tilde{\mathcal{L}}_L &:= \{(u, v) : v + B(S_L) \left(\frac{A(S_R)}{A(S_L)} \right)^{\frac{\gamma-1}{2\gamma}} u^{-\frac{\gamma-1}{2}} = c_L\}. \end{aligned}$$

The inequality (3.13) implies that $\tilde{\mathcal{L}}_L$ and $\mathcal{L}_R$ do not intersect for u between u_R and $\tilde{u}_L := \left(\frac{A(S_R)}{A(S_L)} \right)^{\frac{1}{\gamma}} u_L$, $\mathcal{L}_R$ lying above $\tilde{\mathcal{L}}_L$ for u on this interval. Observe that $p(\tilde{u}_L, S_R) = p(u_L, S_L)$. Then, there are two possibilities: either $c_R < C_L$ and the curve $\tilde{\mathcal{L}}_L$ intersects the curve $\mathcal{L}_R$ when u grows from $\max\{u_R, \tilde{u}_L\}$ to $+\infty$; or $c_L \leq c_R$ and these curves never intersect, when u grows from $\min\{u_R, \tilde{u}_L\}$ to $+\infty$.

In the first case, let $\tilde{\mathcal{L}}_L$ and $\mathcal{L}_R$ intersect at the point (u_{*r}, v_*) in the (u, v) -plane, and let u_{*l} be such that $(u_{*l}, v_*) \in \mathcal{L}_L$. Clearly, we must have $u_L < u_{*l}$. The Riemann solution is then defined by

$$\bar{W}(x, t) = \begin{cases} W_L, & \xi < -\sqrt{-p_u(u_L, S_L)}, \\ (u_{l\xi}, v_{l\xi}, S_L), & -\sqrt{-p_u(u_L, S_L)} \leq \xi < -\sqrt{-p_u(u_{*l}, S_L)}, \\ (u_{*l}, v_*, S_L), & -\sqrt{-p_u(u_{*l}, S_L)} \leq \xi < 0, \\ (u_{*r}, v_*, S_R), & 0 < \xi \leq \sqrt{-p_u(u_{*r}, S_R)}, \\ (u_{r\xi}, v_{r\xi}, S_R), & \sqrt{-p_u(u_{*r}, S_R)} < \xi \leq \sqrt{-p_u(u_R, S_R)}, \\ W_R, & \xi > \sqrt{-p_u(u_R, S_R)}, \end{cases} \quad (3.14)$$

where $u_{l\xi}, u_{r\xi}$ are given by

$$\sqrt{-p_u(u_{l\xi}, S_L)} = |\xi|, \quad \sqrt{-p_u(u_{r\xi}, S_R)} = \xi,$$

and $v_{l\xi}, v_{r\xi}$ are such that $(u_{l\xi}, v_{l\xi}) \in \mathcal{L}_L$ and $(u_{r\xi}, v_{r\xi}) \in \mathcal{L}_R$. So, in this case, the solution of the Riemann problem is composed by a rarefaction wave of the first family, followed by a contact discontinuity, and then another rarefaction wave of the third family.

In the second case, the Riemann solution is defined by

$$\bar{W}(x, t) = \begin{cases} W_L, & \xi < -\sqrt{-p_u(u_L, S_L)}, \\ (u_{l\xi}, v_{l\xi}, S_L), & -\sqrt{-p_u(u_L, S_L)} \leq \xi < 0, \\ ((c_R - c_L)t \delta_0(x) \otimes dt, \frac{c_L + c_R}{2}, \frac{S_L + S_R}{2}), & \xi = 0, \\ (u_{r\xi}, v_{r\xi}, S_R), & 0 < \xi \leq \sqrt{-p_u(u_R, S_R)}, \\ W_R, & \xi > \sqrt{-p_u(u_R, S_R)}, \end{cases} \quad (3.15)$$

where $(u_{l\xi}, v_{l\xi})$ and $(u_{r\xi}, v_{r\xi})$ are as above. Observe also that, as $\xi \rightarrow 0-$, $v_{l\xi} \rightarrow c_L$, and, as $\xi \rightarrow 0+$, $v_{r\xi} \rightarrow c_R$, while $u_{l\xi} \rightarrow +\infty$, and $u_{r\xi} \rightarrow +\infty$, respectively. So, in this case, the Riemann solution is composed of a rarefaction wave of the first family and a rarefaction wave of the third family, both of which touch the line $x = 0$, where $\bar{u}(x, t)$ becomes singular with respect to the two-dimensional Lebesgue measure, and both $\bar{v}(x, t)$ and $\bar{S}(x, t)$ have a jump discontinuity. As in the example on isentropic gas dynamics given in the beginning of the section, the Riemann solution loses the self-similarity at the line $x = 0$, where the measure $\bar{u}(x, t)$ is singular.

We then consider the initial-value problem for (3.1)-(3.3), formed by prescribing initial data

$$W(x, 0) = W_0(x) = (u_0(x), v_0(x), S_0(x)), \quad (3.16)$$

and we assume

$$u_0, v_0, S_0 \in L^\infty(\mathbb{R}), \quad u_0 > \delta_0 > 0, \text{ a.e. in } \mathbb{R}. \quad (3.17)$$

DEFINITION 3.1. We say that $W(x, t) = (u(x, t), v(x, t), S(x, t))$ is a weak solution to (3.1)-(3.3), (3.16), (3.17) in $\Pi_T = \mathbb{R} \times (0, T)$ if the following conditions are satisfied:

- (i) $u \in \mathcal{M}(\Pi_T)$, and there exists $c_0 > 0$ such that $u \geq c_0 \, dx \, dt$, in Π_T ;
- (ii) $v, S \in L^\infty(\Pi_T)$;
- (iii) for all $\phi \in C_0^1(\mathbb{R} \times (-\infty, T))$,

$$\int_{\Pi_T} \phi_t \, du(x, t) - v(x, t) \phi_x \, dx \, dt + \int_{\mathbb{R}} u_0(x) \phi(x, 0) \, dx = 0, \quad (3.18)$$

$$\begin{aligned} \int_{\Pi_T} (v(x, t), \frac{1}{2}v^2(x, t) + \mathcal{E}(x, t)) \phi_t + (p(x, t), pv(x, t)) \phi_x \, dx \, dt \\ + \int_{\mathbb{R}} (v_0(x), \frac{1}{2}v_0^2(x) + \mathcal{E}_0(x)) \phi(x, 0) \, dx = 0, \end{aligned} \quad (3.19)$$

where we denote $p(x, t) = p(u_{\text{abs}}(x, t), S(x, t))$, $\mathcal{E}(x, t) = \mathcal{E}(u_{\text{abs}}(x, t), S(x, t))$, $\mathcal{E}_0(x) = \mathcal{E}(u_0(x), S_0(x))$;

- (iv) for all $0 \leq \psi \in C_0^1(\mathbb{R} \times (0, T))$,

$$\int_{\Pi_T} S(x, t) \psi_t \, dx \, dt + \int_{\mathbb{R}} S_0(x) \psi(x, 0) \, dx \leq 0. \quad (3.20)$$

We recall that the inequality (3.20) expresses the thermodynamical law of the increase of the entropy. Actually, in the context of smooth solutions $(u, v, S)(x, t)$, this inequality is a trivial consequence of the equation $S_t = 0$, which, in this case, may be easily deduced from (3.1)-(3.3) and (3.6).

We also observe that the concept of a weak solution as defined by Definition 3.1 still allows nonphysical solutions as remarked by Wagner [13].

The stability of the Riemann solution given either by (3.14) or (3.15) was established in [3], as an application of the theory of the (extended) divergence-measure fields, reviewed in the last section. Actually, the introduction of the pairing $\overline{\nabla g \cdot F}$, for $g \in W^{1,\infty}(U)$ and $F \in \mathcal{DM}^{ext}(U)$, allows several simplifications in the proof of the stability result as given in [3], as we will see in what follows. Namely, using the properties of this pairing, recalled in the last section, we may dispense with the most difficult parts of Theorem 4.1 of [3].

Indeed, from Theorem 4.1 of [3] we only need the trivial fact that, if $(u, v, S)(x, t)$ is a weak solution of (3.1)-(3.3), (3.16), (3.17), in the sense of Definition 3.1, then u admits a slicing as $du(x, t) = du_t(x) \otimes dt$, with $u_t \in \mathcal{M}(\mathbb{R})$, that is, for all $\phi \in C_0(\mathbb{R} \times (0, \infty))$,

$$\int \phi(x, t) du(x, t) = \int \left(\int \phi(x, t) du_t(x) \right) dt.$$

This follows from the fact that, by (3.18), there is a function $y(x, t) \in BV_{\text{loc}}(\Pi_T)$, such that $u = \frac{\partial y}{\partial x}$, and, by the properties of BV functions (see, e.g., [8]), we know that $y \in L^1((0, T); BV_{\text{loc}}(\mathbb{R}))$, so that the slicing holds with $u_t = \frac{\partial y}{\partial x}(\cdot, t)$.

Before we recall the statement and proof of the stability result just mentioned, we state a lemma which will be needed and is a simplified form of Corollary 4.1 of [3].

LEMMA 3.1. Let $(u, v, S)(x, t)$ be a weak solution of (3.1)-(3.3), (3.16), (3.17), in Π_T , and let $\bar{p}(x, t) = p(\bar{u}_{\text{abs}}(x, t), \bar{v}(x, t))$, where $\bar{W}(x, t) = (\bar{u}, \bar{v}, \bar{S})(x, t)$ is the Riemann solution given either by (3.14) or (3.15). Then, for any $0 \leq \zeta \in C_0^1(\mathbb{R})$,

$$\limsup_{t \rightarrow 0+} \int_{\mathbb{R}} \zeta(x) \bar{p}(x, t) du_t(x) \leq \int_{\mathbb{R}} \zeta(x) \bar{p}_0(x) u_0(x) dx, \quad (3.21)$$

where $\bar{p}_0(x) = p(\bar{u}_0(x), \bar{S}_0(x))$.

Proof. We provide an outline of the proof. We first observe that, for any $\delta > 0$, $\bar{p}_\delta(x, t) := \bar{p}(x, t + \delta) \in W^{1,\infty}(\Pi_T)$. It is then possible to define $\overline{\nabla_{x,t} \bar{p}_\delta \cdot (-v, u)}$. An important observation is that $\bar{p}_\delta$ is C^1 in $\mathbb{R} \times (0, \infty)$, except on at most 4 rays $\frac{x}{t} = \text{const.}$, each representing the beginning or the end of one of the rarefaction regions, on any of which we have $\bar{p}_\delta = \text{const.}$ This observation will also be helpful for the proof of the stability result. Therefore, with no loss of acuity, we can write

$$\overline{\nabla_{x,t} \bar{p}_\delta \cdot (-v, u)} = \frac{\partial \bar{p}_\delta}{\partial t} du - \frac{\partial \bar{p}_\delta}{\partial x} v dx dt.$$

As in [3], we consider a $\psi^\varepsilon(x, t) := \sigma^\varepsilon(t - t_0)\zeta(x)$, with $\zeta(x) \geq 0$, $0 < t_0 < T$, and

$$0 \leq \sigma^\varepsilon \in C_0^1(\mathbb{R}),$$

$$\sigma^\varepsilon(t - t_0) \rightarrow \chi_{(-\infty, t_0)}(t), \quad \text{a.e. } t \in \mathbb{R},$$

$$\delta^\varepsilon(t - t_0) := -\frac{d}{dt} \sigma^\varepsilon(t - t_0) \rightarrow \delta_{t_0}, \quad \text{in the sense of distributions in } \mathbb{R},$$

where, as usual, δ_{t_0} denotes the Dirac measure concentrated at t_0 . From (3.18), we obtain

$$\begin{aligned} &= \int_{\Pi_T} (\psi^\varepsilon \bar{p}_\delta)_t du_t dt - (\psi^\varepsilon \bar{p}_\delta)_x v dx dt + \int_{\mathbb{R}} \psi^\varepsilon(x, 0) \bar{p}(x, \delta) u_0(x) dx \\ &= - \int \delta^\varepsilon \left(\int \zeta \bar{p}_\delta du_t \right) dt + \int (\bar{p}_\delta)_t \psi^\varepsilon du - \int (\psi^\varepsilon \bar{p}_\delta)_x dx dt + \int_{\mathbb{R}} \psi^\varepsilon(x, 0) \bar{p}(x, \delta) u_0(x) dx. \end{aligned}$$

Now, the integration in du above may be taken only over the region where $(\bar{p}_\delta)_t$ is continuous, by the observation made above, and we also have $(\bar{p}_\delta)_t \leq 0$ on this continuity region. Using this to get rid of the integral in du , by switching from the equality to an

inequality, and making $\delta \rightarrow 0$, through dominated convergence, we obtain

$$\begin{aligned} & - \int \delta^\varepsilon(t - t_0) \left(\int \zeta \bar{p} du_t \right) dt \\ & + C_0 \iint_{\substack{0 \leq t \leq t_0 \\ x \in \text{supp } \zeta}} (|\bar{p}_x| + |\zeta'|) dx dt + \int_{\mathbb{R}} \psi^\varepsilon(x, 0) \bar{p}_0(x) u_0(x) dx \geq 0. \end{aligned}$$

Now, we send $\varepsilon \rightarrow 0$, assuming that t_0 is a Lebesgue point of $g(t) = \int \zeta(x) \bar{p}(x, t) du_t(x)$, to obtain

$$\int \zeta \bar{p} du_t \leq C_0 \iint_{\substack{0 \leq t \leq t_0 \\ x \in \text{supp } \zeta}} (|\bar{p}_x| + |\zeta'|) dx dt + \int_{\mathbb{R}} \zeta(x) \bar{p}_0(x) u_0(x) dx,$$

and we get (3.21) by making $t_0 \rightarrow 0$, observing that $\bar{p}_x$ is in $L^1(\Pi_T)$. $\square$

We now recall the theorem on the stability of the shock-free Riemann solutions proved in [3].

THEOREM 3.1 (Chen and Frid [3]). Let $\bar{W}(x, t)$ be the Riemann solution given either by (3.14) or (3.15), and let $W(x, t)$ be a weak solution of (3.1)-(3.3), (3.16), (3.17), in the sense of Definition 3.1. Then there exist positive constants C and K_0 , and functions $\omega \in L^\infty(\Pi_T)$, positive a.e. in Π_T , and $\omega_0 \in L^\infty(\mathbb{R})$, positive a.e. in $\mathbb{R}$, such that, for any $X > 0$ and a.e. $t > 0$,

$$\begin{aligned} & \int_{|x| \leq X} (|v(x, t) - \bar{v}(x, t)|^2 + |(u_{\text{abs}} - \bar{u}_{\text{abs}}, S - \bar{S})(x, t)|^2 \omega(x, t)) dx \\ & \leq C \int_{|x| \leq X + K_0 t} (|v_0(x) - \bar{v}_0(x)|^2 + |(u_0 - \bar{u}_0, S_0 - \bar{S}_0)(x)|^2 \omega_0(x)) dx. \end{aligned} \quad (3.22)$$

Proof. We give here just a brief outline of the proof, showing the points where simplifications are obtained from the more general statement of the Gauss-Green theorem and the definition of normal traces for (extended) divergence-measure fields provided by Theorem 2.2 and Theorem 2.3.

The proof is based on DiPerna's symmetry breaking method introduced in [7], whose main tool is the use of Dafermos' relative entropy dissipation measure. More specifically, recalling that $W = (u, v, S)$, we define

$$f(W) = (-v, p(u, S), 0), \quad \eta(W) = \mathcal{E}(u, S) + \frac{1}{2}v^2, \quad q(W) = vp(u, S),$$

and

$$\begin{aligned} \alpha(W, \bar{W}) &= \eta(W) - \eta(\bar{W}) - \nabla \eta(\bar{W}) \cdot (W - \bar{W}), \\ \beta(W, \bar{W}) &= q(W) - q(\bar{W}) - \nabla \eta(\bar{W}) \cdot (f(W) - f(\bar{W})). \end{aligned}$$

In fact, $(\alpha(W, \bar{W}), \beta(W, \bar{W}))$ is not a usual Dafermos' relative entropy pair since S is not a conserved (dependent) variable (cf. [6]), but, instead, it satisfies the inequality $S_t \geq 0$, in the weak sense (cf. (3.20)). Nevertheless, it works equally well, as first observed in [4] and [5].

As already said, the basic tool is the relative entropy dissipation measure ϖ , defined by

$$\varpi := \alpha(W, \bar{W})_t + \beta(W, \bar{W})_x.$$

Given $X > 0$ and $T > t_* > t_0 > 0$, we define the trapezium

$$U_{t_0, t_*} := \{(x, t) : |x| < X + K_0(t_* - t), t_0 < t < t_*\},$$

where $K_0 > 0$ is to be suitably chosen. The method consists in using the fact that $F := (\alpha(W, \bar{W}), \beta(W, \bar{W}))$ is a divergence-measure field over Π_T and so applies the divergence theorem (Theorem 2.2), which gives

$$\varpi(U_{t_0, t_*}) = \langle F \cdot \nu | \partial U_{t_0, t_*}, 1 \rangle. \quad (3.23)$$

Since we can slightly modify t_0, t_*, X if needed, by Remark 2.3, we may assume that $F \cdot \nu | \partial U_{t_0, t_*}$ is a measure on $\partial U_{t_0, t_*}$, and so we can write

$$\begin{aligned} \langle F \cdot \nu | \partial U_{t_0, t_*}, 1 \rangle &= \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_{top}} \rangle + \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_{bot}} \rangle \\ &\quad + \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_l} \rangle + \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_r} \rangle, \end{aligned} \quad (3.24)$$

where $\chi_{L_{top}}, \chi_{L_{bot}}, \chi_{L_l}, \chi_{L_r}$ are the characteristic functions of the top side, L_{top} , the bottom side, L_{bot} , the left lateral side L_l , and the right lateral side, L_r , of the trapezium U_{t_0, t_*} , respectively. To be precise, we take these sides as half-open line segments, so that each vertex of the trapezium belongs to one and only one of these line segments.

By using calculus arguments, it is shown in [3] that we may choose K_0 large enough so that (we refer to [3] for the details)

$$\langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_l} \rangle \geq 0, \quad \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_r} \rangle \geq 0. \quad (3.25)$$

The main point which remains is thus to show that

$$\varpi(U_{t_0, t_*}) \leq 0, \quad (3.26)$$

for then we easily have that

$$\begin{aligned} \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_{top}} \rangle &= \int_{(-X, X]} \alpha(W(x, t), \bar{W}(x, t)) \\ &\geq \int_{|x| \leq X} (|v(x, t_*) - \bar{v}(x, t_*)|^2 + |(u_{\text{abs}} - \bar{u}_{\text{abs}}, S - \bar{S})(x, t_*)|^2 \omega(x, t_*)) dx, \end{aligned} \quad (3.27)$$

for some function $\omega \in L^\infty(\Pi_T)$, positive a.e. in Π_T , while

$$\begin{aligned} \langle F \cdot \nu | \partial U_{t_0, t_*}, \chi_{L_{bot}} \rangle &= - \int_{|x| \leq X + K_0(t_* - t_0)} \{ \eta(W(x, t_0)) - \eta(\bar{W}(x, t_0)) - \bar{v}(v - \bar{v})(x, t_0) \\ &\quad - \bar{\theta}(S - \bar{S})(x, t_0) \} dx dt - \int_{L_{bot}} \bar{p} (d\bar{u}_{t_0} - du_{t_0}) \\ &\geq - \int_{|x| \leq X + K_0(t_* - t_0)} \{ (\eta(W(x, t_0)) - \eta(\bar{W}(x, t_0)) - \bar{v}(v - \bar{v})(x, t_0) \\ &\quad - \bar{\theta}(S - \bar{S})(x, t_0) - \bar{p} \bar{u}_{\text{abs}}) dx dt + \bar{p} du_{t_0} \}, \end{aligned} \quad (3.28)$$

where $\bar{\theta} = \mathcal{E}_S(\bar{u}, \bar{S}) = c_u \mathcal{E}(\bar{u}, \bar{S})$. Hence, using Lemma 3.1, when taking the $\liminf$ as $t_0 \rightarrow 0$ in the inequality obtained from (3.23)–(3.28), we obtain (3.22).

We now make some brief comments on the proof of (3.23). We consider first the case where the Riemann solution is given by (3.14).

We first observe that on U_{t_0, t_*} ,

$$\varpi = -\operatorname{div}_{x,t}(F_1 + F_2 + F_3),$$

where

$$F_1 = \bar{p}(\bar{v} - v, u - \bar{u}), \quad F_2 = -\bar{v}(p - \bar{p}, v - \bar{v}), \quad F_3 = \bar{\theta}(0, S - \bar{S}),$$

since we have

$$\eta(W)_t + q(W)_x = \eta(\bar{W})_t + q(\bar{W})_x = 0,$$

in the sense of distributions, by Definition 3.1. By the product rule, Theorem 2.2, each of the fields, F_1, F_2, F_3 , is a divergence-measure field, since, for $\bar{W}$ given by (3.14), $\bar{p}$, $\bar{v}$ and $\bar{\theta}$ are Lipschitz functions on U_{t_0, t_*} . Moreover, by Theorem 2.2, we have

$$-\operatorname{div}_{x,t} F_1 = -\overline{\nabla_{x,t} \bar{p} \cdot (\bar{v} - v, u - \bar{u})}, \quad (3.29)$$

$$-\operatorname{div}_{x,t} F_2 = -\overline{\nabla_{x,t} \bar{v} \cdot (p - \bar{p}, v - \bar{v})}, \quad (3.30)$$

$$-\operatorname{div}_{x,t} F_3 \leq -\overline{\nabla_{x,t} \bar{\theta} \cdot (0, S - \bar{S})}, \quad (3.31)$$

since we have

$$\operatorname{div}_{x,t}(\bar{v} - v, u - \bar{u}) = \operatorname{div}_{x,t}(p - \bar{p}, v - \bar{v}) = 0, \quad \operatorname{div}_{x,t}(0, S - \bar{S}) \geq 0, \quad \bar{\theta} \geq 0,$$

in the sense of distributions, by Definition 3.1. We can then apply Lemma 2.3 to deduce that

$$\operatorname{div}_{x,t} F_i \llcorner U_{t_0, t_*}(l_j) = 0, \quad \operatorname{div}_{x,t} F_3 \llcorner U_{t_0, t_*}(l_j) \leq 0, \quad i = 1, 2, \quad j = 1, 2, 3, 4, \quad (3.32)$$

where l_1 – l_4 are the lines bounding the two rarefaction regions of $\bar{W}(x, t)$ in $\mathbb{R} \times (0, \infty)$, since $\bar{p}$, $\bar{v}$ and $\bar{\theta}$ are constant on these lines. We remark that, in this point, the use of Lemma 2.3 here greatly simplifies the argument in [3] for the proof of (3.32).

We observe that, if we denote by Ω_i , $i = 1, 2$, the interiors of the two rarefaction regions of $\bar{W}$, then the fact that

$$\varpi \llcorner U_{t_0, t_*}(\Pi_T \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)) \leq 0$$

follows trivially from the fact that $\varpi \llcorner U_{t_0, t_*}$ reduces to $-\bar{\theta} S_t$ on $\Pi_T \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)$.

The proof that

$$\varpi \llcorner U_{t_0, t_*}(\Omega_i) \leq 0, \quad i = 1, 2,$$

is by now standard and follows from the fact that, on these regions, ϖ_{abs} reduces to $-\bar{v}_x(x, t)$, which is nonpositive, times the quadratic part of p at $(\bar{u}(x, t), \bar{S}(x, t))$, evaluated on $(u_{\text{abs}}(x, t), S(x, t))$, which is nonnegative by convexity, while ϖ_{sing} equals $\bar{v}_x \bar{p}_u u_{\text{sing}}$, which is nonpositive, since $\bar{p}_u \leq 0$.

The proof of (3.23) when the Riemann solution is given by (3.15), presenting a vacuum on the line $x = 0$, is similar, the only difference being the fact that $\bar{v}$ this time is no longer Lipschitz, but, instead, it is a function in $BV_{\text{loc}}(\mathbb{R} \times (0, \infty))$, whose jump set consists exactly of the line $x = 0$, across which $\bar{v}$ jumps from c_L , on the left, to c_R , on the right. So the pairing in (3.30) should be replaced by the one given by Theorem 2.1, $\overline{F \cdot \nabla g}$, between a $\mathcal{DM}^\infty$ -field and the gradient of a BV -function, so

$$-\operatorname{div}_{x,t} F_2 = -\overline{(p - \bar{p}, v - \bar{v}) \cdot \nabla_{x,t} \bar{v}}. \quad (3.33)$$

So, by the properties of the $\mathcal{DM}^\infty$ -fields, and the fact that $(u, v, S)(x, t)$ satisfies (3.1)-(3.3) in the sense of distributions, we deduce that

$$-\operatorname{div}_{x,t} F_2[U_{t_0,t_*}(\{x=0\})] = -(c_R - c_L) \int_{t_0}^{t_*} p(0, t) dt,$$

which is clearly nonpositive, and this concludes the verification of (3.23) also for $\bar{W}$ given by (3.15) (see [3] for details). $\square$

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