MAXIMUM PRINCIPLE FOR CERTAIN GENERALIZED TIME AND SPACE FRACTIONAL DIFFUSION EQUATIONS

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Abstract. We present an inequality for fractional derivatives related to the Leibniz rule that not only answers positively a conjecture raised by J. I. Diaz, T. Pierantozi, and L. Vázquez but also helps us to obtain a modern proof of the maximum principle for fractional differential equations. The inequality turns out to be versatile in nature as it can be used to obtain a priori estimates for many fractional differential problems.

1. Introduction. The aim of this work is to answer affirmatively a conjecture raised by J. I. Diaz, T. Pierantozi, and L. Vázquez in their interesting paper [11]. Based on a huge number of simulated examples, they conjectured that for $\alpha \in (0, 1)$ and $u \in C([0, T]; \mathbb{R})$, $u' \in L^1((0, T); \mathbb{R})$, D_{0+}^α the following inequality involving the Riemann-Liouville fractional derivative (see the definition in Section 2) holds true:

$$2u(t)D_{0+}^\alpha u(t) \geq D_{0+}^\alpha u^2(t), \quad t \in [0, T]. \quad (1)$$

In [11] the authors have used the former inequality for the finite time extinction phenomenon for a nonlinear fractional evolution equation with the restriction that u is monotone.

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We will develop a more general inequality by relaxing the monotonicity requirement that leads to (1) in a particular situation.

Moreover, relying on the inequality obtained, we will present maximum principle results for three important time and time-space fractional diffusion equations.

The first equation considered is

$$D_{0+}^{\alpha} u = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) + \sum_{j=1}^N b_j(x, t) \frac{\partial u}{\partial x_j} + C(x, t)u + f(x, t),$$

which in its simplest form

$$D_{0+}^{\alpha} u = \nabla(a(x, t)\nabla u) + f(x, t)$$

has been considered, for example, by Benson et al. [3] to describe the dispersion through fractured rock.

The second equation we consider is the fractional porous medium equation

$$D_{0+}^{\alpha} u = \Delta(|u|^{m-1}u),$$

which is a generalization of the equation considered by Bologna et al. [5] (cf. also the references in [5]) which describes anomalous diffusion of Lévy-like and correlated-like in porous media, and transport of fluid in porous media and in viscous fingering [14].

The third equation is

$$D_{0+}^{\alpha} u + (-\Delta)^{\beta} u = 0,$$

which has been considered by del-Castillo-Negrete et al. [8] and Bakunin [2] to describe the transport in magnetically confined plasmas (examples include confinement time scaling L-mode plasmas, fast propagation and nonlocal transport phenomena, non-Gaussianity and long-range correlations of fluctuations, anomalous diffusion and non-Gaussian pdfs in tracer transport studies).

Let us mention that in the references, one may find fractional differential equations close to those we consider in applied sciences.

2. Preliminaries.

Notation. Let Ω be a bounded open domain in $\mathbb{R}^N$, $N \geq 1$ with measure dx . $\partial\Omega$ is the boundary of Ω , which is a smooth $(N-1)$ -dimensional manifold, with measure $d\hat{x}$. We define by $Q = \Omega \times (0, T]$. $|\Omega|$ is the standard Lebesgue measure of the domain Ω .

Function spaces. We will consider the following real Banach spaces of measurable functions with the norms

$$\begin{aligned} L^p(\Omega) &= \left\{ u(x) / \|u\|_p = \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p} < \infty \right\}; \\ L^{\infty}(\Omega) &= \left\{ u(x) / \|u\|_{\infty, \Omega} = \operatorname{ess\,max}_{x \in \Omega} |u(x)| < \infty \right\}; \\ L^{\infty}(Q) &= \left\{ u(x, t) / \|u\|_{\infty, Q} = \operatorname{ess\,max}_{(x, t) \in Q} |u(x, t)| < \infty \right\}; \\ C([0, T]) &= \{u : [0, T] \rightarrow \mathbb{R} / u \text{ continuous}\}; \end{aligned}$$

$$C^\sigma([0, T]) = \left\{ u : [0, T] \rightarrow \mathbb{R} / |u(t) - u(t-h)| = O(h^\sigma) \right. \\ \left. \text{uniformly for } 0 \leq t-h < t \leq T \right\}.$$

Inequalities. We shall need the following inequalities.

Young's inequality:

$$ab \leq \frac{\varepsilon^p}{p} a^p + \frac{b^q}{\varepsilon^q q}, \quad p+q=pq, \quad \varepsilon > 0, \quad (2)$$

where a, b are positive real numbers.

Hölder's inequality:

$$\int_{\Omega} |f||g|dx \leq \left(\int_{\Omega} |f|^p dx \right)^{1/p} \left(\int_{\Omega} |g|^q dx \right)^{1/q}, \quad p+q=pq, \quad p > 1, \quad q > 1. \quad (3)$$

For an integrable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, the *Riemann-Liouville fractional integral* of order $0 < \alpha < 1$ is defined by

$$J_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \quad (4)$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

For an integrable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, the *Riemann-Liouville fractional derivative* of order $0 < \alpha < 1$ is defined by

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad t > 0. \quad (5)$$

For an absolutely continuous (derivable) function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, the *Caputo fractional derivative* of order $0 < \alpha < 1$ is defined by

$${}^c D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau, \quad t > 0. \quad (6)$$

The relation between the Riemann-Liouville and the Caputo fractional derivatives for an absolutely continuous (derivable) function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is

$${}^c D_{0+}^\alpha f(t) = D_{0+}^\alpha (f(t) - f(0)), \quad 0 < \alpha < 1. \quad (7)$$

The Mittag-Leffler function of two parameters, defined by

$$E_{\xi, \beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + \beta)} \quad (z, \beta \in \mathbb{C}, \operatorname{Re}(\xi) > 0), \quad (8)$$

admits the following properties (see Prabhakar [26], page 9):

$$E_{\alpha, \alpha}(\mu t^\alpha) \leq M, \quad t \in [a, b], \quad a \geq 0, \quad \mu > 0, \quad 0 < \alpha \leq 1, \quad (9)$$

and

$$\int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(\mu \tau^\alpha) d\tau < \infty. \quad (10)$$

3. Main results.

LEMMA 1. Let one of the following conditions be satisfied:

- $u \in C([0, T])$ and $v \in C^\beta([0, T])$, $\alpha < \beta \leq 1$;
- $v \in C([0, T])$ and $u \in C^\beta([0, T])$, $\alpha < \beta \leq 1$;
- $u \in C^\beta([0, T])$, $v \in C^\delta([0, T])$, $\alpha < \beta + \delta$, $0 < \beta < 1$, $0 < \delta < 1$.

Then we have

$$D_{0+}^\alpha(uv)(t) = u(t)D_{0+}^\alpha v(t) + v(t)D_{0+}^\alpha u(t) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{(u(s) - u(t))(v(s) - v(t))}{(t-s)^{\alpha+1}} ds - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^\alpha} \quad (11)$$

pointwise.

Proof. Observe that

$$u(s)v(s) = (u(s) - u(t))(v(s) - v(t)) + u(t)v(s) + u(s)v(t) - u(t)v(t). \quad (12)$$

Now, by definition

$$D_{0+}^\alpha(uv)(t) = \frac{1}{\Gamma(1-\alpha)} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_0^{t+\varepsilon} \frac{u(s)v(s)}{(t+\varepsilon-s)^\alpha} ds - \int_0^t \frac{u(s)v(s)}{(t-s)^\alpha} ds \right], \quad (13)$$

which, using (12), can be written in the form

$$\begin{aligned} D_{0+}^\alpha(uv)(t) &= \frac{1}{\Gamma(1-\alpha)} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_0^{t+\varepsilon} \frac{(u(s) - u(t))(v(s) - v(t))}{(t+\varepsilon-s)^\alpha} ds \right. \\ &\quad - \int_0^t \frac{(u(s) - u(t))(v(s) - v(t))}{(t-s)^\alpha} ds \\ &\quad + u(t) \left\{ \int_0^{t+\varepsilon} \frac{v(s)}{(t+\varepsilon-s)^\alpha} ds - \int_0^t \frac{v(s)}{(t-s)^\alpha} ds \right\} \\ &\quad + v(t) \left\{ \int_0^{t+\varepsilon} \frac{u(s)}{(t+\varepsilon-s)^\alpha} ds - \int_0^t \frac{u(s)}{(t-s)^\alpha} ds \right\} \\ &\quad \left. - u(t)v(t) \left\{ \int_0^{t+\varepsilon} \frac{1}{(t+\varepsilon-s)^\alpha} ds - \int_0^t \frac{1}{(t-s)^\alpha} ds \right\} \right] \\ &= u(t)D_{0+}^\alpha v(t) + v(t)D_{0+}^\alpha u(t) - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^\alpha} \\ &\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{(u(s) - u(t))(v(s) - v(t))}{(t-s)^{\alpha+1}} ds. \end{aligned} \quad (14)$$

Consequences. The immediate consequences are listed below.

1. In the case where u and v have the same sign and are both increasing or both decreasing, then

$$D_{0+}^\alpha(uv)(t) \leq u(t)D_{0+}^\alpha v(t) + v(t)D_{0+}^\alpha u(t). \quad (15)$$

By setting $u = v$ in inequality (15) and requiring only $u \in C^\beta([0, T])$, $\alpha < 2\beta$, $\beta \leq 1$, we obtain the inequality conjectured by J. I. Diaz, T. Pierantozzi, and L. Vázquez:

$$2u(t)D_{0+}^\alpha u(t) \geq D_{0+}^\alpha u^2(t). \quad (16)$$

In the case $\beta < 1$, our requirement on u is weaker than the one of [11] as u is not differentiable. However, in the case $\beta = 1$, by Rademacher's theorem u is almost everywhere differentiable [12].

2. By induction, one can show that, for any integer $p \geq 2$,

$$pu^{(p-1)}(t)D_{0+}^\alpha u(t) \geq D_{0+}^\alpha u^p(t) \quad \begin{cases} \text{for } p \text{ even,} \\ \text{for } p \text{ odd whenever } u \geq 0. \end{cases} \quad (17)$$

It is inequality (17) for p even that we will use hereafter.

REMARK 1. For the Caputo derivative, inequality (15) reads

$$\begin{aligned} {}^c D_{0+}^\alpha (uv)(t) &\leq u(t){}^c D_{0+}^\alpha v(t) + v(t){}^c D_{0+}^\alpha u(t) \\ &\quad + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} (u(t)v(0) + v(t)u(0) - u(0)v(0)). \end{aligned}$$

4. Applications.

1. The main result in [11] remains true without requiring the solution to be monotone increasing.

2. We create a modern proof of the maximum principle for the fractional diffusion equation.

Maximum principle for partial fractional differential equations. The maximum principle for fractional-in-time diffusion equations has been dealt with in a series of very interesting papers by Luchko [21], [22], and [23]. Our lines of proofs for the different problems under consideration differ from the one used by Luchko, for example, in [23].

1. *Iteration proof of the maximum principle for the fractional heat equation.*

Here, we use the iteration technique to prove a version of the maximum principle for the fractional heat problem in divergence form. We consider

$$D_{0+}^\alpha u = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) + \sum_{j=1}^N b_j(x, t) \frac{\partial u}{\partial x_j} + C(x, t)u + f(x, t) \quad (18)$$

in $Q = \Omega \times (0, T]$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, supplemented either with the von Neumann boundary condition

$$\partial u / \partial \eta = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (19)$$

where η is the outward normal, or the Dirichlet boundary condition

$$u = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (20)$$

and the initial condition

$$\lim_{t \rightarrow 0} \Gamma(\alpha) t^{1-\alpha} u(t, x) = \lim_{t \rightarrow 0} (J_{0+}^{1-\alpha} u)(x, t) = u_0(x), \quad x \in \Omega. \quad (21)$$

Assume that

(H_1) the function u satisfying (18) is in $\mathcal{C}(\bar{Q})$ and has derivatives $\partial u / \partial x_i$, $\partial^2 u / \partial x_i \partial x_j$ in $\mathcal{C}(Q)$, $\frac{t^{-\alpha}}{\Gamma(1-\alpha)} * u \in C^1((0, T]; C(\Omega))$;

(H₂) a_{ij} , $\partial/\partial x_i(a_{ij})$ are in $\mathcal{C}(\bar{Q})$ and for every $(x, t) \in Q$ and every $\xi \in \mathbb{R}^N$

$$\sum_{i,j=1}^N a_{ij}(x, t)(\xi_i - \xi_j) \geq 0;$$

(H₃) for a positive constant d , $C(x, t) \leq -d$ in Q ;

(H₄) f is uniformly bounded, $|f(x, t)| \leq M$ in Q .

The aim is to obtain a bound on $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ in terms of $\|u(\cdot, 0)\|_{L^\infty}$, $\mathcal{N}(t) = \max_{t \geq 0} \|f(\cdot, t)\|_{L^\infty(\Omega)}$ and $m = \max_{(x,t)} |u(x, s)|$ for $(x, t) \in \partial\Omega \times (0, T]$.

THEOREM 1. If b_j , $(b_j)_{x_j}$ are continuous in Q for $j = 1, \dots, N$ and u satisfies (18), then for every $0 < t \leq T$, it holds that

$$\|u(t, \cdot)\|_{\infty, \Omega} \leq T \|f(\cdot, \cdot)\|_{\infty, Q} + C_0 \|u(\cdot, 0)\|_{\infty, \Omega} + \frac{3}{2} \|u(\cdot, \cdot)\|_{\infty, \partial\Omega \times (0, T]}, \quad (22)$$

where C_0 is a positive constant.

Proof. We proceed by multiplying each member of (18) by u^{2^k-1} ($k \in \mathbb{N}, k \gg 1$) throughout and then integrate over Ω . Thus,

$$\begin{aligned} \int_{\Omega} (D_{0+}^{\alpha} u) u^{2^k-1} dx &= \int_{\Omega} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) u^{2^k-1} dx \\ &+ \int_{\Omega} \sum_{j=1}^N b_j(x, t) \frac{\partial u}{\partial x_j} u^{2^k-1} dx + \int_{\Omega} C(x, t) u^{2^k} dx \\ &+ \int_{\Omega} f(x, t) u^{2^k-1} dx. \end{aligned} \quad (23)$$

After integrating by parts, using inequality (17) and the Hölder inequality with $q = 2^k$, $p = 2^k(2^k - 1)^{-1}$, $(p + q = pq)$ for the fourth term, we obtain

$$\begin{aligned} 2^{-k} D_{0+}^{\alpha} \int_{\Omega} u^{2^k} dx &\leq \int_{\partial\Omega} \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \cos(\eta, x_j) u^{2^k-1} d\hat{x} \\ &- (2^k - 1) \int_{\Omega} \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} u^{2^k-2} dx \\ &+ 2^{-k} \int_{\partial\Omega} \sum_{j=1}^N b_j \cos(\eta, x_j) u^{2^k} d\hat{x} - 2^{-k} \int_{\Omega} \sum_{j=1}^N \frac{\partial b_j}{\partial x_j} u^{2^k} dx \\ &+ \int_{\Omega} C(x, t) u^{2^k} dx + \left(\int_{\Omega} |f|^{2^k} dx \right)^{2^{-k}} \left(\int_{\Omega} u^{2^k} dx \right)^{1-2^{-k}}. \end{aligned} \quad (24)$$

Using (H₂), (H₃) and the fact that a_{ij}, b_j , u , $\partial a_{ij}/\partial x_i$, $\partial b_j/\partial x_j$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$ are continuous in $\bar{Q}$, we obtain

$$\begin{aligned} D_{0+}^{\alpha} \int_{\Omega} u^{2^k} dx &\leq 2^k m^{2^k-1} C_1 + m^{2^k} C_2 + \int_{\Omega} C_3(x, t) u^{2^k} dx \\ &+ 2^k \mathcal{N}(t) |\Omega|^{2^{-k}} \left(\int_{\Omega} u^{2^k} dx \right)^{1-2^{-k}}, \end{aligned} \quad (25)$$

where C_1 and C_2 are positive constants such that

$$\begin{aligned} C_1 &\geq \max_{t_0 \leq t \leq T} \int_{\partial\Omega} \left| \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \cos(\eta, x_j) \right| d\hat{x}, \\ C_2 &\geq \max_{t_0 \leq t \leq T} \int_{\partial\Omega} \sum_{j=1}^N b_j \cos(\eta, x_j) d\hat{x}, \\ C_3(x, t) &= 2^k C(x, t) - \sum_{j=1}^N \frac{\partial b_j}{\partial x_j}. \end{aligned}$$

Since $\partial b_j / \partial x_j$ is bounded and $C(x, t) \leq -d < 0$, then $C_3(x, t)$ is negative for sufficiently large k .

Now we set

$$y_k(t) = \|u(\cdot, t)\|_{L^{2^k}(\Omega)}^{2^k},$$

$$b_k(t) = \mathcal{N}(t) |\Omega|^{2^{-k}},$$

$$\beta = 2^{-k}, \quad a_k = 2m^{2^k-1}(2^k C_1 + m C_2),$$

and fix $\tilde{a} > 0$ such that $C_3(x, t) \leq -\tilde{a}$ for all $(x, t) \in Q$.

• If $m = 0$ (equivalently $a_k = 0$), then from (25) we obtain the fractional differential inequality:

$$D_{0+}^\alpha y_k(t) \leq -\tilde{a} y_k(t) + \beta^{-1} b_k(t) (y_k(t))^{1-\beta}. \quad (26)$$

Next we attain a bound on $y_k(t)$.

Using (2) with $a = b_k(t)$, $b = (y_k(t))^{1-\beta}$ and $\frac{1}{1-\beta} = q > 1$, $p = \frac{1}{\beta}$, we find that

$$\beta^{-1} b_k(t) (y_k(t))^{1-\beta} \leq \varepsilon^p b_k^p(t) + \frac{1}{\varepsilon^q q} y_k(t) \leq \varepsilon^p b_k^p(t) + \frac{1-\beta}{\beta \varepsilon^{\frac{1}{1-\beta}}} y_k(t).$$

Whereupon

$$D_{0+}^\alpha y_k(t) \leq -\left(\tilde{a} - \frac{1-\beta}{\beta \varepsilon^{\frac{1}{1-\beta}}}\right) y_k(t) + \varepsilon^p b_k^p(t). \quad (27)$$

Inequality (27) can be rewritten in the form

$$D_{0+}^\alpha (y_k(t) - y_k(0)) \leq \lambda_k (y_k(t) - y_k(0)) + \lambda_k y_k(0) + \varepsilon^p b_k^p(t), \quad (28)$$

where we have set

$$\lambda_k = -\tilde{a} + \frac{1-\beta}{\beta \varepsilon^{\frac{1}{1-\beta}}}.$$

According to [18, p. 230], the solution of (28) can be estimated as

$$\begin{aligned} y_k(t) \leq & y_k(0) + \lambda_k y_k(0) \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda_k(t-s)^\alpha) ds \\ & + \varepsilon^p \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda_k(t-s)^\alpha) b_k^p(s) ds. \end{aligned} \quad (29)$$

Using the inequality $\sqrt[n]{A+B} \leq \sqrt[n]{A} + \sqrt[n]{B}$, $A, B > 0, n > 1, n \in \mathbb{N}$, we can transform (29) into

$$\begin{aligned} y_k^\beta(t) &\leq y_k^\beta(0) + \lambda_k^\beta y_k^\beta(0) \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}^\beta(\lambda_k(t-s)^\alpha) ds \right)^\beta \\ &\quad + \varepsilon^{p\beta} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda_k(t-s)^\alpha) ds \right)^\beta |\mathcal{N}(t)|^\beta |\Omega|^{2-k} |p\beta|. \end{aligned} \quad (30)$$

Observe that $p\beta = 1$ and $\beta \rightarrow 0$, $y_k^\beta(t) \rightarrow \|u(., t)\|_{L^\infty(\Omega)}$ as $k \rightarrow \infty$. We set $\varepsilon = t \geq \varepsilon_0 > 0$. Thus, taking the limit as $k \rightarrow +\infty$ in inequality (30), we obtain

$$\|u(., t)\|_{\infty, \Omega} \leq T \|f(., .)\|_{\infty, Q} + C_0 \|u(., 0)\|_{\infty, \Omega},$$

for $0 < t \leq T$ and $C_0 = 1 + \varepsilon_0^{-1}$.

- If $m \neq 0$, proceeding as before we obtain the inequality

$$D_{0+}^\alpha(y_k(t) - y_k(0)) \leq \lambda_k(y_k(t) - y_k(0)) + \Lambda_k + \varepsilon^p b_k^p(t), \quad (31)$$

where we have set

$$\Lambda_k = \lambda_k y_k(0) + a_k.$$

An integration of the inequality (31) leads to the estimate

$$\|u(t, .)\|_{\infty, \Omega} \leq T \|f(., .)\|_{\infty, Q} + C_0 \|u(., 0)\|_{\infty, \Omega} + \frac{3}{2} \|u(., .)\|_{\infty, \partial\Omega \times (0, T]}$$

for $0 < t \leq T$.

2. The maximum principle for the fractional heat equation with a nonlinear diffusion.

For this part we need the Stroock-Varopoulos inequality, which is well known in the theory of sub-Markovian operators (cf. for example [28], [30]). We recall it in the following lemma.

LEMMA 2. Let $0 < \beta \leq 1$, $1 < p$. Then

$$\int_D (|\varphi|^{p-2} \varphi) (-\Delta)^\beta \varphi \geq \frac{4(p-1)}{p^2} \int_D |(-\Delta)^{\beta/2} \varphi|^{p/2} \varphi^2 \quad (32)$$

for all $\varphi \in L^p(D)$ such that $(-\Delta)^{\beta/2} \varphi \in L^p(D)$ (D a bounded domain or $\mathbb{R}^d$).

Let us consider the following porous medium equation with time fractional derivative

$$D_{0+}^\alpha u = \Delta(|u|^{m-1} u), \quad 0 < \alpha < 1 < m, \quad x \in \Omega, t > 0, \quad (33)$$

$$\lim_{t \rightarrow 0} \Gamma(\alpha) t^{1-\alpha} u(t, x) = \lim_{t \rightarrow 0} (J_{0+}^{1-\alpha} u)(x, t) = u(x, 0), \quad x \in \Omega, \quad (34)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0. \quad (35)$$

THEOREM 2. Let $u_0 \geq 0$ and u be the solution to problem (33), (34) and (35). Then, we have

$$\|u(., t)\|_{\infty, \Omega} \leq \|u(., 0)\|_{\infty, \Omega}. \quad (36)$$

Proof. First, as $u(x, 0) \geq 0$, then $u(x, t) \geq 0$. Multiplying each term of (33) by pu^{p-1} ($p \in \mathbb{N}^*$), and integrating over Ω , we obtain

$$\int_{\Omega} pu^{p-1} D_{0+}^{\alpha} u \, dx - p \int_{\Omega} u^{p-1} \Delta u^m \, dx = 0. \quad (37)$$

But

$$\begin{aligned} \int_{\Omega} u^{p-1} \Delta u^m \, dx &= \int_{\partial\Omega} u^{p-1} u^{m-1} \frac{\partial}{\partial \eta} u \, d\sigma - \int_{\Omega} (p-1) u^{p-2} u^{m-1} |\nabla u|^2 \, dx \\ &= - \int_{\Omega} (p-1) u^{p-2} u^{m-1} |\nabla u|^2 \, dx. \end{aligned} \quad (38)$$

If $p > 1$, then

$$p \int_{\Omega} u^{p-1} D_{0+}^{\alpha} u \, dx + p(p-1) \int_{\Omega} u^{p-2} u^{m-1} |\nabla u|^2 \, dx = 0. \quad (39)$$

Now, using the inequality (17), we get

$$\int_{\Omega} D_{0+}^{\alpha} u^p \, dx + \frac{4p(p-1)}{(p+m-1)^2} \int_{\Omega} |\nabla u^{\frac{p+m-1}{2}}|^2 \, dx \leq 0. \quad (40)$$

As a first consequence, we obtain

$$D_{0+}^{\alpha} \int_{\Omega} u^p \, dx \leq 0 \implies D_{0+}^{\alpha} \left(\int_{\Omega} u^p \, dx - \int_{\Omega} u_0^p \, dx \right) \leq 0.$$

Whereupon

$$\int_{\Omega} u^p \, dx \leq \int_{\Omega} u_0^p \, dx. \quad (41)$$

We finally conclude that

$$\|u(\cdot, t)\|_{\infty, \Omega} \leq \|u(\cdot, 0)\|_{\infty, \Omega}.$$

REMARK 2. In the case where $\Omega = \mathbb{R}^N$, we still have inequality (41). So,

$$\|u(\cdot, t)\|_p \leq \|u(\cdot, 0)\|_p$$

for any $1 < p < \infty$. The limit cases $p = 1$ and $p = \infty$ can be obtained by an elliptic approximation.

3. The maximum principle for the time and space fractional heat equation.

In [23] Luchko raised the problem of the maximum principle for time-space fractional diffusion equations. However, he did not fix the type of space fractional derivatives that have to be considered as several options were available. One may consider the regional fractional Laplacian [4] or a fractional power of the Laplacian from spectral theory or a sum of space fractional derivatives in the sense of Riemann-Liouville with respect to each direction such as $\sum_{i=1}^{i=N} \partial^{\alpha_i} / \partial x_i^{\alpha_i}$, where $\partial^{\alpha_i} / \partial x_i^{\alpha_i}$ stands for the Riemann-Liouville fractional derivative in the direction x_i .

Here we give some results in certain situations.

Let us consider the problem

$$\begin{cases} D_{0+}^{\alpha} u + (-\Delta)^{\beta} u = 0, & 0 < \alpha < 1, \quad 0 < \beta \leq 1, \quad x \in \Omega, t > 0, \\ u = 0 \text{ or } \partial u / \partial \eta = 0, & x \in \partial\Omega, t > 0, \\ \lim_{t \rightarrow 0} \Gamma(\alpha) t^{1-\alpha} u(t, x) = \lim_{t \rightarrow 0} (J_{0+}^{1-\alpha} u)(x, t) = u_0(x), \end{cases} \quad (42)$$

where Ω is a bounded domain. Here, Δ is the Laplace operator, the fractional power $(-\Delta)^\beta$ is a positive self-adjoint operator; it can be described by the spectral theorem. Let $\{\varphi_k, \lambda_k\}$ denote an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions and their corresponding eigenvalues. The operator $(-\Delta)^\beta$ is defined for $u \in C_0^\infty(\Omega)$, $u = \sum_{k=1}^\infty u_k \varphi_k$, by

$$(-\Delta)^\beta u = \sum_{k=1}^\infty \lambda_k^\beta u_k \varphi_k.$$

This operator can be extended by density for u in the Hilbert space

$$H_0^\beta(\Omega) = \left\{ u \in L^2(\Omega) : \|u\|_{H_0^\beta}^2 = \sum_{k=1}^\infty \lambda_k^\beta u_k^2 < \infty \right\}.$$

By the spectral theorem, one has

$$e^{-t(-\Delta)^\beta} = \int_0^t \Phi_{t,\beta}(s) e^{s\Delta} ds$$

for $t > 0$, where $\Phi_{t,\beta}$ has the property

$$e^{-t\lambda^\beta} = \int_0^t \Phi_{t,\beta}(s) e^{-s\lambda} ds, \quad \lambda > 0. \quad (43)$$

The fact that

$$(-1)^k \frac{\partial^k}{\partial \lambda^k} e^{-t\lambda^\beta} \geq 0, \quad \lambda, t > 0, k \in \mathbb{Z}^+$$

implies

$$\Phi_{t,\beta}(s) \geq 0, \quad 0 \leq \lambda < \infty, t > 0. \quad (44)$$

As the solution of (42) is given by

$$u(t) = E_{\alpha,\alpha}(-t^\alpha(-\Delta)^\beta)u_0, \quad (45)$$

where the linear operator $E_{\alpha,\alpha}(-t^\alpha(-\Delta)^\beta)$ is given by the standard operator calculus for self-adjoint operators. As derived in (A.37) of [24], one has

$$E_{\alpha,\alpha}(-s) = \int_0^\infty M_\alpha(r) e^{-rs} dr, \quad s > 0,$$

with

$$M_\alpha(r) = \frac{1}{2\pi i} \int_\gamma e^{\sigma-r\sigma^\alpha} \frac{d\sigma}{\sigma^{1-\alpha}},$$

where γ can be taken as a vertical line $\{i\varrho + \varepsilon : \varrho \in \mathbb{R}\}$, with small $\varepsilon > 0$. It follows that

$$E_{\alpha,\alpha}(-t^\alpha(-\Delta)^\beta) = \int_0^\infty M_\alpha(r) e^{-rt^\alpha} dr, \quad t > 0. \quad (46)$$

As noted in [24, (A.41)],

$$\alpha \int_0^\infty r^{-1-\alpha} M_\alpha(r^{-\alpha}) e^{-rs} dr = e^{-s^\alpha},$$

which compared with (43) gives

$$\alpha r^{-1-\beta} M_\alpha(r^{-\alpha}) = \Phi_{1,\alpha}(r).$$

Thus it follows that $M_\alpha(r) \geq 0$.

THEOREM 3. Let u be the solution to (42) with $0 \leq u_0(x)$. Then we have

$$\|u(\cdot, t)\|_{\infty, \Omega} \leq \|u_0\|_{\infty, \Omega}.$$

Proof. As $0 \leq u_0(x)$, then $0 \leq u(x, t)$. Now, multiplying each term of the first equation of (42) by pu^{p-1} and then integrating over Ω , we obtain

$$D_{0+}^{\alpha} \int_{\Omega} u^p dx + \frac{4(p-1)}{p} \int_{\Omega} (A^{\alpha/2}(\phi^{p/2}))^2 dx \leq 0,$$

thanks to the Stroock-Varopoulos inequality (Lemma 2).

Consequently, we have

$$D_{0+}^{\alpha} \int_{\Omega} u^p dx \leq 0.$$

As in the proof of the previous theorem, we conclude that

$$\|u(\cdot, t)\|_{\infty, \Omega} \leq \|u_0\|_{\infty, \Omega}.$$

REMARK 3. In the case where $\Omega = \mathbb{R}^N$, we have

$$\|u(\cdot, t)\|_p \leq \|u(\cdot, 0)\|_p$$

for $1 < p < \infty$. However in this case, the nonlocal operator $(-\Delta)^{\beta}$ is defined as follows for any function u in the Schwartz class through the Fourier transform: if $(-\Delta)^{\beta}u = v$, then

$$\hat{v}(\xi) = |\xi|^{2\beta} \hat{u}(\xi),$$

where ξ is the Fourier variable.

If $0 < \beta < 1$, the nonlocal operator $(-\Delta)^{\beta}$ is defined by its Riesz representation

$$(-\Delta)^{\beta}u(x) = C_{N,\beta} P.V \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{d+2\beta}},$$

where $C_{d,\beta}$ is a normalization constant (cf. [20]).

For example, following the same lines as before, a maximum principle can be obtained for the equation

$$D_{0+}^{\alpha}u + (-\Delta)^{\beta}u^m = f(x, t), \quad 0 < \alpha < 1, \quad x \in \Omega, t > 0.$$

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