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BIFURCATION OF COUPLED-MODE RESPONSES BY MODAL COUPLING IN CUBIC NONLINEAR SYSTEMS

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Abstract. When a stable normal mode loses stability in nonlinear conservative 2-degree-of-freedom systems, the phenomenon of internal resonance occurs involving rigorous energy exchange between modes and generating a stable coupled mode (called a *modal coupling*). Based on this observation, bifurcation of the coupled-mode responses is studied when the system is weakly damped and under a small sinusoidal excitation applied to one mode. The motions are not necessarily assumed to be small throughout. To analyze the stability of the driving mode in the underlying conservative system, a procedure is formulated to construct the stability curve in a stability chart. It is found that if the driving mode loses stability, then a stable coupled-mode response is formed and can be expressed in Fourier series. Assuming that the stability curve of the driving mode enters the pth unstable region with $p = 2, 3, 4, \ldots$, the coupled-mode response for $p = 2, 3, 4, \ldots$ can be determined with two terms as the first-order approximation; i.e., each coordinate is expressed by the sum of two predominant harmonic terms. One-term approximation of coupled-mode response is plausible if $p = 1$, which may result in 1:1 internal resonance. If the stability curve passes through the pth unstable region with $p = 2, 3, 4, \ldots$ and if the coupled-mode responses are expressed in the first-order approximation form, then the frequency response curve of the stable coupled-mode response is overlapped with the curve of the unstable response. As the order of approximation increases, two curves are separated from each other. The proposed method is compared with other perturbation techniques in the systems that exhibit 1:1 and 3:1 internal resonances.

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1. Introduction. Consider a conservative 2-degree-of-freedom (DOF) system whose kinetic energy T and potential energy V are written as

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right), \; V = \frac{1}{2}\left(\omega_1^2 x^2 + \omega_2^2 y^2\right) + N(x, y) \tag{1.1}$$

where x and y are the modal coordinates; ω_1 and ω_2 are the linearized natural frequencies; $N(x, y)$ represents the nonlinear restoring term; and $T + V = h$ denotes the total energy. Assume that the nonlinear restoring term is positive definite in $\mathbb{R}^2$ and can be written as

$$N(x, y) = \sum_{n=0}^{4} \alpha_n x^{4-n} y^n. \tag{1.2}$$

When the system is damped and under a harmonic excitation, the equations of motion can be written as

$$\begin{aligned} \ddot{x} + \omega_1^2 x + N_x(x, y) &= -c_1 \dot{x} + F \cos \omega t \\ \ddot{y} + \omega_2^2 y + N_y(x, y) &= -c_2 \dot{y} \end{aligned} \tag{1.3}$$

where N_x and N_y are the partial differentiations of $N(x, y)$ with respect to x and y, respectively. It is assumed that c_1, c_2, and F are small.

Since the origin O of the configuration space for the underlying conservative system is a stable equilibrium, there exist two normal modes (called x-mode and y-mode, respectively) in the neighborhood of O due to Liapunov's Theorem [1–3]. These modes are called the fundamental nonlinear normal modes to distinguish them from any bifurcating nonlinear normal modes, as will be described in Section 3.

The idea of nonlinear normal modes was proposed by Rosenberg [4] with the view that a nonlinear resonance occurs when a nonlinear normal mode exists in the system. And many others (e.g., see Rand [5], Vakakis et al. [6] and Shaw and Pierre [7]) made significant contributions to understanding nonlinear normal modes. Bifurcation of coupled modes has been extensively investigated, mostly in the systems with 1:1 and 3:1 internal resonances [8–21]. It was recently found that, by using numerical methods, the coupled modes exhibiting various geometrical shapes bifurcate off the fundamental nonlinear normal modes. Furthermore, it was demonstrated that each bifurcation point corresponds to the change in stability of the fundamental nonlinear normal modes.

The primary interest in this work is to show that, if the driving mode loses stability in the underlying conservative system, then a stable coupled-mode response is formed. Also, an analytical procedure of computing the coupled-mode responses is formulated.

The stability of the driving mode is analyzed by using Synge's concept [22, 23], which requires beforehand the solution of the driving mode. The driving mode is expressed in Fourier series. In Section 2, the driving mode is properly approximated to construct the stability curve in the stability chart. Some properties of transition curves (eigenvalues) and eigenfunctions are described.

A procedure is formulated in Section 3 to compute the coupled modes which bifurcate off the driving mode. These modes are also expressed in Fourier series and are properly approximated. A criterion is established to distinguish bifurcating nonlinear normal modes from the fundamental nonlinear normal modes. In Section 4, a procedure is formulated, by using the approximated coupled modes, to compute the undamped coupled-mode responses and to estimate the stability of the responses.

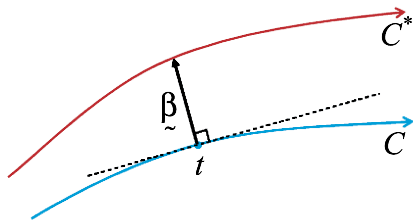


FIG. 1. Disturbance vector β.

In Section 5, the effect of damping is considered to derive the functional form of damped coupled-mode responses as the first-order approximation. In Section 6, some examples are given to demonstrate the procedure of the proposed method, which is compared with perturbation techniques in the systems with 1:1 and 3:1 internal resonances.

2. Stability analysis of a driving mode. The stability of the driving mode is analyzed by using Synge's concept [22, 23], which defines the disturbance vector $\underset{\sim}{\beta}$ between configurations of the unperturbed trajectory C and a perturbed trajectory C^* by the condition that $\underset{\sim}{\beta}$ is orthogonal to C (cf. Figure 1). The equation governing the magnitude $\beta = |\underset{\sim}{\beta}|$ is derived, for a fixed h, as

$$\ddot{\beta}(t) + \left(Kv^2 + 3\kappa^2 v^2 + \sum_{i,j=1}^{2} V_{ij} n_i n_j \right) \beta(t) = 0 \tag{2.1}$$

where K is the Gaussian curvature; v the velocity of C; κ the curvature of C; $V_{ij} = \nabla\nabla V$; and $\vec{n} = \{n_1, n_2\}$, the unit vector of $\underset{\sim}{\beta}$. Then, C is said to be stable in kinematico-statical sense if β, as every solution of Eq. (2.1), is permanently small. It is shown that this concept is equivalent to the orbital stability [24].

To analyze the stability of the driving mode, the solution of C should be known. The driving mode is a nonlinear normal mode in the underlying conservative system. It is (i) a periodic trajectory passing through and symmetric with respect to the origin, (ii) possesses two rest points, and (iii) can be expressed in Fourier series,

$$\begin{aligned} x(t) &= \sum_{j=0}^{\infty} A_j \cos j\omega t + \sum_{j=1}^{\infty} C_j \sin j\omega t \\ y(t) &= \sum_{j=0}^{\infty} B_j \cos j\omega t + \sum_{j=1}^{\infty} D_j \sin j\omega t. \end{aligned} \tag{2.2}$$

Setting $t = 0$ at a rest point, we can deduce that $x(-t) = x(t)$ and $y(-t) = y(t)$, which implies that $C_j = D_j = 0$, $j = 1, 2, \ldots$. Furthermore, due to the symmetry, we can write $x(\pi/2\omega - t) = -x(\pi/2\omega + t)$ and $y(\pi/2\omega - t) = -y(\pi/2\omega + t)$, which leads to additional conditions that $A_{2j} = B_{2j} = 0$, $j = 0, 1, 2, \ldots$. The driving mode can be simply approximated by

$$x(t) \approx \sum_{j=1}^{s} A_j \cos(2j-1)\omega t,\ y(t) \approx \sum_{j=1}^{s} B_j \cos(2j-1)\omega t \tag{2.3}$$

where s is the order of approximation so that the modal curve in the configuration space is expressed as

$$y(x) \approx \sum_{j=1}^{s} k_j x^{2j-1}. \tag{2.4}$$

The driving mode (i.e., the x-mode) is said to be simple if

$$y(t) \equiv 0,\ \ddot{x} + \omega_1^2 x + 4\alpha_0 x^3 = 0. \tag{2.5}$$

If the y-mode is also simple, then the system is said to be simple. For a simple system, we have that $\alpha_1 = \alpha_3 = 0$. Simple systems are frequently observed if a certain symmetry exists. Based on strongly nonlinear analysis (i.e., motions are not necessarily assumed to be small), the solution of Eq. (2.5) is approximated as

$$y(t) \equiv 0,\ x(t) = A\cos\omega t \tag{2.6}$$

where $\omega^2 = \omega_1^2 + 3\alpha_0 A^2$.

On the same line of approximating the driving mode in simple systems, the driving mode, as a fundamental nonlinear normal mode in general systems, can be approximated as

$$x(t) = A\cos\omega t,\ y(t) = B\cos\omega t. \tag{2.7}$$

Justification of this approximation is presented in Appendix A. To obtain the solution of the driving mode, the harmonic balance method is used to yield

$$\begin{aligned}(\omega_1^2 - \omega^2)A + \tfrac{3}{4}N_x(A,B) = 0\\ (\omega_2^2 - \omega^2)B + \tfrac{3}{4}N_y(A,B) = 0.\end{aligned} \tag{2.8}$$

Eliminating ω in Eq. (2.8), we obtain

$$(\omega_1^2 - \omega_2^2)AB + \frac{3}{4}\left[BN_x(A,B) - AN_y(A,B)\right] = 0. \tag{2.9}$$

Introducing a polar coordinate system,

$$A = R\cos\theta,\ B = R\sin\theta,\ q = \tan\theta. \tag{2.10}$$

Equation (2.9) can be rewritten as the fourth-order polynomial equation

$$f(q) + \frac{3}{4}R^2 g(q) = 0 \tag{2.11}$$

where

$$\begin{aligned}f(q) &= (\omega_1^2 - \omega_2^2)q(q^2+1)\\ g(q) &= \alpha_3 q^4 + (2\alpha_2 - 4\alpha_4)q^3 + (3\alpha_1 - 3\alpha_3)q^2 + (4\alpha_0 - 2\alpha_2)q - \alpha_1.\end{aligned}$$

It is found that there are at least two district real roots in Eq. (2.11). Thus, a procedure is established to obtain the solution of the driving mode; that is, we solve Eq. (2.11) for

q by increasing R gradually from zero so that A, B and ω are obtained accordingly. The effective nonlinear frequency is

$$\omega^2 = \omega_1^2 \cos^2\theta + \omega_2^2 \sin^2\theta + 3R^2 N(\cos\theta, \sin\theta). \tag{2.12}$$

Now, the stability curve of the driving mode will be constructed in the stability chart. It is found that $K = 0$, $\kappa = 0$, $n_1 = -\sin\theta$, and $n_2 = \cos\theta$. By substituting the solution of the driving mode into Eq. (2.1), we obtain the Mathieu equation,

$$\frac{d^2\beta}{d\tau^2} + (\delta + 2\epsilon\cos 2\tau)\beta = 0 \tag{2.13}$$

where $\tau = \omega t$, and

$$\begin{aligned} \epsilon &= \tfrac{R^2}{4\omega^2}\left(N_{xx}\sin^2\theta - 2N_{xy}\sin\theta\cos\theta + N_{yy}\cos^2\theta\right) \\ \delta &= \tfrac{1}{\omega^2}\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right) + 2\epsilon \end{aligned} \tag{2.14}$$

where N_{xx}, N_{xy} and N_{yy} are the functions of $\sin\theta$ and $\cos\theta$ in the place of x and y, respectively. Thus, a procedure is established to construct the stability curve in the well-known Strutt chart; that is, as R increases from zero to infinity, ϵ and δ are obtained. It is found that the stability curve begins from $(\delta, \epsilon) = (\omega_2^2/\omega_1^2, 0)$ when $R = 0$.

In the stability chart, two transition curves emanate from $(\delta, \epsilon) = (p^2, 0)$, $p = 1, 2, \ldots$, so that the pth unstable region may be constructed. On every transition curve, there is one, but only one, periodic solution of Eq. (2.13) with the period π or 2π, called the eigenfunction, denoted by $\beta^*(t)$. It is noted that the period of $\beta^*(t)$ is either a half of or equal to that of the driving mode.

Assume that the stability curve enters the pth unstable region with $p = 1, 2, \ldots$. Then the driving mode loses stability and the eigenfunction is expressed in a Fourier series if p is odd,

$$\beta^*(t) = \sum_{j=1}^{\infty} a_j \cos(2j-1)\omega t \tag{2.15}$$

or

$$\beta^*(t) = \sum_{j=1}^{\infty} b_j \sin(2j-1)\omega t; \tag{2.16}$$

and if p is even,

$$\beta^*(t) = \sum_{j=0}^{\infty} c_j \cos 2j\omega t \tag{2.17}$$

or

$$\beta^*(t) = \sum_{j=1}^{\infty} d_j \sin 2j\omega t. \tag{2.18}$$

The eigenfunction is, as the first-order approximation, written as

$$\beta^*(t) = \cos p\omega t + \mathcal{O}(\epsilon) \tag{2.19}$$

or

$$\beta^*(t) = \sin p\omega t + \mathcal{O}(\epsilon). \tag{2.20}$$

3. The solution of coupled modes. Assume that $c_1 = c_2 = F = 0$ in the system (1.3). If the stability curve of the driving mode enters an unstable region, then the driving mode loses stability, and hence in the Poincaré map the elliptic center is replaced by a saddle point. It is found that two elliptic centers bifurcate off the saddle point, giving rise to the stable coupled mode which may be two periodic orbits of period-one (through a pitchfork bifurcation) or one periodic orbit of period-two (through a period-doubling bifurcation [25]).

We will describe the functional form of coupled modes. It is evident, for the formation of an elliptic center, that one of the perturbed trajectories bifurcates off the saddle point (i.e., the driving mode C). Let C^* be the perturbed trajectory whose solution for Eq. (2.13) is $\beta^*(t)$. Then, at the outset of bifurcation, C^* must be the coupled mode.

Lemma 3.1. The modal curve of the coupled mode is, at the outset of bifurcation, in the form that a small disturbance $k\beta^*(t)$ is superimposed on the driving mode C. Therefore, the coupled mode is approximated by

$$\underset{\sim}{z}(t) = \underset{\sim}{z}^0(t) + k\beta^*(t)\underset{\sim}{n} \tag{3.1}$$

where $\underset{\sim}{z}^0(t) = (x(t), y(t))$ is the solution of C given by Eq. (2.7); $\underset{\sim}{n} = (-\sin\theta, \cos\theta)$; and k is a constant to be determined by the harmonic balance method [25].

Corollary 3.2. The bifurcation of coupled mode in Lemma 3.1 is pitchfork, and two elliptic centers in the Poincaré map correspond to two eigenfunctions $\pm\beta^*(t)$.

Proof. Suppose on the contrary that the bifurcation is period-doubling. Then, the period of $\beta^*(t)$ would be twice the period of C. But, the period of $\beta^*(t)$ is either a half of or equal to that of C. This contradiction leads to the pitchfork bifurcation. Due to the symmetry of Eq. (2.13), $-\beta^*(t)$ is also a periodic solution, leading to the fact that two periodic solutions, $\pm\beta^*(t)$, correspond to two elliptic centers in the Poincaré map. $\square$

If the stability curve of the driving mode enters the first unstable region, the stable coupled mode is, as the first-order approximation, written as

$$\begin{aligned} x(t) &= \bar{A}\cos\omega t + \bar{C}\cos\omega t \equiv A\cos\omega t \\ y(t) &= \bar{B}\cos\omega t + \bar{D}\cos\omega t \equiv B\cos\omega t \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} x(t) &= \bar{A}\cos\omega t + \bar{C}\sin\omega t \equiv A\cos(\omega t + \theta_1) \\ y(t) &= \bar{B}\cos\omega t + \bar{D}\sin\omega t \equiv B\cos(\omega t + \theta_2). \end{aligned} \tag{3.3}$$

If the stability curve enters the pth unstable region with $p = 2, 3, \ldots$, then the stable coupled mode is, as the first-order approximation, written as

$$\begin{aligned} x(t) &= A\cos\omega t + C\cos p\omega t \\ y(t) &= B\cos\omega t + D\cos p\omega t \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} x(t) &= A\cos\omega t + C\sin p\omega t \\ y(t) &= B\cos\omega t + D\sin p\omega t. \end{aligned} \tag{3.5}$$

It is noted that bifurcating nonlinear normal modes occur, as a part of coupled modes, when $p = 1, 3, 5, \ldots$ in Eq. (3.4), called the nongeneric bifurcation of nonlinear normal modes [26, 27].

The coupled modes given by Eqs. (3.2)-(3.5) can be computed by using the harmonic balance method, except when $p = 3$ in Eq. (3.4). The exceptional coupled mode is written as

$$\begin{aligned} x(t) &= A\cos\omega t + C\cos 3\omega t \\ y(t) &= B\cos\omega t + D\cos 3\omega t. \end{aligned} \tag{3.6}$$

This is in the second-order approximated form for the driving mode, i.e., $s = 2$ in Eq. (2.3). Therefore, a criterion is required to distinguish the coupled mode given by Eq. (3.6) from the driving mode.

To this end, the properties of the coupled modes will be derived. It is observed that two period-one orbits generated by the disturbances $\pm\beta^*(t)$ are expressed in terms of one set of modulation equations when the harmonic balance method is used, called *the property of symmetry*. It is found that the property of symmetry holds valid when the coupled modes are given by Eqs. (3.2)–(3.5) except when $p = 3$ in Eq. (3.4), and the exceptional case leads to Eq. (3.6) (see, for example, Eqs. (6.38)–(6.43) and (6.56)–(6.60) in Section 6). For the second-order or higher approximation, the coupled modes generated by the eigenfunctions given by Eqs. (2.15)–(2.18) are tested to find whether the property of symmetry holds valid. It is found that the property of symmetry holds valid except Eq. (2.15). This exceptional case leads to the third-order or higher approximation of the driving mode.

Another property of the coupled modes will be derived. When the pth unstable region with $p = 2, 3, \ldots$ is constructed, two approximated eigenfunctions, which generate two coupled modes given by Eqs. (3.4) and (3.5), do correspond to one, but only one, approximated eigenvalue (i.e., the transition curve, $\delta = p^2$). As a result, it is found that two coupled modes given by Eqs. (3.4) and (3.5) for fixed p are expressed in terms of one set of modulation equations when the harmonic balance method is used, called *the property of overlap*, except when $p = 3$ in Eq. (3.4).

We assume without loss of generality that the coupled mode given by Eq. (3.6) enjoys both properties of symmetry and of overlap, as every other coupled mode does. For the coupled mode given by Eq. (3.6) to meet the property of symmetry, it is required that firstly the set of modulation equations is obtained by using the harmonic balance method, and then the set is symmetrized by deleting asymmetric terms from the set. Here the asymmetric terms imply that, due to the presence of these terms, the property of symmetry does not hold. It is found that the property of overlap holds valid when the symmetrizing process is applied to the coupled mode given by Eq. (3.6).

Lemma 3.3. In order to have the solution of the coupled mode given by Eq. (3.6), the set of modulation equations is firstly obtained and then the asymmetric terms are deleted from the set at every order of approximation.

In simple systems, if the stability curve of the driving mode enters the first unstable region, then the stable coupled mode is, as the first-order approximation, written as

$$x(t) = A\cos\omega t,\ y(t) = B\cos\omega t \tag{3.7}$$

or

$$x(t) = A\cos\omega t,\ y(t) = B\sin\omega t. \tag{3.8}$$

If the stability curve enters the pth unstable region with $p = 2, 3, \ldots$, then the stable copled mode is, as the first-order approximation, written as

$$x(t) = A\cos\omega t,\ y(t) = B\cos p\omega t + \mathcal{O}(\epsilon) \tag{3.9}$$

or

$$x(t) = A\cos\omega t,\ y(t) = B\sin p\omega t + \mathcal{O}(\epsilon) \tag{3.10}$$

where $\mathcal{O}(\epsilon)$ denotes a harmonic term in $\mathcal{O}(\epsilon)$ of $\beta^*(t)$ (cf. Section 5).

To obtain the solution of coupled modes, for example, given by Eq. (3.4), the harmonic balance method is used to yield

$$X_j(A, B, C, D;\ \omega) = 0,\ Y_j(A, B, C, D;\ \omega) = 0 \tag{3.11}$$

where $j = 1, p$. Given A, other variables can be computed. Then, the backbone curve can be constructed. The set of ω in the backbone curve is called the effective frequency interval of the coupled mode. One limit of the interval is the bifurcation frequency ω_b, and the other depends on the type of bifurcation, either supercritical or subcritical.

Corollary 3.4. The backbone curve of the coupled mode expressed by Eq. (3.4) is overlapped with the curve by Eq. (3.5) for given p. As the order of approximation increases, two backbone curves are separated.

Proof. The property of overlap is applicable so that two coupled modes are expressed in terms of one set of modulation equations. Therefore, for given A the same values of B, C, D and ω are obtained for two coupled modes. For the second-order approximation, harmonic terms in $\mathcal{O}(\epsilon)$ for $\beta^*(t)$, such as $\cos(p \pm 2)\omega t$ and $\sin(p \pm 2)\omega t$, are added to Eqs. (3.4) and (3.5). Then the property of overlap no longer holds, implying that two curves are separated. □

In what follows, we consider one coupled mode instead of two coupled modes generated by $\pm\beta^*(t)$.

4. Undamped coupled-mode responses. Assume that $c_1 = c_2 = 0$ in the system (1.3) and that the stability curve of the driving mode enters an unstable region. The bifurcation point is denoted by $(\omega, A) = (\omega_b, A_b)$. Then, it will be shown that a stable coupled-mode response bifurcates off a single-mode response, and a procedure is described to compute the response.

Definition 4.1. A forced response is said to be a single-mode response if its functional form is equal to that of the driving mode.

The frequency response equations of single-mode response can be derived, by using Eq. (2.8), as

$$\begin{aligned} (\omega_1^2 - \omega^2)A + \tfrac{3}{4}N_x(A, B) &= F \\ (\omega_2^2 - \omega^2)B + \tfrac{3}{4}N_y(A, B) &= 0. \end{aligned} \tag{4.1}$$

There are two branches, the upper S^+ and the lower S^-, in the frequency response curve. The existence of two branches can be verified, for instance, by using Eqs. (6.1)–(6.4) in Section 6. Indeed, it is found that, for $c_1 = c_2 = 0$, $\sin\varphi = 0$ by Eq. (6.4), and then $\sin\theta_1 = 0$ by Eq. (6.2). Then, $\cos\theta_1 = \pm 1$ leads to S^+ and S^-, respectively.

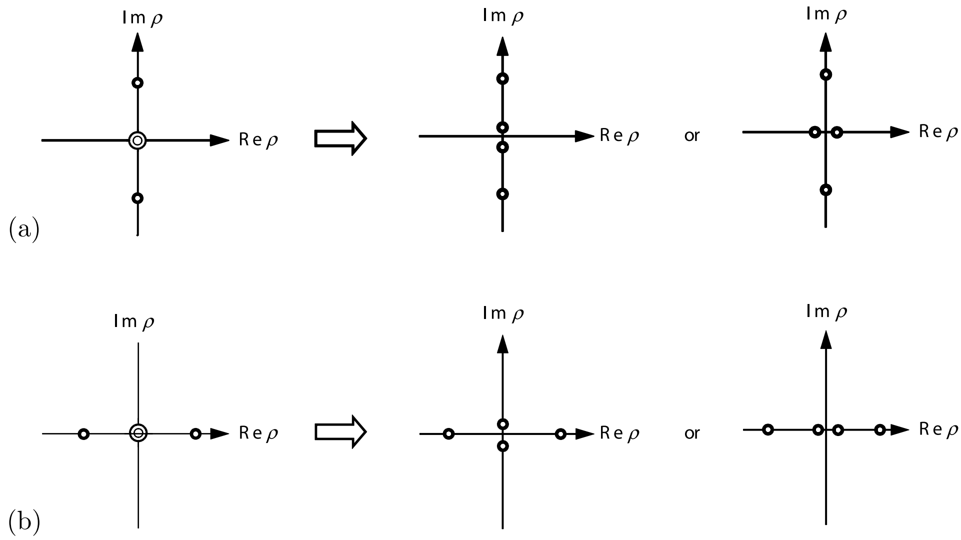


FIG. 2. Characteristic exponents of undamped forced responses: (a) stable driving mode and (b) unstable driving mode.

DEFINITION 4.2. A forced response C_1 is said to be the coupled-mode response of a coupled mode C_2 if the functional forms of C_1 and C_2 are equal to each other.

To derive the frequency response equations of the coupled-mode response of a coupled mode, for example, given by Eq. (3.4), Eq. (3.11) is used with

$$\begin{aligned} &X_1(A,B,C,D;\omega) = F \\ &X_p(A,B,C,D;\omega) = 0 \\ &Y_j(A,B,C,D;\omega) = 0,\ j = 1, p. \end{aligned} \tag{4.2}$$

If ω is not in the effective frequency interval, it is obvious that $C = D = 0$, implying that the coupled-mode response is not formed. On the other hand, if ω is in the effective frequency interval, then nonzero values of C and D can be obtained. In the latter case, it is found that a coupled-mode response S_b^+ (and S_b^-) bifurcates off the branch S^+ (and S^-). To observe this bifurcation, the values of C and D can be computed by using equations $X_p(A,B,C,D;\omega) = 0$, $Y_p(A,B,C,D;\omega) = 0$ in Eq. (4.2) when the values of A and B are chosen from S^+ (or S^-) for given ω so that the approximate solution of S_b^+ (or S_b^-) can be obtained.

The stability of coupled-mode responses is analyzed in the sense of Liapunov. Since an undamped forced system is Hamiltonian, the symplectic property is applicable so that the characteristic exponents ρ_j, $j = 1, \ldots, 4$, are grouped into two pairs,

$$\rho_1 + \rho_2 = 0,\ \rho_3 + \rho_4 = 0. \tag{4.3}$$

The first pair is denoted by ρ_{12}, and the second by ρ_{34}. If $F = 0$, we have

$$\rho_3 = \rho_4 \equiv 0. \tag{4.4}$$

As F increases from zero, ρ_{34} splits either in the real or imaginary axis (cf. Figure 2).

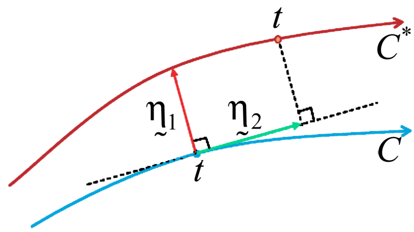


FIG. 3. Two kinds of disturbance.

To estimate the pairs of exponents, two kinds of disturbance are defined in the configuration space (Figure 3). Let C be the trajectory of a single-mode response, and C^* a perturbed trajectory. The disturbance $\underset{\sim}{\eta}_1$ (or $\underset{\sim}{\eta}_2$) is orthogonal to C (or in the direction of C). For small F, the disturbance $\underset{\sim}{\eta}_1$ is close to $\underset{\sim}{\beta}$ of Synge's concept, and hence ρ_{12} is close to that obtained by $\underset{\sim}{\beta}$.

The second pair ρ_{34} of a single-mode response is estimated by using the disturbance $\underset{\sim}{\eta}_2$ in the following.

LEMMA 4.3. The second pair ρ_{34} of characteristic exponents for a single-mode response is imaginary on one branch (say, S^+), but is real on the other branch S^-. This property persists for large A.

Proof. See Appendix B. □

It is found in S_b^+ that ρ_{12} is zero and ρ_{34} is imaginary at the bifurcation point, and thereafter ρ_{12} becomes imaginary, whereas ρ_{34} continues to remain imaginary, giving rise to the elliptical stability (cf. Figure 2(a)). On the other hand, it is found in S_b^- that ρ_{12} is zero and ρ_{34} is real at the bifurcation, and after bifurcation ρ_{12} becomes imaginary, whereas ρ_{34} continues to remain real, giving rise to instability (cf. Figure 2(a)).

Similarly, coupled-mode responses may be formed when the unstable driving mode recovers stability (i.e., the stability curve of the driving mode leaves an unstable region). The stability of the coupled-mode response can be estimated and is found to be unstable, as depicted in Figure 2(b).

5. The effects of damping. The solution of damped coupled-mode responses can be obtained by using the harmonic balance method. To this end, the coupled mode should be, as the first-order approximation, properly chosen both in general systems ($\alpha_1\alpha_3 \neq 0$) and in simple systems ($\alpha_1 = \alpha_3 = 0$). By the presence of damping, the phase of every harmonic term in the coupled mode is shifted. It is found that the properties of symmetry and of overlap hold valid in damped systems.

In general systems, the solution of damped coupled-mode response can be computed without difficulty by using the coupled modes given by Eqs. (3.2)–(3.5) except when $p = 3$ in the coupled mode given by Eq. (3.4), written as

$$\begin{aligned} x(t) &= A\cos(\omega t + \theta_1) + C\cos(3\omega t + \theta_3) \\ y(t) &= B\cos(\omega t + \theta_2) + D\cos(3\omega t + \theta_4). \end{aligned} \tag{5.1}$$

This functional form gives the second-order approximation of a single-mode response. In this case, Lemma 3.3 is applicable.

PROPOSITION 5.1. If the stability curve of the driving mode enters the first unstable region in general systems, then the functional form of stable coupled-mode response is, as the first-order approximation, written as

$$\begin{aligned} x(t) &= A\cos(\omega t+\theta_1)+C\cos(\omega t+\theta_3) \\ y(t) &= B\cos(\omega t+\theta_2)+D\cos(\omega t+\theta_4) \end{aligned} \tag{5.2}$$

or

$$\begin{aligned} x(t) &= A\cos(\omega t+\theta_1)+C\sin(\omega t+\theta_3) \\ y(t) &= B\cos(\omega t+\theta_2)+D\sin(\omega t+\theta_4). \end{aligned} \tag{5.3}$$

The functional form (5.2) or (5.3) may be reduced to the form

$$x(t) = A\cos(\omega t+\theta_1), \quad y(t) = B\cos(\omega t+\theta_2) \tag{5.4}$$

which is that of the single-mode response. The coupled-mode response requires an additional harmonic term in each coordinate; that is, the two-term approximation in general systems is implemented.

The presence of damping in simple systems generates important effects on approximating the functional form of coupled-mode responses. The equations of motion are written as

$$\begin{aligned} \ddot{x}+\omega_1^2 x+4\alpha_0 x^3+2\alpha_2 xy^2 &= -c_1\dot{x}+F\cos\omega t \\ \ddot{y}+\omega_2^2 y+2\alpha_2 x^2 y+4\alpha_4 y^3 &= -c_2\dot{y}. \end{aligned} \tag{5.5}$$

The driving mode is written as

$$y(t)\equiv 0,\ x(t)=A\cos\omega t,\ \omega^2=\omega_1^2+3\alpha_0A^2. \tag{5.6}$$

Synge's concept is used to analyze the stability of the driving mode. It is found that $K=0$, $\kappa=0$, $n_1=0$, and $n_2=1$. Then, the Mathieu equation (2.13) is obtained with $\tau=\omega t$, and

$$\epsilon=\frac{\alpha_2A^2}{2(\omega_1^2+3\alpha_0A^2)},\ \delta=\frac{\omega_2^2}{\omega_1^2+3\alpha_0A^2}+2\epsilon. \tag{5.7}$$

By eliminating A, the stability curve is written as

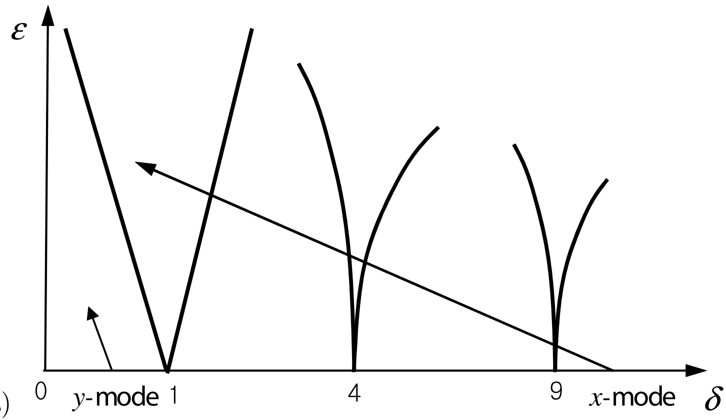
$$\delta=\frac{\omega_2^2}{\omega_1^2}-2\left(3\frac{\omega_2^2}{\omega_1^2}\frac{\alpha_0}{\alpha_2}-1\right)\epsilon. \tag{5.8}$$

When the system parameters are given as

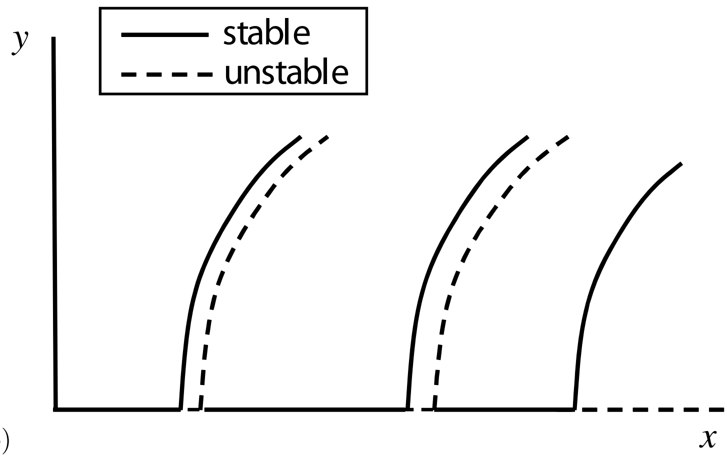
$$\omega_1=5,\ \omega_2=20,\ \alpha_0=1,\ \alpha_2=3,\ \alpha_4=12,\ \alpha_1=\alpha_3=0 \tag{5.9}$$

the stability curve begins from $(\delta,\epsilon)=(16,0)$ at $A=0$, sequentially passes through the third and the second unstable regions, and finally enters the first unstable region, as depicted in Figure 4. We shall be interested in approximating the coupled-mode response when the stability curve enters the third unstable region with $\beta^*(t)=\cos 3\omega t+\mathcal{O}(\epsilon)$. The bifurcation point is found as $(\omega_b,A_b)=(5.68,2.7)$. If the coupled mode is approximated by

$$x(t)=A\cos\omega t,\ y(t)=B\cos 3\omega t, \tag{5.10}$$



(a)



(b)

FIG. 4. Result of stability loss of the driving mode with the parameters given by Eq. (5.5): (a) stability loss and (b) bifurcation of coupled modes.

then the frequency response equations are given by

$$\begin{aligned} A(\omega_1^2 - \omega^2 + 3\alpha_0 A^2 + \alpha_2 B^2) &= F \\ B(\omega_2^2 - 9\omega^2 + \alpha_2 A^2 + 3\alpha_4 B^2) &= 0. \end{aligned} \tag{5.11}$$

The nonzero value of B is obtained when ω is slightly greater than ω_b, implying that the undamped coupled-mode response is formed. In damped systems, the functional form of the coupled-mode response is written as

$$x(t) = A\cos(\omega t + \theta_1),\ \ y(t) = B\cos(3\omega t + \theta_2). \tag{5.12}$$

Then, the frequency response equations are obtained as

$$A(\omega_1^2 - \omega^2 + 3\alpha_0 A^2 + \alpha_2 B^2) = F\cos\theta_1 \tag{5.13}$$

$$0 = c_1\omega A + F\sin\theta_1 \tag{5.14}$$

$$B(\omega_2^2 - 9\omega^2 + \alpha_2 A^2 + 3\alpha_4 B^2) = 0 \tag{5.15}$$

$$0 = 3c_2\omega B. \tag{5.16}$$

It is found from Eq. (5.16) that $B = 0$, implying that the damped coupled-mode response is not formed. It can be shown that if the stability curve enters the pth unstable region with $p = 2, 3, \ldots$ and if the functional form of the coupled-mode response is approximated by

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\cos(p\omega t + \theta_2) \tag{5.17}$$

or

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\sin(p\omega t + \theta_2), \tag{5.18}$$

then it is found that $B = 0$, implying that the functional form is not properly approximated. However, if the functional form of the coupled-mode response is approximated by

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\cos(p\omega t + \theta_2) + \mathcal{O}(\epsilon) \tag{5.19}$$

where $\mathcal{O}(\epsilon)$ is a harmonic term in $\mathcal{O}(\epsilon)$ of $\beta^*(t)$, such as $\cos[(p \pm 2)\omega t + \theta_3]$, then the damped coupled-mode response is formed (see, for example, Eq. (6.37) in Section 6.2).

Proposition 5.2. If the stability curve of the driving mode in simple systems enters the first unstable region, then the functional form of the stable coupled-mode response is, as the first-order approximation, written as

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\cos(\omega t + \theta_2) \tag{5.20}$$

or

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\sin(\omega t + \theta_2). \tag{5.21}$$

If the stability curve enters the pth unstable region with $p = 2, 3, \ldots$, then the functional form of the stable coupled-mode response is, as the first-order approximation, written as

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\cos(p\omega t + \theta_2) + \mathcal{O}(\epsilon) \tag{5.22}$$

or

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B\sin(p\omega t + \theta_2) + \mathcal{O}(\epsilon) \tag{5.23}$$

where $\mathcal{O}(\epsilon)$ denotes a harmonic term in $\mathcal{O}(\epsilon)$ of $\beta^*(t)$.

If a pair of characteristic exponents is imaginary in an undamped system, then the pair becomes complex conjugates with negative real part in a damped system. If a pair of exponents is real in an undamped system, then the pair remains real in a damped system. Therefore, an asymptotically stable coupled-mode response is formed if both pairs of exponents are imaginary in an undamped system.

Theorem 5.3. If the stability curve of the driving mode enters the pth unstable region with $p = 1, 2, \ldots$, then a stable damped coupled-mode response is formed. The response can be computed by using the harmonic balance method; the functional form of the response is, as the first-order approximation, given by Propositions 5.1 and 5.2.

THEOREM 5.4. If the stability curve of the driving mode passes through the pth unstable region with $p = 2, 3, \ldots$, and if the coupled-mode responses are expressed by the first-order approximation, then the frequency response curve of the stable coupled-mode response is overlapped with the curve of the unstable response. As the order of approximation increases, two curves are separated from each other.

Proof. Since the stability curve passes through the pth unstable region, two coupled-mode responses are formed: one is stable, but the other is unstable. Due to the property of overlap, two responses are expressed in terms of a set of modulation equations. Given ω, the same values of A, B, C, D, and θ's are obtained for two responses. For the remaining part of the proof, refer to the proof for Corollary 3.4. □

6. Examples.

6.1. 1:1 *internal resonance.* Assume that $\omega_1 \approx \omega_2$ and that the stability curve of the driving mode enters the first unstable region. Then, the coupled-mode response can be written as that in Eq. (5.20). Substitute into Eq. (1.3), and the harmonic balance method will yield

$$\begin{aligned}(\omega_1^2 - \omega^2)A + 3\alpha_0 A^3 + \frac{9\alpha_1}{4}A^2 B\cos\varphi \\ + \frac{\alpha_2}{2}AB^2(2+\cos 2\varphi) + \frac{3\alpha_3}{4}B^3\cos\varphi = F\cos\theta_1\end{aligned} \tag{6.1}$$

$$\frac{3\alpha_1}{4}A^2 B\sin\varphi + \frac{\alpha_2}{2}AB^2\sin 2\varphi + \frac{3\alpha_3}{4}B^3\sin\varphi = c_1\omega A + F\sin\theta_1 \tag{6.2}$$

$$\begin{aligned}(\omega_2^2 - \omega^2)B + \frac{3\alpha_1}{4}A^3\cos\varphi + \frac{\alpha_2}{2}A^2B(2+\cos 2\varphi) \\ + \frac{9\alpha_3}{4}AB^2\cos\varphi + 3\alpha_4 B^3 = 0\end{aligned} \tag{6.3}$$

$$\frac{3\alpha_1}{4}A^3\sin\varphi + \frac{\alpha_2}{2}A^2B\sin 2\varphi + \frac{3\alpha_3}{4}AB^2\sin\varphi = c_2\omega B \tag{6.4}$$

where $\varphi = \theta_2 - \theta_1$.

Assume that the motions are small. Then, the method of multiple scales is applicable. Let μ be a small parameter with $0 < \mu \ll 1$. Define $x = \mu^{1/2}\bar{x}$, $y = \mu^{1/2}\bar{y}$, $c_j = \mu\bar{c_j}$, $j = 1, 2$ and $F = \mu^{3/2}f$. Assume also that

$$\omega_2 = \omega_1 + \mu\sigma_1, \ \omega = \omega_1 + \mu\sigma_2. \tag{6.5}$$

Then, Eq. (1.3) can be rewritten as

$$\begin{aligned}\ddot{\bar{x}} + \omega_1^2\bar{x} &= -\mu\left[N_x(\bar{x},\bar{y}) + \bar{c}_1\dot{\bar{x}} - f\cos\omega t\right] \\ \ddot{\bar{y}} + \omega_2^2\bar{y} &= -\mu\left[N_y(\bar{x},\bar{y}) + \bar{c}_2\dot{\bar{y}}\right].\end{aligned} \tag{6.6}$$

The solutions are written as

$$\begin{aligned}\bar{x}(t,\mu) &= x_0(T_0,T_1) + \mu x_1(T_0,T_1) + \mathcal{O}(\mu^2) \\ \bar{y}(t,\mu) &= y_0(T_0,T_1) + \mu y_1(T_0,T_1) + \mathcal{O}(\mu^2)\end{aligned} \tag{6.7}$$

where $T_0 = t$ and $T_1 = \mu t$. Through the usual procedures [28], the approximate solutions can be obtained as

$$\begin{aligned} x_0(T_0, T_1) &= a(T_1)\cos(\omega_1 T_0 + \gamma_1(T_1)) \\ y_0(T_0, T_1) &= b(T_1)\cos(\omega_2 T_0 + \gamma_2(T_1)). \end{aligned} \tag{6.8}$$

The modulation equations for the amplitudes and phases can be written as

$$\begin{aligned} \omega_1 a' = -\frac{1}{2}\omega_1 \bar{c}_1 a + \frac{3}{8}\alpha_1 a^2 b \sin\varphi_1 + \frac{1}{4}\alpha_2 a b^2 \sin 2\varphi_1 \\ + \frac{3}{8}\alpha_3 b^3 \sin\varphi_1 + \frac{1}{2} f \sin\varphi_2 \end{aligned} \tag{6.9}$$

$$\begin{aligned} \omega_2 b' = -\frac{\omega_2}{2}\bar{c}_2 b + \frac{3}{8}\alpha_1 a^3 \sin\varphi_1 + \frac{1}{4}\alpha_2 a^2 b \sin 2\varphi_1 \\ + \frac{3}{8}\alpha_3 a b^2 \sin\varphi_1 \end{aligned} \tag{6.10}$$

$$\begin{aligned} \omega_1 a \gamma_1' = \frac{3}{2}\alpha_0 a^3 + \frac{9}{8}\alpha_1 a^2 b \cos\varphi_1 + \frac{1}{4}\alpha_2 a b^2 (2 + \sin 2\varphi_1) \\ + \frac{3}{8}\alpha_3 b^3 \cos\varphi_1 - \frac{1}{2} f \sin\varphi_2 \end{aligned} \tag{6.11}$$

$$\begin{aligned} \omega_2 b \gamma_2' = \frac{3}{8}\alpha_1 a^3 \cos\varphi_1 + \frac{1}{4}\alpha_2 a^2 b (2 + \sin 2\varphi_1) \\ + \frac{9}{8}\alpha_3 a b^2 \cos\varphi_1 + \frac{3}{2}\alpha_4 b^3 \end{aligned} \tag{6.12}$$

where $\varphi_1 = \gamma_2 - \gamma_1 + \sigma_1 T_1$, $\varphi_2 = \sigma_2 T_1 - \gamma_1$, and the prime denotes the derivative with respect to T_1. The above equations can be transformed into an autonomous system by defining

$$\varphi_1' = \gamma_2' - \gamma_1' + \sigma_1, \ \ \varphi_2' = \sigma_2 - \gamma_1'. \tag{6.13}$$

Imposing the steady-state conditions

$$a' = b' = \varphi_1' = \varphi_2' = 0 \tag{6.14}$$

we approximate the steady-state solutions as

$$\begin{aligned} x_0(t) &= \hat{a}\cos(\omega t - \hat{\varphi}_2) \\ y_0(t) &= \hat{b}\cos(\omega t - \hat{\varphi}_2 + \hat{\varphi}_1) \end{aligned} \tag{6.15}$$

where the hat denotes the steady-state value for the amplitudes and phase differences determined from Eqs. (6.9)–(6.14).

To obtain the coupled-mode solution, the single-mode response is firstly computed. By increasing the amplitude from zero, the stability of the single-mode response is analyzed to find a bifurcation point and to determine the type of bifurcation, such as saddle-node, pitchfork, etc., so that the form of the coupled-mode response may be found.

The steady-state solution of the coupled-mode response obtained by the proposed method is compared with that by the method of multiple scales. It is found that they are agreeable. In fact, Eq. (6.1) agrees in $\mathcal{O}(\mu)$ with the condition $\varphi_1' = 0$, Eq. (6.2) in $\mathcal{O}(\mu)$ with $a' = 0$, Eq. (6.3) in $\mathcal{O}(\mu)$ with $\varphi_1' - \varphi_2' = 0$, and Eq. (6.4) in $\mathcal{O}(\mu)$ with $b' = 0$.

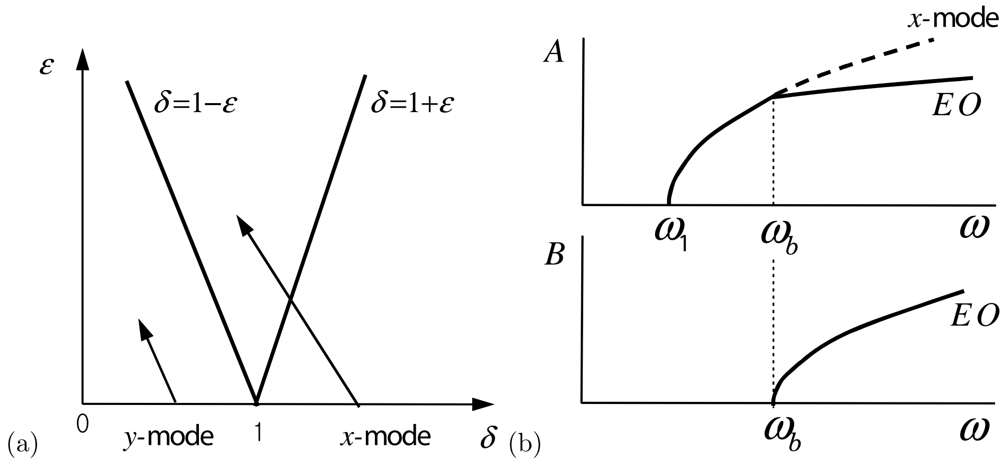


FIG. 5. Result of stability loss of the driving mode with the parameters given by Eq. (6.16): (a) stability chart and (b) backbone curves.

To demonstrate the procedure of the proposed method, we consider a simple system ($\alpha_1 = \alpha_3 = 0$) with parameters

$$\omega_1 = 5,\ \omega_2 = 7,\ \alpha_0 = 1,\ \alpha_2 = 3,\ \alpha_4 = 4. \tag{6.16}$$

The driving mode can be written as in Eq. (2.6),

$$y(t) \equiv 0,\ x(t) = A\cos\omega t \tag{6.17}$$

where $\omega^2 = \omega_1^2 + 3\alpha_0 A^2$.

The stability is analyzed by the procedure in Section 5. It is found that the stability curve enters the first unstable region through the transition curve, $\delta = 1+\epsilon+\mathcal{O}(\epsilon^2)$ with $\beta^*(t) = \sin\omega t + \mathcal{O}(\epsilon)$ as depicted in Figure 5(a). Thus, a stable coupled mode bifurcates off the driving mode in the form

$$x(t) = A\cos\omega t,\ y(t) = B\sin\omega t. \tag{6.18}$$

The corresponding modal curve is an elliptic orbit (EO) in the configuration space. To compute the solution of the coupled mode, the harmonic balance method is used to obtain

$$\begin{aligned} A(\omega_1^2 - \omega^2 + 3\alpha_0 A^2 + \tfrac{1}{2}\alpha_2 B^2) = 0 \\ B(\omega_2^2 - \omega^2 + \tfrac{1}{2}\alpha_2 A^2 + 3\alpha_4 B^2) = 0. \end{aligned} \tag{6.19}$$

Eliminate ω to obtain

$$B^2 = \frac{1}{7}\left(A^2 - 16\right), \tag{6.20}$$

and the effective nonlinear natural frequency of the EO is

$$\omega^2 = 25 + 3A^2 + 1.5B^2 = 49 + 1.5A^2 + 12B^2. \tag{6.21}$$

The bifurcation point is $(\omega_b, A_b) = (8.54, 4)$, and the backbone curve is shown in Figure 5(b).

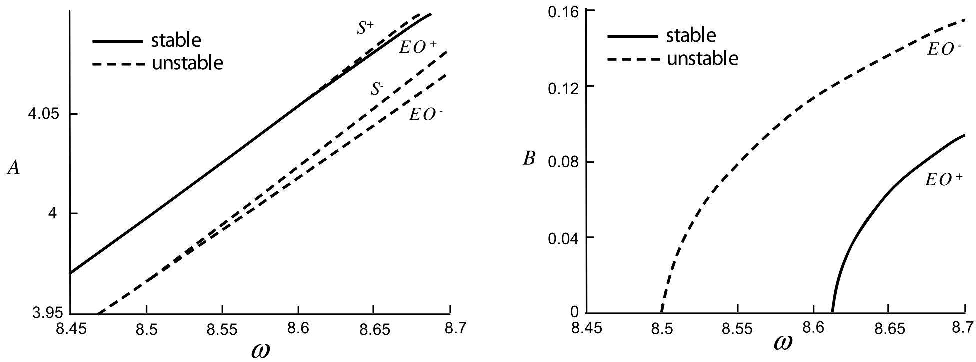


FIG. 6. Bifurcation of coupled-mode responses in the system of Fig. 5 when $F = 2$ and $c_1 = c_2 = 0$.

When a harmonic excitation is applied, the frequency response equations of single-mode response are

$$y(t) \equiv 0,\ x(t) = A\cos\omega t,\ A(\omega_1^2 - \omega^2 + 3\alpha_0 A^2) = F. \tag{6.22}$$

There are two branches of frequency response curve: stable S^+ and unstable S^-. Thus, the second pair of exponents are imaginary in S^+ and real in S^-. To obtain the first pair the disturbance η_1 in used. Writing $\eta_1 = y(t)$, we obtain the Mathieu equation (2.13) with $\tau = \omega t$ and

$$\epsilon = \frac{\alpha_2 A^2}{2\omega^2},\ \delta = \frac{\omega_2^2}{\omega^2} + 2\epsilon. \tag{6.23}$$

Since for given ω the values of A in S^+ and S^- are close to the backbone curve of the driving mode, the stability curves of S^+ and S^- are close to that of the driving mode. Therefore, the first pair of exponents in S^+ or S^- are close to that obtained by β of Synge's concept. The frequency response equations of the coupled-mode responses are written as

$$\begin{aligned} A(\omega_1^2 - \omega^2 + 3\alpha_0 A^2 + \tfrac{1}{2}\alpha_2 B^2) &= F \\ B(\omega_2^2 - \omega^2 + \tfrac{1}{2}\alpha_2 A^2 + 3\alpha_4 B^2) &= 0. \end{aligned} \tag{6.24}$$

When $F = 2$, the stable coupled-mode response (EO+) bifurcates off S^+ at $(\omega_b', A_b') = (8.61, 4.07)$, and the unstable coupled-mode response (EO-), bifurcates off S^- at $(\omega_b', A_b') = (8.5, 3.94)$, as depicted in Figure 6.

6.2. 3:1 *internal resonance.* Assume that $\omega_2 \approx 3\omega_1$ and that the motions are small. Then, the method of multiple scales is applicable. Let μ be a small parameter with $0 < \mu \ll 1$. Define $x = \mu^{1/2}\bar{x}$, $y = \mu^{1/2}\bar{y}$, $c_j = \mu\bar{c_j}$, $j = 1, 2$ and $F = \mu^{3/2} f$. Assume that

$$\omega_2 = 3\omega_1 + \mu\sigma_1,\ \omega = \omega_1 + \mu\sigma_2. \tag{6.25}$$

The solutions are written as

$$\begin{aligned} \bar{x}(t,\mu) &= x_0(T_0, T_1) + \mu x_1(T_0, T_1) + \mathcal{O}(\mu^2) \\ \bar{y}(t,\mu) &= y_0(T_0, T_1) + \mu y_1(T_0, T_1) + \mathcal{O}(\mu^2) \end{aligned} \tag{6.26}$$

where $T_0 = t$ and $T_1 = \mu t$. Through the usual procedures [28], the approximate solutions can be obtained as

$$\begin{aligned} x_0(T_0, T_1) &= a(T_1)\cos(\omega_1 T_0 + \gamma_1(T_1)) \\ y_0(T_0, T_1) &= b(T_1)\cos(\omega_2 T_0 + \gamma_2(T_1)). \end{aligned} \tag{6.27}$$

The modulation equations for the amplitudes and phases can be written as

$$\omega_1 a' = -\bar{c}_1\omega_1 a - \frac{3\alpha_1}{8}a^2 b\sin\varphi_1 - \frac{f}{2}\sin\varphi_2 \tag{6.28}$$

$$\omega_2 b' = -\bar{c}_2\omega_2 b + \frac{\alpha_1}{8}a^3\sin\varphi_1 \tag{6.29}$$

$$\omega_1 a\gamma_1' = \frac{3\alpha_0}{2}a^3 + \frac{3\alpha_1}{8}a^2 b\cos\varphi_1 + \frac{\alpha_2}{2}ab^2 - \frac{f}{2}\cos\varphi_2 \tag{6.30}$$

$$\omega_2 b\gamma_2' = \frac{\alpha_1}{8}a^3\cos\varphi_1 + \frac{\alpha_2}{2}a^2 b + \frac{3\alpha_4}{2}b^3 \tag{6.31}$$

where $\varphi_1 = \gamma_2 - 3\gamma_1 + \sigma_1 T_1$, $\varphi_2 = \sigma_2 T_1 - \gamma_1$, and the prime denotes the derivative with respect to T_1. The above equations can be transformed into an autonomous system by defining

$$\varphi_1' = \gamma_2' - 3\gamma_1' + \sigma_1, \;\; \varphi_2' = \sigma_2 - \gamma_1'. \tag{6.32}$$

The steady-state coupled-mode responses can be obtained by the conditions, $a' = b' = \varphi_1' = \varphi_2' = 0$, such that

$$\begin{aligned} x_0(t) &= \hat{a}\cos(\omega t - \hat{\varphi}_2) \\ y_0(t) &= \hat{b}\cos(3\omega t - 3\hat{\varphi}_2 + \hat{\varphi}_1) \end{aligned} \tag{6.33}$$

where the hats denote the steady-state solutions for Eqs. (6.28)–(6.31).

To compute the coupled-mode response by using the proposed method, we assume that the stability curve of the driving mode enters the third unstable region. Then, it is found by Eq. (5.3) or (5.4) that the coupled-mode response is written in the form of the two-term approximation

$$\begin{aligned} x(t) &= A\cos(\omega t + \theta_1) + C\cos(3\omega t + \theta_3) \\ y(t) &= B\cos(\omega t + \theta_2) + D\cos(3\omega t + \theta_4), \end{aligned} \tag{6.34}$$

which turns out to contradict the one-term approximation given by Eq. (6.33).

It is found by the condition $b' = 0$ in Eq. (6.29) that if $\alpha_1 = 0$, then $b = 0$, which implies that a coupled-mode response cannot be formed in a simple system. On the contrary, it can be shown by using the proposed method that a coupled-response is formed in a simple system. For example, we consider the system with the parameters given by Eq. (5.9). The stability curve enters the third unstable region, as depicted in Figure 4, and the transition curves and eigenfunctions can be expressed as

$$\delta = 9 + \frac{\epsilon^2}{16} - \frac{\epsilon^3}{64} + \ldots; \quad \beta^*(t) = \cos 3\omega t - \tfrac{\epsilon}{8}\left(\cos\omega t - \cos 5\omega t\right) + \ldots \tag{6.35}$$

$$\delta = 9 + \frac{\epsilon^2}{16} + \frac{\epsilon^3}{64} + \ldots; \quad \beta^*(t) = \sin 3\omega t - \tfrac{\epsilon}{8}\left(\sin\omega t - \sin 5\omega t\right) + \ldots. \tag{6.36}$$

Then, the coupled-mode response, due to Eq. (5.19), is written as

$$x(t) = A\cos(\omega t + \theta_1), \;\; y(t) = B_1\cos(\omega t + \theta_2) + B_2\cos(3\omega t + \theta_3). \tag{6.37}$$

Applying the harmonic balance method yields

$$(\omega_1^2-\omega^2)A+3\alpha_0A^3+\frac{\alpha_2}{2}AB_1^2(2+\cos\varphi_2) \\ +\alpha_2AB_2^2+\alpha_2AB_1B_2\cos\varphi_1=F\cos\theta_1 \tag{6.38}$$

$$\alpha_2A(\frac{1}{4}B_1^2\sin\varphi_2+\frac{1}{2}B_1B_2\sin\varphi_1)=c_1\omega A+F\sin\theta_1 \tag{6.39}$$

$$(\omega_2^2-\omega^2)B_1+\alpha_2A^2\left(\frac{1}{2}B_1(2+\cos\varphi_2)+B_2\cos\varphi_1\right) \\ +\alpha_4(3B_1^3+2B_1^2B_2\cos\varphi_3+6B_1B_2^2)=0 \tag{6.40}$$

$$-\frac{\alpha_2}{2}A^2(B_1\sin\varphi_2+B_2\sin\varphi_1)+3\alpha_4B_1B_2^2=c_2\omega B_1 \tag{6.41}$$

$$(\omega_2^2-9\omega^2)B_2+\alpha_2A^2(B_2+B_1\cos\varphi_1)+\alpha_4B_1B_2^2\cos\varphi_3 \\ +6\alpha_4B_1^2B_2+3\alpha_4B_2^3=0 \tag{6.42}$$

$$-\frac{\alpha_2}{2}A^2B_1\sin\varphi_1-\alpha_4B_1^3\sin\varphi_3=3c_2\omega B_2 \tag{6.43}$$

where $\varphi_1=2\theta_1+\theta_2-\theta_3$, $\varphi_2=2\theta_1-2\theta_2$ and $\varphi_3=3\theta_2-\theta_3$. It is found by Eqs. (6.41) and (6.43) that B_1 and B_2 are not identically zero, in contrast to the condition $b'=0$ in Eq. (6.29), where $\alpha_1=0$. Since six equations in Eqs. (6.38)–(6.43) are independent, six variables, $A,\ B_1,\ B_2,\ \theta_1,\ \theta_2$ and θ_3, can be solved so that the coupled-mode response may be formed.

It is observed that if the stability curve of the driving mode intersects the transition curve given by Eq. (6.35), a coupled mode is generated in the form of Eq. (2.3), giving rise to a bifurcating nonlinear normal mode. On the other hand, a coupled mode is also obtained in the form

$$x_0(t)=a\cos(\omega t),\ y_0(t)=b\cos(3\omega t) \tag{6.44}$$

as a steady-state solution when the method of multiple scales is used for $F=c_1=c_2=0$. Eliminating t, we obtain

$$y_0=k_1x_0+k_3x_0^3,\ k_3=-\frac{4}{3a^2}k_1. \tag{6.45}$$

This form is the first-order approximation of the bifurcating nonlinear normal mode. A procedure is formulated by Vakakis et al. [6] to compute the second-order or higher approximation based on the form (6.45).

6.3. *Conditions away from internal resonances.* Some examples are given to demonstrate that the coupled-mode responses are formed in systems which are neither 1:1 nor 3:1 internally resonant.

6.3.1. *Thin elastica (zeroth unstable region).* When a thin elastica is twisted and bent simultaneously, a two-mode model is derived theoretically and verified experimentally [8, 9]. By twisting the clamped end sinusoidally, periodic and chaotic motions are observed. The system is simple, and the kinetic energy T and potential energy V are written as

$$T=\frac{1}{2}(1+\alpha y^2)\dot{x}^2+\frac{1}{2}\dot{y}^2,\ V=\frac{1}{2}(\Omega^2x^2+y^2),\ T+V=h \tag{6.46}$$

where x and y are the torsional and the bending modes, respectively. Assume that $\Omega = 20$ and $\alpha = 0.1$. When the system is forced and damped, the equations of motion become

$$\begin{aligned}&(1+\alpha y^2)\ddot{x} + 2\alpha\dot{x}y\dot{y} + \Omega^2 x = -c_1\dot{x} + F\cos\omega t\\&\ddot{y} - \alpha\dot{x}^2 y + y = -c_2 y.\end{aligned} \tag{6.47}$$

The solution of the driving mode is

$$y(t) \equiv 0,\ x(t) = A\cos\Omega t. \tag{6.48}$$

To analyze stability of the driving mode, Synge's concept [22, 23] is used. It is found that $K = 0$, $\kappa = 0$, $n_1 = 0$, and $n_2 = 1$. Then, the Mathieu equation (2.13) is obtained with

$$\tau = \Omega t,\ \epsilon = \frac{\alpha}{4}A^2,\ \delta = \frac{1}{\Omega^2} - 2\epsilon. \tag{6.49}$$

As depicted in Figure 7(a), the stability curve enters the zeroth unstable region. The bifurcation point is found at $(\omega_b, A_b) = (20, 0.22)$. The corresponding eigenfunction is

$$\beta^*(t) = 1 + \frac{\epsilon}{2}\cos 2\Omega t + \mathcal{O}(\epsilon^2). \tag{6.50}$$

The stable coupled mode can be approximated by

$$x(t) = A\cos\omega t,\ y(t) = B_0 + B_1\cos 2\omega t \tag{6.51}$$

where $|B_0| \gg |B_1|$. By using the harmonic balance method, the solution of coupled mode is obtained, and the backbone curve is depicted in Figure 7(b).

When $F\cos\omega t$ is added, the single-mode response is written as

$$y(t) \equiv 0,\ x(t) = A\cos\omega t,\ A = \frac{F}{\Omega^2 - \omega^2}. \tag{6.52}$$

Stability of the response is analyzed by using the disturbances η_1 and η_2. Write $\eta_1(t) = y(t)$. Then, the Mathieu equation (2.13) is obtained with

$$\tau = \Omega t,\ \epsilon = \frac{\alpha}{2}A^2,\ \delta = \frac{1}{\omega^2} - 2\epsilon. \tag{6.53}$$

Bifurcation of the undamped coupled-mode response is computed. For instance, when $F = 10$, we compute $(\omega_b', A_b') = (18.9, 0.24)$, as depicted in Figure 8. To compute the second pair of exponents, we write $x(t) = A\cos\omega t + \eta_2(t)$. Then, we derive

$$\ddot{\eta}_2 + \Omega^2\eta_2 = 0 \tag{6.54}$$

which leads to $\rho_3 = -\rho_4 = i\Omega$.

When damping terms are added, the coupled-mode response becomes

$$x(t) = A\cos(\omega t + \theta_1),\ y(t) = B_0 + B_1\cos(2\omega t + \theta_2). \tag{6.55}$$

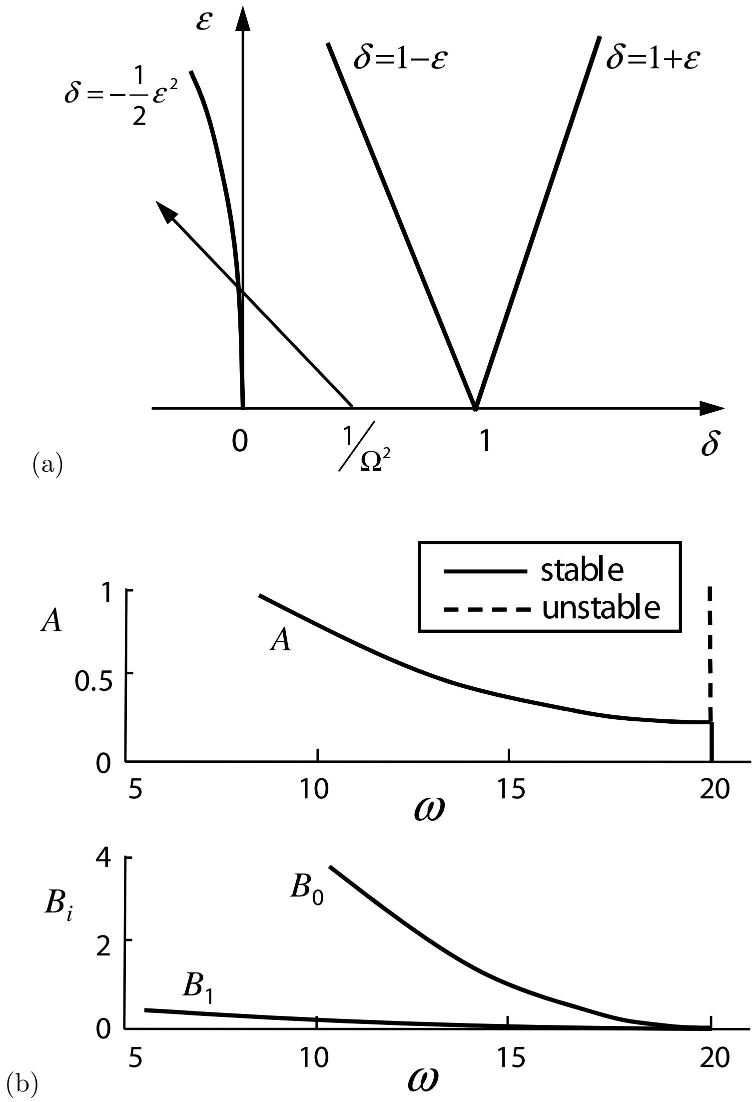


FIG. 7. Result of stability loss of the driving mode in the thin elastica system: (a) stability chart and (b) backbone curves of the coupled mode.

Using the harmonic balance method, we derive

$$\left[\Omega^2 - \omega^2 - \alpha\omega^2(B_0^2 + \frac{1}{2}B_1^2 - B_0 B_1 \cos\varphi)\right] A = F\cos\theta_1 \tag{6.56}$$

$$-\alpha\omega^2 A B_0 B_1 = F\sin\theta_1 + c_1\omega A \tag{6.57}$$

$$B_0 - \frac{1}{2}\alpha A^2\omega^2(B_0 - \frac{1}{2}B_1\cos\varphi) = 0 \tag{6.58}$$

$$(1 - 4\omega^2)B_1 + \frac{1}{2}\alpha A^2\omega^2(B_0\cos\varphi - B_1) = 0 \tag{6.59}$$

$$\frac{\alpha}{2}\omega^2 A^2 B_0 \sin\varphi = -2c_2\omega B_1 \tag{6.60}$$

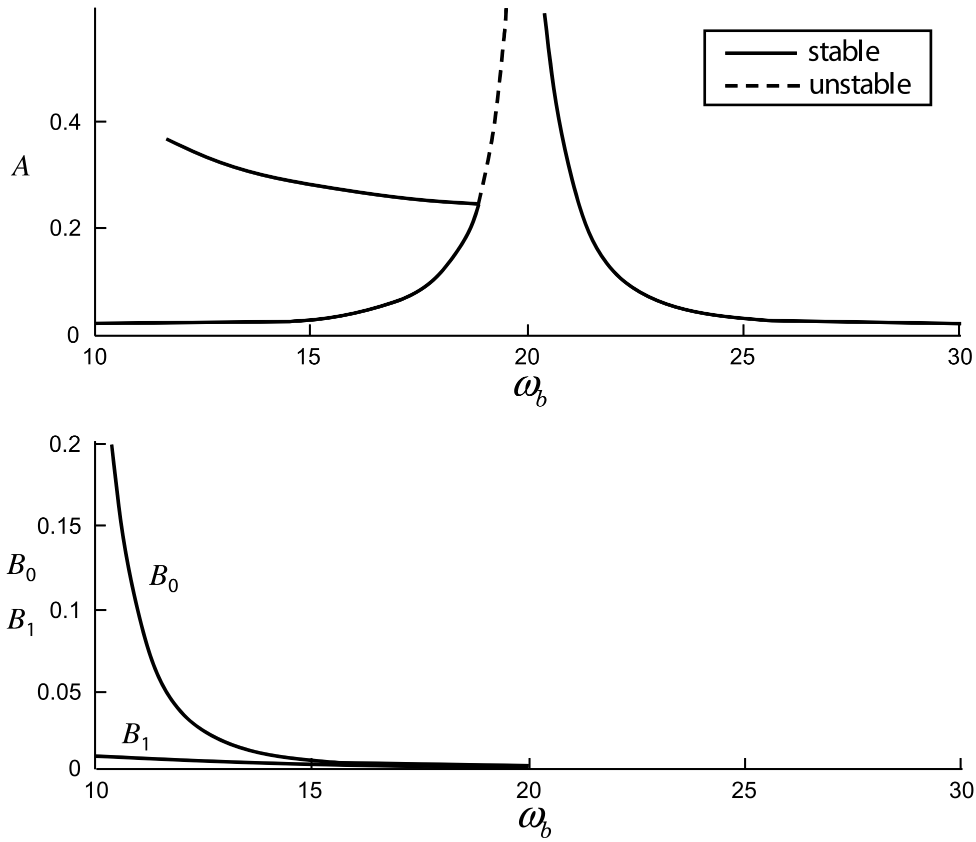


FIG. 8. Bifurcation of coupled-mode responses in the thin elastica when $F = 10$ and $c_1 = c_2 = 0$.

where $\varphi = \theta_2 - 2\theta_1$. It is found by Eq. (6.60) that B_0 and B_1 are not identically zero, in contrast to Eq. (6.29) when $\alpha_1 = 0$. Since the five equations in (6.56)–(6.60) are independent, five unknown variables, A, B_0, B_1, θ_1 and θ_2, can be computed to solve for the coupled-mode response.

6.3.2. *Successive losses of stability.* As depicted in Figure 4, the stability curve of the driving mode passes through the third and second unstable regions, finally entering the first unstable region. It is observed that the phenomena of 3:1, 2:1 and 1:1 internal resonances are successively founded as the amplitude of the driving mode increases.

In a nonlinear system of coupled oscillators [15–17, 29], the stability curves of fundamental nonlinear normal modes pass through infinitely many unstable regions in the underlying conservative system. It is expected due to damping that a finite number of $p : 1$ internal resonances are formed.

7. Summary and discussions. The results in this work can be summarized in the following.

a. It is shown that if the driving mode loses stability in the underlying conservative system, then a stable coupled-mode response is formed in the damped forced system.

b. An analytical procedure is, as the first-order approximation, formulated to compute the coupled-mode response. If the stability curve of the driving mode enters the first unstable region, then the stable coupled-mode response is expressed in the form of one-term approximation. If the stability curve enters the pth unstable region with $p = 2, 3, \ldots$, then the stable coupled-mode response is expressed in the form of two-term approximation (i.e., each coordinate is written by two dominant harmonic terms).

c. If the stability curve of the driving mode passes through the pth unstable region with $p = 2, 3, \ldots$ in general systems, and if the coupled-mode responses are expressed in the form of the first-order approximation, then the frequency response curve of the stable coupled-mode response is overlapped with the curve of the unstable response. As the order of approximation increases, two curves are separated from each other.

d. The proposed method is compared with other perturbation techniques in the systems with 1:1 and 3:1 internal resonances. By using perturbation techniques, the coupled-mode response is expressed in the form of one-term approximation. Both methods result in a good agreement between approximation for the case of 1:1 internal resonance, but not for the case of 3:1 internal resonance. Necessity of the second- or higher-order approximation of the latter case is discussed.

8. Appendix A. Justification of Eq. (2.7). It will be demonstrated that a one-term approximation of the fundamental nonlinear normal modes, given by Eq. (2.7), is reasonable. To design a device called a nonlinear energy sink (see, for example, [30] for more information), the stable periodic motions are sought in special types of nonlinear mass-spring systems where kinetic energy T and potential energy V are expressed as, in System 1,

$$T = \frac{1}{2}(\dot{x}^2 + \mu\dot{y}^2),\ V = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{2}\epsilon(x-y)^2 + \frac{1}{4}\kappa y^4 \tag{8.1}$$

where $\mu = 1$, $\omega_0 = 1$, $\epsilon = 0.1$ and $\kappa = 1$; and, in System 2,

$$T = \frac{1}{2}(\dot{x}^2 + \mu\dot{y}^2),\ V = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\kappa(x-y)^4 \tag{8.2}$$

where $\mu = 0.05$, $\omega_0 = 1$ and $\kappa = 1$ [29].

Two linearized natural frequencies are $\omega_1 = 0.3$ and $\omega_2 = 1.054$ in System 1, and $\omega_1 = 0$ and $\omega_2 = 1$ in System 2. By using the linear orthogonal transformation of coordinate system, T and V are written in the form of Eq. (1.1). Following the procedures provided in Section 2, the stability curves of fundamental nonlinear normal modes can be constructed. It is found that the stability curve passes through infinitely many unstable regions and that infinitely many kinds of coupled modes are formed. The bifurcation points and the geometrical forms of coupled modes are reasonably agreeable with those obtained earlier by using numerical methods in a wide range of frequency and total energy [29].

9. Appendix B. Proof of Lemma 4.3. By utilizing the stability behavior of the motions in free vibrations, the stability of the forced responses will be derived in an undamped system. The driving mode is considered as an oscillation where a mass point is moving along the modal curve in the configuration space. Given the amplitude R of motion, the effective nonlinear natural frequency ω is given by Eq. (2.10). Since the period of motions depends on R, the motion is orbitally stable but parabolically unstable in Liapunov's sense.

The equation of motion can be written in the form

$$\ddot{z} + f(z) = 0, \ f(z) = \omega_1^2 z + k_3 z^3 + k_5 z^5 + \cdots . \tag{9.1}$$

Then, the solution can be approximated as

$$z(t) = R\cos\omega t \tag{9.2}$$

where ω is given by Eq. (2.12). To analyze the stability, we introduce a small perturbation such that $z(t) = R\cos\omega t + \eta_2(t)$ into Eq. (9.1). Then, Hill's equation [31] is obtained:

$$\eta_2'' + \left(\delta + \sum_{i=1} \epsilon^i \sum_{j=1} b_{ij} \cos 2j\tau\right)\eta_2 = 0 \tag{9.3}$$

where the differentiation is with respect to a nondimensional time $\tau = \omega t$ and

$$\epsilon = \frac{R^2}{\omega^2}, \ \delta = \frac{\omega_1^2}{\omega^2} + \sum_{j=1} aj\epsilon^j. \tag{9.4}$$

Since the set of parabolic instability is the totality of transition curves in the stability chart, and since it is found that $\delta = 1$ when $\epsilon = 0$, the stability curve of the driving mode lies on a transition curve emanating from $(\delta, \epsilon) = (1, 0)$. Therefore, the backbone curve of the driving mode corresponds to that transition curve.

When a small force is applied, the response is written as Eq. (9.2), and two branches, $S^{\pm}$, of the frequency response curve lie closely on the right- and left-hand sides of the backbone curve, respectively. To analyze the stability of forced responses, write $z(t) = R\cos\omega t + \eta_2(t)$. Then, Hill's equation (9.3) is obtained in which R and ω are taken from $S^{\pm}$. This implies that the stability curves of $S^{\pm}$ lie closely on the right- and left-hand sides of that transition curve. Therefore, the second pair of exponents is imaginary in one branch and real in the other branch. This property persists for large R (unlimited). Since $N(x, y)$ is positive definite, ω^2 in Eq. (2.12) is positive, and, moreover, $\nabla V \neq 0$ except at the origin of the configuration space. Thus, ϵ and δ are well defined for all R.

References

[1] A. Liapounoff, *Problème général de la stabilité du mouvement* (French), Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2) **9** (1907), 203–474. MR1508297

[2] Alan Weinstein, *Normal modes for nonlinear Hamiltonian systems*, Invent. Math. **20** (1973), 47–57. MR0328222 (48 #6564)

[3] V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Mathematical Series, No. 22, Princeton University Press, Princeton, N.J., 1960. MR0121520 (22 #12258)

[4] R. M. Rosenberg, *On nonlinear vibrations of systems with many degrees of freedom*, Adv. Appl. Mech. **9** (1966), 155–242.

[5] R. H. Rand, *A higher-order approximation for nonlinear normal modes in two-degree-of-freedom systems*, Int. J. Non-linear Mech. **6** (1971), 545–547.

[6] Alexander F. Vakakis, Leonid I. Manevitch, Yuri V. Mikhlin, Valery N. Pilipchuk, and Alexandr A. Zevin, *Normal modes and localization in nonlinear systems*, Wiley Series in Nonlinear Science, John Wiley & Sons, Inc., New York, 1996. MR1397270 (97i:70038)

[7] S. W. Shaw and C. Pierre, *Normal modes for nonlinear vibratory systems*, J. Sound Vibration **164** (1993), no. 1, 85–124, DOI 10.1006/jsvi.1993.1198. MR1231018 (94e:70019)

[8] J. P. Cusumano and F. C. Moon, *Chaotic non-planar vibrations of thin elastica: Part I: Experimental observation of planar instability*, J. Sound Vib. **179** (1995), 185–208.

[9] J. P. Cusumano and F. C. Moon, *Chaotic non-planar vibrations of thin elastica: Part II: Derivation and analysis of a low-dimensional model*, J. Sound Vib. **179** (1995), 209–226.

[10] Ali H. Nayfeh, *Nonlinear interactions*, Analytical, computational, and experimental methods, Wiley Series in Nonlinear Science, Wiley-Interscience [John Wiley & Sons], New York, 2000. MR1889696 (2003c:34002)

[11] A. F. Vakakis and R. H. Rand, *Normal modes and global dynamics of a two-degree-of-freedom nonlinear system. I. Low energies*, Internat. J. Non-Linear Mech. **27** (1992), no. 5, 861–874, DOI 10.1016/0020-7462(92)90040-E. MR1183125 (93i:70026)

[12] A. F. Vakakis and R. H. Rand, *Normal modes and global dynamics of a two-degree-of-freedom nonlinear system. II. High energies*, Internat. J. Non-Linear Mech. **27** (1992), no. 5, 875–888, DOI 10.1016/0020-7462(92)90041-5. MR1183126 (93i:70027)

[13] Fengxia Wang, Anil K. Bajaj, and Keisuke Kamiya, *Nonlinear normal modes and their bifurcations for an inertially coupled nonlinear conservative system*, Nonlinear Dynam. **42** (2005), no. 3, 233–265, DOI 10.1007/s11071-005-3582-5. MR2186329 (2006f:70048)

[14] Young Sup Lee, Gaetan Kerschen, Alexander F. Vakakis, Panagiotis Panagopoulos, Lawrence Bergman, and D. Michael McFarland, *Complicated dynamics of a linear oscillator with a light, essentially nonlinear attachment*, Phys. D **204** (2005), no. 1-2, 41–69, DOI 10.1016/j.physd.2005.03.014. MR2147471 (2005m:70079)

[15] G. Kerschen, M. Peeter, J. C. Golinval, and A. F. Vakakis, *Nonlinear normal modes. Part I: A useful framework for the structural dynamist*, Mech. Sys. Sig. Processing **23** (2009), 170–194.

[16] M. Peeter, R. Viguie, G. Serandour, G. Kerschen, and J. C. Golinval, *Nonlinear normal modes. Part II: Toward a practical computation using numerical continuation techniques*, Mech. Sys. Sig. Processing **23** (2009), 195–216.

[17] G. Serandour, M. Peeter, G. Kerschen, and J. C. Golinval, *Computation of nonlinear normal modes, Part II: Numerical continuation in AUTO*, ENOC-2008, St. Peterberg, Russia, June 30–July 4, 2008.

[18] D. Jiang, C. Pierre, and S. W. Shaw, *The construction of nonlinear normal modes for systems with internal resonance*, Int. J. Non-linear Mech. **40** (2005), 729–746.

[19] A. H. Nayfeh, C. M. Chin, and S. A. Nayfeh, *On nonlinear normal modes of system with internal resonance*, Trans. ASME, J. Vib. Acoustics **118** (1996), 340–345.

[20] Xinye Li, Yushu Chen, and Zhiqiang Wu, *Non-linear normal modes and their bifurcation of a class of systems with three double of pure imaginary roots and dual internal resonances*, Internat. J. Non-Linear Mech. **39** (2004), no. 2, 189–199, DOI 10.1016/S0020-7462(02)00124-5. MR2009250 (2004h:70029)

[21] Ali H. Nayfeh, Walter Lacarbonara, and Char-Ming Chin, *Nonlinear normal modes of buckled beams: three-to-one and one-to-one internal resonances*, Nonlinear Dynam. **18** (1999), no. 3, 253–273, DOI 10.1023/A:1008389024738. MR1686099

[22] J. L. Synge, *On the geometry of dynamics*, Phil. Trans. Roy. Soc. London, Series A **226** (1926), 33–106.

[23] R. H. Rand, *The geometrical stability of nonlinear normal modes in two degree-of-freedom systems*, Int. J. Non-Linear Mech. **8** (1973), 161–168.

[24] C. H. Pak, *Synge's concept of stability applied to non-linear normal modes*, Internat. J. Non-Linear Mech. **41** (2006), no. 5, 657–664, DOI 10.1016/j.ijnonlinmec.2006.01.007. MR2245744 (2007i:70030)

[25] C. H. Pak, *On the coupling of non-linear normal modes*, Internat. J. Non-Linear Mech. **41** (2006), no. 5, 716–725, DOI 10.1016/j.ijnonlinmec.2006.04.003. MR2245749 (2007h:70030)

[26] T. L. Johnson and R. H. Rand, *On the existence and bifurcation of minimal normal modes* (English, with French and German summaries), Internat. J. Non-Linear Mech. **14** (1979), no. 1, 1–12. MR533419 (80e:70028)

[27] C. H. Pak, *On the stability behavior of bifurcated normal modes in coupled nonlinear systems*, Trans. ASME J. Appl. Mech. **56** (1989), no. 1, 155–161, DOI 10.1115/1.3176037. MR991105 (90b:70031)

[28] Ali Hasan Nayfeh and Dean T. Mook, *Nonlinear oscillations*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1979. MR549322 (80m:70002)

[29] Y. S. Lee and H. Chen, *Bifurcation of Nonlinear Normal Modes by Means of Synge's Stability (DETC2011-48690)*, ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, Washington, DC, August 28–31, 2011.

[30] A. F. Vakakis, O. Gendelman, L. A. Bergman, D. M. McFarland, G. Kerschen, and Y. S. Lee, *Passive Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems: I and II*, Springer-Verlag: Berlin and New York, 2008.

[31] Wilhelm Magnus and Stanley Winkler, *Hill's equation*, Dover Publications, Inc., New York, 1979. Corrected reprint of the 1966 edition. MR559928 (80k:34001)