

DILUTE EMULSIONS WITH SURFACE TENSION

BY

GRIGOR NIKA (*Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, Massachusetts 01609*)

AND

BOGDAN VERNESCU (*Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, Massachusetts 01609*)

Abstract. We consider an emulsion formed by two newtonian fluids in which one fluid is dispersed under the form of droplets of arbitrary shape in the presence of surface tension. We consider both cases of droplets with fixed centers of mass and of convected droplets. In the non-dilute case, for spherical droplets of radius a_ϵ of the same order as the period length ϵ , the two models were studied by Lipton-Avellaneda (1990) and Lipton-Vernescu (1994). Here we are interested in the time-dependent, dilute case when the characteristic size of the droplets a_ϵ , of arbitrary shape, is much smaller than ϵ . We study the limit behavior when $\epsilon \rightarrow 0$ in each of these two models. We establish a Brinkman type law for the critical size $a_\epsilon = O(\epsilon^3)$ in the first case, whereas in the second case there is no “strange” term, and in the limit the flow is unperturbed by the droplets.

1. Introduction. The literature on emulsions, and in particular the study of their effective properties, is vast and starts with the work on dilute emulsions by Taylor [19], who considered an emulsion formed by two newtonian, incompressible fluids, one of which is dispersed in the other in the form of spherical droplets, with fixed centers of mass, and derived the form of its effective viscosity:

$$\mu^{eff} = \left(1 + \frac{5\mu_1 + 2\mu_2}{2(\mu_1 + \mu_2)}\phi + \mathcal{O}(\phi^2)\right)\mu_2 \quad (1.1)$$

in the case of droplets that have fixed centers of mass (i.e. are not convected with the flow). Here μ_1 and ϕ are the viscosity and respectively the volume fraction of the droplets, and μ_2 the viscosity of the continuous liquid phase. The formula generalizes Einstein’s celebrated formula for the viscosity of suspensions of spherical, rigid particles (as $\mu_1 \rightarrow \infty$), and considers the so-called “zero-th order” approximation, i.e. the case

Received May 6, 2014.

2010 *Mathematics Subject Classification.* Primary 35J25, 35J20, 35K10; Secondary 76D07, 76T20.

Key words and phrases. Stokes flow, Brinkman equations, surface tension, Γ -convergence, Mosco-convergence, emulsions.

E-mail address: gnika@wpi.edu

E-mail address: vernescu@wpi.edu

when the droplets remain spherical. The “first order” approximation of the droplet deformation was considered in the case of dilute emulsions by Schowalter, Chaffey and Brenner [16] and Frankel and Acrivos [10], who derived a non-newtonian behavior of the emulsions that exhibit “fluid memory” effects.

Another model of dilute fluid droplets was introduced by Ammari et al. [2]. While for the limit case of rigid particles, Einstein’s formula can be recovered from this work, Taylor’s formula (1.1) cannot be obtained, as Ammari et al.’s model, while using the equations in the Eulerian frame, imposes the boundary conditions in the Lagrangian frame. The same model is used by Bonnetier, Manceau and Triki [5], who extend to the case with surface tension for droplets of known curvature.

In the non-dilute case, constitutive equations for emulsions have been derived by Choi and Schowalter [8], who considered “first order” approximation of the droplet deformation. In the framework of periodic homogenization, the effective behavior of emulsions was studied by Lipton and Vernescu [14] in the case of spherical droplets that are convected with the fluid, results that extended the case of spherical droplets with fixed centers studied by Lipton and Avellaneda [13]. In the former an effective viscosity was derived consistent with the effective stress formula of Batchelor [4]. In the latter, neglecting the bubble velocity, the problem yielded a Darcy flow, since it is equivalent to a flow around fixed obstacles.

The present paper focuses on the dilute case of droplets in the periodic homogenization framework. We consider the time-dependent, slow motion of a two-fluid dilute emulsion formed by two newtonian, incompressible fluids, one of which is dispersed in the other in the form of droplets, and derive its effective behavior. We consider only the “zero-th order” problem, in which the effects of droplet deformation are not taken into account. Thus if we denote by Ω_1 the domain occupied by the droplets of viscosity μ_1 , and by Ω_2 the domain occupied by the continuous liquid phase of viscosity μ_2 , and by S the union of the bubble surfaces (i.e. $S = \overline{\Omega_1} \cap \overline{\Omega_2}$), the problem is described by

$$\frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} (-pI + 2\mu(\mathbf{x})e(\mathbf{v})) = \mathbf{f} \quad \text{in } \Omega_1 \cup \Omega_2, \quad (1.2)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.3)$$

where $\mu(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_1$ and $\mu(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_2$.

The droplets are periodically distributed, and the size of the period is much larger than the characteristic length of the droplets. This corresponds to a zero limit concentration of droplets. We assume that the fluid velocity is continuous across the droplet surface and both a kinematic and a dynamic condition are imposed on the fluid interface. In addition we impose the condition that the droplets are neutrally buoyant.

The formulation of the stationary problem is discussed in Section 2, where, for the reader’s convenience, we give details on the boundary conditions that need to be imposed on the fluid interface S , the droplets’s boundary: (i) a no-slip condition, (ii) a kinematic condition (that expresses the fact that droplet boundary is a material boundary) and (iii) a dynamic condition (expressing the stress jump in terms of the surface tension). In addition, the balance equations for the forces and torques on each droplet need to be imposed, a condition expressing the fact that the droplets are neutrally buoyant.

More details on boundary conditions can be found in the monographs of Leal [11] and Zapryanov and Tabakova [21].

In Section 3 we formulate the periodic homogenization problem for droplets of arbitrary shapes as a variational problem. While two interesting cases can be derived, the case of convected or non-convected droplets, we further detail the case of droplets with fixed centers of mass and give its weak formulation. In the case of non-zero center of mass velocity, the computations of the limit problem are similar but easier, so we do not detail them here; we will merely state the final result.

Section 4 is dedicated to the so-called “local problem”. We identify a critical size of droplets for which the sequence of solutions loses compactness.

In Section 5 we give the main result, the Γ -convergence of the functionals describing the periodic problem to a limit functional, which has in the critical case an extra term, the limit in this case being a Brinkman type equation. The limit problem is of the form

$$-\operatorname{div} (-pI + 2\mu_2 e(\mathbf{v})) + \mathcal{M}\mathbf{v} = \mathbf{f} \quad \text{in } \Omega, \quad (1.4)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.5)$$

where \mathcal{M} is a symmetric second order tensor that depends on the geometry of the droplets and on the two fluid viscosities. The Brinkman law obtained by Brillard [7] and Allaire [1] can be obtained from here when $\mu_1 \rightarrow +\infty$. For the problem of convected droplets, the limit corresponds to the unperturbed flow, i.e. $\mathcal{M} \equiv 0$.

The particular case of spherical droplets is considered in Section 6, and in this case the explicit form of the tensor \mathcal{M} is found:

$$\mathcal{M}_{mk} = m\mu_2 \frac{\pi}{8} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \delta_{mk},$$

with $m = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon^3}{\epsilon^3}$. Let us remark here that in the case of suspensions of spherical rigid particles, the tensor reduces to

$$\mathcal{M}_{mk} = m\mu_2 \frac{3\pi}{8} \delta_{mk}.$$

The time-dependent case is treated via Mosco-convergence in Section 7. The Appendix contains the derivation of the weak formulation and some technical results regarding the local problems.

2. Problem statement.

2.1. Balance of mass and momentum. Let us denote by Ω the domain occupied by the emulsion, by Ω_1 the domain occupied by the droplets of viscosity μ_1 , and by Ω_2 the domain occupied by the continuous liquid phase of viscosity μ_2 and $\Omega = \Omega_1 \cup \overline{\Omega_2}$. The droplets are denoted by T_ℓ and their surface by S_ℓ ; the union of the bubble surfaces $S = \overline{\Omega_1} \cap \overline{\Omega_2}$. The problem is described by the balance of momentum and mass equations

$$-\operatorname{div} (-pI + 2\mu(\mathbf{x}) e(\mathbf{v})) = \mathbf{f} \quad \text{in } \Omega_1 \cup \Omega_2, \quad (2.1)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad (2.2)$$

where \mathbf{v} and p represent the fluid velocity and pressure, \mathbf{f} denotes the body forces, and the viscosity $\mu(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_1$ and $\mu(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_2$. The stress tensor will be

denoted by $\boldsymbol{\sigma} = -pI + 2\mu(\mathbf{x})e(\mathbf{v})$, where $e(\mathbf{v}) = 1/2(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ is the strain rate tensor.

2.2. Boundary conditions on droplet surfaces. (i) A kinematic boundary condition needs to be imposed on the droplet boundary, which expresses the fact that the boundary remains an interphase boundary.

Let us assume that the shape of the droplets is given by the surface $F(t, \mathbf{x}) = 0$. Then

$$0 = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla_x F \cdot \mathbf{v},$$

and thus the normal velocity of the droplet boundary is given by

$$\mathbf{v} \cdot \mathbf{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}.$$

The kinematic boundary condition on the droplet surface imposes the normal velocity of both fluids to be equal to the normal velocity of the surface:

$$[\![\mathbf{v} \cdot \mathbf{n}]\!] = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}. \quad (2.3)$$

Let us now assume that although it moves, the droplet shape does not change in time; thus $F(t, \mathbf{x}) = G(\mathbf{x}')$, where \mathbf{x}' are the coordinates of a point on the droplet surface in a moving frame, with orthonormal base $\{\mathbf{e}'_i\}$, centered at the center of mass \mathbf{x}_C^ℓ of the droplet T_ℓ . Thus

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_C^\ell \text{ and } \frac{d\mathbf{e}'_i}{dt} = A_{ij}\mathbf{e}'_j,$$

with $A = (A_{ij})$ an antisymmetric matrix. Then

$$0 = \frac{dG}{dt} = \frac{\partial G}{\partial x'_i} \frac{dx'_i}{dt} = n'_i((\mathbf{v} - \mathbf{v}_C^\ell) \cdot \mathbf{e}'_i + (\mathbf{x} - \mathbf{x}_C^\ell) \cdot (A_{ik}\mathbf{e}'_k)),$$

and thus the normal velocity of the interface S_ℓ is given by

$$\mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + A(\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (2.4)$$

Thus the kinematic boundary condition (2.3) becomes

$$[\![\mathbf{v} \cdot \mathbf{n}]\!] = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + A(\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (2.5)$$

The angular velocity \mathbf{c} can be defined in \mathbb{R}^3 as $\mathbf{c} = (c_1, c_2, c_3)$ as $c_1 = -A_{23}$, $c_2 = -A_{31}$, and $c_3 = -A_{12}$, and

$$\frac{d\mathbf{e}'_i}{dt} = \mathbf{c} \times \mathbf{e}'_i,$$

and the kinematic boundary condition (2.5) becomes

$$[\![\mathbf{v} \cdot \mathbf{n}]\!] = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (2.6)$$

REMARK 2.1. Let us observe that the kinematic boundary condition (2.5) implies

$$\mathbf{v}_C^\ell = \frac{1}{|T_\ell|} \int_{T_\ell} \mathbf{v} d\mathbf{x}.$$

Indeed using the incompressibility condition we have

$$\int_{S_\ell} x_i v_j n_j ds = \int_{T_\ell} (v_j \delta_{ij} + x_i \operatorname{div} \mathbf{v}) d\mathbf{x} = \int_{T_\ell} v_i d\mathbf{x},$$

and using the interphase velocity equation (2.4) we obtain

$$\begin{aligned} \int_{S_\ell} x_i v_j n_j ds &= \int_{S_\ell} ((v_C^\ell)_j x_i n_j + x_i (x_l - x_{Cl}^\ell) A_{lj} n_j) ds \\ &= |T_\ell| (v_C^\ell)_i + \int_{T_\ell} (x_l - x_{Cl}^\ell) A_{li} d\mathbf{x} + \int_{T_\ell} x_i A_{li} d\mathbf{x}. \end{aligned} \quad (2.7)$$

The last integral above cancels because of the antisymmetry of A , and the one before last from the definition of the center of mass. In this case the kinematic condition (2.4) becomes

$$\mathbf{v} \cdot \mathbf{n} = \left(\frac{1}{|T_\ell|} \int_{T_\ell} \mathbf{v} d\mathbf{x} + A(\mathbf{x} - \mathbf{x}_C^\ell) \right) \cdot \mathbf{n}.$$

In the particular case of spherical droplets, $\mathbf{x} - \mathbf{x}_C^\ell$ is parallel to \mathbf{n} , and thus the kinematic condition (2.5) reduces to

$$\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = \frac{1}{|T_\ell|} \left(\int_{T_\ell} \mathbf{v} d\mathbf{x} \right) \cdot \mathbf{n}. \quad (2.8)$$

(ii) A second type of boundary condition connects the stress in each fluid at the boundary. Indeed on the droplet surface there is a stress jump $\llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket \neq 0$, and (2.1) is only valid in Ω_1 and Ω_2 , and therefore we have

$$-\operatorname{div} \boldsymbol{\sigma} = \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket \delta_{S_\ell} + \mathbf{f} \text{ in } \Omega, \quad (2.9)$$

$$\boldsymbol{\sigma} = -pI + 2\mu e(\mathbf{v}), \quad (2.10)$$

with δ_{S_ℓ} the Dirac measure on S_ℓ . The stress jump can be obtained from the principle that the forces on an element of interfacial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass. One can thus obtain [11]

$$\llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket = s(\nabla_s \cdot \mathbf{n}) \mathbf{n} - \nabla_s s,$$

where s is the surface tension and $\nabla_s = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$ is the surface gradient operator. If the surface tension is uniform, the stress has only a normal jump across the interface, which is proportional to the surface tension and the mean curvature.

(iii) A third type of boundary condition needs to be imposed if the droplets do not change shape. In this case the droplet surface acts as a rigid surface that needs to be in equilibrium as the viscous stresses act on it, and thus the balance of forces and torques needs to be satisfied:

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0 \text{ and } \int_{S_\ell} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0. \quad (2.11)$$

If the droplets are not allowed to translate, the balance of forces and torques becomes

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds + \mathbf{F}_\ell \delta_{C_\ell} = 0 \text{ and } \int_{S_\ell} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0, \quad (2.12)$$

where \mathbf{F}_ℓ is a pointwise force centered at the center of mass of each droplet, which keeps the droplet from translating with the fluid but which thus gives no extra torque; δ_{C_ℓ} is

the Dirac mass at the center of the droplet. In this case the balance of forces equation (2.12) can be used to determine \mathbf{F}_ℓ .

Let us observe that in the case of spherical droplets of radius R with constant surface tension $s = \text{const.}$ the stress jump becomes

$$\llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket = \frac{s}{R} \mathbf{n},$$

and the balance of forces on the droplet surface is automatically satisfied:

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0.$$

2.3. Boundary conditions on the exterior boundary. For simplicity on the exterior boundary we will consider a no-slip condition:

$$\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega. \quad (2.13)$$

3. Periodic homogenization for droplets of arbitrary shape. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary $\Gamma = \partial\Omega$, and let $Y = (-\frac{1}{2}, \frac{1}{2})^3$ be the unit cube in \mathbb{R}^3 . For every $\epsilon > 0$, let N^ϵ be the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with Lipschitz boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subseteq \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon\ell + a_\epsilon T$, where $a_\epsilon \ll \epsilon$. The set T_ℓ^ϵ represents one of the droplets suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface. We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the droplets of viscosity μ_1 , and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid, of viscosity μ_2 . Let \mathbf{n} be the unit normal on the boundary of $\Omega_{2\epsilon}$ that points outside the domain.

The problem describing the flow of the emulsion is described by

$$-\operatorname{div} \boldsymbol{\sigma}^\epsilon = \mathbf{f} \text{ in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \quad (3.1a)$$

$$\boldsymbol{\sigma}^\epsilon = -p^\epsilon I + 2\mu^\epsilon e(\mathbf{v}^\epsilon), \quad (3.1b)$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0 \text{ in } \Omega, \quad (3.1c)$$

with boundary conditions (see (2.5) and (2.12)) on the surface of each droplet T_ℓ^ϵ , $\ell \in N^\epsilon$:

$$\llbracket \mathbf{v}^\epsilon \rrbracket = \mathbf{0} \quad \text{on } S_\ell^\epsilon, \quad (3.2a)$$

$$\mathbf{v}^\epsilon = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) \quad \text{on } S_\ell^\epsilon, \quad (3.2b)$$

$$\int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma}^\epsilon \mathbf{n} \rrbracket ds = 0, \quad (3.2c)$$

and, for simplicity, a zero velocity condition on the exterior boundary

$$\mathbf{v}^\epsilon = \mathbf{0} \quad \text{on } \Gamma, \quad (3.3)$$

where $\llbracket \cdot \rrbracket$ denotes the jump across S_ℓ^ϵ , \mathbf{c} is an unknown, constant vector in \mathbb{R}^3 , \mathbf{x}_C^ℓ is the position vector of the center of mass of the droplet T_ℓ^ϵ , and the viscosity μ^ϵ is defined as

$$\mu^\epsilon(\mathbf{x}) = \begin{cases} \mu_1 & \text{if } x \in \Omega_{1\epsilon}, \\ \mu_2 & \text{if } x \in \Omega_{2\epsilon}. \end{cases}$$

Condition (3.2a) is the no-slip condition at the interface of the two fluids, and (3.2b), (3.2c), and (3.3) follow from (2.5), (2.12) and (2.13).

3.1. *Weak formulation.* The emulsion flow problem in (3.1) – (3.3) has the equivalent variational formulation:

For any $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $\mathbf{v}^\epsilon \in V^\epsilon$ such that

$$\int_\Omega 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, d\mathbf{x}, \text{ for any } \mathbf{w} \in V^\epsilon, \quad (3.4)$$

where V^ϵ is the closed subspace of $\mathbf{H}_0^1(\Omega)$ given by

$$V^\epsilon = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \mathbf{c} \in \mathbb{R}^3\}.$$

A weak solution to (3.1) – (3.3) is any \mathbf{v}^ϵ that satisfies (3.4). Conversely, (3.1) – (3.3) can be obtained from (3.4) in the sense of distributions; the details are presented in the Appendix. The existence and uniqueness of a weak solution of the emulsion flow problem follow from the Lax-Milgram lemma.

Furthermore, any \mathbf{v}^ϵ solution to (3.4) is the unique solution of the problem

$$\begin{cases} \text{Find } \mathbf{v}^\epsilon \in \mathbf{H}_0^1(\Omega) \text{ such that} \\ J^\epsilon(\mathbf{v}^\epsilon) = \min_{\mathbf{u} \in \mathbf{H}_0^1(\Omega)} J^\epsilon(\mathbf{u}), \end{cases} \quad (3.5)$$

where

$$J^\epsilon(\mathbf{u}) = \int_\Omega \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) \, d\mathbf{x} - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}), \quad (3.6)$$

and I_S represents the indicator function of the set S , defined by

$$I_S(s) = \begin{cases} 0 & \text{if } s \in S, \\ +\infty & \text{if } s \notin S. \end{cases}$$

We are interested in studying the Γ -convergence of the sequence $\{J^\epsilon\}$ when $\epsilon \rightarrow 0$.

4. The local problem. Let us consider the local problem for a reference cell $Y_\ell^\epsilon = \epsilon(\ell + Y)$ for some $\ell \in N^\epsilon$.

$$\left\{ \begin{array}{ll} -\operatorname{div} \boldsymbol{\sigma}^{k\epsilon} = \mathbf{0} & \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \boldsymbol{\sigma}^{k\epsilon} = -q^{k\epsilon} \mathbf{I} + 2\mu e(\mathbf{w}^{k\epsilon}), & \\ \operatorname{div} \mathbf{w}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon, \\ \llbracket \mathbf{w}^{k\epsilon} \rrbracket = \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} = \mathbf{c} \times (\mathbf{y} - \mathbf{y}_C^\ell) & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} = \mathbf{e}_k & \text{on } \partial B_\ell^\epsilon, \\ \int_{S_\ell^\epsilon} (\mathbf{y} - \mathbf{y}_C^\ell) \times \llbracket \boldsymbol{\sigma}^{k\epsilon} \mathbf{n} \rrbracket \, ds = 0, & \end{array} \right. \quad (4.1)$$

where B_ℓ^ϵ is the ball with center the center of cell $\ell \in N^\epsilon$ and radius $\epsilon/2$, \mathbf{e}_k is the k -th - unit vector of the cartesian base and $\mu = \mu_1$ in T_ℓ^ϵ and $\mu = \mu_2$ in $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$.

Define $\hat{\mathbf{w}}^{k\epsilon} = \mathbf{w}^{k\epsilon} - \mathbf{e}_k$. Then (4.1) becomes

$$\left\{ \begin{array}{ll} -\operatorname{div} \hat{\boldsymbol{\sigma}}^{k\epsilon} = \mathbf{0} & \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \hat{\boldsymbol{\sigma}}^{k\epsilon} = -q^{k\epsilon} I + 2\mu e(\hat{\mathbf{w}}^{k\epsilon}), & \\ \operatorname{div} \hat{\mathbf{w}}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon, \\ \llbracket \hat{\mathbf{w}}^{k\epsilon} \rrbracket = \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \hat{\mathbf{w}}^{k\epsilon} = -\mathbf{e}_k + \mathbf{c} \times (\mathbf{y} - \mathbf{y}_C^\ell) & \text{on } S_\ell^\epsilon, \\ \hat{\mathbf{w}}^{k\epsilon} = \mathbf{0} & \text{on } \partial B_\ell^\epsilon, \\ \int_{S_\ell^\epsilon} (\mathbf{y} - \mathbf{y}_C^\ell) \times \llbracket \hat{\boldsymbol{\sigma}}^{k\epsilon} \mathbf{n} \rrbracket ds = 0. \end{array} \right.$$

Applying a change of variable we get $\hat{\mathbf{w}}^{k\epsilon}(a_\epsilon \mathbf{x})$ and $a_\epsilon q^{k\epsilon}(a_\epsilon \mathbf{x})$ are solutions for a problem of type (9.6), where $\chi^{k\frac{\epsilon}{a_\epsilon}} = \hat{\mathbf{w}}^{k\epsilon}(a_\epsilon \mathbf{x})$ and $\eta^{k\frac{\epsilon}{a_\epsilon}} = a_\epsilon q^{k\epsilon}(a_\epsilon \mathbf{x})$. Hence, using our results in the Appendix, there exists a unique solution to (4.1).

4.1. Properties of the local solution.

LEMMA 4.1. The solution $(\mathbf{w}^{k\epsilon}, q^{k\epsilon})$ of (4.1) has the following properties:

1. If $a_\epsilon = o(\epsilon^3)$, then $\mathbf{w}^{k\epsilon} \rightarrow \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, $q^{k\epsilon} \rightarrow 0$ in $L^2(\Omega)$.
2. If $a_\epsilon = O(\epsilon^3)$, then $\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, $q^{k\epsilon} \rightharpoonup 0$ in $L^2(\Omega)$.

Proof. First, we extend $\mathbf{w}^{k\epsilon}$ by periodicity to all of \mathbb{R}^3 . Since the number of microscopic cells, Y_ℓ^ϵ , included in $|\Omega|$ is equivalent to $|\Omega|/\epsilon^3$, we have

$$\int_\Omega \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \simeq \frac{|\Omega|}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} = |\Omega| \frac{a_\epsilon}{\epsilon^3} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\chi^{k\frac{\epsilon}{a_\epsilon}}) : e(\chi^{k\frac{\epsilon}{a_\epsilon}}) d\mathbf{x}.$$

From Remark 9.1 in the Appendix, the corresponding limit of the last term above exists as $\epsilon \rightarrow 0$. Hence, for a positive constant C (independent of ϵ), we have

$$\int_\Omega e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \leq C.$$

Furthermore, by Korn's inequality on B_ℓ^ϵ and the fact that $e(\mathbf{e}_k) = 0$, we note that

$$\|\mathbf{w}^{k\epsilon} - \mathbf{e}_k\|_{\mathbf{H}^1(\Omega)} \leq \sum_\ell \|\mathbf{w}^{k\epsilon} - \mathbf{e}_k\|_{\mathbf{H}^1(B_\ell^\epsilon)} \leq \sum_\ell C \int_{B_\ell^\epsilon} e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \leq C.$$

Therefore, we get that $\|\mathbf{w}^{k\epsilon}\|_{\mathbf{H}^1(\Omega)} < C$. Hence, by taking a subsequence still denoted by $\mathbf{w}^{k\epsilon}$ we have

$$\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{w} \quad \text{weakly in } \mathbf{H}^1(\Omega).$$

Since $\chi_{\cup_{\ell \in N^\epsilon} Y_\ell^\epsilon \setminus B_\ell^\epsilon}$ converges in the weak topology of $\mathbf{L}^2(\Omega)$ to the non-zero constant $|\Omega|(1 - \pi/6)$ and $\mathbf{w}^{k\epsilon} = \mathbf{e}_k$ on $Y_\ell^\epsilon \setminus B_\ell^\epsilon$, we get that $\mathbf{w} = \mathbf{e}_k$. \square

5. Convergence results. Using (3.6), let us define the energy functional $E^\epsilon: \mathbf{H}_0^1(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$E^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}): e(\mathbf{u}) \, d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}). \quad (5.1)$$

Our goal is to show that the sequence $(E^\epsilon)_\epsilon$ Γ -converges to E in the weak topology of $\mathbf{H}_0^1(\Omega)$ where

$$E(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}): e(\mathbf{u}) \, d\mathbf{x} + \sum_{k,m=1}^3 \mathcal{M}_{mk} \int_{\Omega} u_m u_k \, d\mathbf{x} + I_V(\mathbf{u}), \quad (5.2)$$

with V the closed subspace of $\mathbf{H}_0^1(\Omega)$ defined by

$$V = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{w} = 0\},$$

and $\mathcal{M} = (\mathcal{M}_{mk})_{mk}$ the positive definite, symmetric matrix defined by

$$\mathcal{M}_{mk} = m \lim_{\epsilon \rightarrow 0} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\chi^{m \frac{\epsilon}{a_\epsilon}}): e(\chi^{k \frac{\epsilon}{a_\epsilon}}) \, d\mathbf{x}, \quad (5.3)$$

and

$$m = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon}{\epsilon^3}.$$

THEOREM 5.1. The sequence $(E^\epsilon)_\epsilon$ defined by (5.1) Γ -converges in the weak topology of $\mathbf{H}_0^1(\Omega)$ to the functional E defined by (5.2).

Proof. We first remark that for every $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ which is not divergence-free in Ω one derives that

$$\Gamma - \liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}) = \Gamma - \limsup_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}) = +\infty.$$

Hence, we only have to deal with divergence-free functions. Specifically, we have to prove the following two assertions:

- (a) For all $\mathbf{v}^0 \in V$ there exists a $\mathbf{v}^\epsilon \in V^\epsilon$, $\mathbf{v}^\epsilon \rightharpoonup \mathbf{v}^0$ in $\mathbf{H}_0^1(\Omega)$ such that $\lim_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{v}^\epsilon) = E(\mathbf{v}^0)$.
- (b) For all $\mathbf{u}^0 \in V$ and for all $\mathbf{u}^\epsilon \in V^\epsilon$, $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}^0$ in $\mathbf{H}_0^1(\Omega)$ such that $\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u}^0)$.

Part (a). Let $\mathbf{v}^0 \in \mathcal{D}(\Omega)$ such that $\operatorname{div} \mathbf{v}^0 = 0$. Define the sequence \mathbf{v}^ϵ in the following way (see [7], [20]):

$$\mathbf{v}^\epsilon(\mathbf{x}) = \begin{cases} \mathbf{v}^0(\mathbf{x}) & \text{in } Y_\ell^\epsilon - B_\ell^\epsilon, \\ \mathbf{v}^0(\mathbf{x}) + (\mathbf{w}^{k_\epsilon}(\mathbf{x}) - \mathbf{e}_k)v_k^0(\mathbf{x}_C^\ell) - \operatorname{curl}(\tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) & \text{in } B_\ell^\epsilon - T_\ell^\epsilon, \\ v_k^0(\mathbf{x}_C^\ell)\mathbf{w}^{k_\epsilon}(\mathbf{x}) & \text{in } T_\ell^\epsilon, \end{cases} \quad (5.4)$$

where $\tilde{\mathbf{v}}_{\epsilon\ell}$ is the vector valued function associated with $\mathbf{v}^0(\mathbf{x}) - \mathbf{v}^0(\mathbf{x}_C^\ell)$ such that

$$\sum_{\ell \in N^\epsilon} \operatorname{curl}(\tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) \rightarrow \mathbf{0} \text{ strongly in } \mathbf{H}_0^1(\Omega) \text{ as } \epsilon \rightarrow 0,$$

with

$$\phi_{\epsilon\ell}(\mathbf{x}) = \phi_\ell(\mathbf{x}/\epsilon), \quad \phi_{\epsilon\ell}(\mathbf{x}) \in \mathcal{D}(B_\ell^\epsilon), \quad \phi_{\epsilon\ell}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in T_\ell^\epsilon, \\ 0 & \text{if } \mathbf{x} \in B_\ell^\epsilon - \hat{B}_\ell^{a_\epsilon}, \end{cases}$$

and the $\text{supp}(\phi_{\ell}) \subset \hat{B}_{\ell}^{a_{\epsilon}}$, where $\hat{B}_{\ell}^{a_{\epsilon}}$ is the ball with center the center of cell $\ell \in N^{\epsilon}$ and radius $a_{\epsilon}^{2/3}$. One can now verify that the sequence \mathbf{v}^{ϵ} is divergence free, belongs in $\mathbf{H}_0^1(\Omega)$, $\mathbf{v}^{\epsilon} = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^{\ell})$ on S_{ℓ}^{ϵ} and $\mathbf{v}^{\epsilon} \rightharpoonup \mathbf{v}^0$ in $\mathbf{H}_0^1(\Omega)$. Hence, computing $E^{\epsilon}(\mathbf{v}^{\epsilon})$ we obtain:

$$\begin{aligned}
E^{\epsilon}(\mathbf{v}^{\epsilon}) &= \sum_{\ell \in N^{\epsilon}} \int_{Y_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon}} \mu v_k^0(\mathbf{x}_C^{\ell}) v_m^0(\mathbf{x}_C^{\ell}) e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} \\
&+ \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} 2 \mu_2 v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{v}^0) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \\
&- \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} 2 \mu_2 e(\mathbf{v}^0) : e(\text{curl } (\tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell})) d\mathbf{x} \\
&- \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} 2 \mu_2 v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{w}^{k\epsilon}) : e(\text{curl } (\tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell})) d\mathbf{x} \\
&+ \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\text{curl } (\tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell})) : e(\text{curl } (\tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell})) d\mathbf{x} \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon}} \mu v_k^0(\mathbf{x}_C^{\ell}) v_m^0(\mathbf{x}_C^{\ell}) e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} + o(1) \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} \\
&+ \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) v_m^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \frac{1}{\epsilon^3} \int_{B_{\ell}^{\epsilon}} \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} + o(1) \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} \\
&+ \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) v_m^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \frac{a_{\epsilon}}{\epsilon^3} \int_{B_{\ell}^{\epsilon}} \mu e(\chi^{k \frac{\epsilon}{a_{\epsilon}}}) : e(\chi^{m \frac{\epsilon}{a_{\epsilon}}}) d\mathbf{x} + o(1).
\end{aligned}$$

By the smoothness of \mathbf{v}^0 we get

$$\lim_{\epsilon \rightarrow 0} E^{\epsilon}(\mathbf{v}^{\epsilon}) = \int_{\Omega} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{k,m=1}^3 \mathcal{M}_{mk} \int_{\Omega} v_m^0 v_k^0 d\mathbf{x}.$$

For $\mathbf{v}^0 \in \mathbf{H}_0^1(\Omega)$ we use a diagonalization process to complete the argument.

Part (b). Let $\mathbf{u}^{\epsilon}, \mathbf{u}^0 \in \mathbf{H}_0^1(\Omega)$ be such that $\text{div } \mathbf{u}^{\epsilon} = \text{div } \mathbf{u}^0 = 0$, $\mathbf{u}^{\epsilon} = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^{\ell})$ on S_{ℓ}^{ϵ} , and $\mathbf{u}^{\epsilon} \rightharpoonup \mathbf{u}^0$ in $\mathbf{H}^1(\Omega)$. Let $\mathbf{v}^0 \in \mathcal{D}(\Omega)$ be such that $\text{div } \mathbf{v}^0 = 0$ and define the sequence \mathbf{v}^{ϵ} as before. Using a sub-differential type inequality we get

$$E^{\epsilon}(\mathbf{u}^{\epsilon}) \geq E^{\epsilon}(\mathbf{v}^{\epsilon}) + \int_{\Omega} 2 \mu^{\epsilon} e(\mathbf{v}^{\epsilon}) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x}. \quad (5.5)$$

In order to pass to the limit in the last term we make use of Lemma 9.3 and obtain the following bound for the stress:

$$\|\sigma^{k\epsilon}\|_{L^2(\partial B_{\ell}^{\epsilon})} \leq C \epsilon^3 \|\mathbf{e}_k + \mathbf{c} \times (\mathbf{y} - \mathbf{y}_C^{\ell})\|_{L^2(S_{\ell})}.$$

Therefore, the last term of (5.5) becomes

$$\begin{aligned}
\int_{\Omega} \mu_{\epsilon} e(\mathbf{v}^{\epsilon}) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} &= \sum_{\ell \in N^{\epsilon}} \int_{Y_{\ell}^{\epsilon} - B_{\ell}^{\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} \\
&+ \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \\
&- \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\operatorname{curl}(\tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell})) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{T_{\ell}^{\epsilon}} \mu_1 u_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon}} \mu v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} + o(1).
\end{aligned}$$

We rewrite the last term as

$$\begin{aligned}
\frac{1}{2} \sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \int_{B_{\ell}^{\epsilon}} \boldsymbol{\sigma}^{k\epsilon} : \nabla(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} &= \frac{1}{2} \sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \int_{\partial B_{\ell}^{\epsilon}} \boldsymbol{\sigma}^{k\epsilon} \mathbf{n} \cdot (\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} \\
&\geq -C \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \cdot \|\mathbf{-e}_k + \mathbf{c} \times (\mathbf{y} - \mathbf{y}_C^{\ell})\|_{\mathbf{L}^2(S_{\ell})} \|\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}\|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Putting everything together we get

$$\begin{aligned}
E^{\epsilon}(\mathbf{u}^{\epsilon}) &\geq E^{\epsilon}(\mathbf{v}^{\epsilon}) + \int_{\Omega \setminus \Omega_{1\epsilon}} 2 \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{v}^0) d\mathbf{x} \\
&- C \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \|\mathbf{-e}_k + \mathbf{c} \times (\mathbf{y} - \mathbf{y}_C^{\ell})\|_{\mathbf{L}^2(S_{\ell})} \|\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}\|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Passing to the limit first as $\epsilon \rightarrow 0$, then using a diagonalization argument to make $\mathbf{v}^0 \rightarrow \mathbf{u}^0$ in the strong topology of $\mathbf{H}_0^1(\Omega)$, we get

$$\liminf_{\epsilon \rightarrow 0} E^{\epsilon}(\mathbf{u}^{\epsilon}) \geq E(\mathbf{u}^0).$$

□

The immediate consequence is

COROLLARY 5.2. The sequence $\{\mathbf{v}^{\epsilon}\}$ of solutions to (3.5) is weakly convergent in the weak topology of $\mathbf{H}_0^1(\Omega)$ to the \mathbf{v} solution to

$$\begin{cases} \text{Find } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \text{ such that} \\ J(\mathbf{v}) = \min_{\mathbf{u} \in \mathbf{H}_0^1(\Omega)} J(\mathbf{u}), \end{cases} \quad (5.6)$$

where

$$J(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathcal{M} \mathbf{u} \cdot \mathbf{u} d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_V(\mathbf{u}) \quad (5.7)$$

and

$$V = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}.$$

The Brinkman law obtained here takes into account the geometry of the droplets and their viscosity. For $\mu_1 \rightarrow \infty$ one can show that the result yields the Brinkman law for rigid particles obtained in [1] and [7].

6. Spherical droplets. In this section we determine the matrix $\mathcal{M} = (\mathcal{M}_{mk})_{mk}$ in (5.3) for the case of spherical droplets. In accordance with Section 2.2 the emulsion problem simplifies in the following way:

$$\left\{ \begin{array}{ll} -\operatorname{div} \boldsymbol{\sigma}^\epsilon = \mathbf{f} & \text{in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \\ \boldsymbol{\sigma}^\epsilon = -p^\epsilon I + 2\mu^\epsilon e(\mathbf{v}^\epsilon), & \\ \operatorname{div} \mathbf{v}^\epsilon = 0 & \text{in } \Omega, \\ \llbracket \boldsymbol{\sigma}^\epsilon \mathbf{n} \rrbracket = (\llbracket \boldsymbol{\sigma}^\epsilon \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} & \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon \cdot \mathbf{n} = 0 & \text{on } S_\ell^\epsilon, \\ \llbracket \mathbf{v}^\epsilon \rrbracket = \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon = \mathbf{0} & \text{on } \Gamma. \end{array} \right. \quad (6.1)$$

The balance of forces and torque is given by (2.12). The balance of torque is automatically satisfied in this case, while the balance of forces we do not write down since it is de-coupled from problem 6.1. Moreover, the variational formulation is the same as in (3.4) where V^ϵ simplifies to

$$V^\epsilon = \{ \mathbf{w} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } S_\ell^\epsilon \}.$$

6.1. *The local problem for spheres.* Let us consider the local transmission problem for cell Y_ℓ^ϵ for some $\ell \in N^\epsilon$:

$$\left\{ \begin{array}{ll} -\operatorname{div} \boldsymbol{\sigma}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \boldsymbol{\sigma}^{k\epsilon} = -p^{k\epsilon} I + 2\mu e(\mathbf{w}^{k\epsilon}), & \\ \operatorname{div} \mathbf{w}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon, \\ \llbracket \boldsymbol{\sigma}^{k\epsilon} \mathbf{n} \rrbracket = (\llbracket \boldsymbol{\sigma}^{k\epsilon} \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} \cdot \mathbf{n} = 0 & \text{on } S_\ell^\epsilon, \\ \llbracket \mathbf{w}^{k\epsilon} \rrbracket = \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} = \mathbf{e}_k & \text{on } \partial B_\ell^\epsilon. \end{array} \right. \quad (6.2)$$

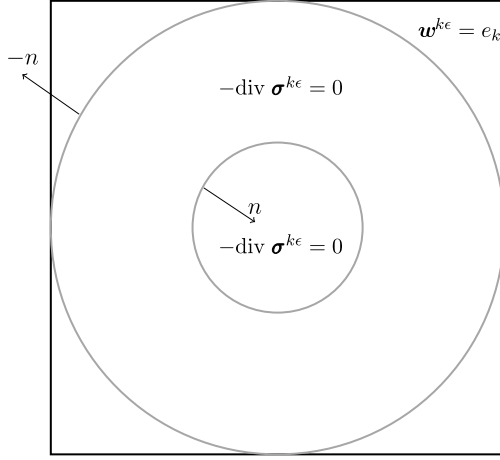
Problem (6.2) admits an explicit solution. Indeed due to the spherical symmetry we look for a solution in spherical coordinates (r, θ, ϕ) with θ being the angle between \mathbf{x} and \mathbf{e}_k and with no dependence on ϕ :

$$\mathbf{w}^{k\epsilon} = f(r) \cos(\theta) \mathbf{e}_r + g(r) \sin(\theta) \mathbf{e}_\theta, \quad (6.3a)$$

$$q^{k\epsilon} = h(r) \cos(\theta). \quad (6.3b)$$

Here $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ denote the unit vectors in spherical coordinates. Substituting (6.3a), (6.3b) into (6.2) we obtain the following system of differential equations:

$$\begin{aligned} \mu(f'' + \frac{2}{r}f' - \frac{4}{r^2}(f+g)) + h' &= 0, \\ \mu(g'' + \frac{2}{r}g' - \frac{2}{r^2}(f+g)) - \frac{1}{4}h &= 0, \\ f' + \frac{2}{r}(f+g) &= 0, \end{aligned} \quad (6.4)$$



in each domain $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$ and T_ℓ^ϵ . By eliminating g and h we obtain an Euler equation for f :

$$\frac{r^2}{2} f^{(4)} + 4r f^{(3)} + r f'' - \frac{4}{r} f' = 0. \quad (6.5)$$

Thus the solution for (6.2) is given by

$$f(r) = C_1 r^2 + C_2 + \frac{C_3}{r} + \frac{C_4}{r^3}, \quad (6.6a)$$

$$g(r) = -2C_1 r^2 - C_2 - \frac{C_3}{2r} + \frac{C_4}{2r^3}, \quad (6.6b)$$

$$h(r) = -\mu \left(10r C_1 + \frac{C_3}{r^2} \right) \quad (6.6c)$$

inside B_ℓ^ϵ and similar solutions, with different constants, say K_1 , K_2 , K_3 , K_4 , for $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$. Also, by requiring $\mathbf{w}^{k\epsilon}$, $q^{k\epsilon}$ to be in $\mathbf{L}^2(B_\ell^\epsilon)$ we get $C_3 = C_4 = 0$. Furthermore, the boundary conditions yield the system:

$$\begin{cases} C_1 a_\epsilon^2 + C_2 = 0, \\ K_1 a_\epsilon^2 + K_2 + \frac{K_3}{a_\epsilon} + \frac{K_4}{a_\epsilon^3} = 0, \\ -2C_1 a_\epsilon^2 - C_2 = -2K_1 a_\epsilon^2 - K_2 - \frac{K_3}{2a_\epsilon} + \frac{K_4}{2a_\epsilon^3}, \\ \mu_1 C_1 a_\epsilon = \mu_2 \left(K_1 + \frac{K_4}{a_\epsilon^4} \right), \\ K_1 \epsilon^2 + K_2 + \frac{K_3}{\epsilon} + \frac{K_4}{\epsilon^4} = 1, \\ -2K_1 \epsilon^2 - K_2 - \frac{K_3}{2\epsilon} + \frac{K_4}{2\epsilon^4} = -1, \end{cases}$$

which determines uniquely C_1 , C_2 , K_1 , K_2 , K_3 and K_4 . For instance,

$$K_1 = \frac{(3\mu_1 a_\epsilon^2 - \epsilon^2(3\mu_1 + 2\mu_2)) a_\epsilon \epsilon}{(a_\epsilon - \epsilon)^3 (4a_\epsilon^3(\mu_1 - \mu_2) + 3a_\epsilon^2 \epsilon(\mu_1 - 2\mu_2) - 3a_\epsilon \epsilon^2(\mu_1 + 2\mu_2) - 4\epsilon^3(\mu_1 + \mu_2))}.$$

We will next study the behavior of $\mathbf{w}^{k\epsilon}$, $q^{k\epsilon}$ as $\epsilon \rightarrow 0$.

LEMMA 6.1. The solution $(\mathbf{w}^{k\epsilon}, q^{k\epsilon})$ of (6.2) has the following properties:

1. If $a_\epsilon = o(\epsilon^3)$, then $\mathbf{w}^{k\epsilon} \rightarrow \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, $q^{k\epsilon} \rightarrow 0$ in $L^2(\Omega)$.

2. If $a_\epsilon = O(\epsilon^3)$, then $\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{e}_k$ in $\mathbf{H}^1(\Omega)$, $q^{k\epsilon} \rightharpoonup 0$ in $L^2(\Omega)$, and there exists $\gamma \in H^{-1}(\Omega)$ such that

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \phi \mathbf{u}^\epsilon \rangle \rightarrow \langle \gamma \mathbf{e}_k , \phi \mathbf{u} \rangle ,$$

for any $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $\mathbf{H}_0^1(\Omega)$ with $\mathbf{u}^\epsilon \cdot \mathbf{n} = 0$ on S_ℓ^ϵ and for any $\phi \in \mathcal{D}(\Omega)$. Moreover, if $a_\epsilon = m\epsilon^3$, then

$$\gamma = -m \frac{\pi}{4} \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}.$$

Proof. Take $\boldsymbol{\psi} \in \mathbf{H}_0^1(\Omega)$ with $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ on S_ℓ^ϵ . From (6.2) one gets

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \boldsymbol{\psi} \rangle = \int_{\partial B^\epsilon} (\mu_2 \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) \boldsymbol{\psi} \cdot \mathbf{n} \, ds, \quad (6.7)$$

where $\partial B^\epsilon = \cup_{\ell \in N^\epsilon} \partial B_\ell^\epsilon$ and the right-hand side can be estimated since

$$\begin{aligned} (\mu_2 \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) \mathbf{e}_r &= (-h(\epsilon) + \mu_2 f'(\epsilon))(\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r + \mu_2 g'(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \\ &= F(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r + G(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta. \end{aligned} \quad (6.8)$$

Therefore (6.8) becomes

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \boldsymbol{\psi} \rangle = F(\epsilon) \int_{\partial B^\epsilon} (\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r \, ds + G(\epsilon) \int_{\partial B^\epsilon} (\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \, ds. \quad (6.9)$$

On the other hand, one has (see [9], [1]) the following convergences:

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon \mathbf{e}_k \rightarrow \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (6.10)$$

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon (\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r \rightarrow \frac{1}{3} \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (6.11)$$

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon (\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \rightarrow \frac{2}{3} \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (6.12)$$

where δ_ℓ^ϵ is the measure supported on ∂B^ϵ and S_3 is the surface of the unit sphere. Thus in order to pass the limit in (6.9) we have to compute the limit of $F(\epsilon)/\epsilon$ and $G(\epsilon)/\epsilon$. Therefore, we have

$$\frac{F(\epsilon)}{\epsilon} = \frac{-h(\epsilon) + \mu_2 f'(\epsilon)}{\epsilon} = \mu_2 \left(10 K_1 + \frac{K_3}{\epsilon^3} \right) + \mu_2 \left(2 K_1 - \frac{K_3}{\epsilon^3} - 3 \frac{K_4}{\epsilon^4} \right),$$

which implies

$$\frac{F(\epsilon)}{\epsilon} \sim \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \left(-10 \frac{10 a_\epsilon}{\epsilon^3} - \frac{a_\epsilon}{2\epsilon^3} - \frac{a_\epsilon}{2\epsilon^3} + \frac{a_\epsilon}{2\epsilon^3} + 0 \right),$$

and hence

$$\lim_{\epsilon \rightarrow 0} \frac{F(\epsilon)}{\epsilon} = -3 m \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}. \quad (6.13)$$

Similarly,

$$\frac{G(\epsilon)}{\epsilon} = \frac{\mu_2}{\epsilon} g'(\epsilon) = \mu_2 \left(-4 K_1 + \frac{K_3}{2\epsilon^3} - \frac{3 K_4}{2\epsilon^5} \right),$$

which implies

$$\frac{G(\epsilon)}{\epsilon} \sim \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \left(\frac{a_\epsilon}{\epsilon^3} - \frac{a_\epsilon}{4\epsilon^3} \right),$$

and hence

$$\lim_{\epsilon \rightarrow 0} \frac{G(\epsilon)}{\epsilon} = m\mu_2 \frac{3(3\mu_1 + 2\mu_2)}{4(\mu_1 + \mu_2)}. \quad (6.14)$$

Relations (6.9) – (6.14) yield

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I), \boldsymbol{\psi} \rangle \rightarrow -m\mu_2 \frac{\pi}{4} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \int_{\Omega} \mathbf{e}_k \cdot \phi \mathbf{u} \, d\mathbf{x}, \quad (6.15)$$

for any $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $\mathbf{H}_0^1(\Omega)$ with $\mathbf{u}^\epsilon \cdot \mathbf{n} = 0$ on S_ℓ^ϵ , and for any $\phi \in \mathcal{D}(\Omega)$. \square

Using Lemma 6.1 we can now compute (5.3). Indeed, we have

$$\begin{aligned} \frac{a_\epsilon}{\epsilon^3} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\boldsymbol{\chi}^{m\frac{\epsilon}{a_\epsilon}}) : e(\boldsymbol{\chi}^{k\frac{\epsilon}{a_\epsilon}}) \, d\mathbf{x} &= \frac{1}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\mathbf{w}^{m\epsilon}) : e(\mathbf{w}^{k\epsilon}) \, d\mathbf{x} \\ &= \frac{1}{2|\Omega|} \int_{B^\epsilon} 2\mu e(\mathbf{w}^{m\epsilon}) : e(\mathbf{w}^{k\epsilon}) \, d\mathbf{x} = \frac{1}{2|\Omega|} \int_{B^\epsilon} \sigma_{ij}^{m\epsilon} \frac{\partial w_i^{k\epsilon}}{\partial x_j} \, d\mathbf{x} = \frac{1}{2|\Omega|} \langle -\operatorname{div} \boldsymbol{\sigma}^{m\epsilon}, \mathbf{w}^{k\epsilon} \rangle, \end{aligned} \quad (6.16)$$

where the last term was obtained using integration by parts and properties of the local problem (6.2). As $\epsilon \rightarrow 0$ we obtain

$$\mathcal{M}_{mk} = m\mu_2 \frac{\pi}{8} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \delta_{mk}.$$

7. Time-dependent case. The results obtained for the time stationary Stokes equation can be extended to the time-dependent case by using the fact that Γ -convergence of E^ϵ to E in the weak topology $\mathbf{H}_0^1(\Omega)$ is equivalent to Mosco-convergence of E^ϵ to E in $\mathbf{L}^2(\Omega)$. Then using the connection between Mosco-convergence and the convergence of solutions for a class of evolution problems we can get convergence in the time-dependent case. We begin with some auxiliary results.

Let $\mathbf{H} = \mathbf{L}^2(\Omega)$ and for any $\epsilon > 0$ let E^ϵ be the convex functional on \mathbf{H} , with $\operatorname{dom}(E^\epsilon) = V^\epsilon$, defined by

$$E^\epsilon(\mathbf{u}) = \begin{cases} \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) \, d\mathbf{x} & \mathbf{u} \in V^\epsilon, \\ +\infty & \mathbf{u} \notin V^\epsilon. \end{cases}$$

The sub-differential of E^ϵ , ∂E^ϵ , is

$$\partial E^\epsilon(\mathbf{u}) = \{ \boldsymbol{\xi} \in \mathbf{H} : (\boldsymbol{\xi}, \mathbf{v} - \mathbf{u}) \leq E^\epsilon(\mathbf{v}) - E^\epsilon(\mathbf{u}) \text{ for all } \mathbf{v} \in \operatorname{dom}(E^\epsilon) \}.$$

Thus, if $\mathbf{u}^\epsilon \in V^\epsilon$, then $\boldsymbol{\xi} \in \partial E^\epsilon(\mathbf{u}^\epsilon)$ if and only if for every $\mathbf{v}^\epsilon \in \operatorname{dom}(E^\epsilon)$,

$$E^\epsilon(\mathbf{v}^\epsilon) \geq E^\epsilon(\mathbf{u}^\epsilon) + (\boldsymbol{\xi}, \mathbf{v}^\epsilon - \mathbf{u}^\epsilon). \quad (7.1)$$

Select $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon + \lambda \boldsymbol{\phi}$ where $\boldsymbol{\phi} \in \mathbf{D}(\Omega_{1\epsilon})$ with $\operatorname{div} \boldsymbol{\phi} = 0$ and $\lambda \in \mathbb{R}$. Substitute \mathbf{v}^ϵ in (7.1) to obtain

$$\int_{\Omega_{1\epsilon}} 2\mu^\epsilon e(\mathbf{u}^\epsilon) : e(\boldsymbol{\phi}) \, d\mathbf{x} = (\boldsymbol{\xi}, \boldsymbol{\phi}). \quad (7.2)$$

This implies that there exists a distribution $p_1^\epsilon \in \mathbf{L}^2(\Omega_{1\epsilon})$ such that $-\operatorname{div} \boldsymbol{\sigma}^{1\epsilon} = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{1\epsilon}$. In a similar manner if we select $\boldsymbol{\phi} \in \mathbf{D}(\Omega_{2\epsilon})$ with $\operatorname{div} \boldsymbol{\phi} = 0$ we obtain that $-\operatorname{div} \boldsymbol{\sigma}^{2\epsilon} = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{2\epsilon}$, or we can write $-\operatorname{div} \boldsymbol{\sigma}^\epsilon = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{1\epsilon} \cup \Omega_{2\epsilon}$. Furthermore, in exactly the same way as in the

Appendix, we obtain the remaining boundary conditions and balance of torque on each fluid particle.

Let $I = (0, T)$; the time-dependent problem for the emulsion is

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}^\epsilon(t, \mathbf{x})}{\partial t} - \operatorname{div} \boldsymbol{\sigma}^\epsilon = \mathbf{f}(t, \mathbf{x}) & \text{in } I \times (\Omega_{1\epsilon} \cup \Omega_{2\epsilon}), \\ \operatorname{div} \mathbf{u}^\epsilon(t, \mathbf{x}) = 0 & \text{in } I \times \Omega, \\ \llbracket \mathbf{u}^\epsilon(t, \mathbf{x}) \rrbracket = \mathbf{0} & \text{on } I \times S_\ell^\epsilon, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) & \text{on } I \times S_\ell^\epsilon, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) = \mathbf{0} & \text{on } I \times \partial\Omega, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) = \mathbf{u}^\epsilon(\mathbf{x}) & \text{on } \{0\} \times (\Omega_{1\epsilon} \cup \Omega_{2\epsilon}), \\ \int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma}^\epsilon \mathbf{n} \rrbracket ds = 0. \end{array} \right. \quad (7.3)$$

We associate with $\mathbf{u}^\epsilon(t, \mathbf{x})$ a mapping $\hat{\mathbf{u}}^\epsilon: [0, T] \mapsto \mathbf{H}$ defined by $[\hat{\mathbf{u}}^\epsilon(t)](\mathbf{x}) = \mathbf{u}^\epsilon(t, \mathbf{x})$ where $\mathbf{x} \in \Omega_{1\epsilon} \cup \Omega_{2\epsilon}$ and $0 \leq t \leq T$. In other words we are going to consider $\mathbf{u}^\epsilon(t, \mathbf{x})$ not as a function of \mathbf{x} and t but as a mapping $\hat{\mathbf{u}}^\epsilon$ of t into \mathbf{H} of functions of \mathbf{x} . We interpret $\mathbf{f}(t, \mathbf{x})$ in a similar manner.

The variational formulation of (7.3) is

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^\epsilon \in L^2(0, T; \mathbf{H}) \text{ such that} \\ \frac{d}{dt} \int_{\Omega} \mathbf{u}^\epsilon(t, \mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x} + \int_{\Omega} 2\mu^\epsilon e(\mathbf{u}^\epsilon(t, \mathbf{x})) : e(\boldsymbol{\phi}(\mathbf{x})) d\mathbf{x} \\ = \int_{\Omega} \mathbf{f}(t, \mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x} \text{ for all } \boldsymbol{\phi} \in V^\epsilon. \end{array} \right. \quad (7.4)$$

Equivalently, using (7.2), we can write (7.4) as

$$\left\{ \begin{array}{l} \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} + \partial E^\epsilon(\hat{\mathbf{u}}^\epsilon(t)) \ni \hat{\mathbf{f}}(t), \quad 0 < t < T, \\ \hat{\mathbf{u}}^\epsilon(0) = \mathbf{u}^\epsilon. \end{array} \right. \quad (7.5)$$

Therefore we have obtained an equivalence between (7.3) and (7.5).

In a similar manner, if we define

$$E(\mathbf{u}) = \left\{ \begin{array}{ll} \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathbf{u}^\top \mathcal{M} \mathbf{u} d\mathbf{x} & \text{for } \mathbf{u} \in V, \\ +\infty & \text{for } \mathbf{u} \notin V, \end{array} \right.$$

in \mathbf{H} with $\operatorname{dom}(E) = V$ and \mathcal{M} the matrix defined in equation (5.2), we obtain an equivalence between the time-dependent homogenized suspensions problem,

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}(t, \mathbf{x})}{\partial t} - \mu_2 \Delta \mathbf{u} + \nabla p + \mathcal{M} \mathbf{u} = \mathbf{f}(t, \mathbf{x}) & \text{in } I \times \Omega, \\ \operatorname{div} \mathbf{u}(t, \mathbf{x}) = 0 & \text{in } I \times \Omega, \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{0} & \text{on } I \times \partial\Omega, \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}(\mathbf{x}) & \text{on } \{0\} \times \Omega, \end{array} \right. \quad (7.6)$$

and

$$\begin{cases} \frac{d\hat{\mathbf{u}}(t)}{dt} + \partial E(\hat{\mathbf{u}}(t)) \ni \hat{\mathbf{f}}(t), & 0 < t < T, \\ \hat{\mathbf{u}}(0) = \mathbf{u}. \end{cases} \quad (7.7)$$

Using well-known results from the theory of non-linear semigroups and maximal monotone operators in Hilbert spaces (see [6], [17]), well-posedness as well as existence and uniqueness of solutions of both (7.5) and (7.7) are established.

We are now ready to state the main result of this section:

THEOREM 7.1. Assume that $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$ in \mathbf{H} . Then the solution $\hat{\mathbf{u}}^\epsilon$ converges uniformly to $\hat{\mathbf{u}}$ on $[0, T]$ and

$$\int_0^T t \left\| \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} - \frac{d\hat{\mathbf{u}}(t)}{dt} \right\|_{\mathbf{H}}^2 dt \rightarrow 0.$$

If in addition $\mathbf{u}^\epsilon \in \text{dom}(E^\epsilon)$, $\mathbf{u} \in \text{dom}(E)$ and $E^\epsilon(\mathbf{u}^\epsilon) \rightarrow E(\mathbf{u})$, then

$$\frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} \rightarrow \frac{d\hat{\mathbf{u}}(t)}{dt} \text{ strongly in } L^2(0, T; \mathbf{H})$$

and

$$E^\epsilon(\hat{\mathbf{u}}^\epsilon(t)) \rightarrow E(\hat{\mathbf{u}}(t)) \text{ uniformly on } [0, T].$$

Proof. Straightforward application of Theorem 3.66 and Theorem 3.74 in [3] and the Mosco-convergence of E^ϵ to E . \square

8. Conclusions. The problem of dilute emulsions formed by two newtonian fluids in which one fluid is dispersed in the form of droplets of arbitrary shape, in the presence of surface tension, is formulated in the homogenization framework.

In the case of droplets with fixed centers of mass, we prove, using Γ -convergence in Theorem 5.1, that if $a_\epsilon = O(\epsilon^3)$ the limit behavior is described by a Brinkman type law, while for the case of convected droplets the limit is given by the unperturbed Stokes flow. For $a_\epsilon = o(\epsilon^3)$, in both cases the limit is given by the unperturbed flow. The Brinkman law obtained here takes into account the geometry of the droplets and their viscosity; for $\mu_1 \rightarrow \infty$ one can show that the result yields the Brinkman law for rigid particles obtained in [1] and [7].

For spherical droplets we can actually compute the solutions of the local problems and thus the tensor appearing in the Brinkman law in Lemma 6.1. This also gives the form for the suspension of rigid particles in the limit for $\mu_1 \rightarrow \infty$.

The time-dependent problem is studied using the consequences of the Mosco-convergence for the energy functionals corresponding to the elliptic case.

9. Appendix.

9.1. Equivalence between PDE and variational formulation for the emulsion problem.

We seek a vector function \mathbf{u}^ϵ representing the velocity of the fluid and a scalar function p^ϵ representing the pressure, which are defined in $\Omega_{1\epsilon} \cup \Omega_{2\epsilon}$ and satisfy the following equations and boundary conditions:

$$-\operatorname{div} \boldsymbol{\sigma}^\epsilon = \mathbf{f} \text{ in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \quad (9.1a)$$

$$\boldsymbol{\sigma}^\epsilon = -2\mu^\epsilon e(\mathbf{v}^\epsilon) + p^\epsilon I, \quad (9.1b)$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0 \text{ in } \Omega, \quad (9.1c)$$

with boundary conditions on the surface of each droplet T_ℓ^ϵ , $\ell \in N^\epsilon$,

$$[\![\mathbf{v}^\epsilon]\!] = \mathbf{0} \quad \text{on } S_\ell^\epsilon, \quad (9.2a)$$

$$\mathbf{v}^\epsilon = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) \quad \text{on } S_\ell^\epsilon, \quad (9.2b)$$

$$\mathbf{v}^\epsilon = \mathbf{0} \quad \text{on } \Gamma, \quad (9.2c)$$

and an additional condition that comes from the balance of torques,

$$\int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times [\![\boldsymbol{\sigma}^\epsilon \mathbf{n}]\!] ds = 0. \quad (9.3)$$

If \mathbf{f} , \mathbf{u}^ϵ , and p^ϵ are smooth functions satisfying (9.1) – (9.3), then, taking the scalar product of (9.1a) with a function \mathbf{w} in \mathcal{V}^ϵ , where

$$\mathcal{V}^\epsilon = \{\mathbf{w} \in \mathcal{D}(\Omega) \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \mathbf{c} \in \mathbb{R}^3\},$$

we obtain

$$-\int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \operatorname{div} \boldsymbol{\sigma}^\epsilon \cdot \mathbf{w} d\mathbf{x} = \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \mathbf{f} \cdot \mathbf{w} d\mathbf{x}.$$

Thus we get

$$-\sum_{\ell \in N^\epsilon} \int_{S_\ell^\epsilon} [\![\boldsymbol{\sigma}^\epsilon \mathbf{n}]\!] \cdot \mathbf{w} d\mathbf{x} + \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \boldsymbol{\sigma}^\epsilon : \nabla \mathbf{w} d\mathbf{x} = \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \mathbf{f} \cdot \mathbf{w} d\mathbf{x},$$

and using condition (9.3), properties of symmetric matrices, and the fact that the fluid is incompressible we get

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} d\mathbf{x} \text{ for all } \mathbf{w} \in \mathcal{V}^\epsilon. \quad (9.4)$$

Equality (9.4) is still valid by continuity for each $\mathbf{w} \in V^\epsilon$, the closure of \mathcal{V}^ϵ in $\mathbf{H}_0^1(\Omega)$.

Therefore we have the following conclusion:

$$\mathbf{v}^\epsilon \in V^\epsilon \text{ and satisfies } \int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} d\mathbf{x} \text{ for all } \mathbf{w} \in V^\epsilon. \quad (9.5)$$

Conversely, assume that $\mathbf{v}^\epsilon \in V^\epsilon$ satisfies (9.5). Since $\mathbf{v}^\epsilon \in \mathbf{H}_0^1(\Omega)$ we immediately get $\mathbf{v}^\epsilon = \mathbf{0}$ on Γ in the sense of the traces. Furthermore, since $\mathbf{v}^\epsilon \in V^\epsilon$ we obtain that $\operatorname{div} \mathbf{v}^\epsilon = 0$ in the distributional sense, $[\![\mathbf{v}^\epsilon]\!] = \mathbf{0}$ and $\mathbf{v}^\epsilon = \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell)$ on S_ℓ^ϵ , for $\mathbf{c} \in \mathbb{R}^3$, in the sense of the traces.

Let $\boldsymbol{\phi} \in \mathcal{D}(\Omega_{1\epsilon})$ with $\operatorname{div} \boldsymbol{\phi} = 0$. Then $\boldsymbol{\phi} \in \mathcal{V}^\epsilon$ (pick $\mathbf{c} \in \mathbb{R}^3$ to be the zero vector), and using (9.5), we get

$$\langle -\operatorname{div} (2\mu_1 e(\mathbf{v}^{1\epsilon})) - \mathbf{f}, \boldsymbol{\phi} \rangle = 0 \text{ for all } \boldsymbol{\phi} \in \mathcal{D}(\Omega_{1\epsilon}).$$

Using Propositions 1.1 and 1.2 in [18] we find that there exists a distribution $p_1^\epsilon \in \mathbf{L}^2(\Omega_{1\epsilon})$ such that

$$-\operatorname{div} (2\mu_1 e(\mathbf{v}^{1\epsilon})) - \mathbf{f} = -\nabla p_1^\epsilon,$$

in the sense of distributions in $\Omega_{1\epsilon}$. Similarly if we pick a $\phi \in \mathcal{D}(\Omega_{2\epsilon})$ with $\operatorname{div} \phi = 0$ and proceed the same way as above we will obtain

$$-\operatorname{div} (2\mu_2 e(\mathbf{v}^{2\epsilon})) - \mathbf{f} = -\nabla p_2^\epsilon,$$

in the sense of distributions in $\Omega_{2\epsilon}$.

The last condition that we need to recover is (9.3). Consider $\phi \in \mathcal{V}^\epsilon$; using (9.5)

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\phi) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \phi d\mathbf{x} \text{ for all } \phi \in \mathcal{V}^\epsilon,$$

adding the pressure distribution in the above equation and integrating by parts we obtain

$$\sum_{\ell \in N^\epsilon} \int_{S_\ell^\epsilon} \llbracket \sigma^\epsilon \mathbf{n} \rrbracket \cdot \phi ds + \langle -\operatorname{div} \sigma^\epsilon - \mathbf{f}, \phi \rangle = 0,$$

which implies condition (9.3).

9.2. Auxiliary local problems and bounds on the stresses. Consider the solution $\{\chi^{iR}, \eta^{iR}\}$ of the following auxiliary local problem in B_R , the ball of radius R centered at 0, with T a subset of $Y = (-\frac{1}{2}, \frac{1}{2})^3$, for $R > R_0$ where R_0 is such that $Y \Subset B_{R_0}$:

$$\left\{ \begin{array}{ll} -\mu(\mathbf{y}) \Delta \chi^{iR} + \nabla \eta^{iR} = \mathbf{0} & \text{in } B_R \setminus \partial T, \\ \operatorname{div} \chi^{iR} = 0 & \text{in } B_R, \\ \llbracket \chi^{iR} \rrbracket = \mathbf{0} & \text{on } \partial T, \\ \chi^{iR} = -\mathbf{e}_i + \mathbf{c} \times \mathbf{y} & \text{on } \partial T, \\ \chi^{iR} = \mathbf{0} & \text{on } \partial B_R, \end{array} \right. \quad (9.6)$$

and

$$\int_{\partial T} \mathbf{y} \times \llbracket \sigma(\chi^{iR}, \eta^{iR}) \mathbf{n} \rrbracket ds = 0. \quad (9.7)$$

In similar fashion, we denote by $\{\chi^i, \eta^i\}$ the solution of auxiliary local problem in free space,

$$\left\{ \begin{array}{ll} -\mu(\mathbf{y}) \Delta \chi^i + \nabla \eta^i = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \partial T, \\ \operatorname{div} \chi^i = 0 & \text{in } \mathbb{R}^3, \\ \llbracket \chi^i \rrbracket = \mathbf{0} & \text{on } \partial T, \\ \chi^i = -\mathbf{e}_i + \mathbf{c} \times \mathbf{y} & \text{on } \partial T, \\ \chi^i \rightarrow \mathbf{0} & \text{at } \infty, \end{array} \right. \quad (9.8)$$

and

$$\int_{\partial T} \mathbf{y} \times \llbracket \sigma(\chi^i, \eta^i) \mathbf{n} \rrbracket ds = 0. \quad (9.9)$$

Properties of χ^{iR} and χ^i . Due to [12] (Thm. 8.2), χ^{iR} is defined as the unique minimizer of the bilinear form $\mathfrak{S}_{B_R}(\cdot, \cdot)$ over the closed, convex, non-empty set

$$K_i^R = \{\mathbf{u} \in \mathbf{H}_0^1(B_R) \mid \operatorname{div} \mathbf{u} = 0 \text{ on } B_R, \quad \mathbf{u} = -\mathbf{e}_i + \mathbf{c} \times \mathbf{y} \text{ on } \partial T\},$$

where

$$\mathfrak{S}_{B_R}(\mathbf{v}, \mathbf{v}) = \int_{B_R} \mu e(\mathbf{v}) : e(\mathbf{v}) d\mathbf{x}.$$

Similarly, χ^i is defined as the unique minimizer of the bilinear form $\mathfrak{S}_{\mathbb{R}^3}(\cdot, \cdot)$ over the closed, convex, non-empty set

$$K_i = \{\mathbf{u} \in \mathbf{X} \mid \operatorname{div} \mathbf{u} = 0 \text{ on } \mathbb{R}^3, \quad \mathbf{u} = -\mathbf{e}_i + \mathbf{c} \times \mathbf{y} \text{ on } \partial T\},$$

where the Hilbert space \mathbf{X} is defined as the closure of divergence free vector fields in $\mathcal{D}(\mathbb{R}^3)$.

REMARK 9.1. From the fact that χ^{iR} is a minimum, we observe that if $R_2 > R_1$, then

$$\mathfrak{S}_{B_{R_2}}(\chi^{iR_2}, \chi^{iR_2}) \leq \mathfrak{S}_{B_{R_2}}(\chi^{iR_1}, \chi^{iR_1}) = \mathfrak{S}_{B_{R_1}}(\chi^{iR_1}, \chi^{iR_1}).$$

Hence, the limit as $R \rightarrow \infty$ of $\mathfrak{S}_{B_R}(\chi^{iR}, \chi^{iR})$ exists.

LEMMA 9.2. χ^i and η^i satisfy the following pointwise estimates for $|\mathbf{x}| > R_0$:

$$\begin{aligned} |D^\alpha \chi^i(x)| &\leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{|\mathbf{x}|^{1+|\alpha|}} \text{ for } |\alpha| \leq 2, \\ |D^\alpha \eta^i(x)| &\leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{|\mathbf{x}|^{2+|\alpha|}} \text{ for } |\alpha| \leq 1. \end{aligned}$$

Proof. See [15, Lemma 5.6]. □

LEMMA 9.3. There exists a constant C such that for every $R > R_0$ we have

$$\|\sigma(\chi^{iR}, \eta^{iR})\|_{\mathbf{L}^2(\partial B_R)} \leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R^2}.$$

Proof. Take ϕ to be a smooth cut-off function defined to be zero in the neighborhood of Y and 1 outside B_{R_0} . The pair $\{\phi \chi^{iR}, \phi \eta^{iR}\}$ satisfies the following Stokes system:

$$\begin{cases} -\mu_2 \Delta (\phi \chi^{iR}) + \nabla(\phi \eta^{iR}) = \mathbf{f}^R & \text{in } B_R, \\ \operatorname{div} (\phi \chi^{iR}) = g^R & \text{in } B_R, \\ \phi \chi^{iR} = \mathbf{0} & \text{on } \partial B_R. \end{cases}$$

If $\{\tilde{\chi}^i, \tilde{\eta}^i\}$ is the solution pair for the corresponding Stokes system in free space, then by the previous lemma we have the following pointwise estimates for $|\mathbf{x}| > R_0$:

$$\begin{aligned} |D^\alpha \tilde{\chi}^i(x)| &\leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{|\mathbf{x}|^{1+|\alpha|}} \text{ for } |\alpha| \leq 2, \\ |D^\alpha \tilde{\eta}^i(x)| &\leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{|\mathbf{x}|^{2+|\alpha|}} \text{ for } |\alpha| \leq 1. \end{aligned}$$

Using the above estimates for any $R > R_0$ we get

$$\|D^\alpha \tilde{\chi}^i\|_{\mathbf{L}^2(B_{2R} \setminus B_R)} \leq \frac{C \|\mathbf{e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R^{2|\alpha|-1}},$$

$$\|D^\alpha \tilde{\eta}^i\|_{\mathbf{L}^2(B_{2R} \setminus B_R)} \leq \frac{C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R^{2|\alpha|+1}},$$

and furthermore we obtain the following bound for the stress:

$$\begin{aligned} \|\sigma(\tilde{\chi}^i, \tilde{\eta}^i)\|_{\mathbf{L}^2(\partial B_R)} &= \|2\mu_2 e(\tilde{\chi}^i) - \tilde{\eta}^i I\|_{\mathbf{L}^2(\partial B_R)} \\ &\leq C\{\|D\tilde{\chi}^i\|_{\mathbf{L}^2(\partial B_R)} + \|\tilde{\eta}^i\|_{\mathbf{L}^2(\partial B_R)}\} \\ &\leq \frac{C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R^2}. \end{aligned}$$

After changing variables we obtain

$$\left\| \tilde{\chi}^i \left(\frac{R}{R_0} \cdot \right) \right\|_{\mathbf{H}^2(B_{2R_0} \setminus B_{R_0})} \leq C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)} \frac{1}{R}$$

and

$$\left\| \tilde{\eta}^i \left(\frac{R}{R_0} \cdot \right) \right\|_{\mathbf{H}^1(B_{2R_0} \setminus B_{R_0})} \leq C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)} \frac{1}{R^2}.$$

Define $\mathbf{u}^i = \phi \chi^{iR} \left(\frac{R}{R_0} \mathbf{x} \right) - \phi \tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{x} \right)$ and $p^i = \frac{R}{R_0} \phi \eta^{iR} \left(\frac{R}{R_0} \mathbf{x} \right) - \frac{R}{R_0} \phi \tilde{\eta}^i \left(\frac{R}{R_0} \mathbf{x} \right)$. The pair $\{\mathbf{u}^i, p^i\}$ satisfies the following Stokes system:

$$\begin{cases} -\mu_2 \Delta \mathbf{u}^i + \nabla p^i = \mathbf{0} & \text{in } B_{R_0}, \\ \operatorname{div} \mathbf{u}^i = 0 & \text{in } B_{R_0}, \\ \mathbf{u}^i = -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{x} \right) & \text{on } \partial B_{R_0}. \end{cases}$$

The regularity results in [18, Proposition 2.2] yield

$$\begin{aligned} \|\mathbf{u}^i\|_{\mathbf{H}^2(B_{R_0})} + \|p^i\|_{H^1(B_{R_0})} &\leq \left\| -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{x} \right) \right\|_{\mathbf{H}^{3/2}(\partial B_{R_0})} \\ &\leq C \left\| -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{x} \right) \right\|_{\mathbf{H}^2(B_{2R_0} \setminus B_{R_0})} \leq \frac{C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R}. \end{aligned}$$

Using the results above we can compute

$$\begin{aligned} \|\sigma(\mathbf{u}^i, p^i)\|_{\mathbf{L}^2(\partial B_{R_0})} &= \|2\mu_2 e(\mathbf{u}^i) - p^i I\|_{\mathbf{L}^2(\partial B_{R_0})} \\ &\leq C \left\{ \|\mathbf{u}^i\|_{\mathbf{H}^2(B_{R_0})} + \|p^i\|_{H^1(B_{R_0})} \right\} \leq \frac{C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}}{R}. \end{aligned}$$

Applying a change of variable yet again we obtain

$$\|\sigma(\mathbf{u}^i, p^i)\|_{\mathbf{L}^2(\partial B_{R_0})} = \|\sigma(\chi^{iR}, \eta^{iR}) - \sigma(\tilde{\chi}^i, \tilde{\eta}^i)\|_{\mathbf{L}^2(\partial B_R)}.$$

Hence,

$$\|\sigma(\chi^{iR}, \eta^{iR})\|_{\mathbf{L}^2(\partial B_R)}^2 \leq C \|\mathbf{-e}_i + \mathbf{c} \times \mathbf{y}\|_{\mathbf{L}^2(\partial T)}^2 \frac{1}{R^2}.$$

□

Acknowledgments. The authors were supported by the NSF grant DMS-1109356.

REFERENCES

- [1] Grégoire Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes*, Arch. Rational Mech. Anal. **113** (1990), no. 3, 209–259, DOI 10.1007/BF00375065. MR1079189 (91k:35031a)
- [2] Habib Ammari, Pierre Garapon, Hyeonbae Kang, and Hyundae Lee, *Effective viscosity properties of dilute suspensions of arbitrarily shaped particles*, Asymptot. Anal. **80** (2012), no. 3-4, 189–211. MR3025042
- [3] H. Attouch, *Variational convergence for functions and operators*, Applicable Mathematics Series, Pitman (Advanced Publishing Program), Boston, MA, 1984. MR773850 (86f:49002)
- [4] G. K. Batchelor, *The stress system in a suspension of force-free particles*, J. Fluid Mech. **41** (1970), 545–570.
- [5] Eric Bonnetier, David Manceau, and Faouzi Triki, *Asymptotic of the velocity of a dilute suspension of droplets with interfacial tension*, Quart. Appl. Math. **71** (2013), no. 1, 89–117, DOI 10.1090/S0033-569X-2012-01275-7. MR3075537
- [6] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (French), North-Holland Mathematics Studies, No. 5, Notas de Matemática (50), North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. MR0348562 (50 #1060)
- [7] Alain Brillard, *Asymptotic analysis of incompressible and viscous fluid flow through porous media. Brinkman's law via epi-convergence methods* (English, with French summary), Ann. Fac. Sci. Toulouse Math. (5) **8** (1986/87), no. 2, 225–252. MR928845 (89d:35138)
- [8] S. J. Choi and W. R. Schowalter, *Rheological properties of nondilute suspensions of deformable particles*, Physics of Fluids **18** (1975), 420–427.
- [9] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs*, Nonlinear partial differential equations and their applications, College de France Seminar, II & III, ed. H. Brezis and J. L. Lions, Research Notes in Math. no. 60870, Pitman, 1982, pp. 98–138, pp. 154–178. MR0652509, MR0670272
- [10] N. A. Frankel and A. Acrivos, *The constitutive equation for a dilute emulsion*, J. Fluid Mech. **28** (1970), 657–673.
- [11] L. G. Leal, *Laminar Flow and Convective Transport Processes*, Butterworth-Heinemann, Stoneham, Massachusetts, 1992.
- [12] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (French), Dunod; Gauthier-Villars, Paris, 1969. MR0259693 (41 #4326)
- [13] Robert Lipton and Marco Avellaneda, *Darcy's law for slow viscous flow past a stationary array of bubbles*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), no. 1-2, 71–79, DOI 10.1017/S0308210500024276. MR1051608 (91e:76050)
- [14] Robert Lipton and Bogdan Vernescu, *Homogenisation of two-phase emulsions*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 6, 1119–1134, DOI 10.1017/S0308210500030146. MR1313192 (96j:76144)
- [15] F. Maris and B. Vernescu, *Random homogenization for a fluid flow through a membrane*, Asymptot. Anal. **86** (2014), no. 1, 17–48. MR3177474
- [16] W. R. Schowalter, C. C. Chaffey and H. Brenner, *Rheological behavior of a dilute emulsion*, J. Colloid Int. Sci., **26** (1968), 152–160.
- [17] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997. MR1422252 (98c:47076)
- [18] Roger Temam, *Navier-Stokes equations*, 3rd ed., Studies in Mathematics and its Applications, Theory and numerical analysis, with an appendix by F. Thomasset, vol. 2, North-Holland Publishing Co., Amsterdam, 1984. MR769654 (86m:76003)
- [19] G. I. Taylor, *The viscosity of a fluid containing small drops of another fluid*, Proc. Roy. Soc. London Ser. A **138** (1932), 41–48.
- [20] B. Vernescu, *Convergence results for the homogenization of flow in fractured porous media*, IMA Preprint Series, 732, 1990, available at <http://www.ima.umn.edu/preprints/Nov90Series/Nov90Series.html>

- [21] Z. Zapryanov and S. Tabakova, *Dynamics of bubbles, drops and rigid particles*, Fluid Mechanics and its Applications, vol. 50, Kluwer Academic Publishers Group, Dordrecht, 1999. MR1673353 (99m:76112)