GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS FOR A CLASS OF FOURTH ORDER STRONGLY DAMPED NONLINEAR WAVE EQUATIONS

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Abstract. In this paper we study the initial boundary value problem for a class of fourth order strongly damped nonlinear wave equations $u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = f(u)$. By introducing a family of potential wells we prove the existence of global weak solutions and global strong solutions under some weak growth conditions on $f(u)$. Furthermore we give the asymptotic behaviour of solutions.

1. Introduction. In this paper we study the initial boundary value problem (IBVP) of fourth order strongly damped nonlinear wave equations

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = f(u), \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad x \in \Omega, \quad (1.2)$$

$$u(x, t) = \Delta u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.3)$$

where α is a positive constant and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

The strongly damped nonlinear wave equation

$$u_{tt} - \Delta u - \alpha \Delta u_t = f(u) \quad (1.4)$$

was suggested from the motion of a viscous body. In 1980, Webb [23] first studied the IBVP of (1.4) with the initial boundary value condition (1.2) and

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.5)$$

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for $n \leq 3$. He obtained the global existence and asymptotic behaviour of the strong solution. The authors in [1] further considered the problem (1.4), (1.2) and (1.5). They strengthened the hypotheses and assumptions in [23] by proving the local and global existence of the solution and the exponential decay. Subsequently in [14] the same problem was studied again and the results in [23] were improved. In [13] the global existence of the strong solution to problem (1.4), (1.2), (1.5) was given for $n \geq 4$, which generalized the results obtained in [23] and [14]. Then in [24] the authors further showed asymptotic behaviour of the global strong solution of problem (1.4), (1.2), (1.5). Here we would like to mention that the above results in [14, 13, 23, 24] are based on the assumption that (1.4) has a positive definite energy. In [15], the authors made the first try to consider the problem with nonpositive energy. They showed the global existence of both weak and strong solutions to problem (1.4), (1.2), (1.5) by the potential well method. This technique was taken to treat more complex problems such as (1.4). Newly, Pata and Squassina [19] investigated the following 3-D wave equation:

$$u_{tt} - \Delta u - \alpha \Delta u_t + \phi(u) = f(u) \quad (1.6)$$

with (1.4) and (1.5). They carried out a detailed analysis of the asymptotic behaviour of the solution independent of the damping coefficient. Subsequently Filippo and Squassina [7] considered (1.4) with weak damping u_t . They showed the global existence and the polynomial decay property of solutions provided that the initial data are in the potential well and $E(0) < d$. Recently Gerbi and Said-Houari [8] considered the same problem. They proved that the solution decays exponentially by introducing an appropriate Lyapunov function; hence the results in [7] were improved. Levine and Serrin [10] considered the global nonexistence of solutions for equations with weak damping

$$u_{tt} - \Delta u + u_t = f(u).$$

They proved that the solutions with negative initial energy blow up in finite time. Then the case in [10] was extended to the abstract version by Levine, Pucci and Serrin [9]. Subsequently, Pucci and Serrin [21] improved the results from $E(0) < 0$ to $E(0) < d$. Vitillaro [22] showed that the case also holds when $E(0) = d$. For the strongly damped equation, Ono [18] obtained that the solution of (1.4) blows up in finite time if $E(0) < 0$. Later Ohta [17] improved this result provided that the initial data are out of the potential well and $E(0) < d$. Moreover there are many results about the global existence and asymptotic behaviour of solutions for the strongly damped wave equations; we refer the reader to [20, 2, 3, 4] and the references therein.

Less well known is the fact that there are a few results about the nonlinear fourth order equation (1.1) with $\alpha = 0$. In [6], Esquivel studied the IBVP of equation

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = f(u) \quad (1.7)$$

for $\sigma_i(s) = \pm \beta |s|^{m-2} s$ and $f(u) = \mu |s|^{r-2} s$. By using the potential well method, the author obtained a sharp condition for global existence and nonexistence of the solution. Then in [16] these results were extended to more general nonlinear terms $\sigma_i(s)$ and $f(s)$ by employing a family of potential wells. Subsequently Chen and Ren [5] studied the

existence, nonexistence, uniqueness and energy decay estimate of the solution for the IBVP of (1.1) with the nonlinear weak damping $|u_t|^r u_t$. By the multiplier method, they estimated the energy decay rate of the global solution. Most recently, in [11], by a potential well argument also, Lin, Wu and Lai considered the problem (1.1)–(1.3) and proved the existence of a global weak and strong solution under some assumptions on $f(u)$ and initial data. This is the first try to treat this problem, and they left some other interesting problems open. Next, in order to explain the motivation of this paper, we would like to show some of the results obtained in [11]. First we list some definitions in [11] as follows:

$$(A_1) \quad |f(u)| \leq a|u|^p, \quad 1 < p < \infty \text{ if } n = 1, 2; \quad 1 < p \leq \frac{n-2}{n+2} \text{ if } n \geq 3.$$

$$J(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\Delta u\|^2 - \frac{a}{p+1} \|u\|_{p+1}^{p+1},$$

$$I(u) = \|\nabla u\|^2 + \|\Delta u\|^2 - a \|u\|_{p+1}^{p+1},$$

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\Delta u\|^2 - \int_{\Omega} F(u) dx, \quad F(u) = \int_0^u f(s) ds,$$

$$W = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\},$$

$$V = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | I(u) < 0, J(u) < d\},$$

where

$$d = \inf_{\substack{u \in H^2(\Omega) \cap H_0^1(\Omega) \\ u \neq 0}} \sup_{\lambda \geq 0} J(\lambda u);$$

$$(A_2) \quad |f'(u)| \leq A|u|^{p_1}, \quad \frac{1}{2} < p_1 < \infty \text{ if } n = 1, 2; \quad \frac{2}{n} < p_1 \leq \frac{4}{n-2} \text{ if } n \geq 3,$$

$$B_R = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | (\|\nabla u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}} < R\},$$

where

$$R = (a C_*^{p+1})^{\frac{1}{p-1}}, \quad C_* = \sup_{\substack{u \in H^2(\Omega) \cap H_0^1(\Omega) \\ u \neq 0}} \frac{\|u\|_{p+1}}{(\|\nabla u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}}}.$$

The main results of [11] are the following two theorems.

THEOREM 1.1 ([11]). Let $f \in C$, $f(u)u \geq 0$ and satisfy (A_1) , $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$ and $u_0 \in W$. Then problem (1.1)–(1.3) admits a global weak solution $u \in L^\infty(0, \infty; H^2(\Omega) \cap H_0^1(\Omega))$ with $u_t \in L^\infty(0, \infty; L^2(\Omega))$ and $u \in W$ for $0 \leq t < \infty$.

THEOREM 1.2 ([11]). Let $f \in C^1$, $f(u)u \geq 0$ and satisfy (A_2) , $u_0 \in H^4(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that $E(0) < d$ and $u_0 \in B_R$. Then problem (1.1)–(1.3) admits a unique global strong solution $u \in L^\infty(0, T; H^4(\Omega) \cap H_0^1(\Omega))$ with $u_t \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\forall T > 0$ and $u \in B_R$ for $0 \leq t < \infty$.

The above theorems conclude the global-in-time existence of a solution. Further, for the damped nonlinear hyperbolic equation, we usually desire to know how the solutions behave as the time t approaches infinity, the so-called asymptotic behaviour. To be more clear, we expect the solutions to decay to zero as $t \rightarrow \infty$. So this is one of the main tasks of the present paper. On the other hand, we would also like to focus on weakening

the assumptions (A₁) and (A₂). Before doing these, we need to vary the definition of the strong solution, because by paying attention to the definition of the strong solution in [11] we find that from $u \in H^4(\Omega) \cap H_0^1(\Omega)$ one can only see $u|_{\partial\Omega} = 0$ rather than $\Delta u|_{\partial\Omega} = 0$. Hence the strong solution defined in this way cannot satisfy the boundary condition $\Delta u|_{\partial\Omega} = 0$. In order to solve this problem we introduce the space H_4 defined as below.

$H_4 :=$ closed linear extension of $\{w_j(x)\}_{j=1}^\infty$ in $H^4(\Omega)$, where $\{w_j(x)\}_{j=1}^\infty$ is the eigenfunction system of problem $-\Delta w = \lambda w$ in Ω , $w|_{\partial\Omega} = 0$.

Note that for any $w_j(x)$ we have $\Delta w_j|_{\partial\Omega} = -\lambda_j w_j|_{\partial\Omega} = 0$. Hence for any $u \in H_4$ we have $u \in H^4(\Omega) \cap H_0^1(\Omega)$ and $\Delta u \in H^2(\Omega) \cap H_0^1(\Omega)$, which implies that $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$. Thus we can equivalently define H_4 by

$$H_4 := \{u \mid u \in H^4(\Omega) \cap H_0^1(\Omega), \Delta u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ with } u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}.$$

In addition we denote $H^2(\Omega) \cap H_0^1(\Omega)$ by H . Then we give a new definition of a strong solution.

DEFINITION 1.3. Function $u = u(x, t)$ is called a strong solution of problem (1.1)–(1.3) on $\Omega \times [0, T]$ if $u \in L^\infty(0, T; H_4)$ with $u_t \in L^\infty(0, T; H)$, $u_{tt} \in L^\infty(0, T; L^2(\Omega))$ satisfying

$$\int_0^T (u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t - f(u), \varphi) dt = 0, \quad \forall \varphi \in C([0, T], L^2(\Omega))$$

and $u(x, 0) = u_0$ in H_4 , $u_t(x, 0) = u_1$ in H .

Now we are ready to study the problem (1.1)–(1.3), where α is a positive constant, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, and $f(u)$ satisfies

$$(H_1) \begin{cases} (i) & f \in C^1 \text{ and } u(u f'(u) - f(u)) \geq 0, \\ & \text{where the equality holds only for } u = 0; \\ (ii) & |f(u)| \leq A|u|^q, 1 < q < \infty \text{ for } 1 \leq n \leq 4; 1 < q < \frac{n+4}{n-4} \text{ for } n \geq 5; \\ (iii) & \text{there exists a } p > 1 \text{ such that} \\ & (p+1)F(u) \leq u f(u), \quad \forall u \in \mathbb{R}, \quad F(u) = \int_0^u f(s) ds, \end{cases}$$

or

$$(H_2) \begin{cases} (i) & f \in C^1 \text{ and } u(u f'(u) - f(u)) \geq 0, \\ & \text{where the equality holds only for } u = 0; \\ (ii) & f(0) = 0, |f'(u)| \leq a|u|^{q_1}, \\ & 0 < q_1 < \infty \text{ for } 1 \leq n \leq 4; 0 < q_1 < \frac{n}{n-4} \text{ for } n = 5, 6; \\ & 0 < q_1 < \frac{6}{n-4} \text{ for } n \geq 7; \\ (iii) & \text{there exists a } p > 1 \text{ such that} \\ & (p+1)F(u) \leq u f(u), \quad \forall u \in \mathbb{R}, \quad F(u) = \int_0^u f(s) ds. \end{cases}$$

We organize this paper as follows: in Section 2, we introduce a family of potential wells and discuss the invariant set. In Section 3, we give the existence of global weak solutions under the assumption (H₁), and we also prove the existence of global strong solutions under the assumption (H₂). In Section 4, we obtain the asymptotic behaviour of global weak solutions for problem (1.1)–(1.3). For a clear presentation, we use the following

denotation throughout the paper: $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_2$, $\|u\|_H^2 = \|\nabla u\|^2 + \|\Delta u\|^2$, $\|u\|_{H^4}^2 = \|\Delta^2 u\|^2 + \|\nabla^3 u\|^2$, $\nabla^3 = \nabla(\Delta u)$, $(u, v) = \int_{\Omega} uv dx$.

2. Preliminary lemmas. In this section we give some definitions and lemmas related to the potential wells. First for problem (1.1)–(1.3) we define

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 - \int_{\Omega} F(u) dx = \frac{1}{2}\|u\|_H^2 - \int_{\Omega} F(u) dx, \\ I(u) &= \|\nabla u\|^2 + \|\Delta u\|^2 - \int_{\Omega} uf(u) dx = \|u\|_H^2 - \int_{\Omega} uf(u) dx, \\ I_{\delta}(u) &= \delta(\|\nabla u\|^2 + \|\Delta u\|^2) - \int_{\Omega} uf(u) dx = \delta\|u\|_H^2 - \int_{\Omega} uf(u) dx, \quad \delta > 0, \\ d &= \inf_{u \in \mathcal{N}} J(u), \quad \mathcal{N} = \{u \in H | I(u) = 0, u \neq 0\}, \\ d(\delta) &= \inf_{u \in \mathcal{N}_{\delta}} J(u), \quad \mathcal{N}_{\delta} = \{u \in H | I_{\delta}(u) = 0, u \neq 0\}, \\ E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 - \int_{\Omega} F(u) dx = \frac{1}{2}\|u_t\|^2 + J(u). \end{aligned}$$

Clearly if $f(u)$ satisfies (H_1) (or (H_2)), then all of the above definitions are well defined for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ (or $u \in H^4(\Omega) \cap H_0^1(\Omega)$).

LEMMA 2.1. Let $f(u)$ satisfy (H_1) (or (H_2)), $u \in H$ (or $u \in H_4$) and $g(u) = \frac{f(u)}{u}$, $u \neq 0$. Then

- (i) $\lim_{u \rightarrow 0} g(u) = 0$;
- (ii) $g(u)$ is strictly increasing on $(0, \infty)$, strictly decreasing on $(-\infty, 0)$;
- (iii) $f(u)u > 0$, $\forall u \neq 0$;
- (iv) $f(u)$ is strictly increasing on $(-\infty, \infty)$;
- (v) $0 \leq F(u) \leq A_1|u|^{q+1}$ or $A_2|u|^{q_1+2}$, $\forall u \in \mathbb{R}$.

Proof. (i) The conclusion follows directly from (ii) in (H_1) or (H_2) .

(ii) From (i) in (H_1) or (H_2) we get

$$g'(u) = \frac{uf'(u) - f(u)}{u^2} \begin{cases} > 0 \text{ if } u > 0; \\ < 0 \text{ if } u < 0. \end{cases}$$

(iii) From (i) and (ii) of this lemma we see that $g(u) > 0$ for any $u \neq 0$, which gives $f(u)u > 0$ for any $u \neq 0$.

(iv) From (i) in (H_1) or (H_2) , and (iii) of this lemma we get

$$f'(u) \geq \frac{f(u)}{u} > 0, \quad \forall u \neq 0.$$

(v) From $f(0) = 0$ and (ii) of this lemma we have $F(u) \geq 0$, and $F(u) \leq A_1|u|^{q+1}$ or $A_2|u|^{q_1+2}$ follows from (ii) in (H_1) or (H_2) . $\square$

Next we will give some lemmas to assist us to state and prove the main theorems. We shall not prove them, because we can get them by letting $\sigma_i(s) \equiv 0$ ($1 \leq i \leq n$) and $\alpha = 1$ in Lemma 2.1–Lemma 2.9 in [16].

LEMMA 2.2. Let $f(u)$ satisfy (H_1) , $u \in H$, or $f(u)$ satisfy (H_2) , $u \in H_4$. Then

$$F(u) \geq B|u|^{p+1}, \quad \forall |u| \geq 1 \text{ and some } B > 0,$$

and

$$uf(u) \geq (p+1)B|u|^{p+1}, \quad \forall |u| \geq 1.$$

LEMMA 2.3. Let $f(u)$ satisfy (H_1) , $u \in H$, or $f(u)$ satisfy (H_2) , $u \in H_4$, $\varphi(\lambda) = \frac{1}{\lambda} \int_{\Omega} uf(\lambda u) d\lambda$, $\lambda > 0$. Then

- (i) $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$, $\lim_{\lambda \rightarrow +\infty} \varphi(\lambda) = +\infty$;
- (ii) $\varphi(\lambda)$ is strictly increasing on $(-\infty, +\infty)$.

LEMMA 2.4. Let $f(u)$ satisfy (H_1) , $u \in H$, $u \neq 0$, or $f(u)$ satisfy (H_2) , $u \in H_4$, $u \neq 0$. Then

- (i) $\lim_{\lambda \rightarrow 0} J(\lambda u) = 0$, $\lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty$;
- (ii) there exists a unique $\lambda^* = \lambda^*(u)$ such that

$$\left. \frac{d}{d\lambda} J(\lambda u) \right|_{\lambda=\lambda^*} = 0;$$

- (iii) $J(\lambda u)$ is increasing on $0 < \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$;
- (iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$; $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.

LEMMA 2.5. Let $f(u)$ satisfy (H_1) , $u \in H$, or $f(u)$ satisfy (H_2) , $u \in H_4$. Then:

- (i) If $0 < \|u\|_H < r(\delta)$, then $I_\delta(u) > 0$.
- (ii) If $I_\delta(u) < 0$, then $\|u\|_H > r(\delta)$.
- (iii) If $I_\delta(u) = 0$, $u \neq 0$, then $\|u\|_h \geq r(\delta)$,

$$r(\delta) = \left(\frac{\delta}{AC_*^{q+1}} \right)^{\frac{1}{q-1}}, \quad C_* = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|u\|_{q+1}}{\|u\|_H} \text{ if } f(u) \text{ satisfies } (H_1);$$

$$r(\delta) = \left(\frac{(q_1+1)\delta}{aC_*^{q_1+2}} \right)^{\frac{1}{q_1}}, \quad C_* = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|u\|_{q_1+2}}{\|u\|_H} \text{ if } f(u) \text{ satisfies } (H_2).$$

LEMMA 2.6. Let $f(u)$ satisfy (H_1) , $u \in H$, or $f(u)$ satisfy (H_2) , $u \in H_4$. Then

- (i) $d(\delta) \geq a(\delta)r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$;
- (ii) $\lim_{\delta \rightarrow +\infty} d(\delta) = 0$ and there exists a $0 < \delta_0 \leq \frac{p+1}{2}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 < \delta < \delta_0$;
- (iii) $d(\delta)$ is increasing on $0 < \delta \leq 1$, decreasing on $1 \leq \delta < \delta_0$ and takes the maximum at $\delta = 1$.

Now for problem (1.1)–(1.3) we define the following sets; later we will state the invariance of them.

$$W = \{u \in H | I(u) > 0, J(u) < d\} \cup \{0\},$$

$$W_\delta = \{u \in H | I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \quad 0 < \delta < \delta_0,$$

$$V = \{u \in H | I(u) < 0, J(u) < d\},$$

$$V_\delta = \{u \in H | I_\delta(u) < 0, J(u) < d(\delta)\}, \quad 0 < \delta < \delta_0.$$

In addition we define

$$\begin{aligned} W' &= \{u \in H | I(u) > 0\} \cup \{0\}, \\ W'_\delta &= \{u \in H | I_\delta(u) > 0\} \cup \{0\}, \quad 0 < \delta < \delta_0, \\ V' &= \{u \in H | I(u) < 0\}, \\ V'_\delta &= \{u \in H | I_\delta(u) < 0\}, \quad 0 < \delta < \delta_0. \end{aligned}$$

The above definitions are valid only for $f(u)$ satisfying (H_1) . For $f(u)$ satisfying (H_2) , we can replace $u \in H$ by $u \in H_4$ in all of the above definitions. Now we shall give two theorems to state the invariance of the above manifolds.

THEOREM 2.7. Let $f(u)$ satisfy (H_1) , $u_0 \in H$, $u_1 \in L^2(\Omega)$. Assume that $0 < E(0) < d$, and let (δ_1, δ_2) be the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Then:

- (i) All weak solutions of problem (1.1)–(1.3) belong to W_δ for $\delta \in (\delta_1, \delta_2)$ provided that $u_0 \in W'$.
- (ii) All weak solutions of problem (1.1)–(1.3) belong to V_δ for $\delta \in (\delta_1, \delta_2)$ provided that $u_0 \in V'$.

THEOREM 2.8. Let $f(u)$ satisfy (H_2) , $u_0 \in H_4$, $u_1 \in H$. Assume that $0 < E(0) < d$, and let (δ_1, δ_2) be the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Then:

- (i) All weak solutions of problem (1.1)–(1.3) belong to W_δ for $\delta \in (\delta_1, \delta_2)$ provided that $u_0 \in W'$.
- (ii) All weak solutions of problem (1.1)–(1.3) belong to V_δ for $\delta \in (\delta_1, \delta_2)$ provided that $u_0 \in V'$.

3. Global existence of solutions. In this section we shall give some new existence theorems for global weak and strong solutions respectively for problem (1.1)–(1.3), which are different from those in [11]. First we consider the existence of global weak solutions for problem (1.1)–(1.3) in the following theorem.

THEOREM 3.1. Let $f(u)$ satisfy (H_1) , $u_0 \in H$, $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$ and $u_0 \in W'$. Then problem (1.1)–(1.3) admits a global solution $u \in L^\infty(0, \infty; H)$ with $u_t \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H_0^1(\Omega))$ and $u \in W$ for $0 \leq t < \infty$.

Proof. Let $\{w_j(x)\}_{j=1}^\infty$ be the eigenfunction system of problem $-\Delta w = \lambda w$ in Ω , $w|_{\partial\Omega} = 0$. Construct the approximate solutions of problem (1.1)–(1.3):

$$u_m(x, t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad m = 1, 2, \dots, \quad (3.1)$$

satisfying

$$(u_{mtt}, w_s) + (\nabla u_m, \nabla w_s) + (\Delta u_m, \Delta w_s) + \alpha(\nabla u_{mt}, \nabla w_s) = (f(u_m), w_s), \quad (3.2)$$

$$s = 1, 2, \dots, m,$$

$$u_m(x, 0) = \sum_{j=1}^m a_{jm} w_j(x) \rightarrow u_0 \text{ in } H, \quad (3.3)$$

$$u_{mt}(x, 0) = \sum_{j=1}^m b_{jm} w_j(x) \rightarrow u_1 \text{ in } L^2(\Omega). \quad (3.4)$$

Multiplying (3.2) by $g'_{sm}(t)$ and summing for s we get

$$\frac{d}{dt} E_m(t) + \alpha \|\nabla u_{mt}\|^2 = 0$$

and

$$\frac{1}{2} \|u_{mt}\|^2 + J(u_m) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 d\tau = E_m(0), \quad 0 \leq t < \infty, \quad (3.5)$$

where

$$E_m(t) = \frac{1}{2} \|u_{mt}\|^2 + J(u_m).$$

From $E(0) < d$, $u_0 \in W'$ and (3.3), (3.4) it follows that $E_m(0) < d$ and $u_m(0) \in W'$ for sufficiently large m . Then from (3.5) and the argument in the proof of Theorem 3.1 in [16] we can obtain that $u_m(t) \in W'$ for $0 \leq t < \infty$ and sufficiently large m . Thus from (3.5) and

$$J(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2 + \frac{1}{p+1} I(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2,$$

we get

$$\frac{1}{2} \|u_{mt}\|^2 + \frac{p-1}{2(p+1)} \|u_m\|_H^2 + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 d\tau < d, \quad 0 \leq t < \infty$$

for sufficiently large m . Hence we have

$$\|u_m\|_H^2 \leq \frac{2(p+1)}{p-1} d, \quad \|u_{mt}\|^2 \leq 2d, \quad 0 \leq t < \infty,$$

$$\int_0^t \|\nabla u_{m\tau}\|^2 d\tau \leq \frac{d}{\alpha}, \quad 0 \leq t < \infty,$$

$$\|u_m\|_{q+1}^2 \leq C_*^2 \|u_m\|_H^2 \leq C_*^2 \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty,$$

$$\|f(u_m)\|_{r+1}^{r+1} \leq \int_{\Omega} (A|u_m|^q)^r dx = A_1 \|u_m\|_{q+1}^{q+1} \leq C_*^{q+1} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{q+1}{2}},$$

$$0 \leq t < \infty, \quad r = \frac{q+1}{q}.$$

Hence there exist a u and a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu \rightarrow \infty$,

$$u_\nu \rightarrow u \text{ in } L^\infty(0, \infty; H) \text{ weakly star and a.e. in } Q = \Omega \times [0, \infty),$$

$$u_\nu \rightarrow u \text{ in } L^{q+1}(\Omega) \text{ strongly for each } t > 0,$$

$$u_{\nu t} \rightarrow u_t \text{ in } L^\infty(0, \infty; L^2(\Omega)) \text{ weakly star,}$$

$$f(u_\nu) \rightarrow \chi \text{ in } L^\infty(0, \infty; L^r(\Omega)) \text{ weakly star,}$$

$u_{\nu t} \rightarrow u_t$ in $L^2(0, \infty; H_0^1(\Omega))$ weakly.

By Lemma 1.3 in [12] we have $\chi = f(u)$.

Integrating (3.2) with respect to t we get

$$\begin{aligned} (u_{mt}, w_s) + \int_0^t (\nabla u_m, \nabla w_s) d\tau + \int_0^t (\Delta u_m, \Delta w_s) d\tau + \alpha (\nabla u_m, \nabla w_s) \\ = \int_0^t (f(u_m), w_s) d\tau + (u_{mt}(0), w_s) + \alpha (\nabla u_m(0), \nabla w_s). \end{aligned} \quad (3.6)$$

In (3.6) for fixed s letting $m = \nu \rightarrow \infty$ we obtain

$$\begin{aligned} (u_t, v) + \int_0^t (\nabla u, \nabla v) d\tau + \int_0^t (\Delta u, \Delta v) d\tau + \alpha (\nabla u, \nabla v) \\ = \int_0^t (f(u), v) d\tau + (u_1, v) + \alpha (\nabla u_0, \nabla v), \quad \forall v \in H. \end{aligned} \quad (3.7)$$

On the other hand, from (3.3) and (3.4) we have $u(x, 0) = u_0$ in H and $u_t(x, 0) = u_1$ in $L^2(\Omega)$. Therefore $u(x, t)$ is a global weak solution of problem (1.1)–(1.3). Finally from Theorem 2.7 we get $u \in W$ for $0 \leq t < \infty$. $\square$

Next we try to give the existence theorem of global strong solutions for problem (1.1)–(1.3). But here we would like to prove the main theorem in another way from Theorem 3.1. We shall give some lemmas first. Then we will show the key theorem (Theorem 3.5) as follows readily.

Let $f(u)$ satisfy (H_2) , $u_0 \in H_4$, $u_1 \in H$, $\{w_j(x)\}_{j=1}^\infty$ be the same as those in the proof of Theorem 3.1. Construct the approximate solutions u_m by (3.1) satisfying

$$(u_{mtt} - \Delta u_m + \Delta^2 u_m - \alpha \Delta u_{mt} - f(u_m), w_s) = 0, \quad s = 1, 2, \dots, m, \quad (3.8)$$

$$u_m(x, 0) = \sum_{j=1}^m a_{jm} w_j(x) \rightarrow u_0(x) \text{ in } H^4(\Omega), \quad (3.9)$$

$$u_{mt}(x, 0) = \sum_{j=1}^m b_{jm} w_j(x) \rightarrow u_1(x) \text{ in } H^2(\Omega). \quad (3.10)$$

First by the proof of Theorem 3.1 we can obtain the following lemma.

LEMMA 3.2. Let $f(u)$ satisfy (H_2) , $u_0 \in H_4$, $u_1 \in H$. Assume that $E(0) < d$ and $u_0 \in W'$. Then for the approximate solutions u_m defined by (3.1) and (3.8)–(3.10) we have

$$\|u_m\|_H^2 \leq \frac{2(p+1)}{p-1} d, \quad \|u_{mt}\|^2 \leq 2d, \quad 0 \leq t < \infty. \quad (3.11)$$

LEMMA 3.3. Under the conditions of Lemma 3.2 we further have

$$\|u_{mt}\|_{H^2}^2 + \|u_m\|_{H_4}^2 \leq C(T), \quad 0 \leq t < \infty, \quad \forall T > 0. \quad (3.12)$$

Proof. Multiplying (3.8) by $\lambda_s^2 g'_{sm}(t)$ and summing for s we get

$$\frac{d}{dt} \left(\frac{1}{2} \|\Delta u_{mt}\|^2 + \frac{1}{2} \|u_m\|_{H_4}^2 \right) + \alpha \|\nabla^3 u_{mt}\|^2 = (f(u_m), \Delta^2 u_{mt}), \quad (3.13)$$

$$(f(u_m), \Delta^2 u_{mt}) = - (f'(u_m) \nabla u_m, \nabla^3 u_{mt}) \leq \|f'(u_m)\|_r \|\nabla u_m\|_s \|\nabla^3 u_{mt}\|,$$

$$1 < r, s < \infty, \frac{1}{r} + \frac{1}{s} = \frac{1}{2}.$$

(i) If $n \geq 7$, taking $s = \frac{2n}{n-6}$, $r = \frac{n}{3}$, we get

$$\|\nabla u_m\|_s \leq C \|u_m\|_{H_4},$$

$$\|f'(u_m)\|_r^r \leq \int_{\Omega} (a|u_m|^{q_1})^{\frac{n}{3}} dx = a^{\frac{n}{3}} \int_{\Omega} |u_m|^{q_1 \cdot \frac{n}{3}} dx.$$

From (H₂) we have $q_1 \cdot \frac{n}{3} \leq \frac{2n}{n-4}$. Hence from Lemma 3.2 we get $\|f'(u_m)\|_r \leq C$ and

$$(f(u_m), \Delta^2 u_{mt}) \leq C \|u_m\|_{H_4} \|\nabla^3 u_{mt}\| \leq \frac{C}{2\alpha} \|u_m\|_{H_4}^2 + \frac{\alpha}{2} \|\nabla^3 u_{mt}\|^2. \quad (3.14)$$

(ii) If $n = 5, 6$, take s sufficiently large such that $q_1 r \leq \frac{2n}{n-4}$. Then again we get $\|f'(u_m)\|_r \leq C$ and (3.14).

(iii) If $1 \leq n \leq 4$, take $s = r = 4$. Then we again get (3.14).

Substituting (3.14) into (3.13) and then integrating the inequality with respect to t , by (3.3), (3.4) and the Gronwall inequality we can get (3.12). $\square$

LEMMA 3.4. Under the conditions of Lemma 3.2 we further have

$$\|u_{mtt}\| \leq C(T), \quad 0 \leq t < T, \quad \forall T > 0. \quad (3.15)$$

Proof. Multiplying (3.8) by $g_{sm}''(t)$ and summing for s we get

$$\begin{aligned} \|u_{mtt}\|^2 &= (\Delta u_m - \Delta^2 u_m + \alpha \Delta u_{mt} + f(u_m), u_{mtt}) \\ &\leq (\|\Delta u_m\| + \|\Delta^2 u_m\| + \alpha \|\Delta u_{mt}\| + \|f(u_m)\|) \|u_{mtt}\|, \quad 0 \leq t \leq T. \end{aligned} \quad (3.16)$$

From (ii) in (H₂) we can get

$$|f(u)| \leq \frac{a}{q_1 + 1} |u|^{q_1 + 1}.$$

So by (3.12) it is easy to see that $\|f(u_m)\| \leq C(T)$ for $0 \leq t \leq T$ and $\forall T > 0$, which combining with (3.12) and Lemma 3.3 give (3.15). $\square$

From Lemma 3.2–Lemma 3.4 it is easy to get the following theorem.

THEOREM 3.5. Let $f(u)$ satisfy (H₂), $u_0 \in H_4$, $u_1 \in H$. Assume that $E(0) < d$, $I(u_0) > 0$ or $u = 0$, i.e. $u_0 \in W'$. Then problem (1.1)–(1.3) admits a global strong solution $u \in L^\infty(0, T; H_4) \cap L^\infty(0, \infty; H)$ with $u_t \in L^\infty(0, T; H) \cap L^\infty(0, \infty; L^2(\Omega))$, $u_{tt} \in L^\infty(0, T; L^2(\Omega))$, $\forall T > 0$ and $u \in W$ for $0 \leq t < \infty$.

4. Asymptotic behaviour of solutions. In this section we prove the asymptotic behaviour of global weak solutions for problem (1.1)–(1.3).

LEMMA 4.1. Let $f(u)$ satisfy (H₁), $u_0 \in H$, $u_1 \in L^2(\Omega)$, u_m be the approximate solutions given in the proof of Theorem 3.1. Then:

(i)

$$I(u_m) = \|u_{mt}\|^2 - \frac{d}{dt}(u_{mt}, u_m) - \frac{\alpha}{2} \frac{d}{dt} \|\nabla u_m\|^2, \quad \forall m \text{ and } 0 \leq t < \infty. \quad (4.1)$$

(ii) Furthermore if $0 < E(0) < d$, $u_0 \in W'$, then $u_m \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$ and

$$I(u_m) \geq (1 - \delta_1) \|u_m\|_H^2 \quad (4.2)$$

for $0 \leq t < \infty$ and sufficiently large m , where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$.

Proof. (i) Multiplying (3.2) by $g_{sm}(t)$ and summing for s yields (4.1).

(ii) First from (3.5) we get

$$\frac{1}{2} \|u_{mt}\|^2 + J(u_m) \leq E_m(0) < d(\delta), \quad \delta_1 < \delta < \delta_2, \quad 0 \leq t < \infty \quad (4.3)$$

for sufficiently large m . From (4.3) we can prove that $u_m(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 \leq t < \infty$ and sufficiently large m by an argument similar to that in the proof of Theorem 3.1 in [16]. Thus we have $I_\delta(u_m) \geq 0$ for $\delta \in (\delta_1, \delta_2)$ and $I_{\delta_1}(u_m) \geq 0$. Hence

$$\begin{aligned} I(u_m) &= \|u_m\|_H^2 - \int_{\Omega} u_m f(u_m) dx = (1 - \delta_1) \|u_m\|_H^2 + I_{\delta_1}(u_m) \\ &\geq (1 - \delta_1) \|u_m\|_H^2. \end{aligned}$$

□

THEOREM 4.2. Let $f(u)$ satisfy (H_1) , $u_0 \in H$, $u_1 \in L^2(\Omega)$. Assume that $0 < E(0) < d$ and $u_0 \in W'$. Then for the global weak solution u given in Theorem 3.1,

$$\|u_t\|^2 + \|u\|_H^2 \leq C e^{-\lambda t}, \quad 0 \leq t < \infty \quad (4.4)$$

for some positive constants C and λ .

Proof. First we prove that for the approximate solutions u_m given in the proof of Theorem 3.1,

$$0 \leq E_m(t) \leq C e^{-\lambda t}, \quad 0 \leq t < \infty \quad (4.5)$$

for some positive constants C and λ and sufficiently large m , where

$$E_m(t) = \frac{1}{2} \|u_{mt}\|^2 + \frac{1}{2} \|u_m\|_H^2 - \int_{\Omega} F(u_m) dx.$$

Note that from the proof of Theorem 3.1 we have

$$\frac{d}{dt} E_m(t) + \alpha \|\nabla u_{mt}\|^2 = 0. \quad (4.6)$$

Multiplying (4.6) by $e^{\gamma t}$ ($\gamma > 0$) we get

$$\frac{d}{dt} (e^{\gamma t} E_m(t)) + \alpha e^{\gamma t} \|\nabla u_{mt}\|^2 = \gamma e^{\gamma t} E_m(t)$$

and

$$e^{\gamma t} E_m(t) + \alpha \int_0^t e^{\gamma \tau} \|\nabla u_{m\tau}\|^2 d\tau = E_m(0) + \gamma \int_0^t e^{\gamma \tau} E_m(\tau) d\tau. \quad (4.7)$$

From Lemma 4.1 we have $u_m(t) \in W$ for $0 \leq t < \infty$ and sufficiently large m , which gives

$$E_m(t) = \frac{1}{2} \|u_{mt}\|^2 + J(u_m) \geq \frac{1}{2} \|u_{mt}\|^2 + \frac{p-1}{2(p+1)} \|u_m\|_H^2 + \frac{1}{p+1} I(u_m) \geq 0, \quad 0 \leq t < \infty. \quad (4.8)$$

From Lemma 4.1 we get

$$\begin{aligned}
\int_0^t e^{\gamma\tau} E_m(\tau) d\tau &\leq \frac{1}{2} \int_0^t e^{\gamma\tau} (\|u_{m\tau}\|^2 + \|u_m\|_H^2) d\tau \\
&\leq \frac{1}{2} \int_0^t e^{\gamma\tau} \left(\|u_{m\tau}\|^2 + \frac{1}{1-\delta_1} I(u_m) \right) d\tau \\
&= \frac{1}{2} \left(1 + \frac{1}{1-\delta_1} \right) \int_0^t e^{\gamma\tau} \|u_{m\tau}\|^2 d\tau \\
&\quad - \frac{1}{2(1-\delta_1)} \int_0^t e^{\gamma\tau} \frac{d}{d\tau} \left((u_{m\tau}, u_m) + \frac{\alpha}{2} \|\nabla u_m\|^2 \right) d\tau
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
& - \int_0^t e^{\gamma\tau} \frac{d}{d\tau} \left((u_{m\tau}, u_m) + \frac{\alpha}{2} \|\nabla u_m\|^2 \right) d\tau \\
&= (u_{mt}(0), u_m(0)) + \frac{\alpha}{2} \|\nabla u_m(0)\|^2 - e^{\gamma t} \left((u_{mt}, u_m) + \frac{\alpha}{2} \|\nabla u_m\|^2 \right) \\
&\quad + \gamma \int_0^t e^{\gamma\tau} \left((u_{m\tau}, u_m) + \frac{\alpha}{2} \|\nabla u_m\|^2 \right) d\tau \\
&\leq \frac{1}{2} (\|u_{mt}(0)\|^2 + \|u_m(0)\|^2 + \alpha \|\nabla u_m(0)\|^2) \\
&\quad + \frac{1}{2} e^{\gamma t} (\|u_{mt}\|^2 + \|u_m\|^2 + \alpha \|\nabla u_m\|^2) \\
&\quad + \frac{\gamma}{2} \int_0^t e^{\gamma\tau} (\|u_{m\tau}\|^2 + \|u_m\|^2 + \alpha \|\nabla u_m\|^2) d\tau.
\end{aligned} \tag{4.10}$$

From (4.8) we get

$$E_m(t) \geq \frac{1}{2} \|u_{mt}\|^2 + \frac{p-1}{2(p+1)} \|u_m\|_H^2. \tag{4.11}$$

Hence there exists a $C > 0$ such that

$$\frac{1}{2} (\|u_{mt}\|^2 + \|u_m\|^2 + \alpha \|\nabla u_m\|^2) \leq C E_m(t), \quad 0 \leq t < \infty.$$

Thus from (4.7), (4.9), (4.10) and (4.11) it follows that there exist positive constants C_0 and C_1 such that

$$\begin{aligned}
& e^{\gamma t} E_m(t) + \alpha \int_0^t e^{\gamma\tau} \|\nabla u_{m\tau}\|^2 d\tau \\
&\leq C_0 E_m(0) + \frac{\lambda_0 \gamma}{2} \left(1 + \frac{1}{1-\delta_1} \right) \int_0^t e^{\gamma\tau} \|\nabla u_{m\tau}\|^2 d\tau + C_1 \gamma e^{\gamma t} E_m(t) \\
&\quad + C_1 \gamma^2 \int_0^t e^{\gamma\tau} E_m(\tau) d\tau, \quad 0 \leq t < \infty,
\end{aligned} \tag{4.12}$$

where

$$\lambda_0 = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|u\|^2}{\|\nabla u\|^2}.$$

Take γ such that

$$0 < \gamma < \min \left\{ \frac{1}{2C_1}, \frac{2\alpha}{\lambda_0 \left(1 + \frac{1}{1-\delta_1}\right)} \right\}.$$

Then (4.12) gives

$$e^{\gamma t} E_m(t) \leq 2C_0 E_m(0) + 2C_1 \gamma^2 \int_0^t e^{\gamma \tau} E_m(\tau) d\tau$$

and

$$e^{\gamma t} E_m(t) \leq 2C_0 E_m(0) e^{2C_1 \gamma^2 t} < 2C_0 d e^{2C_1 \gamma^2 t}, \quad 0 \leq t < \infty$$

for sufficiently large m , which gives (4.5), where $C = 2C_0 d$, $\lambda = \gamma(1 - 2C_1 \gamma) > 0$.

From (4.11) we get that for sufficiently large m ,

$$\|u_{mt}\|^2 + \|u_m\|_H^2 \leq \frac{2(p+1)}{p-1} E_m(t), \quad 0 \leq t < \infty.$$

Let $\{u_\nu\}$ be the subsequence of $\{u_m\}$ given in the proof of Theorem 3.1. Then we have

$$\begin{aligned} \|u_t\|^2 + \|u\|_H^2 &\leq \liminf_{\nu \rightarrow \infty} (\|u_{\nu t}\|^2 + \|u_\nu\|_H^2) \\ &\leq \liminf_{\nu \rightarrow \infty} \frac{2(p+1)}{p-1} E_\nu(t) \leq \frac{2(p+1)}{p-1} C e^{-\lambda t}, \quad 0 \leq t < \infty, \end{aligned}$$

which gives (4.4). □

COROLLARY 4.3. Under the conditions of Theorem 4.2,

$$\|u\|_r^2 \leq C e^{-\lambda t}, \quad 0 \leq t < \infty$$

for some positive constants C and λ , where $1 \leq r \leq \frac{2n}{n-4}$ for $n \geq 5$; $1 \leq r < \infty$ for $n = 4$; $1 \leq r \leq \infty$ for $n \leq 3$. In particular, if $n \leq 3$, then

$$\|u\|_\infty^2 \leq C e^{-\lambda t}, \quad 0 \leq t < \infty. \quad (4.13)$$

Inequality (4.13) shows that the amplitude or oscillation of the wave described by problem (1.1)–(1.3) and the solutions of problem (1.1)–(1.3) decay to zero exponentially as time t tends to infinity.

5. Examples. In this section we shall give some explicit examples of nonlinearities f satisfying assumptions (H1) and (H2).

EXAMPLE 5.1. If

$$f(u) = a u^{2k+1}, \quad k = 1, 2, \dots,$$

where $a > 0$, $p = q = 2k + 1$ satisfies the conditions in (H1), then $f(u)$ satisfies (H1).

EXAMPLE 5.2. If

$$f(u) = u^{q_1+1},$$

where $p = q_1 + 1$ satisfies the conditions in (H2), then $f(u)$ satisfies (H2).

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