FROM A DYNAMICAL SYSTEM OF THE KNEE TO NATURAL JET GEOMETRICAL OBJECTS

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Abstract. In this paper, we construct some natural geometrical objects on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$, like a nonlinear connection, a Cartan linear connection (together with its d-torsions and d-curvatures), a jet “electromagnetic” d-field and its geometric “electromagnetic” Yang-Mills energy, starting from a given dynamical system governing the three-dimensional motion of the knee in the mathematical model introduced by Grood and Suntay. The computer-drawn graphics of corresponding Yang-Mills energetic surfaces of constant level (produced by this knee dynamical system) are given.

1. Short introduction. The 1-jet spaces are basic objects in the study of classical and quantum field theories (see Olver [10]). For this reason, a lot of authors (Asanov [2], Saunders [12] and many others) studied the differential geometry of 1-jet spaces. Using the Miron-Anastasiei’s Lagrangian geometrical ideas as a pattern (see [9]), Balan and Neagu have recently developed the *single-time Lagrange geometry on 1-jet spaces* (see [3]), which is very suitable for the geometrical study of the solutions of a given ODE system, via the *least squares variational method* initiated by Udriște (see [13]).

At present, the biomechanics of the lower limb is an extremely fruitful field of research. This research field is of interest to a lot of biomechanists (e.g., see Barbu [4] and Roșca [11]). In this direction, it is important to note that the differential geometrical methods from 1-jet spaces are very useful for studying biomechanics (see Ivancevic [7]). In such a perspective, this paper applies the jet geometrical results from monograph [3] to Euler’s

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dynamical equations that govern the three-dimensional motion of the tibia with respect to the femur, in the biomechanical and mathematical model introduced by Grood and Suntay [5], and studied further by many scholars (e.g., see Hefzy and Abdel-Rahman [6]).

2. Jet single-time Lagrange geometry produced by a nonlinear dynamical system. Let us present now the main jet single-time Lagrangian geometrical results that, in our opinion, may characterize a given nonlinear dynamical system. In this way, let us consider the jet fibre bundle of order one $J^1(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R} \times \mathbb{R}^n$, where $n \geq 1$, whose local coordinates (t, x^i, x_1^i) , $i = \overline{1, n}$, obey the rules

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{x}_1^i = \frac{\partial \widetilde{x}^i}{\partial x^j} \frac{dt}{dt} \cdot x_1^j.$$

Let $\mathfrak{X} = (\mathbf{X}_{(1)}^{(i)}(x^k))$ be an arbitrary distinguished (d-) tensor field on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^n)$, whose local components transform by the rules

$$\widetilde{\mathbf{X}}_{(1)}^{(i)} = \frac{\partial \widetilde{x}^i}{\partial x^j} \frac{dt}{dt} \cdot \mathbf{X}_{(1)}^{(j)}.$$

The d-tensor field $\mathfrak{X}$ produces the jet first-order ODE system (*jet dynamical system*)

$$x_1^i = \mathbf{X}_{(1)}^{(i)}(x^k(t)), \quad \forall i = \overline{1, n}, \quad (2.1)$$

where $c(t) = (x^i(t))$ is an unknown curve on $\mathbb{R}^n$ (i.e., a *jet field line* of the d-tensor field $\mathfrak{X}$), and we have

$$x_1^i := \dot{x}^i = \frac{dx^i}{dt}, \quad \forall i = \overline{1, n}.$$

Let us consider now the *jet least squares Lagrangian function* (attached to the dynamical system (2.1) and to Euclidean manifolds $(\mathbb{R}, 1)$ and $(\mathbb{R}^n, \delta_{ij})$),

$$\mathbf{JLS}^{\text{ODEs}} : J^1(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R}_+,$$

expressed by

$$\begin{aligned} \mathbf{JLS}^{\text{ODEs}}(x^k, x_1^k) &= \sum_{i,j=1}^n \delta_{ij} \left[x_1^i - \mathbf{X}_{(1)}^{(i)}(x) \right] \left[x_1^j - \mathbf{X}_{(1)}^{(j)}(x) \right] \\ &= \sum_{i=1}^n \left[x_1^i - \mathbf{X}_{(1)}^{(i)}(x) \right]^2, \end{aligned} \quad (2.2)$$

where $x = (x^k)_{k=\overline{1, n}}$. It is obvious that the global minimum points of the *jet least squares energy action*

$$\mathcal{E}^{\text{ODEs}}(c(t)) = \int_a^b \mathbf{JLS}^{\text{ODEs}}(x^k(t), \dot{x}^k(t)) dt, \quad t \in [a, b]$$

are exactly the solutions of class C^2 of the jet dynamical system (2.1). In other words, any solution of class C^2 of the system (2.1) verifies the second-order Euler-Lagrange equations

(i.e., the *jet geometric dynamics* associated to the ODE system (2.1)) produced by the jet least squares Lagrangian function (2.2):

$$\frac{\partial [\mathbf{JLS}^{\text{ODEs}}]}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial [\mathbf{JLS}^{\text{ODEs}}]}{\partial \dot{x}^i} \right) = 0, \quad \forall i = \overline{1, n}. \quad (2.3)$$

REMARK 2.1. Conversely, the preceding statement is not true. In other words, there exist solutions of the Euler-Lagrange ODE system of second order (2.3) that are not global minimum points for the jet least squares energy action $\mathcal{E}^{\text{ODEs}}$, and hence they are not solutions for the initial jet dynamical system (2.1).

In such a context, we consider that we may regard the jet least squares Lagrangian function $\mathbf{JLS}^{\text{ODEs}}$ as a natural geometrical substitute on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^n)$ for the dynamical system (2.1). But, an entire single-time Lagrange geometry on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^n)$ (in the sense of nonlinear connection, generalized Cartan linear connection, d-torsions, d-curvatures, jet “electromagnetic” d-field and jet “electromagnetic” Yang-Mills energy), geometry which is produced only by the jet least squares Lagrangian function $\mathbf{JLS}^{\text{ODEs}}$ (via its Euler-Lagrange equations (2.3)), is now completely done in the monograph [3]. For such a reason, we introduce the following concept:

DEFINITION 2.2. Any kind of geometrical object on $J^1(\mathbb{R}, \mathbb{R}^n)$ that is produced by the jet least squares Lagrangian function $\mathbf{JLS}^{\text{ODEs}}$ (via its second-order Euler-Lagrange equations (2.3)), is called a *geometrical object produced by the jet dynamical system* (2.1).

Let us consider the Jacobian matrix

$$J(\mathfrak{X}) = \left(\frac{\partial \mathbf{X}_{(1)}^{(i)}}{\partial x^j} \right)_{i,j=\overline{1,n}}.$$

In such a context, we give the following geometrical result (which is proved in Balan-Neagu’s book [3]):

THEOREM 2.3. (1) The *canonical nonlinear connection* on $J^1(\mathbb{R}, \mathbb{R}^n)$ produced by the *jet dynamical system* (2.1) has the local components

$$\Gamma^{\text{ODEs}} = \left(\mathbf{M}_{(1)1}^{(i)} = 0, \mathbf{N}_{(1)j}^{(i)} \right),$$

where

$$\begin{aligned} \mathbf{N}_{(1)} &= \left(\mathbf{N}_{(1)j}^{(i)} = -\frac{1}{2} \left[\frac{\partial \mathbf{X}_{(1)}^{(i)}}{\partial x^j} - \frac{\partial \mathbf{X}_{(1)}^{(j)}}{\partial x^i} \right] \right)_{i,j=\overline{1,n}} \\ &= -\frac{1}{2} [J(\mathfrak{X}) - {}^T J(\mathfrak{X})]. \end{aligned}$$

(2) All adapted components of the *canonical Cartan linear connection* $\mathbf{c}\Gamma^{\text{ODEs}}$ produced by the *jet dynamical system* (2.1) are zero.

- (3) The *torsion tensor* $\mathbf{T}^{\text{ODEs}}$ of the canonical Cartan linear connection $\mathbf{C}\Gamma^{\text{ODEs}}$ produced by the jet dynamical system (2.1) has the adapted components ($k = \overline{1, n}$)

$$\mathbf{T}_{(1)k} = \left(\mathbf{T}_{(1)jk}^{(i)} = -\frac{1}{2} \left[\frac{\partial^2 \mathbf{X}_{(1)}^{(i)}}{\partial x^k \partial x^j} - \frac{\partial^2 \mathbf{X}_{(1)}^{(j)}}{\partial x^k \partial x^i} \right] \right)_{i,j=\overline{1,n}} = \frac{\partial \mathbf{N}_{(1)}}{\partial x^k}.$$

- (4) All adapted components of the *curvature tensor* $\mathbf{R}^{\text{ODEs}}$ of the canonical Cartan linear connection $\mathbf{C}\Gamma^{\text{ODEs}}$ produced by the jet dynamical system (2.1) cancel.
 (5) The *geometric “electromagnetic” distinguished 2-form produced by the jet dynamical system* (2.1) has the expression

$$\mathcal{F}^{\text{ODEs}} = \mathbf{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \mathbf{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1, n},$$

and we have

$$\mathbf{F}^{(1)} = \left(\mathbf{F}_{(i)j}^{(1)} = \frac{1}{2} \left[\frac{\partial \mathbf{X}_{(1)}^{(i)}}{\partial x^j} - \frac{\partial \mathbf{X}_{(1)}^{(j)}}{\partial x^i} \right] \right)_{i,j=\overline{1,n}} = -\mathbf{N}_{(1)}.$$

- (6) The *jet geometric “electromagnetic” Yang-Mills energy produced by the jet dynamical system* (2.1) is given by the formula

$$\mathbf{EYM}^{\text{ODEs}}(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[\mathbf{F}_{(i)j}^{(1)} \right]^2 = \frac{1}{2} \cdot \text{Trace} \left[\mathbf{F}^{(1)} \cdot {}^T \mathbf{F}^{(1)} \right].$$

REMARK 2.4. The adapted components $\mathbf{F}_{(i)j}^{(1)}$ of the “electromagnetic” distinguished 2-form $\mathcal{F}^{\text{ODEs}}$ produced by the jet dynamical system (2.1) verify the following *geometrical Maxwell equations*:

$$\sum_{\{i,j,k\}} \mathbf{F}_{(i)j||k}^{(1)} = 0,$$

where $\sum_{\{i,j,k\}}$ represents a cyclic sum, and we have

$$\mathbf{F}_{(i)j||k}^{(1)} = \frac{\partial \mathbf{F}_{(i)j}^{(1)}}{\partial x^k}.$$

For more details on the jet “electromagnetic” topic, please consult the work [3].

REMARK 2.5. The jet geometric Yang-Mills energy $\mathbf{EYM}^{\text{ODEs}}$ coincides with the norm of the skew-symmetric “electromagnetic” matrix $\mathbf{F}^{(1)}$ in the Lie algebra

$$\mathfrak{o}(n) = L(O(n)) = \{ A \in M_n(\mathbb{R}) \mid A + {}^T A = 0 \},$$

where $O(n) = \{ A \in M_n(\mathbb{R}) \mid A \cdot {}^T A = I_n \}$ is the corresponding Lie group of the orthogonal matrices.

3. Three-dimensional mathematical model used for studying the tibio-femoral dynamics. The femur and tibia are modeled as two rigid bodies. Cartilage deformation is assumed relatively small compared to joint motions and not to affect relative motions and forces within the tibio-femoral joint. Furthermore, friction forces will be neglected because of the extremely low coefficients of friction of the articular surfaces. Hence, in this model, the resistance to motion is essentially due to the ligamentous structures and the contact forces. The menisci were not taken into consideration in the present model. For more biomechanical details, see Hefzy and Abdel-Rahman [6].

The joint coordinate system, which was introduced by Grood and Suntay [5], is used to define the rotation and translation vectors that describe the three-dimensional patella-femoral and tibio-femoral motions. This joint coordinate system is shown in FIG. 1 and consists of an x -axis that is fixed on the femur ($\bar{i}$ is the unit vector directing the x -axis), a z' -axis that is fixed on the tibia ($\bar{k}'$ is the unit vector directing the z' -axis), and a floating axis perpendicular to these two fixed axes ($\bar{i} \times \bar{k}'$ is the unit vector directing the floating axis). The Grood-Suntay's rotation vector includes three angular components:

- (1) α is the *flexion-extension angle* that characterizes the rotation occurring around the femoral fixed axis O_Fx ;
- (2) $\beta - \pi/2$ is the *adduction-abduction angle* for the right knee that characterizes the rotation around the floating axis (this is the case studied by us). For the left knee, the adduction-abduction angle is given by $\pi/2 - \beta$. In both cases the angle β is called the *varus-valgus angle*.
- (3) γ is the *internal-external tibial angle* that characterizes the rotation that occurs about the tibial fixed axis O_Tz' .

Using this joint coordinate system, the rotation vector (describing the orientation of the tibial coordinate system with respect to the femoral coordinate system) is defined as:

$$\bar{\theta} = -\alpha\bar{i} - \beta[\bar{i} \times \bar{k}'] - \gamma\bar{k}'.$$

This rotation vector can be transformed to the femoral coordinate system, and then, it can be differentiated with respect to time to yield the angular velocity and angular acceleration vectors of the tibia with respect to the femur.

In this analysis, it is assumed that the femur is fixed while the tibia is moving. The transformations of these two Grood-Suntay coordinate systems are given by (see [5], [6])

$$\bar{r} = \bar{r}_{O_T} + \bar{r}' = \bar{r}_{O_T} + \mathfrak{R} \cdot \bar{r}_{t=0},$$

where the vector $\bar{r}_{O_T}$ is the position vector that locates the origin of the tibial coordinate system with respect to the femoral coordinate system (the tibial origin O_T is considered in the center of tibia), $\bar{r}_{t=0}$ describes the position vector of an arbitrary point P with respect to the tibial coordinate system (at the initial moment $t = 0$; note that the vector $\bar{r}_{t=0}$ is constant in time), and $\bar{r}'$ is the position vector (at an arbitrary moment t) of the same point P with respect to the tibial coordinate system. As usual in the mechanics of rigid bodies, the position vector $\bar{r}'$ of the point P is expressed by the rotation formula

$$\bar{r}' = \mathfrak{R} \cdot \bar{r}_{t=0},$$

FIG. 1. Grood-Suntay's mathematical model for dynamics of the knee

where the matrix of rotation

$$\mathfrak{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \quad (3.1)$$

has the entries (see [5], [6])

$$\begin{aligned} R_{11} &= \sin \beta \cos \gamma, & R_{12} &= \sin \beta \sin \gamma, & R_{13} &= \cos \beta, \\ R_{21} &= -\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma, \\ R_{22} &= \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma, & R_{23} &= \sin \alpha \sin \beta, \\ R_{31} &= \sin \alpha \sin \gamma - \cos \alpha \cos \beta \cos \gamma, \\ R_{32} &= -\sin \alpha \cos \gamma - \cos \alpha \cos \beta \sin \gamma, & R_{33} &= \cos \alpha \sin \beta. \end{aligned}$$

The rotation matrix $\mathfrak{R}$ is generated by multiplying the three matrices generated by the rotations around the axes, namely:

- (1) a rotation of angle α around the axis $O_F x$ (from the top of the vector $\bar{i}$, the rotation is clock-wise):

$$\mathcal{R}_{O_F x, \alpha} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix};$$

- (2) a rotation of angle $\frac{\pi}{2} - \beta$ around the floating axis (from the top of the vector $\bar{i} \times \bar{k}'$, the rotation is clock-wise):

$$\mathcal{R}_{\text{floating axis}, \beta} := \begin{pmatrix} \sin \beta & 0 & \cos \beta \\ 0 & 1 & 0 \\ -\cos \beta & 0 & \sin \beta \end{pmatrix};$$

- (3) a rotation of angle γ around the axis $O_T z'$ (from the top of the vector $\vec{k}'$, the rotation is clock-wise):

$$\mathcal{R}_{O_T z', \gamma} := \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In other words, we have the matrix equality

$$\mathfrak{R} = \mathcal{R}_{O_F x, \alpha} \cdot \mathcal{R}_{\text{floating axis}, \beta} \cdot \mathcal{R}_{O_T z', \gamma}.$$

The equations that govern the three-dimensional motion of the tibia with respect to the femur are the second-order differential Newton's and Euler's equations of motion.

The classical Newton's equations are written in scalar form, with respect to the femoral fixed system of axes, as:

$$\begin{cases} F_x^{\text{ext}} + G_x = m\ddot{x}_{O_F} \\ F_y^{\text{ext}} + G_y = m\ddot{y}_{O_F} \\ F_z^{\text{ext}} + G_z = m\ddot{z}_{O_F}, \end{cases}$$

where

- $\vec{F}^{\text{ext}} = F_x^{\text{ext}} \cdot \vec{i} + F_y^{\text{ext}} \cdot \vec{j} + F_z^{\text{ext}} \cdot \vec{k}$ is the sum of all external forces applied to the tibia-femur (contact forces in knee, ligamentous forces in knee, etc.);
- $\vec{G} = G_x \cdot \vec{i} + G_y \cdot \vec{j} + G_z \cdot \vec{k}$ is the weight of the leg of mass m ; the mass of the leg was taken in this experiment as $m = 4.0$ kg (see Hefzy and Abdel-Rahman [6]).
- $\vec{a}_{O_F} = \ddot{x}_{O_F} \cdot \vec{i} + \ddot{y}_{O_F} \cdot \vec{j} + \ddot{z}_{O_F} \cdot \vec{k}$ is the acceleration of the center of mass of the leg (which is fixed in the origin of the femoral coordinate system O_F).

From the perspective of rigid body mechanics, Euler's equations of motion are written in the scalar form as:

$$\begin{cases} I_{x'x'} \dot{\omega}_{x'} + (I_{z'z'} - I_{y'y'}) \omega_{y'} \omega_{z'} = M_{x'}^{\text{ext}} \\ I_{y'y'} \dot{\omega}_{y'} + (I_{x'x'} - I_{z'z'}) \omega_{z'} \omega_{x'} = M_{y'}^{\text{ext}} \\ I_{z'z'} \dot{\omega}_{z'} + (I_{y'y'} - I_{x'x'}) \omega_{x'} \omega_{y'} = M_{z'}^{\text{ext}}, \end{cases} \quad (3.2)$$

where

- $\vec{\omega} = \omega_{x'} \cdot \vec{i}' + \omega_{y'} \cdot \vec{j}' + \omega_{z'} \cdot \vec{k}'$ is the angular velocity vector of the tibia with respect to the femur;
- $\dot{\vec{\omega}} = \dot{\omega}_{x'} \cdot \vec{i}' + \dot{\omega}_{y'} \cdot \vec{j}' + \dot{\omega}_{z'} \cdot \vec{k}'$ is the angular acceleration vector of the tibia with respect to the femur;
- $\vec{M}^{\text{ext}} = M_{x'}^{\text{ext}} \cdot \vec{i}' + M_{y'}^{\text{ext}} \cdot \vec{j}' + M_{z'}^{\text{ext}} \cdot \vec{k}'$ is the sum of the moments (the torques) of all external forces acting on the tibia around the x' -, y' -, and z' -axes;
- $I_{x'x'}$, $I_{y'y'}$, and $I_{z'z'}$ are the principal moments of inertia of the leg about its centroidal principal system of axes. The inertial parameters were estimated using anthropometric data as

$$I_{x'x'} = 0.0672 \text{ kg} \cdot \text{m}^2, \quad I_{y'y'} = 0.0672 \text{ kg} \cdot \text{m}^2, \quad I_{z'z'} = 0.005334 \text{ kg} \cdot \text{m}^2.$$

Note that, in this analysis, the leg was assumed to be a right cylinder. For more details, please see Hefzy and Abdel-Rahman [6] and references therein.

REMARK 3.1. If we use the matrix equality

$$\begin{pmatrix} \vec{i}', & \vec{j}', & \vec{k}' \end{pmatrix} = \begin{pmatrix} \vec{i}, & \vec{j}, & \vec{k} \end{pmatrix} \cdot \mathfrak{R},$$

then we deduce that the rotation vector $\vec{\theta}$ takes the following form in the femoral system of axes:

$$\begin{aligned} \vec{\theta} = & (-\alpha - \gamma \cos \beta) \cdot \vec{i} + (-\beta \cos \alpha - \gamma \sin \alpha \sin \beta) \cdot \vec{j} \\ & + (\beta \sin \alpha - \gamma \cos \alpha \sin \beta) \cdot \vec{k}. \end{aligned} \quad (3.3)$$

Therefore, by differentiating the rotation vector $\vec{\theta}$, we find the angular velocity vector

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} = \omega_x \cdot \vec{i} + \omega_y \cdot \vec{j} + \omega_z \cdot \vec{k}.$$

Now, by rotating the angular velocity vector $\vec{\omega}$ via the formula

$$\begin{pmatrix} \omega_{x'}, & \omega_{y'}, & \omega_{z'} \end{pmatrix} = \begin{pmatrix} \omega_x, & \omega_y, & \omega_z \end{pmatrix} \cdot \mathfrak{R},$$

we obtain the equalities (see also [6]):

$$\begin{aligned} \omega_{x'} &= -\dot{\alpha} \sin \beta \cos \gamma - \dot{\alpha} \beta \cos \beta \cos \gamma + \dot{\alpha} \gamma \sin \beta \sin \gamma + \dot{\beta} \sin \gamma + \dot{\beta} \gamma \cos \gamma, \\ \omega_{y'} &= -\dot{\alpha} \sin \beta \sin \gamma - \dot{\alpha} \beta \cos \beta \sin \gamma - \dot{\alpha} \gamma \sin \beta \cos \gamma - \dot{\beta} \cos \gamma + \dot{\beta} \gamma \sin \gamma, \\ \omega_{z'} &= -\dot{\alpha} \cos \beta + \dot{\alpha} \beta \sin \beta - \dot{\gamma}. \end{aligned}$$

4. Approximate values of the components of the total external torque and angular velocity. Using the three-dimensional mathematical model for the tibio-femoral motion due to Luh et al. (see [8]), Apkarian et al. (see [1]) practically measured in a lab frame (using some specialized devices) the total external torques acting on the tibia of a subject (A) having a stride length of 1.41 m and a walking speed of 1.21 m/s. Obviously, the time of the gait cycle for the subject (A), studied in the paper [1], is $T = 1.1652$ s.

Let us consider for the subject (A) the following gait cycle intermediate moments:

$$\begin{aligned} t_0 &= 0 \text{ (0\%)}, \quad t_1 = T/4 = 0.2913 \text{ (25\%)}, \quad t_2 = T/2 = 0.5826 \text{ (50\%)}, \\ t_3 &= 3T/4 = 0.8739 \text{ (75\%)}, \quad t_4 = T = 1.1652 \text{ (100\%)}. \end{aligned}$$

Looking now at the three graphs of the knee torque components (computed in $\text{N} \cdot \text{m}$ in the paper [1], pp. 153), we see that the total external torque vector of the knee,

$$\begin{aligned} \vec{M}^{\text{ext}} &= M_x^{\text{ext}} \cdot \vec{i} + M_y^{\text{ext}} \cdot \vec{j} + M_z^{\text{ext}} \cdot \vec{k} \\ &:= -M_{\text{valgus}} \cdot \vec{i} - M_{\text{extension}} \cdot \vec{j} + M_{\text{internal}} \cdot \vec{k}, \end{aligned}$$

has the following approximate interpolation components:

$M_x^{\text{ext}}(t_0) \approx 7.5,$	$M_y^{\text{ext}}(t_0) \approx 7.5,$	$M_z^{\text{ext}}(t_0) = 0,$
$M_x^{\text{ext}}(t_1) \approx -40,$	$M_y^{\text{ext}}(t_1) = 0,$	$M_z^{\text{ext}}(t_1) \approx 5,$
$M_x^{\text{ext}}(t_2) \approx -15,$	$M_y^{\text{ext}}(t_2) = 0,$	$M_z^{\text{ext}}(t_2) = 0,$
$M_x^{\text{ext}}(t_3) = 0,$	$M_y^{\text{ext}}(t_3) = 0,$	$M_z^{\text{ext}}(t_3) \approx -5,$
$M_x^{\text{ext}}(t_4) \approx 7.5,$	$M_y^{\text{ext}}(t_4) \approx 15,$	$M_z^{\text{ext}}(t_4) = 0.$

At the same time, looking at the approximate values of the three knee angles appearing in the graphs from [1] (pp. 150), we observe that the rotation vector

$$\bar{\theta} = \theta_x \cdot \bar{i} + \theta_y \cdot \bar{j} + \theta_z \cdot \bar{k} := -\theta_{\text{varus}} \cdot \bar{i} - \theta_{\text{flexion}} \cdot \bar{j} - \theta_{\text{external}} \cdot \bar{k}$$

has the interpolation components (the angles are presented here in radians; from the top of the vectors $\bar{i}, \bar{j}, \bar{k}$, these angles are measured counterclockwise):

$\theta_x(t_0) = 0 (= 0^\circ),$	$\theta_y(t_0) \approx -0.0872 (= -5^\circ),$
$\theta_x(t_1) \approx -0.0872 (= -5^\circ),$	$\theta_y(t_1) \approx -0.3490 (= -20^\circ),$
$\theta_x(t_2) \approx -0.0872 (= -5^\circ),$	$\theta_y(t_2) \approx -0.3490 (= -20^\circ),$
$\theta_x(t_3) \approx -0.1745 (= -10^\circ),$	$\theta_y(t_3) \approx -1.1344 (= -65^\circ),$
$\theta_x(t_4) = 0 (= 0^\circ),$	$\theta_y(t_4) \approx -0.0872 (= -5^\circ),$

$\theta_z(t_0) \approx -0.1745 (= -10^\circ),$
$\theta_z(t_1) \approx -0.0872 (= -5^\circ),$
$\theta_z(t_2) \approx -0.1745 (= -10^\circ),$
$\theta_z(t_3) \approx -0.0872 (= -5^\circ),$
$\theta_z(t_4) \approx -0.1745 (= -10^\circ).$

REMARK 4.1. Solving numerically the angle system (see the rotation vector (3.3))

$$\begin{cases} -\alpha - \gamma \cos \beta = \theta_x \\ -\beta \cos \alpha - \gamma \sin \alpha \sin \beta = \theta_y \\ \beta \sin \alpha - \gamma \cos \alpha \sin \beta = \theta_z, \end{cases}$$

we find the following intermediate Grood-Suntay's angles for knee:

$\alpha(t_0) \approx -0.5786,$	$\beta(t_0) \approx 0.1684,$	$\gamma(t_0) \approx 0.5869,$
$\alpha(t_1) \approx -0.0760,$	$\beta(t_1) \approx 0.3546,$	$\gamma(t_1) \approx 0.1740,$
$\alpha(t_2) \approx -0.1851,$	$\beta(t_2) \approx 0.3751,$	$\gamma(t_2) \approx 0.2927,$
$\alpha(t_3) \approx 0.0859,$	$\beta(t_3) \approx 1.1227,$	$\gamma(t_3) \approx 0.2044,$
$\alpha(t_4) \approx -0.5786,$	$\beta(t_4) \approx 0.1684,$	$\gamma(t_4) \approx 0.5869.$

Now, using the Lagrange polynomial of interpolation generally described by

$$\begin{aligned} P(x) = & y_0 \cdot \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} \cdot \frac{x-x_3}{x_0-x_3} \cdot \frac{x-x_4}{x_0-x_4} \\ & + y_1 \cdot \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} \cdot \frac{x-x_3}{x_1-x_3} \cdot \frac{x-x_4}{x_1-x_4} \\ & + y_2 \cdot \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} \cdot \frac{x-x_3}{x_2-x_3} \cdot \frac{x-x_4}{x_2-x_4} \\ & + y_3 \cdot \frac{x-x_0}{x_3-x_0} \cdot \frac{x-x_1}{x_3-x_1} \cdot \frac{x-x_2}{x_3-x_2} \cdot \frac{x-x_4}{x_3-x_4} \\ & + y_4 \cdot \frac{x-x_0}{x_4-x_0} \cdot \frac{x-x_1}{x_4-x_1} \cdot \frac{x-x_2}{x_4-x_2} \cdot \frac{x-x_3}{x_4-x_3}, \end{aligned}$$

we can construct the approximate Apkarian's angle functions $\theta_x(t)$, $\theta_y(t)$, and $\theta_z(t)$. Afterward, by differentiating the angle functions $\theta_x(t)$, $\theta_y(t)$, and $\theta_z(t)$, we get the components of the angular velocity in our gait cycle moments (rad / s):

$\omega_x(t_0) \approx -1.0981,$	$\omega_y(t_0) \approx -5.6920,$	$\omega_z(t_0) \approx 1.5984,$
$\omega_x(t_1) \approx 0.0999,$	$\omega_y(t_1) \approx 1.1983,$	$\omega_z(t_1) \approx -0.3997,$
$\omega_x(t_2) \approx -0.1998,$	$\omega_y(t_2) \approx -1.7974,$	$\omega_z(t_2) \approx -0.0002,$
$\omega_x(t_3) \approx -0.1997,$	$\omega_y(t_3) \approx -2.0970,$	$\omega_z(t_3) \approx 0.3993,$
$\omega_x(t_4) \approx 1.8975,$	$\omega_y(t_4) \approx 12.8820,$	$\omega_z(t_4) \approx -1.5983.$

Consequently, via the rotation formulas

$$X' = R_{11}X + R_{21}Y + R_{31}Z,$$

$$Y' = R_{12}X + R_{22}Y + R_{32}Z,$$

$$Z' = R_{13}X + R_{23}Y + R_{33}Z,$$

where R_{ij} are given by the expressions from (3.1), we find the following intermediate vector components (with respect to tibia coordinate system):

- components of the total external torque:

$M_{x'}^{\text{ext}}(t_0) \approx 0.9361,$	$M_{y'}^{\text{ext}}(t_0) \approx 8.1637,$	$M_{z'}^{\text{ext}}(t_0) \approx 6.7065,$
$M_{x'}^{\text{ext}}(t_1) \approx -18.3490,$	$M_{y'}^{\text{ext}}(t_1) \approx -2.8400,$	$M_{z'}^{\text{ext}}(t_1) \approx -35.7800,$
$M_{x'}^{\text{ext}}(t_2) \approx -5.2618,$	$M_{y'}^{\text{ext}}(t_2) \approx -1.5857,$	$M_{z'}^{\text{ext}}(t_2) \approx -13.9570,$
$M_{x'}^{\text{ext}}(t_3) \approx 2.0263,$	$M_{y'}^{\text{ext}}(t_3) \approx 0.8581,$	$M_{z'}^{\text{ext}}(t_3) \approx -4.4898,$
$M_{x'}^{\text{ext}}(t_4) \approx 0.8255,$	$M_{y'}^{\text{ext}}(t_4) \approx 15.6310,$	$M_{z'}^{\text{ext}}(t_4) \approx 6.0191.$

- components of the angular velocity:

$\omega_{x'}(t_0) \approx -1.6519,$	$\omega_{y'}(t_0) \approx -5.7721,$	$\omega_{z'}(t_0) \approx -0.3366,$
$\omega_{x'}(t_1) \approx 0.2847,$	$\omega_{y'}(t_1) \approx 1.2324,$	$\omega_{z'}(t_1) \approx -0.0762,$
$\omega_{x'}(t_2) \approx 0.1451,$	$\omega_{y'}(t_2) \approx -1.8010,$	$\omega_{z'}(t_2) \approx -0.0647,$
$\omega_{x'}(t_3) \approx 0.1623,$	$\omega_{y'}(t_3) \approx -2.1350,$	$\omega_{z'}(t_3) \approx 0.1098,$
$\omega_{x'}(t_4) \approx 1.6574,$	$\omega_{y'}(t_4) \approx 13.0050,$	$\omega_{z'}(t_4) \approx 0.4656.$

Now, using the last two tables and the well-known method of multiple regression from statistics, by numerical computations, we get the following linear approximations for the torque components $M_{x'}^{\text{ext}}$, $M_{y'}^{\text{ext}}$, and $M_{z'}^{\text{ext}}$ with respect to the angular velocity components $\omega_{x'}$, $\omega_{y'}$, and $\omega_{z'}$ (via the classical method of least squares):

$$\begin{aligned} M_{x'}^{\text{ext}} &\approx -16.9430 \cdot \omega_{x'} - 0.5003 \cdot \omega_{y'} + 80.9290 \cdot \omega_{z'} - 3.0709, \\ M_{y'}^{\text{ext}} &\approx -15.1720 \cdot \omega_{x'} + 1.5740 \cdot \omega_{y'} + 34.8270 \cdot \omega_{z'} + 3.7511, \\ M_{z'}^{\text{ext}} &\approx -39.7610 \cdot \omega_{x'} + 0.9071 \cdot \omega_{y'} + 140.8000 \cdot \omega_{z'} - 7.1266. \end{aligned} \quad (4.1)$$

5. From Euler's equations of motion of the knee to jet "electromagnetic" Yang-Mills energy. Note that, in our jet geometrical approach, we can regard Euler's equations of motion (3.2) as a dynamical system on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$. The coordinates on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$ are considered as being

$$(t, x^1, x^2, x^3, x_1^1, x_1^2, x_1^3) = (t, \omega_{x'}, \omega_{y'}, \omega_{z'}, \dot{\omega}_{x'}, \dot{\omega}_{y'}, \dot{\omega}_{z'}).$$

In such a context, taking the particular case from Hefzy–Abdel-Rahman’s paper [6], if we have

$$I_{x'x'} = I_{y'y'} = 0.0672, \quad I_{z'z'} = 0.0053,$$

and setting the components of the total external torque by the expressions (4.1), then Euler’s equations of motion (3.2) for the subject (A) can be rewritten in the following particular form:

$$\left\{ \begin{array}{lcl} \dot{\omega}_{x'} & = & 0.9211 \cdot \omega_{y'} \cdot \omega_{z'} - 252.1279 \cdot \omega_{x'} - 7.4449 \cdot \omega_{y'} \\ & & + 1204.3005 \cdot \omega_{z'} - 45.6979 \\ \dot{\omega}_{y'} & = & -0.9211 \cdot \omega_{x'} \cdot \omega_{z'} - 225.7738 \cdot \omega_{x'} + 23.4226 \cdot \omega_{y'} \\ & & + 518.2589 \cdot \omega_{z'} + 55.8199 \\ \dot{\omega}_{z'} & = & -7502.0754 \cdot \omega_{x'} + 171.1509 \cdot \omega_{y'} \\ & & + 26566.0377 \cdot \omega_{z'} - 1344.6415. \end{array} \right. \quad (5.1)$$

Consequently, via Theorem 2.3 applied to the first-order ODE system (5.1), we assert that the Lagrangian geometrical behavior on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$ of the above knee dynamical system can be described by the following result:

COROLLARY 5.1. (1) The *canonical nonlinear connection* on $J^1(\mathbb{R}, \mathbb{R}^3)$ produced by the jet dynamical system of the knee (5.1) has local components

$$\overset{\circ}{\Gamma} = \left(\overset{\circ}{M}_{(1)1}^{(i)} = 0, \overset{\circ}{N}_{(1)j}^{(i)} \right),$$

where the matrix of the spatial nonlinear connection

$$\overset{\circ}{N}_{(1)} = \begin{pmatrix} \overset{\circ}{N}_{(1)1}^{(1)} & \overset{\circ}{N}_{(1)2}^{(1)} & \overset{\circ}{N}_{(1)3}^{(1)} \\ \overset{\circ}{N}_{(1)1}^{(2)} & \overset{\circ}{N}_{(1)2}^{(2)} & \overset{\circ}{N}_{(1)3}^{(2)} \\ \overset{\circ}{N}_{(1)1}^{(3)} & \overset{\circ}{N}_{(1)2}^{(3)} & \overset{\circ}{N}_{(1)3}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & -\overset{\circ}{F}_{(1)2}^{(1)} & -\overset{\circ}{F}_{(1)3}^{(1)} \\ \overset{\circ}{F}_{(1)2}^{(1)} & 0 & -\overset{\circ}{F}_{(2)3}^{(1)} \\ \overset{\circ}{F}_{(1)3}^{(1)} & \overset{\circ}{F}_{(2)3}^{(1)} & 0 \end{pmatrix}$$

has the entries

$$\begin{aligned} \overset{\circ}{F}_{(1)2}^{(1)} &= 0.9211 \cdot \omega_{z'} + 109.1644, & \overset{\circ}{F}_{(1)3}^{(1)} &= 0.4605 \cdot \omega_{y'} + 4353.1879, \\ \overset{\circ}{F}_{(2)3}^{(1)} &= -0.4605 \cdot \omega_{x'} + 173.5540. \end{aligned}$$

- (2) All adapted components of the *canonical Cartan linear connection* $\overset{\circ}{C}\overset{\circ}{\Gamma}$ produced by the jet dynamical system of the knee (5.1) are zero.
- (3) The *torsion tensor* $\overset{\circ}{T}$ of the canonical Cartan linear connection $\overset{\circ}{C}\overset{\circ}{\Gamma}$ produced by the jet dynamical system of the knee (5.1) has as adapted components the entries

of the following torsion matrices:

$$\begin{aligned}\mathring{\mathbf{T}}_{(1)1} &= \frac{\partial \mathring{\mathbf{N}}_{(1)}}{\partial \omega_{x'}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.4605 \\ 0 & -0.4605 & 0 \end{pmatrix}, \\ \mathring{\mathbf{T}}_{(1)2} &= \frac{\partial \mathring{\mathbf{N}}_{(1)}}{\partial \omega_{y'}} = \begin{pmatrix} 0 & 0 & -0.4605 \\ 0 & 0 & 0 \\ 0.4605 & 0 & 0 \end{pmatrix}, \\ \mathring{\mathbf{T}}_{(1)3} &= \frac{\partial \mathring{\mathbf{N}}_{(1)}}{\partial \omega_{z'}} = \begin{pmatrix} 0 & -0.9211 & 0 \\ 0.9211 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

- (4) All adapted components of the *curvature tensor* $\mathring{\mathbf{R}}$ of the canonical Cartan linear connection $\mathring{\mathbf{C}}\Gamma$ produced by the jet dynamical system of the knee (5.1) cancel.
- (5) The *geometric “electromagnetic” adapted components produced by the jet dynamical system of the knee* (5.1) are the entries of the matrix

$$\mathring{\mathbf{F}}^{(1)} = -\mathring{\mathbf{N}}_{(1)} = \begin{pmatrix} 0 & \mathring{\mathbf{F}}_{(1)2}^{(1)} & \mathring{\mathbf{F}}_{(1)3}^{(1)} \\ -\mathring{\mathbf{F}}_{(1)2}^{(1)} & 0 & \mathring{\mathbf{F}}_{(2)3}^{(1)} \\ -\mathring{\mathbf{F}}_{(1)3}^{(1)} & -\mathring{\mathbf{F}}_{(2)3}^{(1)} & 0 \end{pmatrix}.$$

- (6) The *jet geometric “electromagnetic” Yang-Mills energy produced by the jet dynamical system of the knee* (5.1) is given by the formula

$$\mathbf{EYM}^{\text{knee}}(\omega_{x'}, \omega_{y'}, \omega_{z'}) = \left[\mathring{\mathbf{F}}_{(1)2}^{(1)} \right]^2 + \left[\mathring{\mathbf{F}}_{(1)3}^{(1)} \right]^2 + \left[\mathring{\mathbf{F}}_{(2)3}^{(1)} \right]^2.$$

Proof. The particular Euler’s equations of motion (5.1) represent a particular case of the jet first-order ODE system (2.1) for $n = 3$ and $\mathfrak{X} = \left(\mathring{\mathbf{X}}_{(1)}^{(i)}(\omega_{x'}, \omega_{y'}, \omega_{z'}) \right)_{i=\overline{1,3}}$, where

$$\begin{aligned}\mathring{\mathbf{X}}_{(1)}^{(1)}(\omega_{x'}, \omega_{y'}, \omega_{z'}) &= 0.9211 \cdot \omega_{y'} \cdot \omega_{z'} - 252.1279 \cdot \omega_{x'} - 7.4449 \cdot \omega_{y'} \\ &\quad + 1204.3005 \cdot \omega_{z'} - 45.6979, \\ \mathring{\mathbf{X}}_{(1)}^{(2)}(\omega_{x'}, \omega_{y'}, \omega_{z'}) &= -0.9211 \cdot \omega_{x'} \cdot \omega_{z'} - 225.7738 \cdot \omega_{x'} + 23.4226 \cdot \omega_{y'} \\ &\quad + 518.2589 \cdot \omega_{z'} + 55.8199, \\ \mathring{\mathbf{X}}_{(1)}^{(3)}(\omega_{x'}, \omega_{y'}, \omega_{z'}) &= -7502.0754 \cdot \omega_{x'} + 171.1509 \cdot \omega_{y'} \\ &\quad + 26566.0377 \cdot \omega_{z'} - 1344.6415.\end{aligned}$$

Consequently, using the formulas

$$\begin{aligned}\mathring{\mathbf{F}}_{(1)2}^{(1)} &= \frac{1}{2} \left[\frac{\partial \mathring{\mathbf{X}}_{(1)}^{(1)}}{\partial \omega_{y'}} - \frac{\partial \mathring{\mathbf{X}}_{(1)}^{(2)}}{\partial \omega_{x'}} \right], & \mathring{\mathbf{F}}_{(1)3}^{(1)} &= \frac{1}{2} \left[\frac{\partial \mathring{\mathbf{X}}_{(1)}^{(1)}}{\partial \omega_{z'}} - \frac{\partial \mathring{\mathbf{X}}_{(1)}^{(3)}}{\partial \omega_{x'}} \right], \\ \mathring{\mathbf{F}}_{(2)3}^{(1)} &= \frac{1}{2} \left[\frac{\partial \mathring{\mathbf{X}}_{(1)}^{(2)}}{\partial \omega_{z'}} - \frac{\partial \mathring{\mathbf{X}}_{(1)}^{(3)}}{\partial \omega_{y'}} \right], & \mathring{\mathbf{F}}^{(1)} &= -\mathring{\mathbf{N}}_{(1)}^*,\end{aligned}$$

some direct computations lead us to what we were looking for. $\square$

6. Yang-Mills energetic surfaces of constant level produced by the jet dynamical system of the knee. Using the notations $\omega_{x'} := X$, $\omega_{y'} := Y$, $\omega_{z'} := Z$, the jet geometric “electromagnetic” Yang-Mills energy produced by the dynamical system of the knee (5.1) (associated to the subject (A)) takes the form

$$\begin{aligned}\mathbf{EYM}^{\text{knee}}(X, Y, Z) &= [0.9211Z + 109.1644]^2 + [0.4605Y + 4353.1879]^2 \\ &\quad + [-0.4605X + 173.5540]^2.\end{aligned}$$

Now, let us consider the jet “electromagnetic” Yang-Mills energetic surfaces of constant level, which are produced by the dynamical system of the knee of the subject (A). These energetic surfaces are defined by the implicit equations

$$\Sigma_k^{\text{knee}} : \mathbf{EYM}^{\text{knee}}(X, Y, Z) = k,$$

where $k \geq 0$ is a given constant real number. By direct numerical computations, we obtain the following approximate implicit equations:

$$\begin{aligned}\Sigma_k^{\text{knee}} : 0.2120 (X')^2 + 0.2120 (Y')^2 + 0.8484 (Z')^2 &= k \\ \Sigma_k^{\text{knee}} : \frac{(X')^2}{4.7169k} + \frac{(Y')^2}{4.7169k} + \frac{(Z')^2}{1.1786k} - 1 &= 0, \quad k \neq 0,\end{aligned}$$

where we have done the spatial translation

$$X' = X - 376.8816, \quad Y' = Y + 9453.1767, \quad Z' = Z + 118.5152.$$

In conclusion, it follows that the jet “electromagnetic” Yang-Mills energetic surface of constant level $k \geq 0$ (produced by the dynamical system of the knee of the subject (A)) is

- (1) for $k = 0$, it is the point $\mathcal{C}(376.8816, -9453.1767, -118.5152)$.
- (2) for $k > 0$, it is an *ellipsoid* centered in the point $\mathcal{C}$, whose semi-axes are parallel with the axes OX , OY , and OZ , and they have the lengths

$$a = b = \sqrt{4.7169k}, \quad c = \sqrt{1.1786k}.$$

REMARK 6.1. Note that our jet “electromagnetic” Yang-Mills energetic ellipsoid is not the same thing as the well-known inertia ellipsoid from classical mechanics of the rigid body.

7. Relativistic gravito-electromagnetic interpretations of the theoretical results. In our opinion, from a relativistic physical point of view, the jet distinguished geometrical objects appearing in Corollary 5.1 (which characterize the jet dynamical system of the knee (5.1)), show the geometrical and physical behaviour of the motion of the knee for an observer which is not situated in the classical “*isotropic*” Newtonian space-time $\mathbb{R} \times \mathbb{R}^3$ (endowed with the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$, i.e., the “*isotropic gravitational potentials*”), but is situated in the 7-dimensional “*non-isotropic*” relativistic jet space $J^1(\mathbb{R}, \mathbb{R}^3)$ (endowed with a metric d-tensor depending on the “angular velocity”). In such a perspective, the classical 4-dimensional space-time $\mathbb{R} \times \mathbb{R}^3$ is extended to the 7-dimensional jet space of first order $J^1(\mathbb{R}, \mathbb{R}^3)$ by adding the three dimensions related to the “angular acceleration” associated to the motion of the knee. In such a context, the relativistic 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$ is endowed with the “*non-isotropic gravitational potential*” (i.e., the jet Euclidean metrical d-tensor)

$$\mathbb{G} = dt \otimes dt + \delta_{ij} dx^i \otimes dx^j + \delta_{ij} \delta x_1^i \otimes \delta x_1^j,$$

which intrinsically characterizes a sort of “*interaction*” between “*external*” fields (or $(t, \overline{\omega})$ -fields) and “*internal*” fields (or $(\overline{\omega})$ -fields), considered as being “a priori” in the study of the biomechanical motion of the knee (e.g., bioelectrical, biomagnetic, or gravitational fields). In fact, all of these “a priori” physical fields are mathematically condensed in the jet distinguished Euclidean metric $\mathbb{G}$.

REMARK 7.1. In our preceding jet geometrical theory, we can also use general Riemannian manifolds $(\mathbb{R}, h_{11})$ and $(\mathbb{R}^3, \varphi_{ij})$ for mathematical modelling of the given isotropic gravitational potentials (i.e., the pair of Riemannian metrics $\mathcal{P} = (h_{11}, \varphi_{ij})$). For more details, please see the book [3].

REMARK 7.2. To find more details on the gravito-electromagnetic theory on anisotropic spaces, we recommend the reading of monographs [9] and [3]. However, in order to get an idea about our present jet electromagnetic-gravitational Lagrangian geometrical approach, we expose the following additional real physical characteristics of our theory:

– It is known that the Lagrangian function that governs the movement law of a particle of mass $m \neq 0$ and electric charge e , which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is¹

$$L(t, x^k, x_1^k) = mch^{11}(t)\varphi_{ij}(x^k)x_1^ix_1^j + \frac{2e}{m}A_{(i)}^{(1)}(t, x^k)x_1^i + \mathcal{F}(t, x^k), \quad (7.1)$$

where the semi-Riemannian metric $\varphi_{ij}(x)$ represents the *gravitational potential* of the space of events $\mathbb{R}^3$, $A_{(i)}^{(1)}(t, x)$ are the components of a d-tensor on the 1-jet space $J^1(\mathbb{R}, \mathbb{R}^3)$ representing the *electromagnetic potential*, and $\mathcal{F}(t, x)$ is a smooth *potential function* on the product manifold $\mathbb{R} \times \mathbb{R}^3$. In our jet Lagrangian geometrical formalism applied to (7.1) (this geometrical formalism is the same as that used in this research paper), the

¹The jet Lagrangian function (7.1) is a natural extension of the classical Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [9].

electromagnetic d-components become the classical ones, namely

$$F_{(i)j}^{(1)} = -\frac{e}{2m} \left(\frac{\partial A_{(i)}^{(1)}}{\partial x^j} - \frac{\partial A_{(j)}^{(1)}}{\partial x^i} \right).$$

Moreover, in our jet single-time Lagrangian geometrical approach, the corresponding local geometrical Einstein and Maxwell distinguished equations involve as projections the famous classical equations of physical fields, namely (for a detailed exposition of all these local geometrical d-equations of physical fields, please read the book [3], Ch. 5):

- (1) the *classical Einstein equations* (the projection of the local Einstein d-equations on the $\mathbb{R}^3$ -horizontal distribution induced by the nonlinear connection)

$$\mathfrak{R}_{ij} - \frac{\mathfrak{R}}{2} \varphi_{ij} = \mathcal{K} \mathcal{T}_{ij};$$

- (2) the *classical Maxwell equations* (the projection of the local Maxwell d-equations on the $\mathbb{R}^3$ -horizontal distribution induced by the nonlinear connection)

$$\sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0,$$

where

$$F_{(i)j|k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)} \gamma_{ik}^m - F_{(i)m}^{(1)} \gamma_{jk}^m.$$

This fact suggests, in our opinion, some kind of naturalness attached to our jet single-time Lagrangian geometrical-physical theory described in this paper to the biomechanics of the knee.

In such a context, for the observer belonging to the extended 7-dimensional relativistic space of physical fields (more exactly, situated in the 3-dimensional space of $\overset{\bullet}{\omega}$ -fields), the jet “electromagnetic” field $\overset{\bullet}{\mathcal{F}}$ (unobserved from the lab frame) appears, whose “source” is intimately connected to the angular velocity and acceleration of the tibia with respect to femur. In this approach, it is important to note that, for this observer, the corresponding jet “electromagnetic” Yang-Mills energy has a high constant value (because of the big values of the angular velocity components), namely

$$\mathbf{EYM}^{\text{knee}}(t) \approx 19 \text{ MJ}.$$

This is because, by numerical computations, we get

$\mathbf{EYM}^{\text{knee}}(t_0) \approx 1.8969 \times 10^7 \text{ J},$
$\mathbf{EYM}^{\text{knee}}(t_1) \approx 1.8997 \times 10^7 \text{ J},$
$\mathbf{EYM}^{\text{knee}}(t_2) \approx 1.8985 \times 10^7 \text{ J},$
$\mathbf{EYM}^{\text{knee}}(t_3) \approx 1.8984 \times 10^7 \text{ J},$
$\mathbf{EYM}^{\text{knee}}(t_4) \approx 1.9044 \times 10^7 \text{ J},$

and we used the Gauss’s regression line to approximate the function $\mathbf{EYM}^{\text{knee}}(t) := \mathbf{E}(t)$.

REMARK 7.3. A non-constant statistical approximation of the jet Yang-Mills energy function can be done by using the polynomial fit of degree four (MJ):

$$\mathbf{E}(t) \approx -0.0058 \cdot t^4 + 0.3540 \cdot t^3 - 0.5416 \cdot t^2 + 0.2240 \cdot t + 18.9690.$$

FIG. 2. The energetic oblate spheroid $\mathbf{EYM}^{\text{knee}}(X, Y, Z) = 19 \text{ MJ}$.

The fact that the preceding Yang-Mills energy has an approximate constant value suggests a sort of “*conservation law*” for our associated jet “electromagnetic” field. This fact underlines the naturalness of our jet geometrical construction attached to the Euler motion of the knee in Grood-Suntay’s mathematical model. Moreover, in our opinion, the “existence” of a such jet “electromagnetic” field (which is directly and naturally provided by the jet dynamical system of the knee (5.1)), is difficult to be practically discovered by usual experiments and measurements, because this strong jet Yang-Mills energy probably occurs in the 3-dimensional $\dot{\bar{\omega}}$ -space of angular velocities.

8. Biomechanical conclusions of the theoretical results. From a biomechanical point of view, the ellipsoid shapes (oblate spheroids) of the jet “electromagnetic” Yang-Mills energetic surfaces of constant level (produced by the dynamical system of the knee (5.1)) theoretically shows us (see FIG. 2.), via the computed data

$$k = 19 \times 10^6, \quad a = b = \sqrt{4.7169k}, \quad c = \sqrt{1.1786k},$$

that the components of the angular velocity in the motion of the knee can take in the Grood-Suntay’s mathematical model only the following values (rad / s):

$$X = \omega_{x'} \in [-9089.9605, 9843.7237], \quad Y = \omega_{y'} \in [-18920.0188, 13.6654],$$

$$Z = \omega_{z'} \in [-4850.6817, 4613.6513].$$

In our opinion, these are the main practical merits and novelties carried out by our jet geometrical approach from this paper, applied to the Euler’s motion of the knee in the Grood-Suntay’s mathematical model. However, we consider that other possible gravito-electromagnetic and biomechanical interpretations still represent an open problem for specialists in the biomechanics of the knee.

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