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ON A VERSION OF QUATERNIONIC FUNCTION THEORY RELATED TO CHEBYSHEV POLYNOMIALS AND MODIFIED STURM-LIOUVILLE OPERATORS

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Abstract. In the last few years considerable attention has been paid to the role of the prolate spheroidal wave functions (PSWFs) to many practical signal and image processing problems. The PSWFs and their applications to wave phenomena modeling, fluid dynamics and filter design played a key role in this development. It is pointed out in this paper that the operator $\mathcal{W}$ arising in the Helmholtz equation after the prolate spheroidal change of variables is the sum of three operators, $\mathcal{S}_{\xi,\alpha}$, $\mathcal{S}_{\eta,\beta}$ and $\mathcal{T}_{\phi}$, each of which acts on functions of one variable: two of them are modified Sturm-Liouville operators and the other one is, up to a variable coefficient, the Chebyshev operator. We believe that this fact reflects the essence of the separation of variables method in this case. We show that there exists a theory of functions with quaternionic values and of three real variables which is determined by the Moisil–Theodorescu-type operator with quaternionic variable coefficients, and that it is intimately related to the modified Sturm-Liouville operators and to the Chebyshev operator (we call it in this way, since its solutions are related to the classical Chebyshev polynomials). We address all the above and explore some basic facts of the arising quaternionic function theory. We further establish analogues of the basic integral formulae of complex analysis such as those

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of Borel-Pompeiu, Cauchy, and so on, for this version of quaternionic function theory. We conclude the paper by explaining the connections between the null-solutions of the modified Sturm-Liouville operators and of the Chebyshev operator, on one hand, and the quaternionic hyperholomorphic and anti-hyperholomorphic functions on the other.

1. Introduction.

1.1. *Prolate spheroidal wave functions revisited.* The *prolate spheroidal wave functions* (PSWFs) have long been successfully used in many different fields of numerical analysis, nuclear modeling, signal processing and communication theory, electromagnetic modeling and physics [8, 9]. The PSWFs were originally introduced by C. Niven in [20] while studying the conduction of heat in an ellipsoid of revolution, which leads to the Helmholtz operator in spheroidal coordinates. Of course, we refer to the second order partial differential operator that is obtained by the underlying change of variables in the initial Helmholtz operator. PSWFs usually appear in the process of solving Dirichlet problems in spheroidal domains arising in hydrodynamics, elasticity and electromagnetism. For the solvability of boundary value problems of radiation, scattering, and propagation of acoustic signals and electromagnetic waves radiated by sources with spheroidal shapes, PSWFs are frequently encountered. These applications have stimulated a growth of new ideas and methods, both theoretical and applied, and have reawakened an interest in spectrum analysis, approximation theory, potential theory, the theory of partial differential equations, and so forth. The connection between PSWFs and the energy concentration problem was first introduced by D. Slepian and H.O. Pollak [23] in the 1960's. They are also known as Slepian functions and solutions of a Sturm-Liouville problem for solving elliptic boundary value problems in spheroidal geometry. The general theory and background on PSWFs is contained in the monograph by C. Flammer [3].

For a given real number $c > 0$, the PSWFs denoted by $\{\chi_c(x)\}_{n=0}^{\infty}$ have been known since the early 1930's as the eigenfunctions of the Sturm-Liouville operator L_c defined on $C^2([-1,1])$ by

$$L_c(\chi_c) = (1-x^2)\frac{d^2\chi_c}{dx^2} - 2x\frac{d\chi_c}{dx} - c^2x^2\chi_c; \quad x \in [-1,1]. \tag{1.1}$$

The above operator arises via the method of separation of variables for the Helmholtz equation with the use of the prolate spheroidal coordinates (ξ, η, ϕ), which are related to the Cartesian coordinates (x, y, z) by the following transformation (cf. E. Hobson [4], N. Lebedev [10]):

$$\begin{aligned} x &= f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \\ y &= f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \\ z &= f\xi\eta, \end{aligned}$$

where $f > 0$ is a scale factor; ξ, η and ϕ are parameters such that $1 < \xi < \infty$, $-1 < \eta < 1$ and $0 \le \phi < 2\pi$, which have the following meanings: ξ is the radius, η is the azimuthal angle about the major z-axis, and ϕ is the rotation term.

The family of confocal prolate spheroids is given by surfaces of constant ξ with major axis of length $2f\xi$ and minor axis of length $2f\sqrt{\xi^2-1}$, and corresponding foci at the

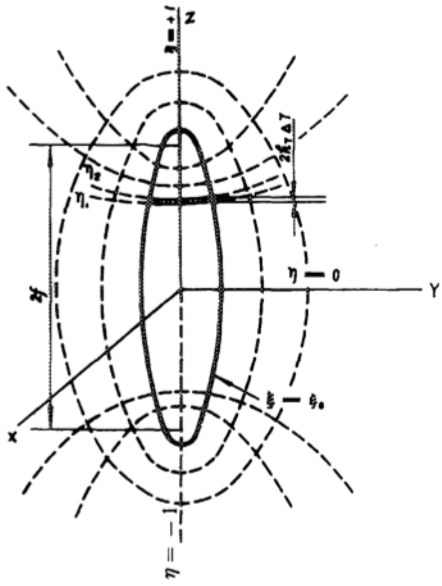

FIG. 1. The prolate spheroidal coordinate system (ξ, η, ϕ).

points $(0, 0, \pm f)$:

$$\frac{x^2 + y^2}{f^2(\xi^2 - 1)} + \frac{z^2}{f^2\xi^2} = 1.$$

Following [24] (cf. [11]), a family of solutions to the Helmholtz equation

$$\Delta_3 \chi_c + k^2 \chi_c = 0 \tag{1.2}$$

in prolate spheroidal coordinates can be represented in the following form:

$$\chi_c(\xi, \eta, \phi) := R(\xi; c)\, S(\eta; c)\, \Phi(\phi), \qquad c := fk.$$

The previous product sometimes bears the name of “Lamé products”. The separation of variables in (1.2) implies the following three differential equations:

$$(\xi^2 - 1)\frac{d^2R}{d\xi^2} + 2\xi\frac{dR}{d\xi} - \left(\lambda(c) - c^2\xi^2 + \frac{m^2}{\xi^2 - 1}\right) R \;=\; 0, \tag{1.3}$$

$$(1 - \eta^2)\frac{d^2S}{d\eta^2} - 2\eta\frac{dS}{d\eta} + \left(\lambda(c) - c^2\eta^2 - \frac{m^2}{1 - \eta^2}\right) S \;=\; 0, \tag{1.4}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi \;=\; 0, \tag{1.5}$$

where $\lambda(c)$ and m are parameters introduced during the separation of variables method. Equations (1.3) and (1.4) are called, respectively, *radial* and *angular prolate spheroidal equations*; we will call them modified Sturm-Liouville equations (compare with operator (1.1)). The solutions of (1.3) and (1.4) are called *radial* and *angular prolate spheroidal functions*. As a matter of fact, because the bandwidth tuning parameter c (also known in the literature as Slepian frequency) and m are constants, one has two families of ordinary differential equations parameterized by the same parameters c and m. It is worth noting that when the *interfocal distance* $2f$ becomes zero (i.e. $c = 0$) the previous radial and angular equations reduce to the Legendre's differential equations which are satisfied, respectively, by the classical associated Legendre functions of the first and second kinds.

The periodicity of Φ requires that m is a positive integer or zero. Hence solutions of (1.5) are

$$\Phi(\phi) := \begin{cases} \cos(m\phi), \\ \sin(m\phi), \end{cases}$$

where $\cos(m\phi)$ and $\sin(m\phi)$ are trigonometric polynomials that are related to the Chebyshev algebraic polynomials of the first and second kinds, respectively, T_m and U_m, in the following way:

$$\cos(m\phi) =: T_m(\cos\phi) \quad \text{and} \quad \sin(m\phi) =: U_{m-1}(\cos\phi)\sin(\phi).$$

Multidimensional PSWFs were first studied by D. Slepian in [24], which provided many of their analytical properties, as well as properties that support the construction of numerical schemes (cf. A.I. Zayed [30]). Very recently, in [6] the authors introduced the generalized PSWFs for offset linear canonical transform in Clifford analysis. These generalized spheroidal functions were successfully applied for the analysis of Slepian's concentration problem. In this line of research, in [15, 16] two distinct sets of monogenic orthogonal polynomials (in the sense of the usual generalized Cauchy-Riemann operator) were constructed over the interior of prolate spheroids (with a bandwidth parameter $c = 0$) which could be expressed in terms of products of associated Legendre functions multiplied by Chebyshev polynomial factors. Studies showed that the underlying generalized prolate functions play an important role in computing the monogenic Szegö kernel function in prolate spheroidal domains [17]. These results were used to investigate a particular class of approximation properties for monogenic functions over prolate spheroids in terms of special systems [18].

1.2. *Helmholtz equation in prolate spheroidal coordinates.* In three-dimensional Cartesian coordinates the Helmholtz equation has the form

$$\left(\Delta_3 + k^2\right)[u] := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0, \quad u \in \mathbb{R}^3,\ k \in \mathbb{R}. \tag{1.6}$$

In general, the parameter k can be complex or even quaternionic; see, e.g., [7].

The Helmholtz operator $\left(\Delta_3 + k^2\right)$ acts on the space of functions $\mathcal{C}^2(\mathbb{R}^3)$. For convenience, we consider its restriction onto $\mathcal{C}^2(\Omega_{x,y,z})$, where

$$\Omega_{x,y,z} := \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \,|\, x = y = 0,\ z \in \mathbb{R}\}.$$

In the sequel, consider a domain Ξ in a copy of $\mathbb{R}^3$ with the coordinates (ξ, η, ϕ). Now, we define a change of variables in the domain Ξ; i.e., there exists a mapping $\varphi : (\xi, \eta, \phi) \in \Xi \mapsto \varphi(\xi, \eta, \phi) = (x = \varphi_1(\xi, \eta, \phi), y = \varphi_2(\xi, \eta, \phi), z = \varphi_3(\xi, \eta, \phi)) \in \Omega_{x,y,z}$, such that $\varphi \in \mathcal{C}^2(\Xi)$ makes a one-to-one correspondence between both domains.

Assume that $\psi = (\psi_1(x,y,z), \psi_2(x,y,z), \psi_3(x,y,z))$ is the inverse mapping, $\psi : \Omega_{x,y,z} \mapsto \Xi$, i.e., so that $\varphi(\psi(x,y,z)) = (x,y,z)$ for any $(x,y,z) \in \Omega_{x,y,z}$ and $\psi(\varphi(\xi,\eta,\phi)) = (\xi,\eta,\phi)$ for any $(\xi,\eta,\phi) \in \Xi$.

Let us introduce the operators of the change of variables:

$$\begin{aligned} &W_\varphi : u \in \mathcal{C}^2(\Omega_{x,y,z}) \mapsto u \circ \varphi =: \tilde{u} \in \mathcal{C}^2(\Xi), \\ &W_\psi = W_\varphi^{-1} : \tilde{u} \in \mathcal{C}^2(\Xi) \mapsto \tilde{u} \circ \psi =: u \in \mathcal{C}^2(\Omega_{x,y,z}). \end{aligned}$$

Note, in passing, that W_φ is an isomorphism of $\mathcal{C}^2(\Omega_{x,y,z})$ onto $\mathcal{C}^2(\Xi)$, whereas W_ψ is an isomorphism of $\mathcal{C}^2(\Xi)$ onto $\mathcal{C}^2(\Omega_{x,y,z})$.

Let A be an arbitrary linear operator acting on $\mathcal{C}^2(\Omega_{x,y,z})$ and B be an arbitrary operator acting on $\mathcal{C}^2(\Xi)$. Define the operators $\widetilde{A}$ and $\widetilde{B}$ as

$$\begin{aligned} W_\varphi A W_\psi &=: \widetilde{A}, \\ W_\psi B W_\varphi &=: \widetilde{B}. \end{aligned}$$

Obviously, $\widetilde{A}$ acts on $\mathcal{C}^2(\Xi)$ while $\widetilde{B}$ acts on $\mathcal{C}^2(\Omega_{x,y,z})$.

Now, let $\mathcal{L}(\mathcal{C}^2(\Omega_{x,y,z}))$ and $\mathcal{L}(\mathcal{C}^2(\Xi))$ denote the algebras of all linear operators acting on the respective function spaces. Then the mapping

$$A \in \mathcal{L}(\mathcal{C}^2(\Omega_{x,y,z})) \mapsto W_\varphi A W_\psi = \widetilde{A} \in \mathcal{L}(\mathcal{C}^2(\Xi))$$

is an isomorphism of algebras.

Now, let us take $A = \Delta_3 + k^2$. Consider

$$W_\varphi A W_\psi = W_\varphi \Delta_3 W_\psi + W_\varphi k^2 W_\psi. \tag{1.7}$$

First of all, for any $\tilde{u} \in \mathcal{C}^2(\Xi)$, we note that

$$W_\varphi k^2 W_\psi[\tilde{u}] = W_\varphi k^2 [u] = W_\varphi [k^2 u] = \widetilde{k^2 u} = k^2 \tilde{u},$$

and therefore, it follows that

$$W_\varphi k^2 W_\psi = k^2 I,$$

where I is the identity operator.

Furthermore, we call attention to the fact that

$$\begin{aligned} W_\varphi \Delta_3 W_\psi &= W_\varphi \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) W_\psi \\ &= W_\varphi \frac{\partial^2}{\partial x^2} W_\psi + W_\varphi \frac{\partial^2}{\partial y^2} W_\psi + W_\varphi \frac{\partial^2}{\partial z^2} W_\psi \\ &= \left(W_\varphi \frac{\partial}{\partial x} W_\psi \right) \left(W_\varphi \frac{\partial}{\partial x} W_\psi \right) + \left(W_\varphi \frac{\partial}{\partial y} W_\psi \right) \left(W_\varphi \frac{\partial}{\partial y} W_\psi \right) \\ &\quad + \left(W_\varphi \frac{\partial}{\partial z} W_\psi \right) \left(W_\varphi \frac{\partial}{\partial z} W_\psi \right). \end{aligned} \tag{1.8}$$

Now, we apply all the above to the aforementioned prolate spheroidal change of variables, namely,

$$\begin{cases} \varphi_1(\xi,\eta,\phi) &= f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \\ \varphi_2(\xi,\eta,\phi) &= f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \\ \varphi_3(\xi,\eta,\phi) &= f\xi\eta. \end{cases} \tag{1.9}$$

Straightforward computations show that

$$W_\varphi(\Delta_3+k^2)W_\psi = \frac{1}{h_1^2(\xi,\eta)}\Big[\mathcal{W}_{\xi,\eta,\phi}+c^2(\xi^2-\eta^2)\Big], \tag{1.10}$$

where

$$\mathcal{W}_{\xi,\eta,\phi} := (\xi^2-1)\frac{\partial^2}{\partial\xi^2}+(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\xi\frac{\partial}{\partial\xi}-2\eta\frac{\partial}{\partial\eta}+\frac{h_1^2(\xi,\eta)}{h_2^2(\xi,\eta)}\frac{\partial^2}{\partial\phi^2} \tag{1.11}$$

with

$$c=fk, \quad h_1^2(\xi,\eta):=f^2(\xi^2-\eta^2), \quad h_2^2(\xi,\eta):=f^2(\xi^2-1)(1-\eta^2).$$

It is clear that $\ker(\Delta_3+k^2)$ is isomorphic to $\ker\big(\mathcal{W}_{\xi,\eta,\phi}+c^2(\xi^2-\eta^2)\big)$.

The preceding conclusions are true if (1.9) is a one-to-one correspondence. Hence we assume that the operator $\mathcal{W}_{\xi,\eta,\phi}+c^2(\xi^2-\eta^2)$ acts on $C^2(\Omega_{\xi,\eta,\phi})$ where $\Omega_{\xi,\eta,\phi} := (1,\infty)\times(-1,1)\times[0,2\pi)$; more precisely, (1.9) is a C^∞-diffeomorphism between $\Omega_{x,y,z}$ and $\Omega_{\xi,\eta,\phi}$.

2. Modified Sturm-Liouville and Chebyshev operators (MSLCOs).

2.1. *Relations between MSLCOs.* Let us introduce the following notation:

$$\mathcal{W}_1 := \frac{1}{h_1^2(\xi,\eta)}\Big[\mathcal{W}_{\xi,\eta,\phi}+c^2(\xi^2-\eta^2)\Big] =: \frac{1}{h_1^2(\xi,\eta)}\mathcal{W}.$$

Consider the operator $\mathcal{W}$ and assume that it acts on $\mathcal{C}^2(\Xi)$ with Ξ being of a special shape, namely, let Ξ be a Cartesian product of three intervals: $\Xi=\Xi_\xi\times\Xi_\eta\times\Xi_\phi$. It is known that in this case the Cartesian product

$$\mathcal{C}^2(\Xi)=\mathcal{C}^2(\Xi_\xi)\otimes\mathcal{C}^2(\Xi_\eta)\otimes\mathcal{C}^2(\Xi_\phi)$$

holds. Such a decomposition of the space $\mathcal{C}^2(\Xi)$ generates a decomposition of the operator $\mathcal{W}$. Indeed, every element of $\mathcal{C}^2(\Xi_\xi)\otimes\mathcal{C}^2(\Xi_\eta)\otimes\mathcal{C}^2(\Xi_\phi)$ is of the form

$$g_c(\xi,\eta,\phi)=\sum_{i=1}^{\infty}R_i(\xi;c)\otimes S_i(\eta;c)\otimes\Phi_i(\phi)$$

where the elementary tensor

$$R_i(\xi;c)\otimes S_i(\eta;c)\otimes\Phi_i(\phi)$$

is just a pointwise product of the three functions $R_i\in\mathcal{C}^2(\Xi_\xi)$, $S_i\in\mathcal{C}^2(\Xi_\eta)$ and $\Phi_i\in\mathcal{C}^2(\Xi_\phi)$. In particular, if I_η is the identity operator on $\mathcal{C}^2(\Xi_\eta)$ and I_ϕ is the identity operator on $\mathcal{C}^2(\Xi_\phi)$, then the operator $\frac{d^2}{d\xi^2} \otimes I_\eta \otimes I_\phi$ acts, by definition, on $\mathcal{C}^2(\Xi_\xi) \otimes \mathcal{C}^2(\Xi_\eta) \otimes \mathcal{C}^2(\Xi_\phi)$ by the following rule:

$$\begin{aligned}\frac{d^2}{d\xi^2} \otimes I_\eta \otimes I_\phi \left[\sum_{i=1}^{\infty} R_i \otimes S_i \otimes \Phi_i\right] &= \sum_{i=1}^{\infty} \frac{d^2}{d\xi^2} \otimes I_\eta \otimes I_\phi \left[R_i \otimes S_i \otimes \Phi_i\right] \\ &= \sum_{i=1}^{\infty} \frac{d^2 R_i}{d\xi^2} \otimes I_\eta[S_i] \otimes I_\phi[\Phi_i] \\ &= \sum_{i=1}^{\infty} \frac{d^2 R_i}{d\xi^2} \cdot S_i \cdot \Phi_i \\ &= \frac{\partial^2 g_c(\xi,\eta,\phi)}{\partial \xi^2}.\end{aligned}$$

Hence $\dfrac{\partial^2}{\partial\xi^2} = \dfrac{d^2}{d\xi^2} \otimes I_\eta \otimes I_\phi$. In a similar way, we obtain that $\dfrac{\partial^2}{\partial\eta^2} = I_\xi \otimes \dfrac{d^2}{d\eta^2} \otimes I_\phi$ and $\dfrac{\partial^2}{\partial\phi^2} = I_\xi \otimes I_\eta \otimes \dfrac{d^2}{d\phi^2}$. A quite analogous reasoning shows that $2\xi\dfrac{\partial}{\partial\xi} = 2\xi\dfrac{d}{d\xi} \otimes I_\eta \otimes I_\phi$ and $2\eta\dfrac{\partial}{\partial\eta} = I_\xi \otimes 2\eta\dfrac{d}{d\eta} \otimes I_\phi$.

For simplicity of presentation, we set $\alpha(\xi) := \lambda(c) - c^2\xi^2 + \frac{m^2}{\xi^2-1}$ and $\beta(\eta) := \lambda(c) - c^2\eta^2 - \frac{m^2}{1-\eta^2}$. All this means that the operator $\mathcal{W}$ can be seen as

$$\begin{aligned}\mathcal{W} = (\xi^2-1)\frac{d^2}{d\xi^2} \otimes I_\eta \otimes I_\phi + 2\xi\frac{d}{d\xi} \otimes I_\eta \otimes I_\phi - \alpha(\xi) I_\xi \otimes I_\eta \otimes I_\phi \\ + I_\xi \otimes (1-\eta^2)\frac{d^2}{d\eta^2} \otimes I_\phi - I_\xi \otimes 2\eta\frac{d}{d\eta} \otimes I_\phi + I_\xi \otimes \beta(\eta) I_\eta \otimes I_\phi \\ + \frac{1}{\xi^2-1} I_\xi \otimes I_\eta \otimes \frac{d^2}{d\phi^2} + I_\xi \otimes \frac{1}{1-\eta^2} I_\eta \otimes \frac{d^2}{d\phi^2} \\ + \frac{1}{\xi^2-1} I_\xi \otimes I_\eta \otimes m^2 I_\phi + I_\xi \otimes \frac{1}{1-\eta^2} I_\eta \otimes m^2 I_\phi.\end{aligned}$$

In order to state our results, we shall need some further notation:

$$\begin{aligned}\mathcal{S}_{\xi,\alpha} &:= \left[(\xi^2-1)\frac{d^2}{d\xi^2} + 2\xi\frac{d}{d\xi} - \alpha(\xi) I_\xi\right] \otimes I_\eta \otimes I_\phi, \\ \mathcal{S}_{\eta,\beta} &:= I_\xi \otimes \left[(1-\eta^2)\frac{d^2}{d\eta^2} - 2\eta\frac{d}{d\eta} + \beta(\eta) I_\eta\right] \otimes I_\phi, \\ \mathcal{T}_\phi &:= I_\xi \otimes I_\eta \otimes \left(\frac{d^2}{d\phi^2} + m^2 I_\phi\right),\end{aligned}$$

and

$$S_{\xi,\alpha} := (\xi^2 - 1)\frac{d^2}{d\xi^2} + 2\xi\frac{d}{d\xi} - \alpha(\xi)I_\xi,$$

$$S_{\eta,\beta} := (1 - \eta^2)\frac{d^2}{d\eta^2} - 2\eta\frac{d}{d\eta} + \beta(\eta)I_\eta,$$

$$T_\phi := \frac{d^2}{d\phi^2} + m^2 I_\phi,$$

giving, of course, that

$$S_{\xi,\alpha}[R] := (\xi^2 - 1)\frac{d^2R}{d\xi^2} + 2\xi\frac{dR}{d\xi} - \alpha(\xi)R,$$

$$S_{\eta,\beta}[S] := (1 - \eta^2)\frac{d^2S}{d\eta^2} - 2\eta\frac{dS}{d\eta} + \beta(\eta)S,$$

$$T_\phi[\Phi] := \frac{d^2\Phi}{d\phi^2} + m^2\Phi,$$

which immediately leads to the equations (1.3)-(1.5) for arbitrary values of the constants c and m.

For this reason we will use the names of Modified Sturm-Liouville and Chebyshev Operators (MSLCOs) for $S_{\xi,\alpha}$, $S_{\eta,\beta}$ and T_ϕ. We will call them in this way, since their solutions are related, respectively, to the modified Sturm-Liouville equations (1.3)-(1.4) and Chebyshev equation (1.5). Therefore we conclude that

$$\mathcal{W} = \mathcal{S}_{\xi,\alpha} + \mathcal{S}_{\eta,\beta} + \frac{h_1^2(\xi,\eta)}{h_2^2(\xi,\eta)}\,\mathcal{T}_\phi. \tag{2.1}$$

We note that this decomposition has been obtained for parallelepiped domains with sides parallel to the coordinate axes only.

2.2. *On the kernels of the MSLCOs.* For simplicity of presentation, we set

$$C := \frac{h_1^2(\xi,\eta)}{h_2^2(\xi,\eta)} = C_1(\xi) + C_2(\eta),$$

where $C_1(\xi) := \frac{1}{\xi^2-1}$ and $C_2(\eta) := \frac{1}{1-\eta^2}$. On the same parallelepiped we consider now $\ker(\mathcal{S}_{\xi,\alpha} + \mathcal{S}_{\eta,\beta} + C\mathcal{T}_\phi)$ and $\ker S_{\xi,\alpha} \otimes \ker S_{\eta,\beta} \otimes \ker T_\phi$.

Given an elementary tensor $g_1 \otimes g_2 \otimes g_3 \in \ker S_{\xi,\alpha} \otimes \ker S_{\eta,\beta} \otimes \ker T_\phi$, there holds:

$$\begin{aligned}
\mathcal{W}[g_1 \otimes g_2 \otimes g_3] &= (\mathcal{S}_{\xi,\alpha} + \mathcal{S}_{\eta,\beta} + C\mathcal{T}_\phi)[g_1 \otimes g_2 \otimes g_3] \\
&= (S_{\xi,\alpha} \otimes I_\eta \otimes I_\phi)[g_1 \otimes g_2 \otimes g_3] \\
&\quad + (I_\xi \otimes S_{\eta,\beta} \otimes I_\phi)[g_1 \otimes g_2 \otimes g_3] \\
&\quad + [C_1(\xi)I_\xi \otimes I_\eta \otimes T_\phi]\,[g_1 \otimes g_2 \otimes g_3] \\
&\quad + [I_\xi \otimes C_2(\eta)I_\eta \otimes T_\phi]\,[g_1 \otimes g_2 \otimes g_3] \\
&= S_{\xi,\alpha}[g_1] \cdot g_2 \cdot g_3 + g_1 \cdot S_{\eta,\beta}[g_2] \cdot g_3 + [C_1(\xi) + C_2(\eta)]\, g_1 \cdot g_2 \cdot T_\phi[g_3] \\
&= 0,
\end{aligned}$$

that is, $g_1\otimes g_2\otimes g_3 \in \ker(\mathcal{S}_{\xi,\alpha}+\mathcal{S}_{\eta,\beta}+C\mathcal{T}_\phi)$. Since $S_{\xi,\alpha}$, $S_{\eta,\beta}$ and T_ϕ are linear operators, every element

$$\sum_{i=1}^{\infty} a_i\, g_{1i}\otimes g_{2i}\otimes g_{3i} \in \ker S_{\xi,\alpha}\otimes \ker S_{\eta,\beta}\otimes \ker T_\phi, \quad a_i\in\mathbb{R},$$

belongs to $\ker(\mathcal{S}_{\xi,\alpha}+\mathcal{S}_{\eta,\beta}+C\mathcal{T}_\phi)$. We conclude, then, from the foregoing that

$$\ker S_{\xi,\alpha}\otimes \ker S_{\eta,\beta}\otimes \ker T_\phi \subseteq \ker(\mathcal{S}_{\xi,\alpha}+\mathcal{S}_{\eta,\beta}+C\mathcal{T}_\phi)=\ker\mathcal{W}. \tag{2.2}$$

Now take an elementary tensor $g_1\otimes g_2\otimes g_3 \in C^2(\Xi_\xi)\otimes C^2(\Xi_\eta)\otimes C^2(\Xi_\phi)$, where $g_3 = \begin{cases}\cos(m\phi),\\ \sin(m\phi),\end{cases}$ which belongs to $\ker(\mathcal{S}_{\xi,\alpha}+\mathcal{S}_{\eta,\beta}+C\mathcal{T}_\phi)=\ker\mathcal{W}$; then

$$\begin{aligned}
0 &= \mathcal{W}[g_1\otimes g_2\otimes g_3]\\
&= (\mathcal{S}_{\xi,\alpha}+\mathcal{S}_{\eta,\beta}+C\mathcal{T}_\phi)[g_1\otimes g_2\otimes g_3]\\
&= (S_{\xi,\alpha}\otimes I_\eta\otimes I_\phi)[g_1\otimes g_2\otimes g_3]\\
&\quad + (I_\xi\otimes S_{\eta,\beta}\otimes I_\phi)[g_1\otimes g_2\otimes g_3]\\
&\quad + [C_1(\xi)I_\xi\otimes I_\eta\otimes T_\phi]\,[g_1\otimes g_2\otimes g_3] + [I_\xi\otimes C_2(\eta)I_\eta\otimes T_\phi]\,[g_1\otimes g_2\otimes g_3]\\
&= S_{\xi,\alpha}[g_1]\cdot g_2\cdot g_3 + g_1\cdot S_{\eta,\beta}[g_2]\cdot g_3 + C\, g_1\cdot g_2\cdot T_\phi[g_3]\\
&= S_{\xi,\alpha}[g_1]\cdot g_2\cdot g_3 + g_1\cdot S_{\eta,\beta}[g_2]\cdot g_3\,.
\end{aligned}$$

Assuming that $g_1(\xi)\neq 0$ for all $\xi\in\Xi_\xi$ and $g_2(\eta)\neq 0$ for all $\eta\in\Xi_\eta$, the above equality is equivalent to

$$d_1 := \frac{1}{g_1}S_{\xi,\alpha}[g_1] = -\frac{1}{g_2}S_{\eta,\beta}[g_2]\ ,$$

and this implies that

$$d_1 = \frac{1}{g_1}\left[(\xi^2-1)\frac{d^2g_1}{d\xi^2}+2\xi\frac{dg_1}{d\xi}-\alpha(\xi)g_1\right]$$

is equivalent to

$$(\xi^2-1)\frac{d^2g_1}{d\xi^2}+2\xi\frac{dg_1}{d\xi}-[\alpha(\xi)+d_1]\,g_1=0. \tag{2.3}$$

Analogously, we obtain that

$$-d_1 = \frac{1}{g_2}\left[(1-\eta^2)\frac{d^2g_2}{d\eta^2}-2\eta\frac{dg_2}{d\eta}+\beta(\eta)g_2\right]$$

is equivalent to

$$(1-\eta^2)\frac{d^2g_2}{d\eta^2}-2\eta\frac{dg_2}{d\eta}+[\beta(\eta)+d_1]\,g_2=0. \tag{2.4}$$

The formulae (2.3) and (2.4) imply that $g_1\otimes g_2\otimes g_3\in\ker S_{\xi,\alpha+d_1}\otimes\ker S_{\eta,\beta+d_1}\otimes\ker T_\phi$.

We shall observe that different elementary tensors in $\ker(\mathcal{S}_{\xi,\alpha} + \mathcal{S}_{\eta,\beta} + C\mathcal{T}_{\phi})$ give rise to different constants d_1, so we conclude that the set of elementary tensors in $\ker(\mathcal{S}_{\xi,\alpha} + \mathcal{S}_{\eta,\beta} + C\mathcal{T}_{\phi}) = \ker \mathcal{W}$ is contained in the set

$$\bigcup_{\alpha,\beta} \left(\ker S_{\xi,\alpha} \otimes \ker S_{\eta,\beta} \otimes \ker T_{\phi}\right).$$

3. Notion of $\mathcal{D}_k$-hyperholomorphic function. In the previous sections we established a direct relation between the MSLCOs (and, hence, the PSWFs) and the operator $\mathcal{W}$. On the other hand, it turns out that there exists a quaternionic function theory which is related to the operator $\mathcal{W}$ in the same way as complex analysis in one variable, classic quaternionic analysis and Clifford analysis are related to the corresponding Laplace operators. Even though there is no straightforward relation, we still manage to relate the PSWFs to a quaternionic function theory.

We proceed by introducing and developing some basic facts of this version of the quaternionic analysis and its associated function theory. Since most applications related to the PSWFs take a general complex wave number k, we shall also take $k \in \mathbb{C}$ in the forthcoming analysis. This consideration makes sense, however, since the underlying theory is connected with the α-hyperholomorphic function theory developed in [7].

Throughout the paper, let $\mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ be the sets of real and complex quaternions. Each quaternion w is represented in the form

$$w := w_0 + w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}.$$

The set $\{w_k\}$ is in $\mathbb{R}$ for real quaternions and $\{w_k\}$ is in $\mathbb{C}$ for complex quaternions, and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the quaternionic imaginary units which obey the usual laws of multiplication: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, and $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. As usual, we denote the imaginary unit in $\mathbb{C}$ by i; by definition, i commutes with all the quaternionic imaginary units. $\mathbb{H}(\mathbb{R})$ is a skew-field; meanwhile $\mathbb{H}(\mathbb{C})$ is a complex non-commutative, associative algebra with zero divisors.

The scalar and vector parts of w, $\mathrm{Sc}(w)$ and $\mathrm{Vec}(w)$ are defined as the w_0 and $w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ terms, respectively. For a complex quaternion w we consider its quaternionic conjugate $\bar{w}$ defined by

$$\bar{w} := w_0 - \mathrm{Vec}(w).$$

Let Ω be a bounded multi-connected domain in $\Omega_{\xi,\eta,\phi}$ with a piecewise smooth boundary, and denote by $\bar{\Omega}$ its closure. A central notion in quaternionic analysis is that of $\mathcal{D}_k$-*hyperholomorphy* (resp. $\mathcal{D}_k$-*anti-hyperholomorphy*). On the set $C^1(\Omega, \mathbb{H}(\mathbb{C}))$ we consider the following first-order partial differential operators with variable quaternionic coefficients:

$$\begin{aligned}
\mathcal{D}_k := k &+ \left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}\right)\xi + \frac{f}{h_1^2(\xi,\eta)}(\xi^2-1)\eta\mathbf{k}\right]\frac{\partial}{\partial\xi} \\
&+ \left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}\right)\eta + \frac{f}{h_1^2(\xi,\eta)}(1-\eta^2)\xi\mathbf{k}\right]\frac{\partial}{\partial\eta} \qquad (3.1)\\
&+ \left[\frac{1}{h_2(\xi,\eta)}\left(-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}\right)\right]\frac{\partial}{\partial\phi}
\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{D}}_k := k &- \left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\cos\phi\mathbf{i}+\sin\phi\mathbf{j}\right)\xi + \frac{f}{h_1^2(\xi,\eta)}(\xi^2-1)\eta\mathbf{k}\right]\frac{\partial}{\partial\xi} \\ &- \left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\cos\phi\mathbf{i}+\sin\phi\mathbf{j}\right)\eta + \frac{f}{h_1^2(\xi,\eta)}(1-\eta^2)\xi\mathbf{k}\right]\frac{\partial}{\partial\eta} \\ &- \left[\frac{1}{h_2(\xi,\eta)}\left(-\sin\phi\mathbf{i}+\cos\phi\mathbf{j}\right)\right]\frac{\partial}{\partial\phi}.\end{aligned} \tag{3.2}$$

These operators are well-defined on

$$\mathbb{R}^3 \setminus \{(\xi,\eta,\phi)\in\mathbb{R}^3 \mid h_1(\xi,\eta)\neq 0 \text{ and } h_2(\xi,\eta)\neq 0\},$$

but in order to unify the notation and to simplify the calculations further, we choose them acting on $\Omega_{\xi,\eta,\phi}$. These operators can act on the right, in which case they will be written as $\mathcal{D}_{k,r}$ and $\overline{\mathcal{D}}_{k,r}$. We denote by $\mathcal{D}_0$ the operator in (3.1) with $k=0$.

The motivation of introducing both operators is explained in detail in Section 5.

Definition 3.1. Any solution $g\in C^1(\Omega,\mathbb{H}(\mathbb{C}))$ of the equation $\mathcal{D}_k[g]=0$ is called a $\mathcal{D}_k$-hyperholomorphic function. Analogously, for solutions of the equation $\overline{\mathcal{D}}_k[g]=0$ a reasonably natural name is $\mathcal{D}_k$-anti-hyperholomorphic functions.

Denote the set of $\mathcal{D}_k$-hyperholomorphic functions by $\mathfrak{M}_k$:

$$\mathfrak{M}_k := \mathfrak{M}_k(\Omega,\mathbb{H}(\mathbb{C})) := \ker\mathcal{D}_k,$$

and define in a similar way $\mathfrak{M}_{k,r}$ as

$$\mathfrak{M}_{k,r} := \ker\mathcal{D}_{k,r}.$$

Straightforward computations show that

$$\begin{aligned}\mathcal{D}_k\cdot\overline{\mathcal{D}}_k = \overline{\mathcal{D}}_k\cdot\mathcal{D}_k &= \frac{1}{h_1^2(\xi,\eta)}\left[\mathcal{W}_{\xi,\eta,\phi}+c^2(\xi^2-\eta^2)\right] \\ &= \frac{1}{h_1^2(\xi,\eta)}\mathcal{W} = \mathcal{W}_1,\end{aligned} \tag{3.3}$$

where $\mathcal{W}_{\xi,\eta,\phi}$ is given by (1.11).

The above means that $\mathcal{D}_k$-hyperholomorphic functions indeed play the same role for the $\mathcal{W}$ operator as the usual holomorphic functions in one complex variable or hyperholomorphic functions of quaternionic or Clifford analysis play for the corresponding Laplace operators, and they are in striking analogy with related investigations about α-hyperholomorphic functions for the Helmholtz operator in [7]. At the same time, there exists a deep difference since the operators (3.1) and (3.2) have variable, not constant, coefficients, and it is well-known that function theories using such operators are much more sophisticated.

4. Integral formulae for $\mathcal{D}_k$-hyperholomorphic functions. This section presents main integral theorems and formulae for the class of functions introduced in the previous section.

Consider the following function:

$$\mathcal{K}_k(\xi,\eta,\phi) := \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{ik|\zeta|}\left(k+\dfrac{\zeta_{st}}{|\zeta|^2}-ik\dfrac{\zeta_{st}}{|\zeta|}\right), & \operatorname{Im} k>0, \\[2ex] -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{-ik|\zeta|}\left(k+\dfrac{\zeta_{st}}{|\zeta|^2}+ik\dfrac{\zeta_{st}}{|\zeta|}\right), & \operatorname{Im} k<0, \end{cases} \tag{4.1}$$

where

$$\zeta = \left(f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi, f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi, f\xi\eta\right)$$

is such that $|\zeta| = f\sqrt{\xi^2+\eta^2-1}$ and

$$\zeta_{st} := f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi\mathbf{i} + f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi\mathbf{j} + f\xi\eta\mathbf{k}.$$

We note that the function $\mathcal{K}_k$ is well-defined on the set

$$\mathbb{R}^3 \setminus \{(\xi,\eta,\phi)\in\mathbb{R}^3 \mid \xi^2+\eta^2=1,\ 0\le\phi<2\pi\},$$

but for our purposes we choose to define it on $\Omega_{\xi,\eta,\phi}$.

In case $k=0$ both formulae in (4.1) give $-\dfrac{1}{4\pi}\dfrac{\zeta_{st}}{|\zeta|^3}$, which is the fundamental solution of the Moisil-Theodorescu operator. The function $\mathcal{K}_k$ does indeed play the role of the Cauchy kernel for the $\mathcal{D}_k$-hyperholomorphic function theory.

We set

$$\begin{aligned} \omega :&= \mathbf{i}f^2\Bigg[-\sin\phi\left(\frac{\xi^2-\eta^2}{\sqrt{\xi^2-1}\sqrt{1-\eta^2}}\right)d\xi\wedge d\eta \\ &+ \left(\eta\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi\right)d\xi\wedge d\phi + \left(\xi\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi\right)d\eta\wedge d\phi\Bigg] \\ &+ \mathbf{j}f^2\Bigg[\cos\phi\left(\frac{\xi^2-\eta^2}{\sqrt{\xi^2-1}\sqrt{1-\eta^2}}\right)d\xi\wedge d\eta \\ &+ \left(\eta\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi\right)d\xi\wedge d\phi + \left(\xi\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi\right)d\eta\wedge d\phi\Bigg] \\ &+ \mathbf{k}f^2\left[-\xi(1-\eta^2)d\xi\wedge d\phi + \eta(\xi^2-1)d\eta\wedge d\phi\right]. \end{aligned} \tag{4.2}$$

Let ds be the surface element in $\Omega_{\xi,\eta,\phi}$. Then $|\omega| = ds$, and if Γ is a smooth surface, then

$$\omega = \mathbf{i}n_1 ds + \mathbf{j}n_2 ds + \mathbf{k}n_3 ds = n_{st}ds,$$

where

$$n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) = n_1(\Lambda_1,\Lambda_2,\Lambda_3)\mathbf{i} + n_2(\Lambda_1,\Lambda_2,\Lambda_3)\mathbf{j} + n_3(\Lambda_1,\Lambda_2,\Lambda_3)\mathbf{k}$$

is a unit vector of the outward normal to Γ at the point $(\Lambda_1,\Lambda_2,\Lambda_3)\in\Gamma$ seen as a quaternion.

In order to simplify calculations and to present the basic ideas, we will consider piecewise smooth surfaces Γ, although the whole reasoning is valid for more general surfaces as well.

Now consider $\{g_1, g_2\} \subset C^1(\bar{\Omega}, \mathbb{H})$. If we apply the exterior differentiation operator d to the differential form $g_1 \cdot \omega \cdot g_2$ we obtain:

$$\begin{aligned} d(g_1 \cdot \omega \cdot g_2) &= dg_1 \wedge \omega \cdot g_2 - g_1 \cdot \omega \wedge dg_2 \\ &= (\mathcal{D}_{0,r}[g_1] \cdot g_2 + g_1 \cdot \mathcal{D}_0[g_2])\, f h_1^2(\xi, \eta)\, d\xi \wedge d\eta \wedge d\phi. \end{aligned}$$

It then follows that

$$d(g_1 \cdot \omega \cdot g_2) = (\mathcal{D}_{k,r}[g_1] \cdot g_2 + g_1 \cdot \mathcal{D}_k[g_2] - 2k g_1 g_2)\, f h_1^2(\xi, \eta)\, d\xi \wedge d\eta \wedge d\phi. \tag{4.3}$$

We are now ready to formulate the main results of this section.

THEOREM 4.1 (Stokes' formula compatible with $\mathcal{D}_k$-hyperholomorphy). Let Ω be a domain in $\Omega_{\xi,\eta,\phi}$ and let its boundary Γ be a piecewise smooth surface such that $\partial\Omega = \Gamma$. Then for any $g_1, g_2 \in C^1(\bar{\Omega}) \bigcap C(\Omega \cup \Gamma)$ the following formula holds:

$$\int_\Gamma g_1 \cdot \omega \cdot g_2 = \int_\Omega (\mathcal{D}_{k,r}[g_1] \cdot g_2 + g_1 \cdot \mathcal{D}_k[g_2] - 2k g_1 g_2)\, f h_1^2(\xi, \eta)\, d\xi \wedge d\eta \wedge d\phi.$$

Proof. The proof is a direct consequence of the usual "real" Stokes theorem and formula (4.3). □

COROLLARY 4.1 (Analogue of the Cauchy integral theorem). Under the conditions of the preceding theorem and if, in addition, $g_2 \in \mathfrak{M}_k(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ and $g_1 \in \mathfrak{M}_{k,r}(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, then

$$\int_\Gamma g_1 \cdot \omega \cdot g_2 = -2k \int_\Omega g_1 \cdot g_2 \cdot f h_1^2(\xi, \eta)\, d\xi \wedge d\eta \wedge d\phi.$$

We proceed by generalizing two classical one-dimensional complex operators: the Cauchy-type operator and the T-operator. Consider the Cauchy kernel $\mathcal{K}_k$ given by formula (4.1). Let T_k and K_k be the operators defined by the formulae:

$$\begin{aligned} &T_k[g](\xi, \eta, \phi) \\ &\quad := \int_\Omega \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho) g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \end{aligned} \tag{4.4}$$

for $(\xi, \eta, \phi) \in \Omega_{\xi,\eta,\phi}$ and

$$\begin{aligned} &K_k[g](\xi, \eta, \phi) \\ &\quad := -\int_\Gamma \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho) n_{st}(\mu, \nu, \varrho) g(\mu, \nu, \varrho)\, d\Gamma_{(\mu,\nu,\varrho)} \end{aligned} \tag{4.5}$$

for $(\xi, \eta, \phi) \in \Omega_{\xi,\eta,\phi} \setminus \Gamma$ where Ω is as above with a piecewise smooth boundary Γ.

For the integral operators introduced above, we now deduce some theorems which are the exact structural analogues of the corresponding facts of one-dimensional complex analysis and which express profound properties of the $\mathcal{D}_k$-hyperholomorphic function theory, as well as important relations between this theory and operator theory.

THEOREM 4.2 (Quaternionic Borel-Pompeiu formula). Let Ω be a domain in $\Omega_{\xi,\eta,\phi}$ and Γ be a piecewise smooth surface such that $\partial\Omega = \Gamma$. Let $k \in \mathbb{C}$ and $g \in C^1\left(\bar{\Omega}, \mathbb{H}(\mathbb{C})\right) \bigcap C\left(\bar{\Omega}, \mathbb{H}(\mathbb{C})\right)$. Then

$$g(\xi, \eta, \phi) = K_k[g](\xi, \eta, \phi) + T_k \cdot \mathcal{D}_k[g](\xi, \eta, \phi), \quad \forall\, (\xi, \eta, \phi) \in \Omega.$$

Proof. Let $w = (\mu, \nu, \varrho)$ and $\rho = (\xi, \eta, \phi)$. By definition of the operator T_k we have:

$$\begin{aligned} T_k \cdot \mathcal{D}_k[g](\xi, \eta, \phi) & \\ &= \int_\Omega \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{k,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \qquad (4.6) \\ &= \lim_{\epsilon \to 0} \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{k,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho, \end{aligned}$$

where the subindex w in $\mathcal{D}_{k,w}$ means differentiation with respect to the variable w, and

$\vartheta_\epsilon := \Big\{ (\mu,\nu,\varrho) \Big| \Big| f\sqrt{(\xi-\mu)^2-1}\sqrt{1-(\eta-\nu)^2}\cos(\phi-\varrho), f\sqrt{(\xi-\mu)^2-1}\sqrt{1-(\eta-\nu)^2}\sin(\phi-\varrho), f(\xi-\mu)(\eta-\nu) \Big| \leq \epsilon \Big\}.$

Now

$$\begin{aligned} &\int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{k,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &= \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{0,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\quad + k \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho) g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &= \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{0,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\quad + k \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho) g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\quad - \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{D}_{0,r,w}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\quad + \int_{\Omega \setminus \vartheta_\epsilon} \mathcal{D}_{0,r,w}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho. \end{aligned}$$

It is a simple matter to verify that

$$\begin{aligned} -\mathcal{D}_{0,r,w}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] &= -\,\mathcal{D}_{0,w}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] \\ &= \mathcal{D}_{0,\rho}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right]. \end{aligned}$$

Hence we get:

$$\begin{aligned} &\int_{\Omega \setminus \vartheta_\epsilon} \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{k,w}[g](\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\quad = \int_{\Omega \setminus \vartheta_\epsilon} \Big\{ \mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\mathcal{D}_{0,w}\left[g(\mu, \nu, \varrho)\right] \\ &\qquad + \mathcal{D}_{0,r,w}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] g(\mu, \nu, \varrho) \Big\} f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho \\ &\qquad + \int_{\Omega \setminus \vartheta_\epsilon} \Big\{ k\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho) \\ &\qquad\qquad + \mathcal{D}_{0,\rho}\left[\mathcal{K}_k(\xi - \mu, \eta - \nu, \phi - \varrho)\right] \Big\} g(\mu, \nu, \varrho) f h_1^2(\mu, \nu)\, d\mu \wedge d\nu \wedge d\varrho. \end{aligned}$$

Applying Stokes' formula in Theorem 4.1 we have:

$$
\begin{aligned}
&\int_{\Omega\setminus\vartheta_\epsilon} \mathcal{K}_k(\xi-\mu,\eta-\nu,\phi-\varrho)\mathcal{D}_{k,w}[g](\mu,\nu,\varrho) f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho \\
=&\int_{\gamma^\epsilon} \mathcal{K}_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\gamma^\epsilon \\
&+\int_{\Omega\setminus\vartheta_\epsilon} \mathcal{D}_k\left[\mathcal{K}_k(\xi-\mu,\eta-\nu,\phi-\varrho)\right] g(\mu,\nu,\varrho) f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho,
\end{aligned}
$$

where $\gamma^\epsilon := \partial(\Omega\setminus\vartheta_\epsilon)$. The second integral equals zero, and turning back to (4.6) we finally obtain:

$$
\begin{aligned}
&T_k\cdot\mathcal{D}_k[f](\xi,\eta,\phi) \\
=&\lim_{\epsilon\to 0}\int_{\gamma^\epsilon} \mathcal{K}_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\gamma^\epsilon \\
=&\lim_{\epsilon\to 0}\int_{\Gamma} \mathcal{K}_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma \\
&+\lim_{\epsilon\to 0}\int_{\partial\vartheta_\epsilon} \mathcal{K}_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\partial\vartheta_\epsilon \\
=&-K_k[g](\xi,\eta,\phi)+g(\xi,\eta,\phi).
\end{aligned}
$$

This concludes the proof. □

As an immediate consequence of this theorem we obtain the quaternionic version of Cauchy's integral formula.

THEOREM 4.3 (Quaternionic Cauchy integral formula). Let Ω be a domain in $\Omega_{\xi,\eta,\phi}$ and let Γ be a piecewise smooth surface with $\Gamma=\partial\Omega$. Let $g\in\mathfrak{M}_k(\bar\Omega)\bigcap C\left(\bar\Omega\right)$ and $k\in\mathbb{C}$. Then

$$g(\xi,\eta,\phi)=K_k[g](\xi,\eta,\phi),\quad \forall\,(\xi,\eta,\phi)\in\Omega. \tag{4.7}$$

Let $L^p\left(\Omega,\mathbb{H}(\mathbb{C})\right)$ be the set of $\mathbb{H}(\mathbb{C})$-valued functions such that each component is in the usual $L^p(\Omega,\mathbb{C})$; this set forms a right $\mathbb{H}(\mathbb{C})$ module. We are now in a position to give a quaternionic version of Morera's Theorem.

THEOREM 4.4 (Quaternionic Morera theorem). Let $k\in\mathbb{C}$, $g\in C^1\left(\Omega,\mathbb{H}(\mathbb{C})\right)$, and $\mathcal{D}_k[g]\in L^p\left(\Omega,\mathbb{H}(\mathbb{C})\right)$ for some $p>1$. If for any piecewise smooth surface Γ such that $\overline{I(\Gamma)}\subset\Omega$ with $\Omega\subset\Omega_{\xi,\eta,\phi}$ and $I(\Gamma)$ the interior region, we have that

$$\int_\Gamma n_{st}\cdot g\, d\Gamma=-\int_\Omega gkfh_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi; \tag{4.8}$$

then g is $\mathcal{D}_k$-hyperholomorphic in Ω.

Proof. Let $\{\Omega_\ell\}_{\ell\in\mathbb{N}}$ be a regular sequence of domains converging to the point $(\xi_0,\eta_0,\phi_0)\in\Omega$, and let Γ_ℓ be the boundary of Ω_ℓ. Then by Lebesgue's theorem (cf. [25]) for any $h\in L^p(\Omega,\mathbb{H}(\mathbb{C}))$ $(p>1)$ there holds

$$\lim_{\ell\to\infty}\frac{1}{|\Omega_\ell|}\int_{\Omega_\ell} h(\xi,\eta,\phi)\, d\xi\wedge d\eta\wedge d\phi =: \tilde h(\xi_0,\eta_0,\phi_0)$$

and $h = \tilde{h}$ in $L^p(\Omega, \mathbb{H}(\mathbb{C}))$. If we choose $h := \mathcal{D}_0[g] f h_1^2(\xi,\eta)$, then, by hypothesis, it follows that

$$\begin{aligned}
&\frac{1}{|\Omega_\ell|}\int_{\Omega_\ell} \mathcal{D}_k[g](\xi,\eta,\phi) f h_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi \\
&= \frac{1}{|\Omega_\ell|}\int_{\Omega_\ell} \left(\mathcal{D}_0[g](\xi,\eta,\phi) + k g(\xi,\eta,\phi)\right) f h_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi \\
&= \frac{1}{|\Omega_\ell|}\int_{\Omega_\ell} \mathcal{D}_0[g](\xi,\eta,\phi) f h_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi \\
&\quad - \frac{1}{|\Omega_\ell|}\int_{\Gamma_\ell} n_{st}(\Lambda_1,\Lambda_2,\Lambda_3)\cdot g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\ell.
\end{aligned}$$

From Stokes' formula we have:

$$\int_{\Omega_\ell} \mathcal{D}_0[g](\xi,\eta,\phi) f h_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi = \int_{\Gamma_\ell} n_{st}(\Lambda_1,\Lambda_2,\Lambda_3)\cdot g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\ell.$$

Therefore

$$\frac{1}{|\Omega_\ell|}\int_{\Omega_\ell} \mathcal{D}_k[g](\xi,\eta,\phi) f h_1^2(\xi,\eta)\, d\xi\wedge d\eta\wedge d\phi = 0, \quad \ell\in\mathbb{N}\cup\{0\}.$$

Taking the limit as $\ell\to\infty$ we obtain $\mathcal{D}_k[g](\xi_0,\eta_0,\phi_0) = 0$, and since (ξ_0,η_0,ϕ_0) is an arbitrary point in Ω the result follows. $\square$

Let us end this section with one last result.

THEOREM 4.5 (Right inverse for the quaternionic Cauchy-Riemann operator). Let $k\in\mathbb{C}$, $\Omega\subset\Omega_{\xi,\eta,\phi}$ and $g\in C^1\left(\bar{\Omega}, \mathbb{H}(\mathbb{C})\right)\bigcap C\left(\bar{\Omega},\mathbb{H}(\mathbb{C})\right)$. Then the following representation for g holds:

$$g(\xi,\eta,\phi) = \mathcal{D}_k\cdot T_k[g](\xi,\eta,\phi), \quad \forall\,(\xi,\eta,\phi)\in\Omega. \tag{4.9}$$

Proof. Let $(\xi,\eta,\phi)\in\Omega_{\xi,\eta,\phi}$. The fundamental solution of the Helmholtz operator in prolate spheroidal coordinates is

$$\theta_k(\xi,\eta,\phi) := \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{ik|\zeta|}, & \operatorname{Im} k > 0, \\[2ex] -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{-ik|\zeta|}, & \operatorname{Im} k < 0, \end{cases} \tag{4.10}$$

where $\zeta = (f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi, f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi, f\xi\eta)$. It follows from Theorem 4.1 that if $\Gamma = \partial\Omega$ is a piecewise smooth surface and $\Lambda = (\Lambda_1,\Lambda_2,\Lambda_3)$, then

$$\begin{aligned}
&\int_\Gamma \theta_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\Lambda \\
&= \int_\Omega \Big\{\mathcal{K}_k(\xi-\mu,\eta-\nu,\phi-\varrho) g(\mu,\nu,\varrho) \\
&\qquad - \theta_k(\xi-\mu,\eta-\nu,\phi-\varrho)\overline{\mathcal{D}}_k\left[g(\mu,\nu,\varrho)\right]\Big\} f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho.
\end{aligned}$$

It can be seen that

$$T_k[g](\xi,\eta,\phi) = \int_\Omega \theta_k(\xi-\mu,\eta-\nu,\phi-\varrho)\overline{\mathcal{D}}_k[g](\mu,\nu,\varrho) f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho$$
$$+ \int_\Gamma \theta_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\Lambda.$$

If we apply $\mathcal{D}_k$ and the Borel-Pompeiu formula we get:

$$\begin{aligned}
&\mathcal{D}_k\cdot T_k[g](\xi,\eta,\phi)\\
&= -\int_\Omega \mathcal{D}_k\left[\theta_k(\xi-\mu,\eta-\nu,\phi-\varrho)\right]\mathcal{D}_{-k}[g](\mu,\nu,\varrho) f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho\\
&\quad + \int_\Gamma \mathcal{D}_k\left[\theta_k(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3)\right] n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\Lambda\\
&= \int_\Omega \mathcal{K}_{-k}(\xi-\mu,\eta-\nu,\phi-\varrho)\mathcal{D}_{-k}[g](\mu,\nu,\varrho) f h_1^2(\mu,\nu)\, d\mu\wedge d\nu\wedge d\varrho\\
&\quad - \int_\Gamma \mathcal{K}_{-k}(\xi-\Lambda_1,\eta-\Lambda_2,\phi-\Lambda_3) n_{st}(\Lambda_1,\Lambda_2,\Lambda_3) g(\Lambda_1,\Lambda_2,\Lambda_3)\, d\Gamma_\Lambda\\
&= \left(T_{-k}\cdot\mathcal{D}_{-k} + K_{-k}\right)[g](\xi,\eta,\phi)\\
&= g(\xi,\eta,\phi). \qquad \square
\end{aligned}$$

5. Relation with the α-hyperholomorphic function theory. This section describes the direct relation between the obtained results and their analogues constructed in the framework of α-hyperholomorphic function theory.

For the Helmholtz operator in three-dimensional Cartesian coordinates, the following factorization holds:

$$\begin{aligned}
\Delta_3 + k^2 \quad &= \quad \left(k + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(k - \mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} - \mathbf{k}\frac{\partial}{\partial z}\right)\\
&=: \quad \mathbf{D}_k \circ \overline{\mathbf{D}}_k.
\end{aligned}$$

Therefore, the Helmholtz operator in prolate spheroidal coordinates can be written as

$$\begin{aligned}
&\frac{1}{h_1^2(\xi,\eta)}\left[\mathcal{W}_{\xi,\eta,\phi} + c^2(\xi^2-\eta^2)\right]\\
&\quad = W_\varphi\left(\Delta_3 + k^2\right) W_\psi\\
&\quad = W_\varphi\left(\mathbf{D}_k\right) W_\psi W_\varphi\left(\overline{\mathbf{D}}_k\right) W_\psi\\
&\quad = W_\varphi\left(k + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) W_\psi W_\varphi\left(k - \mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} - \mathbf{k}\frac{\partial}{\partial z}\right) W_\psi.
\end{aligned}$$

A direct computation shows that

$$W_\varphi\frac{\partial}{\partial x}W_\psi = \frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\cos\phi - \frac{\sin\phi}{h_2(\xi,\eta)}\frac{\partial}{\partial\phi};$$

similarly,

$$W_\varphi\frac{\partial}{\partial y}W_\psi = \frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\sin\phi + \frac{\cos\phi}{h_2(\xi,\eta)}\frac{\partial}{\partial\phi},$$

and

$$W_\varphi \frac{\partial}{\partial z} W_\psi = \frac{f}{h_1^2(\xi,\eta)} \left[(\xi^2-1)\eta \frac{\partial}{\partial \xi} + (1-\eta^2)\xi \frac{\partial}{\partial \eta} \right].$$

With these computations at hand we obtain

$$\begin{aligned}
&W_\varphi \circ \mathbf{D}_k \circ W_\psi \qquad (5.1)\\
&= W_\varphi \left(k + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) W_\psi \\
&= k + \left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi \mathbf{i} + \sin\phi \mathbf{j})\, \xi + \frac{f}{h_1^2(\xi,\eta)} (\xi^2-1)\eta \mathbf{k} \right] \frac{\partial}{\partial \xi} \\
&\quad + \left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi \mathbf{i} + \sin\phi \mathbf{j})\, \eta + \frac{f}{h_1^2(\xi,\eta)} (1-\eta^2)\xi \mathbf{k} \right] \frac{\partial}{\partial \eta} \\
&\quad + \left[\frac{1}{h_2(\xi,\eta)} (-\sin\phi \mathbf{i} + \cos\phi \mathbf{j}) \right] \frac{\partial}{\partial \phi} \\
&= \mathcal{D}_k. \qquad (5.2)
\end{aligned}$$

This means that the image of the operator $\mathbf{D}_k$, after the prolate spheroidal change of variables, is the operator $\mathcal{D}_k$ defined in (3.1). Hence, $\ker \mathbf{D}_k$ is isomorphic to $\ker \mathcal{D}_k$; that is, if $f \in \ker \mathcal{D}_k$ after applying the operator W_φ, we obtain a function $\tilde{f} \in \ker \mathbf{D}_k$ and vice versa. Therefore all the results obtained for the α-hyperholomorphic function theory (see [7]) can be achieved for the function theory generated by the $\mathcal{D}_k$ operator. This also means that an alternative way of obtaining the preceding results is to apply the prolate spheroidal change of variables to the results produced in the α-hyperholomorphic function theory, as we will see next.

Let $t = (x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. The fundamental solution of the Helmholtz operator is given by (see [7])

$$\Theta_k(t) = \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|t|} e^{ik|t|}, & \operatorname{Im} k > 0, \\[2ex] -\dfrac{1}{4\pi}\dfrac{1}{|t|} e^{-ik|t|}, & \operatorname{Im} k < 0. \end{cases} \qquad (5.3)$$

Formula (5.3) leads to the fundamental solution of the operator $\mathbf{D}_k$, which plays an important role as an analogue of the classical Cauchy kernel. It is defined as follows:

$$\begin{aligned}
\mathbf{K}_k(t) :&= \left(k - \mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} - \mathbf{k}\frac{\partial}{\partial z} \right) [\Theta_k](t) \\
&= \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|t|} e^{ik|t|} \left(k + \dfrac{t_{st}}{|t|^2} - ik\dfrac{t_{st}}{|t|} \right), & \operatorname{Im} k > 0, \\[2ex] -\dfrac{1}{4\pi}\dfrac{1}{|t|} e^{-ik|t|} \left(k + \dfrac{t_{st}}{|t|^2} + ik\dfrac{t_{st}}{|t|} \right), & \operatorname{Im} k < 0, \end{cases}
\end{aligned}$$

where $t_{st} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Having in mind (4.1), after the prolate spheroidal change of variables it follows that

$$\begin{aligned} W_{\varphi}(\mathbf{K}_k) &= \mathbf{K}_k \circ \varphi \\ &= \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{ik|\zeta|}\left(k+\dfrac{\zeta_{st}}{|\zeta|^2}-ik\dfrac{\zeta_{st}}{|\zeta|}\right), & \operatorname{Im} k>0, \\ -\dfrac{1}{4\pi}\dfrac{1}{|\zeta|}e^{-ik|\zeta|}\left(k+\dfrac{\zeta_{st}}{|\zeta|^2}+ik\dfrac{\zeta_{st}}{|\zeta|}\right), & \operatorname{Im} k<0 \end{cases} \\ &= \mathcal{K}_k, \end{aligned}$$

where

$$\zeta = \left(f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi, f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi, f\xi\eta\right)$$

is such that $|\zeta| = f\sqrt{\xi^2+\eta^2-1}$ and

$$\zeta_{st} = f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\phi\mathbf{i} + f\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\phi\mathbf{j} + f\xi\eta\mathbf{k}.$$

For the variable $t=(x,y,z)$ we define the differential form σ of the surface area as follows:

$$\sigma := -\mathbf{i}dy\wedge dz + \mathbf{j}dx\wedge dz - \mathbf{k}dx\wedge dy.$$

Direct computations show that in prolate spheroidal coordinates, σ has the form of (4.2). This means we have the tools necessary to construct all the theory developed above.

6. Relation between $\mathcal{D}_k$-hyperholomorphic functions and solutions of the prolate spheroidal and Chebyshev equations. In this section we show how to directly relate the solutions of the prolate spheroidal equations and Chebyshev equation with the $\mathcal{D}_k$-hyperholomorphic function theory. As we will see, our approach allows the generation of $\mathcal{D}_k$-hyperholomorphic and $\mathcal{D}_k$-anti-hyperholomorphic functions from the solutions of the prolate spheroidal wave equations and Chebyshev equation.

Let $\tilde{u}(\xi,\eta,\phi)$ be a null-solution to the operator $\frac{1}{h_1^2(\xi,\eta)}\mathcal{W}$. We set

$$g_1 := \frac{1}{2k}\overline{\mathcal{D}}_k\tilde{u} \tag{6.1}$$

and

$$g_2 := \frac{1}{2k}\mathcal{D}_k\tilde{u}. \tag{6.2}$$

Then

$$\mathcal{D}_k g_1 = \mathcal{D}_k\left(\frac{1}{2k}\overline{\mathcal{D}}_k\tilde{u}\right) = \frac{1}{2k}\mathcal{D}_k\cdot\overline{\mathcal{D}}_k\tilde{u} = \frac{1}{2k}\frac{1}{h_1^2(\xi,\eta)}(\mathcal{W})\tilde{u} = 0$$

and

$$\overline{\mathcal{D}}_k g_2 = \overline{\mathcal{D}}_k\left(\frac{1}{2k}\mathcal{D}_k\tilde{u}\right) = \frac{1}{2k}\overline{\mathcal{D}}_k\cdot\mathcal{D}_k\tilde{u} = \frac{1}{2k}\frac{1}{h_1^2(\xi,\eta)}(\mathcal{W})\tilde{u} = 0,$$

so that $g_1+g_2=\tilde{u}$. Hence the function $\tilde{u}$ is decomposed into $\tilde{u}=g_1+g_2$, where g_1 is a $\mathcal{D}_k$-hyperholomorphic function and g_2 is a $\mathcal{D}_k$-anti-hyperholomorphic function. This is related to the fact that if $\tilde{u}$ is a metaharmonic function (i.e., $\Delta_3\tilde{u}+k^2\tilde{u}=0$), then there exist (uniquely) two functions $\tilde{u}_1$ and $\tilde{u}_2$ from the conjugate classes of $\mathcal{D}_k$-hyperholomorphy such that $\tilde{u}=\tilde{u}_1+\tilde{u}_2$ (see [7]). Let us compare this with the harmonic case; it is known that $\ker \frac{\partial}{\partial z} \bigcap \ker \frac{\partial}{\partial \bar{z}} = \mathbb{C}$. This implies that the general representation for each harmonic function w is $w = F + G$ with $F = f + c$, $G = g - c$ where $f \in \ker \frac{\partial}{\partial z}$, $g \in \ker \frac{\partial}{\partial \bar{z}}$ and c is an arbitrary complex constant.

Now, let $R(\xi)$ be a solution of the radial equation (1.3), $S(\eta)$ be a solution of the angular equation (1.4), and $\Phi(\phi)$ be a solution of the Chebyshev equation (1.5). Consider

$$\tilde{u}(\xi, \eta, \phi) := R(\xi)S(\eta)\Phi(\phi).$$

The operator $\mathcal{D}_0 = D_{MT}$ can be seen as the three-dimensional gradient in prolate spheroidal coordinates since

$$\begin{aligned}
\mathbf{grad}_{\xi,\eta,\phi} :&= W_\varphi \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) W_\psi \\
&= \left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\xi + \frac{f}{h_1^2(\xi,\eta)}(\xi^2-1)\eta\mathbf{k} \right] \frac{\partial}{\partial \xi} \\
&\quad + \left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\eta + \frac{f}{h_1^2(\xi,\eta)}(1-\eta^2)\xi\mathbf{k} \right] \frac{\partial}{\partial \eta} \\
&\quad + \left[\frac{1}{h_2(\xi,\eta)} (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) \right] \frac{\partial}{\partial \phi}.
\end{aligned}$$

Then for g_1 in (6.1) and g_2 in (6.2) there holds:

$$\begin{aligned}
g_1 &= \frac{1}{2k}(k - \mathbf{grad}_{\xi,\eta,\phi})[R(\xi)S(\eta)\Phi(\phi)], \\
g_2 &= \frac{1}{2k}(k + \mathbf{grad}_{\xi,\eta,\phi})[R(\xi)S(\eta)\Phi(\phi)].
\end{aligned}$$

Hence,

$$\begin{aligned}
g_1(\xi,\eta,\phi) &= \frac{1}{2}R(\xi)S(\eta)\Phi(\phi) \\
&\quad - \frac{1}{2k}\left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\xi + \frac{f}{h_1^2(\xi,\eta)}(\xi^2-1)\eta\mathbf{k} \right] \frac{dR}{d\xi}\, S(\eta)\, \Phi(\phi) \\
&\quad - \frac{1}{2k}\left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\eta + \frac{f}{h_1^2(\xi,\eta)}(1-\eta^2)\xi\mathbf{k} \right] R(\xi)\, \frac{dS}{d\eta}\, \Phi(\phi) \\
&\quad - \frac{1}{2k}\left[\frac{1}{h_2(\xi,\eta)} (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) \right] R(\xi)\, S(\eta)\, \frac{d\Phi}{d\phi},
\end{aligned}$$

and

$$\begin{aligned}
g_2(\xi,\eta,\phi) &= \frac{1}{2}R(\xi)S(\eta)\Phi(\phi) \\
&\quad + \frac{1}{2k}\left[\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\xi + \frac{f}{h_1^2(\xi,\eta)}(\xi^2-1)\eta\mathbf{k} \right] \frac{dR}{d\xi}\, S(\eta)\, \Phi(\phi) \\
&\quad + \frac{1}{2k}\left[-\frac{h_2(\xi,\eta)}{h_1^2(\xi,\eta)} (\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\,\eta + \frac{f}{h_1^2(\xi,\eta)}(1-\eta^2)\xi\mathbf{k} \right] R(\xi)\, \frac{dS}{d\eta}\, \Phi(\phi) \\
&\quad + \frac{1}{2k}\left[\frac{1}{h_2(\xi,\eta)} (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) \right] R(\xi)\, S(\eta)\, \frac{d\Phi}{d\phi}.
\end{aligned}$$

The last two equations show us that each triple of solutions of the prolate spheroidal wave and Chebyshev equations generate a $\mathcal{D}_k$-hyperholomorphic function and a $\mathcal{D}_k$-anti-hyperholomorphic function. Hence prolate spheroidal wave functions generate $\mathcal{D}_k$-hyperholomorphic and $\mathcal{D}_k$-anti-hyperholomorphic functions. All this departing from the fact that every metaharmonic function can be decomposed into the direct sum of two functions from the conjugate classes of $\mathcal{D}_k$-hyperholomorphy.

7. Concluding remarks and perspectives. We have announced that the operators defining the left-hand sides of (1.3)-(1.5) are tightly related to the operator $\mathcal{W}$. This operator can be decomposed into the product of two first order linear partial differential operators (p.d.o.), $\mathcal{D}_k$ and $\overline{\mathcal{D}}_k$, with variable quaternionic coefficients, and for each of the factorizing p.d.o. the corresponding analysis can be developed in a very similar manner to that of complex analysis in one variable or classic quaternionic analysis for the Fueter and the Moisil-Theodorescu operators or quaternionic analysis for the Helmholtz operator; see [7].

The primary purpose of this study was to give a wide and detailed description of the above ideas together with a number of related questions. In Subsection 1.2, it is shown that the prolate spheroidal change of variables generates two mutually inverse operators that realize a relation of similarity between the initial Helmholtz operator (in Cartesian coordinates) and its image $\frac{1}{h_1^2(\xi,\eta)}\mathcal{W}$ under the change of variables. Section 2 deals with the operator $\mathcal{W}$, and it establishes that this operator can be seen as the sum of two modified Sturm-Liouville operators and Chebyshev operator (MSLCOs) (we have assigned these names since they are generated directly by the operators in (1.3)-(1.5)). We find also some relations between the null-solutions of the operator $\mathcal{W}$, on one hand, and of the MSLCOs, on the other. Section 3 introduces the basic definitions of the quaternionic analysis for the operator $\mathcal{W}$; the corresponding functions are called $\mathcal{D}_k$-hyperholomorphic, and one finds here an explanation of the reasons for introducing such an analysis. Section 4 is a continuation of the previous one and it establishes the main integral formulae of the introduced quaternionic analysis: Stokes' formula compatible with $\mathcal{D}_k$-hyperholomorphy, Cauchy's integral theorem and the formulae based on an analogue of the Cauchy kernel. The obtained statements strongly resemble their analogues constructed in the framework of α-hyperholomorphic function theory which serves for the Helmholtz equation. Thus, in Section 5 we explain that, indeed, there exists a direct relation between both. Finally, in Section 6 we show how to directly relate the solutions of the MSLCOs with the $\mathcal{D}_k$-hyperholomorphic function theory. To be more precise, we can produce $\mathcal{D}_k$-hyperholomorphic and $\mathcal{D}_k$-anti-hyperholomorphic functions from the solutions of the MSLCOs.

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