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A UNIFORM ESTIMATE FOR THE INCOMPRESSIBLE MAGNETO-HYDRODYNAMICS EQUATIONS WITH A SLIP BOUNDARY CONDITION

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Abstract. In this paper, we derive a uniform estimate of the strong solution to the incompressible magneto-hydrodynamic (MHD) system with a slip boundary condition in a conormal Sobolev space with viscosity weight. As a consequence of this uniform estimate, we obtain that the solution of the viscous MHD system converges strongly to a solution of the ideal MHD system from a compactness argument.

Contents

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1. Introduction. The homogeneous incompressible magneto-hydrodynamic systems (MHD) model the interaction between a magnetic field and a viscous incompressible fluid of moving electrically charged particles. It is governed by the following equations:

$$\partial_t u - \varepsilon\Delta u + (u\cdot\nabla)u - (H\cdot\nabla)H + \nabla p = 0, \tag{1.1}$$

$$\partial_t H - \varepsilon\Delta H + (u\cdot\nabla)H - (H\cdot\nabla)u = 0, \tag{1.2}$$

$$\nabla\cdot u = 0, \qquad \nabla\cdot H = 0, \tag{1.3}$$

in the domain $t\in[0,T)$, $x=(x_1,x_2,x_3)\in\Omega\subset\mathbb{R}^3$, where $u\in\mathbb{R}^3$ and $H\in\mathbb{R}^3$ denote the fluid velocity and magnetic field respectively, $p(t,x)$ is the fluid pressure, and ε is the viscosity coefficient. Here we have assumed the viscosites for the fluid and magnetic field are the same. We impose the following impermeable and friction boundary conditions for the equations (1.1)-(1.3):

$$u\cdot n = 0, \qquad (D(u)\cdot n)_\tau + \beta u_\tau = 0 \quad \text{on} \quad [0,T)\times\partial\Omega, \tag{1.4}$$

$$H\cdot n = 0, \qquad (D(H)\cdot n)_\tau + \gamma H_\tau = 0 \quad \text{on} \quad [0,T)\times\partial\Omega, \tag{1.5}$$

where $D(u)=\frac{1}{2}(\nabla u + (\nabla u)^T)$ and $D(H)=\frac{1}{2}(\nabla H + (\nabla H)^T)$ are the rate of the strain tensors, n and τ are unit normal and tangent vectors on the boundary $\partial\Omega$, with u_τ, H_τ denoting the tangential components of u, H respectively, e.g. $u_\tau = u\cdot\tau$. The boundary condition (1.4) means that the slip velocity is proportional to the shear stress; it was introduced by Navier [18] in studying the large eddy simulation, so it is also called the Navier friction boundary condition. The condition (1.5) is adaptable to the systems, since it ensures the boundary balance of the quantities on the boundary, as explained in [26].

One important question in fluid mechanics is, when the viscosities tend to zero whether the viscous flows can be approximated by the inviscid flows. In general, this approximation does not hold; as Prandtl observed in [21], there are boundary layers near the physical boundary as the viscosities vanish. The behavior of boundary layers in problems with the non-slip boundary condition was formally studied by Prandtl in [21], and the well-posedness of the Prandtl boundary layer equations has been studied by many mathematicians; cf. [1, 3–5, 16, 19, 20, 28] and references therein. One of the most challenging problems is to establish a rigorous theory on the Prandtl boundary layer theory, i.e. whether the viscous flows converge to the superposition of the inviscid flows away from the boundary and the Prandtl boundary layers in a neighborhood of boundary when the viscosities go to zero. This problem is wide open except in the cases where the data is analytically studied [22], the flows are circularly symmetric ([13]) or, the recent work [17] that the initial vorticity is supported away from the boundary in two space variables. Another approach had also been given by Kato [10], and improved later by Temam, Wang [23] and Kelliher [11], for seeking necessary and sufficient conditions on vorticity to have the vanishing viscosity limit of solutions to the Navier-Stokes equations with no-slip boundary conditions converging to a strong solution of the Euler equations. There are also many interesting works on problems with the Navier slip boundary conditions. For the two-dimensional problem, Yodovich [29] and Lions [12] studied the vanishing viscosity limit for the incompressible Navier-Stokes equations with a free boundary condition, $u\cdot n = 0$ and curl $u = 0$ on $\partial\Omega$ with curl u denoting the vorticity. For the two-dimensional

Navier-Stokes equations with the Navier friction condition, Clopeau, Mikelic and Robert ([2]), Lopes Filho, Nussenzveig Lopes and Planas ([14]) obtained the vanishing viscosity limit under certain boundedness assumptions on the initial vorticity when the slip length is a constant. Recently, Iftimie and Sueur in [9], and Wang, Wang and Xin [25] have investigated the boundary layer behavior of the Navier-Stokes equations with the Navier boundary condition for different scales of slip length; other related works can be found in [11, 26, 27] and references therein. Recently, Masmoudi and Rousset in [15] have obtained a uniform estimate of solutions to the Navier-Stokes equations with the Navier boundary condition in the conormal spaces, from which follows the small viscosity limit result immediately.

Till now, there have been few works on the small viscosity limit for the initial boundary value problem of the magneto-hydrodynamic system, except that the compressible MHD equations with a non-characteristic boundary condition have been studied in [7], and Xiao and Xin [26] have studied the problem of the incompressible magneto-hydrodynamic equations (1.1)-(1.5) with the complete slip case, $\beta = \gamma = 0$, and showed that any regularity solution of the viscous incompressible MHD system converges to a corresponding solution of the ideal MHD system. The main goal of this work is to use the idea of [15] to study the problem of the incompressible magneto-hydrodynamic system (1.1)-(1.3) with the general Navier friction boundary conditions (1.4)-(1.5).

For simplicity of presentation, we shall consider only the problem (1.1)-(1.5) in the half plane $\Omega = \mathbb{R}^3_+ = \{x = (x_1, x_2, x_3), x_3 > 0\}$. In this case, the boundary conditions can be simplified as

$$u_3 = 0, \quad \partial_3 u_i = 2\beta u_i, \quad i = 1, 2 \quad \text{on} \, [0, T) \times \partial\Omega, \tag{1.6}$$

$$H_3 = 0, \quad \partial_3 H_i = 2\gamma H_i, \quad i = 1, 2 \quad \text{on} \, [0, T) \times \partial\Omega. \tag{1.7}$$

In this paper, we are interested in establishing a uniform bound in a time interval independent of $\varepsilon \in (0, 1]$ for the strong solution of the problem (1.1)-(1.3) with the boundary conditions (1.6)-(1.7), from which we deduce that as the viscosity ϵ goes to zero, this solution converges strongly to a solution of the ideal MHD system,

$$\begin{cases} \partial_t u + u \cdot \nabla u - H \cdot \nabla H + \nabla p = 0, \quad \text{in} \quad [0, T) \times \Omega \\ \partial_t H + u \cdot \nabla H - H \cdot \nabla u = 0, \quad \text{in} \quad [0, T) \times \Omega \\ \nabla \cdot u = 0, \quad \nabla \cdot H = 0, \quad \text{in} \quad [0, T) \times \Omega \\ u \cdot n = 0, \quad H \cdot n = 0 \quad \text{on} \quad [0, T) \times \partial\Omega. \end{cases} \tag{1.8}$$

The remainder is organized as follows. In Section 2, we state the main result on the uniform bound of solutions to the equations (1.1)-(1.3), (1.6) and (1.7). Section 3 is devoted to the proof of the main result, in which we first give an energy estimate in conormal spaces; then we turn to estimating the normal derivatives of velocity and magnetic field, and the pressure. Finally, by deriving an L^∞ estimate by using the maximal principle and combining all estimates, we conclude the uniform a priori estimate.

2. Statement of main results. Before stating the main results, we first recall the notation of conormal spaces as given in [6] and [15]. Introduce the tangential vector fields of the boundary $\{x_3 = 0\}$, $Z_i = \partial_i$, $i = 1, 2$, $Z_3 = \varphi(x_3)\partial_{x_3}$, where $\varphi(x_3)$ is a smooth bounded function such that $\varphi(0)=0$, $\varphi'(0)\neq 0$. Define the conormal Sobolev spaces $H^m_{co}(\Omega)$ for an integer m as

$$H^m_{co}(\Omega)=\{f\in L^2(\Omega)|\ Z^\alpha f\in L^2(\Omega),\forall|\alpha|\le m\}$$

equipped norms $\|f\|_m^2=\sum\limits_{|\alpha|\le m}\|Z^\alpha f\|^2_{L^2}$, with $Z^\alpha=Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}$. Similarly, we define

$$W^{m,\infty}_{co}(\Omega)=\{f\in L^\infty(\Omega)|\ Z^\alpha f\in L^\infty(\Omega),\forall|\alpha|\le m\}$$

with $\|f\|_{m,\infty}=\sum\limits_{|\alpha|\le m}\|Z^\alpha f\|_{L^\infty}$.

Throughout this paper, we denote by $\|\cdot\|_{H^s}$, $\|\cdot\|_{W^{k,\infty}}$ the usual Sobolev norms, and by $\|\cdot\|$ and $(\cdot,\cdot)$ the L^2 norm and scalar product respectively for functions defined in Ω, while $|\cdot|_{H^m(\partial\Omega)}$ denotes the standard Sobolev norms of functions defined on $\partial\Omega$. We shall also use the notation $z=x_3$ and $x=(x_1,x_2,z)\in\mathbb{R}^3_+$, $u=(u_h,u_3)^T\in\mathbb{R}^3$ and $H=(H_h,H_3)^T\in\mathbb{R}^3$ with $u_h=(u_1,u_2)^T, H_h=(H_1,H_2)^T$, and $E^m=\{u\in H^m_{co},\nabla u\in H^{m-1}_{co}\}$.

The main results of this paper can be stated as follows:

THEOREM 2.1. For a fixed integer $m\ge 6$, assume that $(u_0,H_0)\in(E^m(\mathbb{R}^3_+))^2$ satisfy $(\nabla u_0,\nabla H_0)\in(W^{1,\infty}_{co}(\mathbb{R}^3_+))^2$, $\nabla\cdot u_0=0,\nabla\cdot H_0=0$ and $(u_3^0,H_3^0)|_{z=0}=0$, with u_3^0 and H_3^0 denoting the third components of u_0 and H_0 respectively. Then, for any smooth solution of the problems (1.1)-(1.3), (1.6)-(1.7) with the initial data

$$u|_{t=0}=u_0,\quad H|_{t=0}=H_0 \tag{2.1}$$

in $(t,x)\in[0,T]\times\mathbb{R}^3_+$, there exists $C>0$ independent of $\varepsilon\in(0,1]$ and β,γ such that the a priori estimate

$$Q_m(t)\le C\Big(Q_m(0)+(1+t+\varepsilon^3t^2)\int_0^t(Q_m(s)+Q_m^2(s))ds\Big),\quad \forall t\in[0,T] \tag{2.2}$$

holds, where

$$Q_m(t)=\|u(t)\|_m^2+\|\nabla u(t)\|^2_{m-1}+\|\nabla u\|^2_{1,\infty}+\|H(t)\|^2_m+\|\nabla H(t)\|^2_{m-1}+\|\nabla H\|^2_{1,\infty}.$$

Furthermore, we have

THEOREM 2.2. For a fixed integer $m\ge 6$, let the initial data u_0 and H_0 satisfy the same conditions as given in Theorem 2.1. Then there exists $T>0$ independently of $\varepsilon\in(0,1]$, such that the problems (1.1), (1.2), (1.3), (1.6), (1.7) and (2.1) have a unique solution $(u^\varepsilon,H^\varepsilon)\in(C([0,T];E^m))^2$. Moreover, there exists $C>0$ independently of ε such that the following estimate holds:

$$\begin{aligned}\sup_{t\in[0,T]}&(\|(u^\varepsilon,H^\varepsilon)(t)\|_m+\|(\nabla u^\varepsilon,\nabla H^\varepsilon)(t)\|_{m-1}+\|(\nabla u^\varepsilon,\nabla H^\varepsilon)(t)\|_{1,\infty})\\&+\varepsilon\int_0^T(\|\Delta u^\varepsilon(s)\|^2_{m-1}+\|\Delta H^\varepsilon(s)\|^2_{m-1})ds\le C.\end{aligned} \tag{2.3}$$

REMARK 2.3. (1) The estimate (2.3) shall be obtained from (2.2) directly. The existence of a strong solution to the initial boundary value problems (1.1), (1.2), (1.3), (1.6), (1.7) and (2.1) can be obtained in a standard manner as given in [15, 23] by smoothing the initial data and using the a priori estimate given in Theorem 2.1 and the strong compactness argument. The uniqueness of the solution is clear by noting the Lipschitz regularity of the solutions.

(2) By using Theorem 2.2 and a strong compact argument, and passing to the limit as $\varepsilon \to 0$, one also can obtain the existence of a local strong solution to the ideal MHD equations and the convergence from the problem of viscous MHD equations (1.1), (1.2), (1.3), (1.6), (1.7) and (2.1) to the corresponding problem of the ideal MHD equations (1.8).

In the following discussion, we shall always use the notation $A \lesssim B$ to denote the inequality $A \leq CB$ holding for an absolute constant $C > 0$. In the proof of Theorem 2.1, we shall frequently use the following elementary inequalities on the product of two functions, the commutators and the Sobolev embedding in the conormal Sobolev spaces; their proofs can be found in [6, 15]:

LEMMA 2.4. (1) For any given $u, v \in L^\infty \cap H^m_{co}$, $m \geq 0$, we have

$$\|Z^\alpha u \cdot Z^\beta v\| \lesssim \|u\|_{L^\infty}\|v\|_m + \|v\|_{L^\infty}\|u\|_m, \quad |\alpha| + |\beta| = m. \tag{2.4}$$

(2) For any integer $m \geq 1$, $g \in H^{m-1}_{co} \cap L^\infty$ and $f \in H^m_{co}$ with $Zf \in L^\infty$, we have

$$\|[Z^\alpha, f]g\| \lesssim \|Zf\|_{L^\infty}\|g\|_{m-1} + \|g\|_{L^\infty}\|f\|_m, \quad 1 \leq |\alpha| \leq m. \tag{2.5}$$

(3) For $m > 1$, it holds that

$$\|f\|^2_{L^\infty(\mathbb{R}^3_+)} \lesssim \|\partial_z f\|_m \|f\|_m + \|f\|^2_m. \tag{2.6}$$

3. Proof of main results. The remaining main task is to prove the a priori estimate given in Theorem 2.1. It will be divided into the following four subsections.

3.1. *Energy estimates in conormal Sobolev spaces.* In this subsection, we establish energy estimates in the conormal Sobolev spaces. First, we have the following identity.

PROPOSITION 3.1. Assume that (u, H) is a smooth solution of the problem (1.1)-(1.3), (1.6)-(1.7). Then the following energy identity holds:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|H(t)\|^2) + \varepsilon(\|\nabla u(t)\|^2 + \|\nabla H(t)\|^2)&\\ +2\varepsilon(\beta|u_h(t)|^2_{L^2(\partial\Omega)} + \gamma|H_h(t)|^2_{L^2(\partial\Omega)}) = 0.&\end{aligned} \tag{3.1}$$

Proof. Multiplying (1.1) by u and (1.2) by H, respectively, and integrating the resulting equations with respect to the space variables over $\mathbb{R}^3_+$, we get the identity (3.1) immediately by using the divergence free condition (1.3) and the boundary conditions (1.6), (1.7). □

In what follows, we study the conormal energy estimates. They will involve the estimates of pressure and normal derivatives, which will be discussed in the next two subsections.

PROPOSITION 3.2. Assume that (u, H) is a smooth solution of the systems (1.1)-(1.3), (1.6)-(1.7). Then for any $m \geq 0$, $\beta, \gamma > 0$, the following estimate holds:

$$\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_m^2 + \|H(t)\|_m^2) + C_0\varepsilon(\|\nabla u(t)\|_m^2 + \|\nabla H(t)\|_m^2) \\
&\lesssim (1 + \|u(t)\|_{W^{1,\infty}} + \|H(t)\|_{W^{1,\infty}}) \times (\|u(t)\|_m^2 + \|H(t)\|_m^2 \\
&\quad + \|\partial_z u(t)\|_{m-1}^2 + \|\partial_z H(t)\|_{m-1}^2) + \|\nabla p\|_{m-1}\|u(t)\|_m,
\end{aligned} \tag{3.2}$$

for a constant $C_0 > 0$ independent of ε.

Proof. The case $m = 0$ is easily obtained from Proposition 3.1 by using the convention $\| \cdot \|_k = 0$ as $k \leq -1$.

For $m \geq 1$, applying the operator $Z^\alpha (|\alpha| = m)$ to the equations (1.1) and (1.2), one gets

$$\partial_t Z^\alpha u - \varepsilon \Delta Z^\alpha u + (u \cdot \nabla) Z^\alpha u - (H \cdot \nabla) Z^\alpha H + \nabla Z^\alpha p = R_1, \tag{3.3}$$

$$\partial_t Z^\alpha H - \varepsilon \Delta Z^\alpha H + (u \cdot \nabla) Z^\alpha H - (H \cdot \nabla) Z^\alpha u = S_1, \tag{3.4}$$

where

$$R_1 = \sum_{j=1}^{4} R_{1j}, \qquad S_1 = \sum_{j=1}^{3} S_{1j},$$

with

$$R_{11} = \varepsilon[Z^\alpha, \Delta]u, \quad R_{12} = -[Z^\alpha, u \cdot \nabla]u, \quad R_{13} = [Z^\alpha, H \cdot \nabla]H, \quad R_{14} = -[Z^\alpha, \nabla]p,$$

and

$$S_{11} = \varepsilon[Z^\alpha, \Delta]H, \quad S_{12} = -[Z^\alpha, u \cdot \nabla]H, \quad S_{13} = [Z^\alpha, H \cdot \nabla]u.$$

Multiplying (3.3) by $Z^\alpha u$, (3.4) by $Z^\alpha H$, respectively, integrating with respect to the space variables and summing up the resulting equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|Z^\alpha u\|^2 + \|Z^\alpha H\|^2) + \varepsilon(\|\nabla Z^\alpha u\|^2 + \|\nabla Z^\alpha H\|^2) \\
&\quad + \varepsilon\left(\int_{\partial\Omega} \partial_3 Z^\alpha u \cdot Z^\alpha u ds + \int_{\partial\Omega} \partial_3 Z^\alpha H \cdot Z^\alpha H ds\right) \\
&\qquad - \int_{\partial\Omega} Z^\alpha p Z^\alpha u_3 ds - \int_\Omega Z^\alpha p (\nabla \cdot (Z^\alpha u)) dx \\
&\qquad\qquad = (R_1, Z^\alpha u) + (S_1, Z^\alpha H).
\end{aligned} \tag{3.5}$$

Now, we study (3.5) in the following three steps.

STEP 1. Study of the boundary terms.

Notice that from the boundary conditions (1.6) and (1.7), we have

$$Z^\alpha u_3 = 0, \partial_z Z^\alpha u_h = 2\beta Z^\alpha u_h - [Z^\alpha, \partial_z]u_h, \quad x \in \partial\Omega,$$

$$Z^\alpha H_3 = 0, \partial_z Z^\alpha H_h = 2\gamma Z^\alpha H_h - [Z^\alpha, \partial_z]H_h, \quad x \in \partial\Omega,$$

which imply

$$\int_{\partial\Omega} \partial_3 Z^\alpha u \cdot Z^\alpha u ds = 2\beta \int_{\partial\Omega} |Z^\alpha u_h|^2 ds - \int_{\partial\Omega} [Z^\alpha, \partial_z] u_h \cdot Z^\alpha u_h ds, \tag{3.6}$$

$$\int_{\partial\Omega} \partial_3 Z^\alpha H \cdot Z^\alpha H ds = 2\gamma \int_{\partial\Omega} |Z^\alpha H_h|^2 ds - \int_{\partial\Omega} [Z^\alpha, \partial_z] H_h \cdot Z^\alpha H_h ds, \tag{3.7}$$

$$\int_{\partial\Omega} Z^\alpha p Z^\alpha u_3 ds = 0. \tag{3.8}$$

Thanks to the trace theorem and the Young inequality, one gets

$$\begin{aligned} \varepsilon|\int_{\partial\Omega} [Z^\alpha, \partial_z] u_h \cdot Z^\alpha u_h ds| &\le \varepsilon |[Z^\alpha, \partial_z] u_h|_{L^2(\partial\Omega)} |Z^\alpha u_h|_{L^2(\partial\Omega)} \\ &\le \frac{1}{2}\varepsilon \|\nabla Z^\alpha u_h\|^2 + C\|u\|_m^2 + C\varepsilon |[Z^\alpha, \partial_z] u_h|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{3.9}$$

On the other hand, since $[Z_i, \partial_z] = 0$ $(i = 1, 2)$ and $[Z_3, \partial_z] u_h|_{\partial\Omega} = -\varphi'(z)\partial_z u_h|_{\partial\Omega} = -2\beta\varphi'(z) u_h|_{\partial\Omega}$, it follows that

$$\varepsilon |[Z^\alpha, \partial_z] u_h|^2_{L^2(\partial\Omega)} \lesssim \varepsilon |u_h|^2_{H^{m-1}(\partial\Omega)} \lesssim \varepsilon \|\partial_z u\|_{m-1} \|u\|_{m-1} \tag{3.10}$$

by using the trace theorem. Consequently,

$$\varepsilon|\int_{\partial\Omega} [Z^\alpha, \partial_z] u_h \cdot Z^\alpha u_h ds| \le \frac{1}{2}\varepsilon \|\nabla Z^\alpha u_h\|^2 + C\|u\|_m^2 + C\varepsilon \|\partial_z u\|_{m-1} \|u\|_{m-1}. \tag{3.11}$$

Similarly, we have

$$\varepsilon|\int_{\partial\Omega} [Z^\alpha, \partial_z] H_h \cdot Z^\alpha H_h ds| \le \frac{1}{2}\varepsilon \|\nabla Z^\alpha H_h\|^2 + C\|H\|_m^2 + C\varepsilon \|\partial_z H\|_{m-1} \|H\|_{m-1}. \tag{3.12}$$

STEP 2. Estimate of the term $\int_\Omega Z^\alpha p \cdot (\nabla \cdot (Z^\alpha u)) dx$.

It follows from the divergence free conditions (1.3) that

$$\nabla \cdot (Z^\alpha u) = [\partial_z, Z^\alpha] u_3. \tag{3.13}$$

Since $[\partial_z, Z_i] u_3 = 0$ $(i = 1, 2)$, $[\partial_z, Z_3] u_3 = \varphi'(z)\partial_z u_3 = -\varphi'(z)\nabla_h \cdot u_h$, we easily deduce that

$$\|[\partial_z, Z^\alpha] u_3\| \lesssim \|u\|_m. \tag{3.14}$$

Consequently, we have

$$|\int_\Omega Z^\alpha p (\nabla \cdot (Z^\alpha u)) dx| \lesssim \|\nabla p\|_{m-1} \|u\|_m. \tag{3.15}$$

Plugging estimates (3.6)-(3.8) and (3.11), (3.12) and (3.15) into (3.5) and noting that $\frac{1}{2}\varepsilon\|\nabla Z^\alpha u_h\|^2$ and $\frac{1}{2}\varepsilon\|\nabla Z^\alpha H_h\|^2$ in (3.11) and (3.12) can be absorbed by the left hand side in (3.5), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|Z^\alpha u\|^2 + \|Z^\alpha H\|^2) + \varepsilon(\|\nabla Z^\alpha u\|^2 + \|\nabla Z^\alpha H\|^2) \\ &+2\varepsilon(\beta |Z^\alpha u_h|^2_{L^2(\partial\Omega)} + \gamma |Z^\alpha H_h|^2_{L^2(\partial\Omega)}) \\ &\lesssim \|\nabla p\|_{m-1}\|u\|_m + \|\partial_z u\|_{m-1}\|u\|_{m-1} + \|\partial_z H\|_{m-1}\|H\|_{m-1} \\ &+\|u\|_m^2 + \|H\|_m^2 + |(R_1, Z^\alpha u)| + |(S_1, Z^\alpha H)|. \end{aligned} \tag{3.16}$$

STEP 3. Estimate of commutators.

Since

$$[Z_i,\Delta]u=0,\quad i=1,2,\qquad [Z_3,\Delta]u=-2\varphi'\partial_z^2u-\varphi''\partial_z u, \tag{3.17}$$

one gets by using this property repeatedly

$$|(R_{11},Z^\alpha u)|\lesssim \tilde{R}_{11}+\varepsilon\|\partial_z u\|_{m-1}^2+\|u\|_m^2 \tag{3.18}$$

with $\tilde{R}_{11}=\sum\limits_{0\le|\nu|\le m-1}\varepsilon|(C_\nu\partial_z^2Z_3^\nu u,Z^\alpha u)|$ for some functions C_ν depending on the derivatives of φ.

By using integration by parts and $\partial_z Z_3^m u=(Z_3^m+\sum_{l=0}^{m-1}C_{l,m}Z_3^l)\partial_z u$ with $C_{l,m}$ only depending on the derivatives of φ, we deduce

$$\begin{aligned}
&|(C_\nu\partial_z^2Z_3^\nu u,Z^\alpha u)|\\
\lesssim\ &\|\partial_z u\|_m\|\partial_z u\|_{m-1}+\|u\|_m\|\partial_z u\|_{m-1}+|\partial_z u_h|_{L^2(\partial\Omega)}|Z^\alpha u|_{L^2(\partial\Omega)}\\
\lesssim\ &\|\partial_z u\|_m\|\partial_z u\|_{m-1}+\|u\|_m\|\partial_z u\|_{m-1}+|u|_{L^2(\partial\Omega)}|Z^\alpha u|_{L^2(\partial\Omega)}\\
\lesssim\ &\|\partial_z u\|_m\|\partial_z u\|_{m-1}+\|u\|_m\|\partial_z u\|_{m-1}+\|u\|_m\|\partial_z u\|_m.
\end{aligned} \tag{3.19}$$

In the third line above, we use the boundary condition $\partial_z u_h=2\beta u_h$, and in the fourth line we apply the trace theorem. Consequently, by plugging (3.19) into (3.18) it follows that

$$|(R_{11},Z^\alpha u)|\lesssim\varepsilon(\|\partial_z u\|_{m-1}+\|u\|_m)\|\partial_z u\|_m+\|\partial_z u\|_{m-1}^2+\|u\|_m^2. \tag{3.20}$$

Performing the same arguments as above, we deduce that

$$|(S_{11},Z^\alpha H)|\lesssim\varepsilon(\|\partial_z H\|_{m-1}+\|H\|_m)\|\partial_z H\|_m+\|\partial_z H\|_{m-1}^2+\|H\|_m^2. \tag{3.21}$$

We go on investigating the estimates of R_{12} and R_{13}. Obviously, we have

$$R_{12}=-[Z^\alpha,u\cdot\nabla]u=\sum_{\lambda+\nu=\alpha,|\lambda|\ge1}C_{\lambda,\nu}Z^\lambda u\cdot Z^\nu(\nabla u)-u\cdot[Z^\alpha,\nabla]u,$$

with

$$\begin{aligned}
\|u\cdot[Z^\alpha,\nabla]u\|=\|u_3\cdot[Z^\alpha,\partial_z]u\|&\lesssim\sum_{|\nu|\le m-1}\|u_3\cdot\partial_z Z^\nu u\|\\
&=\sum_{|\nu|\le m-1}\|\frac{u_3}{\varphi(z)}\cdot Z^{\nu+1}u\|\lesssim\|\partial_z u_3\|_{L^\infty}\|u\|_m
\end{aligned}$$

and

$$\begin{aligned}
\|Z^\lambda u\cdot Z^\nu(\nabla u)\|&\lesssim\|Z^\lambda u_h\cdot Z^\nu(\nabla_h u)\|+\|Z^\lambda u_3\cdot Z^\nu(\partial_z u)\|\\
&\lesssim\|Zu\|_{L^\infty}(\|u\|_m+\|\partial_z u\|_{m-1})+\|\partial_z u\|_{L^\infty}\|Zu_3\|_{m-1}\\
&\lesssim\|\nabla u\|_{L^\infty}(\|u\|_m+\|\partial_z u\|_{m-1}),
\end{aligned}$$

where we use the inequality (2.4) in Lemma 2.4. Thus, we get

$$\|R_{12}\|\lesssim\|\nabla u\|_{L^\infty}(\|u\|_m+\|\partial_z u\|_{m-1}). \tag{3.22}$$

Similarly, one can obtain

$$\|R_{13}\|\lesssim\|\nabla H\|_{L^\infty}(\|H\|_m+\|\partial_z H\|_{m-1}). \tag{3.23}$$

Applying the same argument as above to estimates $\|S_{12}\|$ and $\|S_{13}\|$, we get

$$S_{12} = -[Z^\alpha, u\cdot\nabla]H = \sum_{\lambda+\nu=\alpha, |\lambda|\ge 1} C_{\lambda,\nu} Z^\lambda u \cdot Z^\nu(\nabla H) - u\cdot [Z^\alpha, \nabla]H,$$

with

$$\|u\cdot[Z^\alpha,\nabla]H\| \lesssim \sum_{|\nu|\le m-1} \|u_3\cdot\partial_z Z^\nu H\| \lesssim \|\partial_z u_3\|_{L^\infty}\|H\|_m$$

and

$$\begin{aligned}
&\|Z^\lambda u\cdot Z^\nu(\nabla H)\| \lesssim \|Z^\lambda u_h\cdot Z^\nu(\nabla_h H)\| + \|Z^\lambda u_3 \cdot Z^\nu(\partial_z H)\|\\
&\lesssim \|Zu\|_{L^\infty}\|H\|_m + \|\nabla_h H\|_{L^\infty}\|Zu_h\|_{m-1} + \|Zu_3\|_{L^\infty}\|\partial_z H\|_{m-1} + \|\partial_z H\|_{L^\infty}\|Zu_3\|_{m-1}\\
&\lesssim \|\nabla u\|_{L^\infty}(\|H\|_m + \|\partial_z H\|_{m-1}) + \|\nabla H\|_{L^\infty}\|u\|_m.
\end{aligned}$$

Thus, it follows that

$$\|S_{12}\| \lesssim \|\nabla u\|_{L^\infty}(\|H\|_m + \|\partial_z H\|_{m-1}) + \|\nabla H\|_{L^\infty}\|u\|_m. \tag{3.24}$$

Similarly, one has $S_{13} = [Z^\alpha, H\cdot\nabla]u$ satisfying

$$\|S_{13}\| \lesssim \|\nabla H\|_{L^\infty}(\|u\|_m + \|\partial_z u\|_{m-1}) + \|\nabla u\|_{L^\infty}\|H\|_m. \tag{3.25}$$

For the term R_{14}, obviously we have

$$\|R_{14}\| \lesssim \|\nabla p\|_{m-1}. \tag{3.26}$$

Combining (3.16) and the commutator estimates (3.20)-(3.26), using the Young inequality, and noting that the terms $\varepsilon(\|\partial_z u\|_m + \|\partial_z H\|_m)$ can be absorbed by the left hand side of (3.16), one obtains the estimate (3.2). □

Remark 3.3. In the boundary conditions (1.6) and (1.7), if $\beta,\gamma\le 0$, as in [9], we can apply the trace theorem and the Young inequality to estimate $|u_h|_{L^2(\partial\Omega)}$ and $|H_h|_{L^2(\partial\Omega)}$ as follows:

$$\begin{aligned}
|u_h|_{L^2(\partial\Omega)} &\le C\|\nabla u\|\|u\| \le \delta\|\nabla u\|^2 + C(\delta)\|u\|^2,\\
|H_h|_{L^2(\partial\Omega)} &\le C\|\nabla H\|\|H\| \le \delta\|\nabla H\|^2 + C(\delta)\|H\|^2.
\end{aligned}$$

Then for the third term on the left hand side of (3.16), one has

$$2\varepsilon(\beta|Z^\alpha u_h|^2_{L^2(\partial\Omega)} + \gamma|Z^\alpha H_h|^2_{L^2(\partial\Omega)}) \le \frac{1}{2}\varepsilon(\|\nabla Z^\alpha u\|^2 + \|\nabla Z^\alpha H\|^2) + C(\varepsilon)(\|u\|_m^2 + \|H\|_m^2), \tag{3.27}$$

and the estimate (3.2) still holds.

3.2. *Normal derivative estimates.* In this subsection, our aims are to estimate $\|\partial_z u\|_{m-1}$ and $\|\partial_z H\|_{m-1}$. First from the divergence free condition, we have

$$\|\partial_z u_3\|_{m-1} \le \|u\|_m, \tag{3.28}$$

$$\|\partial_z H_3\|_{m-1} \le \|H\|_m, \tag{3.29}$$

so it is sufficient to study $\partial_z u_h$ and $\partial_z H_h$ to estimate $\partial_z u$ and $\partial_z H$.

As in [15], by introducing $\omega_u = \nabla\times u$, $\omega_H = \nabla\times H$, then (ω_u,ω_H) satisfies

$$\partial_t\omega_u - \varepsilon\triangle\omega_u + u\cdot\nabla\omega_u - \omega_u\cdot\nabla u - H\cdot\nabla\omega_H + \omega_H\cdot\nabla H = 0, \tag{3.30}$$

$$\partial_t\omega_H - \varepsilon\triangle\omega_H + u\cdot\nabla\omega_H - \omega_u\cdot\nabla H - H\cdot\nabla\omega_u + \omega_H\cdot\nabla u = 0. \tag{3.31}$$

From the boundary conditions (1.6), (1.7), we have

$$(\omega_u)_h = 2\beta u_h^\perp, \quad (\omega_H)_h = 2\gamma H_h^\perp, x \in \partial\Omega,$$

where $u_h^\perp = (-u_2, u_1)^t$ and $H_h^\perp = (-H_2, H_1)^t$.

Denote by

$$V = (\omega_u)_h - 2\beta u_h^\perp, \quad W = (\omega_H)_h - 2\gamma H_h^\perp; \tag{3.32}$$

then $V|_{\partial\Omega} = 0$, $W|_{\partial\Omega} = 0$.

It follows from (3.32):

$$\|\partial_z u_h\|_{m-1} \lesssim \|u\|_m + \|V\|_{m-1}, \tag{3.33}$$

$$\|\partial_z H_h\|_{m-1} \lesssim \|H\|_m + \|W\|_{m-1}. \tag{3.34}$$

Combining (3.28), (3.29), (3.33) and (3.34), it is sufficient to estimate $\|V\|_{m-1}$ and $\|W\|_{m-1}$.

From (3.30), (3.31) and (3.32), (V, W) satisfies the following problem:

$$\partial_t V - \varepsilon\triangle V + (u\cdot\nabla)V - (H\cdot\nabla)W = (\omega_u\cdot\nabla)u_h - (\omega_H\cdot\nabla)H_h + 2\beta\nabla_h^\perp p \quad \text{in} \quad \Omega, \tag{3.35}$$

$$\partial_t W - \varepsilon\triangle W + (u\cdot\nabla)W - (H\cdot\nabla)V = (\omega_u\cdot\nabla)H_h - (\omega_H\cdot\nabla)u_h \quad \text{in} \quad \Omega, \tag{3.36}$$

$$V|_{\partial\Omega} = W|_{\partial\Omega} = 0. \tag{3.37}$$

Proposition 3.4. For any $m \geq 1$, every smooth solution (V, W) of the problem (3.35)-(3.37) satisfies the following estimate:

$$\begin{aligned} &\frac{d}{dt}(\|V(t)\|_{m-1}^2 + \|W(t)\|_{m-1}^2) + C_1\varepsilon(\|\nabla V(t)\|_{m-1}^2 + \|\nabla W(t)\|_{m-1}^2) \\ &\lesssim \|\nabla p\|_{m-1}\|V\|_{m-1} + \big(1 + \|u\|_{2,\infty} + \|H\|_{2,\infty} + \|\partial_z u\|_{1,\infty} + \|\partial_z H\|_{1,\infty}\big) \\ &\quad \cdot\big(\|V\|_{m-1}^2 + \|u\|_m^2 + \|W\|_{m-1}^2 + \|H\|_m^2\big), \end{aligned} \tag{3.38}$$

for a constant $C_1 > 0$ independent of ε.

Proof. Multiplying (3.35) and (3.36) by V and W, respectively, and integrating with respect to the space variables and noticing the boundary conditions (1.6) and (1.7), one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|V(t)\|^2 + \|W(t)\|^2) + \varepsilon(\|\nabla V\|^2 + \|\nabla W\|^2) \\ &\lesssim \|\nabla p\|\|V\| + \|(\omega_u\cdot\nabla)u_h\|\|V\| + \|(\omega_H\cdot\nabla)H_h\|\|V\| \\ &\quad + \|(\omega_u\cdot\nabla)H_h\|\|W\| + \|(\omega_H\cdot\nabla)u_h\|\|W\|. \end{aligned} \tag{3.39}$$

We estimate each term of the above inequality as follows:

$$\|(\omega_u\cdot\nabla)u_h\| \lesssim \|\nabla u\|_{L^\infty}\|\omega_u\| \lesssim \|\nabla u\|_{L^\infty}(\|V\| + \|u\|_1),$$

$$\|(\omega_H\cdot\nabla)H_h\| \lesssim \|\nabla H\|_{L^\infty}\|\omega_H\| \lesssim \|\nabla H\|_{L^\infty}(\|W\| + \|H\|_1),$$

$$\|(\omega_u\cdot\nabla)H_h\| \lesssim \|\nabla H\|_{L^\infty}\|\omega_u\| \lesssim \|\nabla H\|_{L^\infty}(\|V\| + \|u\|_1),$$

$$\|(\omega_H\cdot\nabla)u_h\| \lesssim \|\nabla u\|_{L^\infty}\|\omega_H\| \lesssim \|\nabla u\|_{L^\infty}(\|W\| + \|H\|_1).$$

Summing up, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|V(t)\|^2+\|W(t)\|^2)+\varepsilon(\|\nabla V\|^2+\|\nabla W\|^2)\\&\lesssim\|\nabla p\|\|V\|+(\|\nabla u\|_{L^\infty}+\|\nabla H\|_{L^\infty})(\|V\|^2+\|W\|^2+\|u\|_1^2+\|H\|_1^2).\end{aligned}\tag{3.40}$$

Applying $Z^\alpha(|\alpha|=m-1)$ to (3.35) and (3.36), we get

$$\begin{aligned}&\partial_t Z^\alpha V-\varepsilon\Delta Z^\alpha V+(u\cdot\nabla)Z^\alpha V-(H\cdot\nabla)Z^\alpha W\\&=Z^\alpha((\omega_u\cdot\nabla)u_h)-Z^\alpha((\omega_H\cdot\nabla)H_h)+2\beta Z^\alpha\nabla_h^\perp p+R_2\end{aligned}\tag{3.41}$$

and

$$\begin{aligned}&\partial_t Z^\alpha W-\varepsilon\Delta Z^\alpha W+(u\cdot\nabla)Z^\alpha W-(H\cdot\nabla)Z^\alpha V\\&=Z^\alpha((\omega_u\cdot\nabla)H_h)-Z^\alpha((\omega_H\cdot\nabla)u_h)+S_2,\end{aligned}\tag{3.42}$$

where

$$R_2=\sum_{j=1}^3 R_{2j}\triangleq\varepsilon[Z^\alpha,\Delta]V-[Z^\alpha,u\cdot\nabla]V+[Z^\alpha,H\cdot\nabla]W\tag{3.43}$$

and

$$S_2=\sum_{j=1}^3 S_{2j}\triangleq\varepsilon[Z^\alpha,\Delta]W-[Z^\alpha,u\cdot\nabla]W+[Z^\alpha,H\cdot\nabla]V.\tag{3.44}$$

By using a standard energy estimate and noting that $Z^\alpha V$ and $Z^\alpha W$ vanish on the boundary, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|Z^\alpha V\|^2+\|Z^\alpha W\|^2)+\varepsilon(\|\nabla Z^\alpha V\|^2+\|\nabla Z^\alpha W\|^2)\\&\lesssim\big(\|(\omega_u\cdot\nabla)u_h\|_{m-1}+\|(\omega_H\cdot\nabla)H_h\|_{m-1}+\|\nabla p\|_{m-1}\big)\|V\|_{m-1}+|(R_2,Z^\alpha V)|\\&\quad+\big(\|(\omega_u\cdot\nabla)H_h\|_{m-1}+\|(\omega_H\cdot\nabla)u_h\|_{m-1}\big)\|W\|_{m-1}+|(S_2,Z^\alpha W)|.\end{aligned}\tag{3.45}$$

By using the inequality (2.4) in Lemma 2.4 and (3.33) and (3.34), we get

$$\begin{aligned}\|(\omega_u\cdot\nabla)u_h\|_{m-1}&\lesssim\|\omega_u\|_{L^\infty}\|\nabla u_h\|_{m-1}+\|\nabla u_h\|_{L^\infty}\|\omega_u\|_{m-1}\\&\lesssim\|\nabla u\|_{L^\infty}(\|u\|_m+\|V\|_{m-1}),\end{aligned}\tag{3.46}$$

$$\begin{aligned}\|(\omega_H\cdot\nabla)H_h\|_{m-1}&\lesssim\|\omega_H\|_{L^\infty}\|\nabla H_h\|_{m-1}+\|\nabla H_h\|_{L^\infty}\|\omega_H\|_{m-1}\\&\lesssim\|\nabla H\|_{L^\infty}(\|H\|_m+\|W\|_{m-1}),\end{aligned}\tag{3.47}$$

and

$$\begin{aligned}\|(\omega_u\cdot\nabla)H_h\|_{m-1}&\lesssim\|\omega_u\|_{L^\infty}\|\nabla H_h\|_{m-1}+\|\nabla H_h\|_{L^\infty}\|\omega_u\|_{m-1}\\&\lesssim\|\nabla u\|_{L^\infty}(\|H\|_m+\|W\|_{m-1})+\|\nabla H\|_{L^\infty}(\|u\|_m+\|V\|_{m-1}),\end{aligned}\tag{3.48}$$

$$\begin{aligned}\|(\omega_H\cdot\nabla)u_h\|_{m-1}&\lesssim\|\omega_H\|_{L^\infty}\|\nabla u_h\|_{m-1}+\|\nabla u_h\|_{L^\infty}\|\omega_H\|_{m-1}\\&\lesssim\|\nabla H\|_{L^\infty}(\|u\|_m+\|V\|_{m-1})+\|\nabla u\|_{L^\infty}(\|H\|_m+\|W\|_{m-1}).\end{aligned}\tag{3.49}$$

Similarly to (3.20) and (3.21), for R_{21} and S_{21} defined in (3.43) and (3.44), respectively, we have

$$|(R_{21},Z^\alpha V)|\lesssim\varepsilon\|\partial_z V\|_{m-1}(\|V\|_{m-1}+\|\partial_z V\|_{m-2})+\|V\|_{m-1}^2+\|\partial_z V\|_{m-2}^2\tag{3.50}$$

and

$$|(S_{21}, Z^\alpha W)| \lesssim \varepsilon\|\partial_z W\|_{m-1}(\|W\|_{m-1} + \|\partial_z W\|_{m-2}) + \|W\|_{m-1}^2 + \|\partial_z W\|_{m-2}^2. \quad (3.51)$$

The estimates of the terms R_{22}, R_{23}, S_{22} and S_{23} can be studied similarly, but with slightly more complicated computations. For example, let us consider R_{22} in detail:

$$R_{22} = -[Z^\alpha, u\cdot\nabla]V = \sum_{\lambda+\nu=\alpha, |\lambda|\geq 1} C_{\lambda,\nu} Z^\lambda u \cdot Z^\nu \nabla V - u\cdot [Z^\alpha, \nabla]V, \quad (3.52)$$

where $C_{\lambda,\nu}$ depends only on the derivatives of φ. First, we have

$$\|u\cdot[Z^\alpha,\nabla]V\| \lesssim \sum_{|\nu|\leq m-2} \|u_3\partial_z Z_3^\nu V\| \lesssim \|\partial_z u_3\|_{L^\infty}\|V\|_{m-1}, \quad (3.53)$$

by noting $[Z_j, \nabla] = 0$ with $j = 1, 2$, and

$$\begin{aligned} \|C_{\lambda,\nu} Z^\lambda u \cdot Z^\nu(\nabla V)\| &\lesssim \|Z^\lambda u_h \cdot Z^\nu(\nabla_h V)\| + \|Z^\lambda u_3 \cdot Z^\nu \partial_z V\| \\ &\lesssim \|Z u_h\|_{L^\infty}\|V\|_{m-1} + \|V\|_{L^\infty}\|u\|_m + \|Z^\lambda u_3\cdot Z^\nu\partial_z V\|. \end{aligned} \quad (3.54)$$

Since one can't expect to have a uniform estimate in ε on $\|\partial_z V\|_{m-1}$ and $\|\partial_z V\|_{L^\infty}$, the term $\|Z^\lambda u_3 \cdot Z^\nu \partial_z V\|$ can't be estimated by using the inequality (2.4). To this end, we apply the technique in [15] and rewrite

$$Z^\lambda u_3 \cdot Z^\nu \partial_z V = \frac{1}{\varphi(z)} Z^\lambda u_3 \cdot \varphi(z) Z^\nu \partial_z V$$

and

$$\begin{aligned} \frac{1}{\varphi(z)} Z^\lambda u_3 &= [\frac{1}{\varphi(z)}, Z^\lambda]u_3 + Z^\lambda(\frac{1}{\varphi(z)}u_3), \\ \varphi(z) Z^\nu \partial_z V &= [\varphi(z), Z^\nu]\partial_z V + Z^\nu(Z_3 V). \end{aligned}$$

Obviously, one has

$$\begin{aligned} [\frac{1}{\varphi(z)}, Z^\lambda]u_3 &= \sum_{|\lambda_1|<|\lambda|} Z^{\lambda-\lambda_1}(\frac{1}{\varphi(z)}) Z^{\lambda_1} u_3 \\ &= \sum_{|\lambda_1|<|\lambda|} \varphi(z) Z^{\lambda-\lambda_1}(\frac{1}{\varphi(z)})\Big([\frac{1}{\varphi(z)}, Z^{\lambda_1}]u_3 + Z^{\lambda_1}(\frac{1}{\varphi(z)}u_3)\Big) \\ &= \sum_{|\lambda_1|<|\lambda|} C_{\lambda_1}\Big([\frac{1}{\varphi(z)}, Z^{\lambda_1}]u_3 + Z^{\lambda_1}(\frac{1}{\varphi(z)}u_3)\Big), \end{aligned}$$

with C_{λ_1} being a smooth bounded function.

Considering $[\frac{1}{\varphi(z)}, Z^{\lambda_1}]u_3$ by using the same argument as $[\frac{1}{\varphi(z)}, Z^\lambda]u_3$, we finally get

$$[\frac{1}{\varphi(z)}, Z^\lambda]u_3 = \sum_{|\tilde\lambda|\leq|\lambda|} C_{\tilde\lambda} Z^{\tilde\lambda}(\frac{1}{\varphi(z)}u_3);$$

then

$$\frac{1}{\varphi(z)} Z^\lambda u_3 = \sum_{|\tilde\lambda|\leq|\lambda|} C_{\tilde\lambda} Z^{\tilde\lambda}(\frac{1}{\varphi(z)}u_3).$$

Similarly, thanks to the fact that $Z^\nu \varphi(z)$ has a similar property as $\varphi(z)$, one can deduce

$$\varphi(z) Z^\nu(\partial_z V) = \sum_{|\tilde{\nu}|\le|\nu|} C_{\tilde{\nu}} Z^{\tilde{\nu}}(\varphi(z)\partial_z V).$$

Consequently, we get

$$Z^\lambda u_3 \cdot Z^\nu \partial_z V = \sum_{|\tilde{\lambda}|+|\tilde{\nu}|\le|\alpha|, |\tilde{\nu}|\ne|\alpha|} C_{\tilde{\lambda},\tilde{\nu}} Z^{\tilde{\lambda}}(\frac{1}{\varphi(z)}u_3)\cdot Z^{\tilde{\nu}}(Z_3 V). \tag{3.55}$$

When $\tilde{\lambda}=0, |\tilde{\nu}|\le m-2$, we have

$$\begin{aligned} \|Z^{\tilde{\lambda}}(\frac{1}{\varphi(z)}u_3)\cdot Z^{\tilde{\nu}}(Z_3 V)\| &\lesssim \|\frac{1}{\varphi(z)}u_3\|_{L^\infty}\|V\|_{m-1} \\ &\lesssim \|\partial_z u_3\|_{L^\infty}\|V\|_{m-1} \lesssim \|u\|_{1,\infty}\|V\|_{m-1}, \end{aligned} \tag{3.56}$$

thanks to the boundary condition (1.6) and the property of $\varphi(z)$.

When $\tilde{\lambda}\ne 0$, by using the inequality (2.4) we have

$$\|Z^{\tilde{\lambda}}(\frac{1}{\varphi(z)}u_3)\cdot Z^{\tilde{\nu}}(Z_3 V)\| \lesssim \|Z(\frac{1}{\varphi(z)}u_3)\|_{L^\infty}\|V\|_{m-1} + \|Z_3 V\|_{L^\infty}\|Z(\frac{1}{\varphi(z)}u_3)\|_{m-2}. \tag{3.57}$$

Since $Z^\alpha u_3$ vanishes on the boundary, we get from the Hardy inequality and the divergence free condition that

$$\|Z_i(\frac{1}{\varphi(z)}u_3)\|_{m-2} \lesssim \|\partial_z(Z_i u_3)\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1} \lesssim \|u\|_m, \quad i=1,2, \tag{3.58}$$

$$\|Z_3(\frac{1}{\varphi(z)}u_3)\|_{m-2} \lesssim \|\frac{1}{\varphi(z)}u_3\|_{m-2} + \|\partial_z u_3\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1} \lesssim \|u\|_m. \tag{3.59}$$

We also have

$$\|Z(\frac{1}{\varphi(z)}u_3)\|_{L^\infty} \lesssim \|\partial_z u_3\|_{1,\infty} \lesssim \|u\|_{2,\infty}. \tag{3.60}$$

Consequently, noting $\|ZV\|_{L^\infty} \lesssim \|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}$, we obtain from (3.56)-(3.60) that

$$\|Z^{\tilde{\lambda}}(\frac{1}{\varphi(z)}u_3)\cdot Z^{\tilde{\nu}}(Z_3 V)\| \lesssim (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty})(\|u\|_m + \|V\|_{m-1}). \tag{3.61}$$

So, from (3.55) we get

$$\|Z^\lambda u_3 \cdot Z^\nu \partial_z V)\| \lesssim (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty})(\|u\|_m + \|V\|_{m-1}). \tag{3.62}$$

Plugging (3.62) into (3.54) and combining with (3.53), it follows that

$$\|R_{22}\| \lesssim (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty})(\|u\|_m + \|V\|_{m-1}). \tag{3.63}$$

Similarly, we deduce that

$$\|R_{23}\| \lesssim (\|H\|_{2,\infty} + \|\partial_z H\|_{1,\infty})(\|H\|_m + \|W\|_{m-1}), \tag{3.64}$$

$$\|S_{22}\| \lesssim \|u\|_{2,\infty}\|W\|_{m-1} + (\|H\|_{2,\infty} + \|\partial_z H\|_{1,\infty})\|u\|_m, \tag{3.65}$$

and

$$\|S_{23}\| \lesssim \|H\|_{2,\infty}\|V\|_{m-1} + (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty})\|H\|_m. \tag{3.66}$$

Combining (3.45)-(3.51) and (3.63)-(3.66), we obtain (3.38). □

3.3. *Pressure estimate.* In this subsection, we focus on pressure estimate $\|\nabla p\|_{m-1}$. Rewrite (1.1) as

$$\partial_t u - \varepsilon \Delta u + \nabla p = F \triangleq -(u \cdot \nabla)u + (H \cdot \nabla)H, \quad \nabla \cdot u = 0. \tag{3.67}$$

For the above Stokes problem, we quote a result from [15, Theorem 11] as follows:

LEMMA 3.5 ([15]). For every $m \geq 2$, there exists $C > 0$ such that for every $t \geq 0$, the following estimate holds:

$$\|\nabla p\|_{m-1} \lesssim \|F\|_{m-1} + \|\nabla \cdot F\|_{m-2} + \varepsilon \|\nabla u\|_{m-1} + \|u\|_{m-1}. \tag{3.68}$$

Now, we need to study $\|F\|_{m-1}$ and $\|\nabla \cdot F\|_{m-2}$. From (3.67), we get by using the inequality (2.4)

$$\begin{aligned} \|F\|_{m-1} &\lesssim \|u\|_{L^\infty} \|\nabla u\|_{m-1} + \|\nabla u\|_{L^\infty} \|u\|_{m-1} + \|H\|_{L^\infty} \|\nabla H\|_{m-1} + \|\nabla H\|_{L^\infty} \|H\|_{m-1} \\ &\lesssim \|u\|_{W^{1,\infty}} (\|u\|_m + \|\partial_z u\|_{m-1}) + \|H\|_{W^{1,\infty}} (\|H\|_m + \|\partial_z H\|_{m-1}) \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} \|\nabla \cdot F\|_{m-2} &\lesssim \|\nabla u \cdot \nabla u\|_{m-2} + \|\nabla H \cdot \nabla H\|_{m-2} \\ &\lesssim \|\nabla u\|_{L^\infty} \|\nabla u\|_{m-2} + \|\nabla H\|_{L^\infty} \|\nabla H\|_{m-2} \\ &\lesssim \|u\|_{W^{1,\infty}} (\|u\|_{m-1} + \|\partial_z u\|_{m-2}) + \|H\|_{W^{1,\infty}} (\|H\|_{m-1} + \|\partial_z H\|_{m-2}). \end{aligned} \tag{3.70}$$

Plugging (3.69) and (3.70) into (3.5), we get the following:

PROPOSITION 3.6. The smooth solution of (1.1)-(1.3), (1.6) and (1.7) satisfies the estimate

$$\begin{aligned} \|\nabla p\|_{m-1} \lesssim\, & \varepsilon (\|u\|_m + \|\partial_z u\|_{m-1}) + (1 + \|u\|_{W^{1,\infty}})(\|u\|_m + \|\partial_z u\|_{m-1}) \\ & + \|H\|_{W^{1,\infty}} (\|H\|_m + \|\partial_z H\|_{m-1}). \end{aligned} \tag{3.71}$$

3.4. L^∞ *estimates.* To conclude the main estimate (2.2) from Propositions 3.2, 3.4 and 3.6, it remains to estimate the L^∞−norm of the solution. To do this, similarly to [15], we set

$$G_m(t) = \|u(t)\|_m^2 + \|V(t)\|_{m-1}^2 + \|V(t)\|_{1,\infty}^2 + \|H(t)\|_m^2 + \|W(t)\|_{m-1}^2 + \|W(t)\|_{1,\infty}^2.$$

By using simple computations, we have the following estimates:

PROPOSITION 3.7. For any fixed $m_0 \geq 2$, we have

$$\|u\|_{2,\infty} + \|H\|_{2,\infty} + \|\nabla u\|_{1,\infty} + \|\nabla H\|_{1,\infty} \lesssim G_{m_0+3}^{\frac{1}{2}}. \tag{3.72}$$

Proof. By definition and the anisotropic Sobolev embedding inequality (2.6), for $m_0 \geq 2$, we have

$$\begin{aligned} \|u\|_{2,\infty}^2 &= \|u\|_{L^\infty}^2 + \|Zu\|_{L^\infty}^2 + \|Z^2 u\|_{L^\infty}^2 \\ &\lesssim \|\partial_z u\|_{m_0} \|u\|_{m_0} + \|\partial_z Z u\|_{m_0} \|Z u\|_{m_0} + \|\partial_z Z^2 u\|_{m_0} \|Z^2 u\|_{m_0} + \|u\|_{m_0+2}^2. \end{aligned}$$

Since $\|\partial_z Zu\|_{m_0} \lesssim \|\partial_z u\|_{m_0+1}$, $\|\partial_z Z^2 u\|_{m_0} \lesssim \|\partial_z u\|_{m_0+2} + \|u\|_{m_0+1}$, one gets

$$\begin{aligned}\|u\|_{2,\infty}^2 &\lesssim \|\partial_z u\|_{m_0}\|u\|_{m_0} + \|\partial_z u\|_{m_0+1}\|u\|_{m_0+1} + \|\partial_z u\|_{m_0+2}\|u\|_{m_0+2} + \|u\|_{m_0+2}^2\\ &\lesssim \|\partial_z u\|_{m_0+2}\|u\|_{m_0+2} + \|u\|_{m_0+2}^2\\ &\lesssim \|V\|_{m_0+2}^2 + \|u\|_{m_0+3}^2 \lesssim G_{m_0+3}(t).\end{aligned} \tag{3.73}$$

On the other hand, by using the divergence free conditions, (3.32) and (3.73), we deduce that

$$\begin{aligned}\|\nabla u\|_{1,\infty} &\lesssim \|\nabla u\|_{L^\infty} + \|Z\nabla u\|_{L^\infty} \lesssim \|\nabla_h u\|_{L^\infty} + \|\partial_z u\|_{L^\infty} + \|Z\nabla_h u\|_{L^\infty} + \|Z\partial_z u\|_{L^\infty}\\ &\lesssim \|u\|_{2,\infty} + \|\partial_z u_h\|_{L^\infty} + \|Z\partial_z u_h\|_{L^\infty} \lesssim \|u\|_{2,\infty} + \|V\|_{1,\infty} \lesssim G_{m_0+3}^{\frac12}(t).\end{aligned}$$

One can study the estimates of H similarly. Consequently, the estimate (3.72) holds. □

It remains to estimate $\|V\|_{1,\infty}$ and $\|W\|_{1,\infty}$. By using the special structure of the system, we shall see that both $V+W$ and $V-W$ satisfy scalar degenerate parabolic equations, from which one can get estimates of $\|V\|_{L^\infty}$, $\|W\|_{L^\infty}$, $\|Z_iV\|_{L^\infty}$ and $\|Z_iW\|_{L^\infty}$ $(i=1,2)$ by using the maximal principle. But $\|Z_3V\|_{L^\infty}$ and $\|Z_3W\|_{L^\infty}$ cannot be estimated similarly due to the bad commutator $[Z_3,\partial_z]$. To this end, we introduce a cut-off function $\chi(z)$ equal to one near $z=0$ and supported in $[0,1]$. Rewrite V and W as

$$V = \chi V + (1-\chi)V \triangleq V^b + V^{int}, \quad W = \chi W + (1-\chi)W \triangleq W^b + W^{int}.$$

Proposition 3.8. For $m \ge 6$, the following estimate holds:

$$\|V\|_{1,\infty}^2 + \|W\|_{1,\infty}^2 \lesssim G_m(0) + (1+t+\varepsilon^3 t^2)\int_0^t (G_m(s)^2 + G_m(s))ds. \tag{3.74}$$

Proof.

Step 1 (Estimates of $\|V^{int}\|_{1,\infty}$ and $\|W^{int}\|_{1,\infty}$). Since the norm H_{co}^m is equivalent to the usual H^m norm away from the boundary, it follows from the definitions of V and W and the usual Sobolev embedding that for $m \ge 4$,

$$\|V^{int}\|_{1,\infty} + \|W^{int}\|_{1,\infty} \lesssim \|u\|_m + \|H\|_m \lesssim G_m^{\frac12}. \tag{3.75}$$

Step 2 (Estimates of $\|V^b\|_{L^\infty}$, $\|W^b\|_{L^\infty}$, $\|Z_iV^b\|_{L^\infty}$ and $\|Z_iw^b\|_{L^\infty}(i=1,2)$). From (3.35) and (3.36), V^b and W^b solve

$$\begin{aligned}&\partial_t V^b + (u\cdot\nabla)V^b - (H\cdot\nabla)W^b - \varepsilon\partial_z^2 V^b = \varepsilon\Delta_y V^b + \chi F_1 + E_1 \triangleq I_1,\\ &\partial_t W^b + (u\cdot\nabla)W^b - (H\cdot\nabla)V^b - \varepsilon\partial_z^2 W^b = \varepsilon\Delta_y W^b + \chi F_2 + E_2 \triangleq I_2,\end{aligned} \tag{3.76}$$

with the boundary and initial conditions

$$(V^b, W^b)|_{z=0} = 0, \; V^b(0,x) = V_0^b(x), \; W^b(0,x) = W_0^b(x),$$

where

$$\begin{aligned}
&F_1 = (\omega_u \cdot \nabla)u_h - (\omega_H \cdot \nabla)H_h + 2\beta\nabla_h^{\perp} p,\\
&F_2 = (\omega_u \cdot \nabla)H_h - (\omega_H \cdot \nabla)u_h,\\
&E_1 = -[\chi, u\cdot\nabla]V + [\chi, H\cdot\nabla]W + \varepsilon[\chi, \Delta]V,\\
&E_2 = -[\chi, u\cdot\nabla]W + [\chi, H\cdot\nabla]V + \varepsilon[\chi, \Delta]W.
\end{aligned}$$

Let

$$f_1 = V^b + W^b, \quad f_2 = V^b - W^b. \tag{3.77}$$

Then from (3.76) f_1 and f_2 solve

$$\partial_t f_1 + (u\cdot\nabla)f_1 - (H\cdot\nabla)f_1 - \varepsilon\partial_z^2 f_1 = I_1 + I_2 \tag{3.78}$$

and

$$\partial_t f_2 + (u\cdot\nabla)f_2 + (H\cdot\nabla)f_2 - \varepsilon\partial_z^2 f_2 = I_1 - I_2, \tag{3.79}$$

with the boundary conditions $f_1(t, y, 0) = f_2(t, y, 0) = 0$, and the initial conditions

$$f_1(0, x) = f_1^0(x) \triangleq V_0^b + W_0^b, \quad f_2(0, x) = f_2^0(x) \triangleq V_0^b - W_0^b. \tag{3.80}$$

By using the maximum principle for the initial-boundary value problems (3.78), (3.79) and (3.80), one gets

$$\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty} \lesssim \|f_1^0\|_{L^\infty} + \|f_2^0\|_{L^\infty} + \int_0^t (\|I_1(s)\|_{L^\infty} + \|I_2(s)\|_{L^\infty})ds,$$

which implies

$$\|V_b\|_{L^\infty} + \|W_b\|_{L^\infty} \lesssim \|V_0^b\|_{L^\infty} + \|W_0^b\|_{L^\infty} + \int_0^t (\|I_1(s)\|_{L^\infty} + \|I_2(s)\|_{L^\infty})ds. \tag{3.81}$$

Applying the operator ∂_i $(i = 1, 2)$ to (3.78) and (3.79), we have

$$\partial_t\partial_i f_1 + (u\cdot\nabla)\partial_i f_1 - (H\cdot\nabla)\partial_i f_1 - \varepsilon\partial_z^2\partial_i f_1 = \partial_i(I_1 + I_2) - \partial_i u\cdot\nabla f_1 + \partial_i H\cdot\nabla f_1, \tag{3.82}$$

$$\partial_t\partial_i f_2 + (u\cdot\nabla)\partial_i f_2 + (H\cdot\nabla)\partial_i f_2 - \varepsilon\partial_z^2\partial_i f_2 = \partial_i(I_1 - I_2) - \partial_i u\cdot\nabla f_2 - \partial_i H\cdot\nabla f_2. \tag{3.83}$$

By using the maximum principle again, one obtains

$$\begin{aligned}
\|\partial_i f_1\|_{L^\infty} + \|\partial_i f_2\|_{L^\infty} \lesssim{}& \|\partial_i f_1^0\|_{L^\infty} + \|\partial_i f_2^0\|_{L^\infty} + \int_0^t (\|\partial_i I_1\|_{L^\infty} + \|\partial_i I_2\|_{L^\infty}\\
&+\|\partial_i u\cdot\nabla f_1\|_{L^\infty} + \|\partial_i H\cdot\nabla f_1\|_{L^\infty} + \|\partial_i u\cdot\nabla f_2\|_{L^\infty} + \|\partial_i H\cdot\nabla f_2\|_{L^\infty})ds,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\partial_i V^b\|_{L^\infty} + \|\partial_i W^b\|_{L^\infty} \lesssim{}& \|\partial_i V_0^b\|_{L^\infty} + \|\partial_i W_0^b\|_{L^\infty} + \int_0^t (\|\partial_i I_1\|_{L^\infty} + \|\partial_i I_2\|_{L^\infty}\\
&+\|\partial_i u\cdot\nabla V^b\|_{L^\infty} + \|\partial_i H\cdot\nabla W^b\|_{L^\infty} + \|\partial_i u\cdot\nabla W^b\|_{L^\infty} + \|\partial_i H\cdot\nabla V^b\|_{L^\infty})ds.
\end{aligned} \tag{3.84}$$

Next, we need to estimate the terms of I_1 and I_2 appearing in (3.81) and (3.84). By using the anisotropic Sobolev embedding inequality (2.6) and the Cauchy-Schwartz inequality, we have for $m_0 \geq 2$ and $m \geq m_0 + 4 \geq 6$,

$$
\begin{aligned}
\big(\varepsilon\int_0^t \|\Delta_y V^b\|_{1,\infty}\big)^2 \quad & \lesssim \varepsilon^2\Big(\int_0^t (\|\partial_z \Delta_y V^b\|_{m_0}^{\frac{1}{2}} \|\Delta_y V^b\|_{m_0}^{\frac{1}{2}} \\
& \quad + \|\partial_z(Z\Delta_y V^b)\|_{m_0}^{\frac{1}{2}} \|Z\Delta_y V^b\|_{m_0}^{\frac{1}{2}} + \|\Delta_y V^b\|_{m_0+1}) ds\Big)^2 \\
& \lesssim \varepsilon^2\Big(\int_0^t \|\partial_z V^b\|_{m_0+3}^{\frac{1}{2}} \|V^b\|_{m_0+3}^{\frac{1}{2}} ds + \int_0^t \|V^b\|_{m_0+3} ds\Big)^2 \\
& \lesssim \varepsilon \int_0^t \|\nabla V^b\|_{m-1}^2 ds + (\varepsilon^3 t^2 + \varepsilon^2 t)\int_0^t \|V^b\|_{m-1}^2 ds \\
& \lesssim \varepsilon \int_0^t \|\nabla V\|_{m-1}^2 ds + (\varepsilon^3 t^2 + \varepsilon^2 t)\int_0^t G_m(s) ds,
\end{aligned}
\tag{3.85}
$$

thanks to Proposition 3.7. Similarly, we also have

$$
(\varepsilon \int_0^t \|\Delta_y W^b\|_{1,\infty})^2 \lesssim \varepsilon \int_0^t \|\nabla W\|_{m-1}^2 ds + (\varepsilon^3 t^2 + \varepsilon^2 t)\int_0^t G_m(s) ds. \tag{3.86}
$$

Due to Proposition 3.7, one gets

$$
\begin{aligned}
\|\chi F_1\|_{1,\infty} & \lesssim \|\omega_u\|_{1,\infty}\|\nabla u_h\|_{1,\infty} + \|\omega_H\|_{1,\infty}\|\nabla H_h\|_{1,\infty} + \|\nabla_h p\|_{1,\infty} \\
& \lesssim \|u\|_{2,\infty}^2 + \|H\|_{2,\infty}^2 + \|\nabla_h p\|_{1,\infty} \lesssim G_m(t) + \|\nabla_h p\|_{1,\infty}.
\end{aligned}
$$

On the other hand,

$$
\|\nabla_h p\|_{1,\infty} \lesssim \|\partial_z \nabla_h p\|_{m_0}^2 + \|\nabla_h p\|_{m_0}^2 + \|\partial_z(Z\nabla_h p)\|_{m_0}^2 + \|Z\nabla_h p\|_{m_0}^2 \lesssim \|\nabla p\|_{m-1}^2,
$$

for $m - 1 \geq m_0 + 2 \geq 4$, thanks to the anisotropic Sobolev embedding inequality (2.6). Thus, it follows from (3.71) and Proposition 3.7 that

$$
\|\chi F_1\|_{1,\infty} \lesssim G_m(t) + G_m^{\frac{1}{2}}(t). \tag{3.87}
$$

Similarly, we have

$$
\|\chi F_2\|_{1,\infty} \lesssim G_m(t) + G_m^{\frac{1}{2}}(t). \tag{3.88}
$$

Noting $\partial_z \chi$ and $\partial_z^2 \chi$ are supported away from the boundary, by using the usual Sobolev embedding, we obtain

$$
\|E_1\|_{1,\infty} + \|E_2\|_{1,\infty} \lesssim \|u\|_m + \|u\|_m^2 + \|H\|_m^2 \lesssim G_m^{\frac{1}{2}}(t) + G_m(t), \quad \text{for} \quad m \geq 5. \tag{3.89}
$$

Combining (3.85)-(3.89) and the Cauchy-Schwartz inequality, for $m \geq 6$ we get

$$
\begin{aligned}
\Big(\int_0^t (\|I_1(s)\|_{1,\infty} + \|I_2(s)\|_{1,\infty}) ds\Big)^2 \lesssim \quad & \varepsilon \int_0^t (\|\nabla V\|_{m-1}^2 + \|\nabla W\|_{m-1}^2) ds \\
& + (1 + \varepsilon^3 t^2 + t)\int_0^t (G_m(s) + G_m^2(s)) ds.
\end{aligned}
\tag{3.90}
$$

On the other hand, thanks to the Hardy inequality and the divergence free conditions, we have

$$
\begin{aligned}
\|\partial_i u \cdot \nabla V^b\|_{L^\infty} &\lesssim \|\partial_i u_h \cdot \nabla_h V^b\|_{L^\infty} + \|\partial_i u_3 \cdot \partial_z V^b\|_{L^\infty} \\
&\lesssim \|u\|_{1,\infty}\|V^b\|_{1,\infty} + \|\frac{1}{\varphi(z)}\partial_i u_3 \cdot Z_3 V^b\|_{L^\infty} \\
&\lesssim \|u\|_{1,\infty}\|V^b\|_{1,\infty} + \|\partial_z\partial_i u_3\|_{L^\infty}\|V^b\|_{1,\infty} \\
&\lesssim \|u\|_{2,\infty}\|V^b\|_{1,\infty} \lesssim G_m(t), \quad m \geq 5.
\end{aligned}
\tag{3.91}
$$

In a similar way, we also obtain

$$
\|\partial_i H \cdot \nabla W^b\|_{L^\infty} + \|\partial_i u \cdot \nabla W^b\|_{L^\infty} + \|\partial_i H \cdot \nabla V^b\|_{L^\infty} \lesssim G_m(t), \quad m \geq 5. \tag{3.92}
$$

Consequently, combining (3.81), (3.84), (3.90), (3.91) and (3.92), and using the Cauchy-Schwartz inequality, we deduce that

$$
\begin{aligned}
&\|V^b\|_{L^\infty}^2 + \|W^b\|_{L^\infty}^2 + \|\nabla_h V^b\|_{L^\infty}^2 + \|\nabla_h W^b\|_{L^\infty}^2 \\
&\lesssim G_m(0) + \varepsilon\int_0^t (\|\nabla V\|_{m-1}^2 + \|\nabla W\|_{m-1}^2)ds \\
&+(1+t+\varepsilon^3 t^2)\int_0^t (G_m(s) + G_m(s)^2)ds, \quad m \geq 6.
\end{aligned}
\tag{3.93}
$$

STEP 3 (Estimates of $\|Z_3 V^b\|_{L^\infty}$ and $\|Z_3 W^b\|_{L^\infty}$). Rewrite (3.78) and (3.79) as

$$
\begin{aligned}
&\partial_t f_1 + z\partial_z u_3(t,y,0)\partial_z f_1 - z\partial_z H_3(t,y,0)\partial_z f_1 + u_h(t,y,0)\cdot\nabla_h f_1 \\
&\quad -H_h(t,y,0)\cdot\nabla_h f_1 - \varepsilon\partial_z^2 f_1 = I_1 + I_2 - J_1 - J_2
\end{aligned}
\tag{3.94}
$$

and

$$
\begin{aligned}
&\partial_t f_2 + z\partial_z u_3(t,y,0)\partial_z f_2 + z\partial_z H_3(t,y,0)\partial_z f_2 + u_h(t,y,0)\cdot\nabla_h f_2 \\
&\quad +H_h(t,y,0)\cdot\nabla_h f_2 - \varepsilon\partial_z^2 f_2 = I_1 - I_2 - J_1 + J_2,
\end{aligned}
\tag{3.95}
$$

where

$$
\begin{aligned}
J_1 &= (u_h(t,x) - u_h(t,y,0))\cdot\nabla_h V^b + (u_3(t,x) - z\partial_z u_3(t,y,0))\partial_z V^b \\
&-(H_h(t,x) - H_h(t,y,0))\cdot\nabla_h W^b - (H_3(t,x) - z\partial_z H_3(t,y,0))\partial_z W^b,
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
J_2 &= (u_h(t,x) - u_h(t,y,0))\cdot\nabla_h W^b + (u_3(t,x) - z\partial_z u_3(t,y,0))\partial_z W^b \\
&-(H_h(t,x) - H_h(t,y,0))\cdot\nabla_h V^b - (H_3(t,x) - z\partial_z H_3(t,y,0))\partial_z V^b.
\end{aligned}
\tag{3.97}
$$

Let $\mathcal{S}_1(t,\tau)$ and $\mathcal{S}_2(t,\tau)$ be the C^0 evolution operators generated by the left hand side of the equations (3.94) and (3.95), respectively. This means that $g_1(t,y,z) = \mathcal{S}_1(t,\tau)g_1^0(y,z)$ and $g_2(t,y,z) = \mathcal{S}_2(t,\tau)g_2^0(y,z)$ solve

$$
\begin{cases}
\partial_t g_1 + z\partial_z u_3(t,y,0)\partial_z g_1 - z\partial_z H_3(t,y,0)\partial_z g_1 + u_h(t,y,0)\cdot\nabla_h g_1 \\
\qquad\qquad -H_h(t,y,0)\cdot\nabla_h g_1 - \varepsilon\partial_z^2 g_1 = 0, \quad z>0, \quad t>\tau, \\
g_1(\tau,y,z) = g_1^0(y,z), \quad g_1(t,y,0) = 0
\end{cases}
\tag{3.98}
$$

and

$$\begin{cases} \partial_t g_2 + z\partial_z u_3(t,y,0)\partial_z g_2 + z\partial_z H_3(t,y,0)\partial_z g_2 + u_h(t,y,0)\cdot\nabla_h g_2 \\ \qquad\qquad +H_h(t,y,0)\cdot\nabla_h g_2 - \varepsilon\partial_z^2 g_2 = 0, \quad z>0, \quad t>\tau, \\ g_2(\tau,y,z) = g_2^0(y,z), \quad g_2(t,y,0)=0. \end{cases} \tag{3.99}$$

For the problems (3.98) and (3.99), we have the following estimates:

LEMMA 3.9. For $t \ge \tau \ge 0$, the following estimates hold:

$$\|z\partial_z \mathcal{S}_1(t,\tau)g_1^0\|_{L^\infty} \lesssim \|g_1^0\|_{L^\infty} + \|z\partial_z g_1^0\|_{L^\infty} \tag{3.100}$$

and

$$\|z\partial_z \mathcal{S}_2(t,\tau)g_2^0\|_{L^\infty} \lesssim \|g_2^0\|_{L^\infty} + \|z\partial_z g_2^0\|_{L^\infty}. \tag{3.101}$$

This lemma can be proved in the same way as Lemma 15 in [15].

By using the Duhamel formula, we obtain

$$f_1(t) = \mathcal{S}_1(t,0)f_1^0 + \int_0^t \mathcal{S}_1(t,\tau)(I_1+I_2-J_1-J_2)(\tau)d\tau \tag{3.102}$$

and

$$f_2(t) = \mathcal{S}_2(t,0)f_2^0 + \int_0^t \mathcal{S}_2(t,\tau)(I_1-I_2-J_1+J_2)(\tau)d\tau. \tag{3.103}$$

Consequently, we have

$$\begin{aligned} V^b(t) = \frac{1}{2}\Big(&\mathcal{S}_1(t,0)f_1^0 + \mathcal{S}_2(t,0)f_2^0 + \int_0^t (\mathcal{S}_1(t,\tau)(I_1+I_2-J_1-J_2) \\ &+\mathcal{S}_2(t,\tau)(I_1-I_2+J_1-J_2))d\tau\Big) \end{aligned}$$

and

$$\begin{aligned} W^b(t) = \frac{1}{2}\Big(&\mathcal{S}_1(t,0)f_1^0 - \mathcal{S}_2(t,0)f_2^0 + \int_0^t (\mathcal{S}_1(t,\tau)(I_1+I_2-J_1-J_2) \\ &-\mathcal{S}_2(t,\tau)(I_1-I_2+J_1-J_2))d\tau\Big). \end{aligned}$$

It follows from Lemma 3.9 that

$$\begin{aligned} &\|Z_3V^b\|_{L^\infty} + \|Z_3W^b\|_{L^\infty} \\ &\lesssim \|f_1^0\|_{L^\infty} + \|f_2^0\|_{L^\infty} + \|z\partial_z f_1^0\|_{L^\infty} + \|z\partial_z f_2^0\|_{L^\infty} \\ &\quad + \int_0^t (\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty} + \|J_1\|_{L^\infty} + \|J_2\|_{L^\infty})ds \\ &\quad + \int_0^t (\|z\partial_z I_1\|_{L^\infty} + \|z\partial_z I_2\|_{L^\infty} + \|z\partial_z J_1\|_{L^\infty} + \|z\partial_z J_2\|_{L^\infty})ds \\ &\lesssim \|V_0^b\|_{1,\infty} + \|W_0^b\|_{1,\infty} + \int_0^t (\|I_1\|_{1,\infty} + \|I_2\|_{1,\infty} + \|J_1\|_{1,\infty} + \|J_1\|_{1,\infty})ds. \end{aligned} \tag{3.104}$$

From (3.96), noting $u_3(t,y,0) = H_3(t,y,0) = 0$, and Proposition 3.7, and using the Taylor formula and the divergence free condition, we have

$$\begin{aligned}\|J_1\|_{L^\infty} &\lesssim \|u_h\|_{L^\infty}\|V^b\|_{1,\infty} + \|\partial_z u_3\|_{L^\infty}\|Z_3V^b\|_{L^\infty} + \|H_h\|_{L^\infty}\|W^b\|_{1,\infty} \\ &\qquad + \|\partial_z H_3\|_{L^\infty}\|Z_3W^b\|_{L^\infty} \\ &\lesssim \|u\|_{1,\infty}\|V^b\|_{1,\infty} + \|H\|_{1,\infty}\|W^b\|_{1,\infty} \lesssim G_m(t), \quad m \geq 5.\end{aligned} \tag{3.105}$$

Similarly, one gets

$$\|J_2\|_{L^\infty} \lesssim \|u\|_{1,\infty}\|W^b\|_{1,\infty} + \|H\|_{1,\infty}\|V^b\|_{1,\infty} \lesssim G_m(t), \quad m \geq 5. \tag{3.106}$$

On the other hand, we deduce that

$$\begin{aligned}\|ZJ_1\|_{L^\infty} &\lesssim \|u\|_{2,\infty}\|V^b\|_{1,\infty} + \|H\|_{2,\infty}\|W^b\|_{1,\infty} + \|(u_h(t,x) - u_h(t,y,0))Z\nabla_h V^b\|_{L^\infty} \\ &\quad + \|(u_3(t,x) - z\partial_z u_3(t,y,0))Z\partial_z V^b\|_{L^\infty} + \|(H_h(t,x) - H_h(t,y,0))Z\nabla_h W^b\|_{L^\infty} \\ &\quad + \|(H_3(t,x) - z\partial_z H_3(t,y,0))Z\partial_z W^b\|_{L^\infty} \\ &\lesssim \|u\|_{2,\infty}\|V^b\|_{1,\infty} + \|H\|_{2,\infty}\|W^b\|_{1,\infty} + \|\partial_z u_h\|_{L^\infty}\|\varphi(z)Z^2V^b\|_{L^\infty} \\ &\quad + \|\partial_z^2 u_3\|_{L^\infty}\|\varphi^2(z)Z\partial_z V^b\|_{L^\infty} + \|\partial_z H_h\|_{L^\infty}\|\varphi(z)Z^2V^b\|_{L^\infty} \\ &\quad + \|\partial_z^2 H_3\|_{L^\infty}\|\varphi^2(z)Z\partial_z W^b\|_{L^\infty} \\ &\lesssim G_m(t) + \|\partial_z u\|_{1,\infty}\|\varphi(z)Z^2V^b\|_{L^\infty} + \|\partial_z H\|_{1,\infty}\|\varphi(z)Z^2W^b\|_{L^\infty},\end{aligned}$$

by using $\nabla\cdot u = \nabla\cdot H = 0$.

Thanks to the anisotropic Sobolev embedding (2.6), we have, for $m_0 \geq 2$,

$$\|\varphi(z)Z^2V^b\|_{L^\infty} \lesssim \|Z^2V^b\|_{m_0} + \|\partial_z(\varphi(z)Z^2V^b)\|_{m_0} \lesssim \|V^b\|_{m_0+3} \lesssim G_m^{\frac{1}{2}}(t).$$

Similarly, we have

$$\|\varphi(z)Z^2W^b\|_{L^\infty} \lesssim G_m^{\frac{1}{2}}(t),$$

for $m \geq m_0 + 3 \geq 5$. Consequently, for $m \geq 5$ we have

$$\|ZJ_1\|_{L^\infty} + \|ZJ_2\|_{L^\infty} \lesssim G_m(t). \tag{3.107}$$

Combining (3.104)-(3.107) and using the Cauchy-Schwartz inequality, one gets

$$\|Z_3V^b\|_{L^\infty}^2 + \|Z_3W^b\|_{L^\infty}^2 \lesssim G_m(0) + (1 + \varepsilon^3t^2 + t)\int_0^t (G_m^2(s) + G_m(s))ds. \tag{3.108}$$

Finally, collecting (3.75), (3.76), (3.93) and (3.108), we get (3.74). The proof of Proposition 3.8 is complete. □

Proof of Theorem 2.1. Combining Propositions 3.3, 3.4, 3.6, 3.7 and 3.8, the a priori estimate (2.2) given in Theorem 2.1 follows immediately. □

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