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THE SPACE CHARGE PROBLEM AND THE NONLINEAR RESISTOR

By

GIOVANNI CIMATTI

Department of Mathematics, Largo Bruno Pontecorvo, 5, 56100 Pisa, Italy

Abstract. We propose the constitutive equation $\mathbf{J} = f(q)\mathbf{E}$, where $\mathbf{J}$ is the current density, q the charge density, $\mathbf{E}$ the electric field and $f(q)$ a given function to describe the current-voltage laws appearing in certain special materials. A theorem of existence and uniqueness of solution is also given for the related nonlinear boundary value problem.

1. Introduction.

In many cases the simple linear relation

$$V = RI \tag{1.1}$$

between electric current and potential is not verified and more complex current-voltage laws are present. This happens e.g. in the Ovshinsky material [11], in thin films of insulators [13], in semiconductors [6] and in memristors [14]. These space-charge effects are globally described by nonlinear relations of the form

$$V = \mathcal{F}(I), \tag{1.2}$$

which takes the place of (1.1). Multiple conductance states are often present: two or more different currents are possible in correspondence with the same difference of potential and negative resistance, i.e. regions where $\mathcal{F}'(I) < 0$ can be observed [12]. In particular (see [13]), the current in a thin chromium film was found to obey the relationship

$$I = AV + BV^n, \tag{1.3}$$

where A and B are constants at constant temperature and n is a number equal to or greater than 2. Many explanations [8] have been proposed based on the first principle of quantum physics. However, it is difficult in this way to predict the observed current-voltage laws. In this paper we adopt a purely phenomenological approach. We take as starting point a generalization of the classical space-charge constitutive equation

$$\mathbf{J} = q\mathbf{E} \tag{1.4}$$

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E-mail address: cimatti@dm.unipi.it

assuming

$$\mathbf{J} = f(q)\mathbf{E}, \tag{1.5}$$

where $\mathbf{J}$ is the current density, q the charge density, $\mathbf{E}$ the electric field and $f(q)$ a given function. This model gives rise to a wide variety of voltage-current characteristics, and it is also capable of predicting multiple solutions. The space-charge problem based on the classical assumption (1.4) has received great attention, starting with the paper [7]. We quote among others [3], [2], [1], [4], [9], [10] and [5].

If diffusion effects are taken into consideration, instead of (1.5) we have the equation

$$\mathbf{J} = -\kappa\nabla q + f(q)\mathbf{E}, \tag{1.6}$$

where κ is the diffusion coefficient. If (1.5) holds we have from the equation of conservation of charge $\nabla \cdot \mathbf{J} = 0$,

$$\nabla \cdot (f(q)\nabla\varphi) = 0 \tag{1.7}$$

where φ is the electric potential and $\mathbf{E} = -\nabla\varphi$. If we assume (1.6) we have, instead of (1.7),

$$-\kappa\Delta q - \nabla \cdot (f(q)\nabla\varphi) = 0.$$

Let Ω be a bounded and open subset of $\mathbf{R}^3$ with a regular boundary Γ. Assuming (1.6), we have, for the determination of φ and q, the boundary value problem

$$-\Delta\varphi = q \quad \text{in } \Omega, \tag{1.8}$$

$$\varphi = \varphi_b \quad \text{on } \Gamma, \tag{1.9}$$

$$-\kappa\Delta q - \nabla \cdot (f(q)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{1.10}$$

$$q = q_b \quad \text{on } \Gamma, \tag{1.11}$$

where q_b and φ_b are functions given on Γ. If we assume as starting point the constitutive relation (1.5) we have the following boundary value problem which generalizes the classical space-charge problem [3], [4]:

$$-\Delta\varphi = q \quad \text{in } \Omega, \tag{1.12}$$

$$\varphi = \varphi_b \quad \text{on } \Gamma, \tag{1.13}$$

$$\nabla \cdot (f(q)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{1.14}$$

$$q = q_b \text{ on } \Gamma^+ = \left\{\mathbf{x} \in \Gamma,\ \frac{d\varphi}{dn}(\mathbf{x}) > 0\right\},\ \mathbf{x} = (x_1, x_2, x_3). \tag{1.15}$$

This problem when $f(q) = q$ is studied in [5].

In Section 2 we present a theorem of existence and uniqueness of small solutions for problem (1.8)-(1.11) and a theorem of existence under general assumptions on the data for the same problem. In Section 3 we consider the one-dimensional version of problem (1.12)-(1.15) which corresponds to the case of an indefinite slab. If

$$f(q) = q^\alpha, \quad \alpha \geq 1,$$

we obtain the current-voltage relationship (1.3) and one and only one solution for problem (1.12)-(1.15). On the other hand, if

$$f(q) = \frac{1}{1+q^\alpha}, \quad \alpha \geq 1,$$

we have either none or two solutions. Finally, in Section 4 we study a special case of the problem without diffusion adopting a different boundary condition.

2. Existence and uniqueness of solutions for the problem with diffusion. If q_b is small in a suitable norm problem, (1.8)-(1.11) has one and only one solution. In fact, we have

THEOREM 2.1. Let $\varphi_b \in C^{4,\alpha}(\Gamma)$, $q_b \in C^{2,\alpha}(\Gamma)$ and

$$f(q) \in C^2(\mathbf{R}^1), \; f(0) \geq 0. \tag{2.1}$$

Then there exists $N > 0$ such that if

$$\|q_b\|_{C^{2,\alpha}(\Gamma)} \leq N, \tag{2.2}$$

the problem

$$-\Delta\varphi = q \quad \text{in } \Omega, \tag{2.3}$$

$$\varphi = \varphi_b \quad \text{on } \Gamma, \tag{2.4}$$

$$-\kappa\Delta q - \nabla\cdot(f(q)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{2.5}$$

$$q = q_b \quad \text{on } \Gamma \tag{2.6}$$

has one and only one solution.

Proof. We apply the inverse function theorem in Banach spaces. Let φ_0 be the solution of the problem

$$\Delta\varphi_0 = 0 \quad \text{in } \Omega, \quad \varphi_0 = \varphi_b \quad \text{on } \Gamma.$$

Then $(\varphi, q) = (\varphi_0, 0)$ is the solution of the problem (2.1)-(2.6) corresponding to the boundary data φ_b, and $q_b = 0$. Define

$$\mathcal{X} = \{\varphi(\mathbf{x}) \in C^{4,\alpha}(\bar{\Omega}), \; \varphi = \varphi_b \; \text{ on } \Gamma\} \times C^{2,\alpha}(\bar{\Omega}),$$

$$\mathcal{Y} = (C^{2,\alpha}(\bar{\Omega}) \times C^{0,\alpha}(\bar{\Omega})) \times C^{2,\alpha}(\Gamma)$$

and the operator $F : \mathcal{X} \to \mathcal{Y}$:

$$F(\varphi, q) = ((-\Delta\varphi - q, \quad -\kappa\Delta q - f(q)\nabla q\cdot\nabla\varphi + qf(q)), \quad q|_\Gamma).$$

We have

$$F(\varphi_0, 0) = ((0,0), 0).$$

Moreover, in view of (2.1) $F(\varphi, q) \in C^1(\mathcal{X}, \mathcal{Y})$. Computing the first differential of the operator F we find

$$F'(\varphi_0, 0)[\Phi, Q] = ((-\Delta\Phi - Q, \; \kappa\Delta Q - f'(0)\nabla Q\cdot\nabla\varphi_0 + f(0)Q), \quad Q|_\Gamma).$$

Let $\alpha(\mathbf{x}) \in C^{2,\alpha}(\bar{\Omega})$, $\beta(\mathbf{x}) \in C^{0,\alpha}(\bar{\Omega})$, $\gamma(\mathbf{x}) \in C^{2,\alpha}(\Gamma)$. The system

$$-\Delta\Phi - Q = \alpha \quad \text{in } \Omega, \; \Phi = 0 \quad \text{on } \Gamma,$$

$$-\Delta Q - f'(0)\nabla Q\cdot\nabla\varphi_0 + f(0)Q = \beta \quad \text{in } \Omega,$$

$$Q = \gamma \quad \text{on } \Gamma$$

is easily uncoupled, and it has one and only one solution by (2.1). We conclude that there exists N such that if (2.2) holds, the equation $F(\varphi, q) = ((0,0), q_b)$ has one and only one solution which solves (2.3)-(2.6). $\square$

A similar result of existence and uniqueness of small solutions does not hold if $\kappa = 0$ under the sole assumption (2.1). See, in this respect, the second example of Section 3.

The next theorem is based on the a priori estimates for the solutions of elliptic equations, which we quote below for the problem at hand (see [15], page 203). Let $u \in H^{1,2}(\Omega)$ be a solution of the problem

$$-\Delta u + \nabla \mathbf{b} \cdot \nabla u + au = h \text{ in } \Omega, \quad u = u_b \text{ on } \Gamma, \quad \mathbf{b} = (b_1, b_2, b_3).$$

If $a \geq 0$, $u_b \in H^{2,q}(\Omega)$, $q > 3$ and

$$\|\mathbf{b}\|_{L^q(\Omega)} \leq \mu,\ \|a\|_{L^{q/2}(\Omega)} \leq \mu,\ \|u_b\|_{H^{2,q}(\Omega)} \leq \mu,\ \|h\|_{L^q(\Omega)} \leq \mu,$$

then

$$\|u\|_{H^{2,q}(\Omega)} \leq C, \tag{2.7}$$

where the constant C depends only on μ and Ω. Moreover, adopting the Schauder point of view, if

$$\|\mathbf{b}\|_{C^{0,\alpha}(\bar\Omega)} \leq \mu,\ \|a\|_{C^{0,\alpha}(\bar\Omega)} \leq \mu,\ \|h\|_{C^{0,\alpha}(\bar\Omega)} \leq \mu,\ \|u_b\|_{C^{2,\alpha}(\bar\Gamma)} \leq \mu$$

we have

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C, \tag{2.8}$$

where the constant C depends only on μ and Ω.

THEOREM 2.2. Let

$$f(q) \in C^2(\mathbf{R}^1),\ f(q) \geq 0, \tag{2.9}$$

$$q_b \in C^{2,\alpha}(\bar\Omega),\ q_b \geq 0,\ \varphi_b \in H^{2,p}(\Omega),\ p > 3. \tag{2.10}$$

Then there exists at least one solution of the problem

$$-\Delta\varphi = q \text{ in } \Omega, \tag{2.11}$$

$$\varphi = \varphi_b \text{ on } \Gamma, \tag{2.12}$$

$$-\kappa\Delta q + \nabla \cdot (f(q)\nabla\varphi) = 0 \text{ in } \Omega, \tag{2.13}$$

$$q = q_b \text{ on } \Gamma. \tag{2.14}$$

Proof. Define $q_M = \max_{\bar\Omega} q_b$ and

$$\mathcal{A} = \{q(\mathbf{x}) \in C^{0,\alpha}(\bar\Omega),\ q_M \geq q(\mathbf{x}) \geq 0\}.$$

Let $q(\mathbf{x}) \in \mathcal{A}$ and $(v, \bar q)$ be the solution of the problem

$$-\Delta v = q \text{ in } \Omega,\ v = \varphi_b \text{ on } \Gamma, \tag{2.15}$$

$$-\Delta\bar q - f'(q)\nabla\bar q \cdot \nabla v + f(q)\bar q = 0 \text{ in } \Omega,\ \bar q = q_b \text{ on } \Gamma. \tag{2.16}$$

From (2.15) and (2.7) we have, using the Sobolev embedding theorem,

$$\|v\|_{C^{1,\alpha}(\Omega)} \leq C\big(q_M, \|\varphi_b\|_{H^{2,q}(\Omega)}, \Omega\big), \quad q > 3. \tag{2.17}$$

To estimate the solution $\bar q(\mathbf{x})$ of the problem (2.16) we note that $f'(q)\nabla v$ and $f(q)$ are both bounded in the norm $L^q(\Omega)$ with $q > 3$ by a constant which depends only on q_M, $\|\varphi_b\|_{H^{2,q}(\Omega)}$ and Ω by (2.9) since $q(\mathbf{x})$ belongs to $\mathcal{A}$. Thus, by (2.7) and by the Sobolev embedding theorem we have, in addition to (2.17),

$$\|\bar q\|_{C^{1,\alpha}(\Omega)} \leq C(q_M, \|\varphi_b\|_{H^{2,q}(\Omega)}, \Omega), \quad q > 3. \tag{2.18}$$

Moreover, the maximum principle applies to the problem (2.16). Therefore

$$0 \le \bar{q}(\mathbf{x}) \le q_M.$$

If $\bar{q} = T(q)$ is the operator defined by (2.16) we have $T(\mathcal{A}) \subset \mathcal{A}$; moreover $T(\mathcal{A})$ is a compact subset of $C^{0,\alpha}(\bar{\Omega})$ by (2.18). We claim that T is a continuous operator. Let $q \in \mathcal{A}$ and $\bar{q} = T(q)$, i.e.

$$-\Delta v = q \ \text{ in } \Omega,\ v = \varphi_b \ \text{ on } \Gamma,$$

$$-\Delta\bar{q} - f'(q)\nabla\bar{q}\cdot\nabla v + f(q)\bar{q} = 0 \ \text{ in } \Omega,\ \bar{q} = q_b \ \text{ on } \Gamma. \tag{2.19}$$

Let $q_n \in \mathcal{A}$ and $\bar{q}_n = T(q_n)$, i.e.

$$-\Delta v_n = q_n \ \text{ in } \Omega,\ v_n = \varphi_b \ \text{ on } \Gamma, \tag{2.20}$$

$$-\Delta\bar{q}_n - f'(q_n)\nabla\bar{q}_n\cdot\nabla v_n + f(q_n)\bar{q}_n = 0 \ \text{ in } \Omega,\ \bar{q}_n = q_b \ \text{ on } \Gamma. \tag{2.21}$$

Assume $\|q_n - q\|_{C^{0,\alpha}(\bar{\Omega})} \to 0$; we claim that $\|\bar{q}_n - \bar{q}\|_{C^{0,\alpha}(\bar{\Omega})} \to 0$, for from (2.20) we have

$$\|v_n\|_{C^{2,\alpha}(\bar{\Omega})} \le C.$$

On the other hand, $f'(q_n)\nabla v_n$ and $f'(q_n)$ are both bounded in $C^{0,\alpha}(\bar{\Omega})$ independently of n. Hence, by (2.8), we have

$$\|\bar{q}_n\|_{C^{2,\alpha}(\bar{\Omega})} \le C.$$

Setting $w_n = \bar{q}_n - \bar{q}$ we have, by difference from (2.19) and (2.21),

$$\begin{gathered} -\Delta w_n + f'(q_n)\nabla v\cdot\nabla w_n + f(q_n)w_n \\ = f'(q_n)\nabla\bar{q}_n\cdot\nabla(v_n - v) + (f'(q_n) - f'(q))\nabla\bar{q}\cdot\nabla v - \bar{q}(f(q_n) - f(q)), \\ w_n = 0 \ \text{ on } \Gamma. \end{gathered} \tag{2.22}$$

Since $f'(q_n)\nabla v$ and $f(q_n)$ are both bounded in $C^{0,\alpha}(\Omega)$ by a constant not depending on n, the Schauder estimate (2.7) applies to (2.22). We claim that the right hand side of (2.22) tends to zero in $C^{0,\alpha}(\Omega)$, for $\nabla v_n \to \nabla v$ in $C^{1,\alpha}(\bar{\Omega})$. Moreover, by (2.9) we have $f'(q_n) - f'(q) = f''(\xi)(q_n - q)$ with $|f''(\xi)|$ bounded. Since $q_n \to q$ in $C^{0,\alpha}(\bar{\Omega})$ we conclude that $w_n \to 0$ in $C^{2,\alpha}(\bar{\Omega})$. Hence T is continuous. By the Schauder fixed point theorem T has a fixed point, which gives a solution to problem (2.11)-(2.14). $\square$

REMARK 2.3. It appears natural to look at the limit as the diffusion $\kappa \to 0$ in problem (2.11)-(2.14). This is done in [5] for the case $f(q) = q$. However, the proof of [5] does not apply to present more general situations.

3. Nonexistence and nonuniqueness for the problem without diffusion. We consider in this section the one-dimensional counterpart of the problem (1.12)-(1.15), i.e.

$$-\varphi'' = q \text{ in } \ (0, L),\ L > 0, \tag{3.1}$$

$$\varphi(0) = V,\quad V > 0, \tag{3.2}$$

$$\varphi(L) = 0, \tag{3.3}$$

$$(f(q)\varphi')' = 0 \text{ in } \ (0, L), \tag{3.4}$$

$$q(0) = q_b,\ q_b > 0. \tag{3.5}$$

We assume $f(q) \in C^1([0,\infty))$, $f(q) \geq 0$, and $y = f(q)$ globally invertible with inverse denoted $q = g(y)$. Since $E = -\varphi'$ is the electric field we have for the unknown current J,

$$J = f(q)E.$$

Hence, from (3.1)

$$f(E') = \frac{J}{E}$$

and

$$\frac{dE}{dx} = g\Big(\frac{J}{E}\Big). \tag{3.6}$$

On the other hand, we have for $x = 0$,

$$E(0) = \frac{J}{f(q_b)}. \tag{3.7}$$

Separating variables in (3.6) and taking into account (3.7), we get

$$\mathcal{F}(E, J, q_e) = x \tag{3.8}$$

where

$$\mathcal{F}(E, J, q_e) = \int_{\frac{J}{f(q_b)}}^{E} \frac{dt}{g(\frac{J}{t})}. \tag{3.9}$$

If $\mathcal{F}(E, J, q_b) = x$ is solvable with respect to E and we have $E = E(x, J, q_b)$, the current-voltage law is given by

$$V = \int_0^L E(x, J, q_b)dx. \tag{3.10}$$

The total current J in terms of the data V and q_b can be obtained from (3.10). We may well have more than one current J corresponding to the same V. If $\bar{J}$ is one of these values the corresponding potential is given by

$$\varphi(x) = V - \int_0^x E(t, \bar{J}, q_b)dt.$$

As a first example, let us assume $f(q) = q^\alpha$ with $\alpha \geq 1$. Since $g(y) = y^{1/\alpha}$, in this case (3.8) reads

$$\int_{\frac{J}{q_b^\alpha}}^{E} \frac{dt}{(\frac{J}{t})^{1/\alpha}} = x,$$

i.e.

$$\frac{\alpha}{(1+\alpha)^{J^{1/\alpha}}}\Big(E^{\frac{1+\alpha}{\alpha}} - J^{\frac{1+\alpha}{\alpha}} q_b^{-(1+\alpha)}\Big) = x. \tag{3.11}$$

Solving (3.11) with respect to E we obtain

$$E = \Big[\frac{1+\alpha}{\alpha} x J^{1/\alpha} + J^{\frac{1+\alpha}{\alpha}} q_e^{-(1+\alpha)}\Big]^{\frac{\alpha}{1+\alpha}}.$$

Thus (3.10) now reads

$$\int_0^L \Big[\frac{1+\alpha}{\alpha} x J^{1/\alpha} + J^{\frac{1+\alpha}{\alpha}} q_e^{-(1+\alpha)}\Big]^{\frac{\alpha}{1+\alpha}} dx = V,$$

and, after a simple calculation,

$$\frac{\alpha}{(1+2\alpha)q_b^{1+2\alpha}}\Big[\Big(J^{\frac{2(1+\alpha)}{1+2\alpha}}+\frac{1+\alpha}{\alpha}q_b^{1+\alpha}LJ^{\frac{1}{1+2\alpha}}\Big)^{\frac{1+2\alpha}{1+\alpha}}-\Big(J^{\frac{2(1+\alpha)}{1+2\alpha}}\Big)^{\frac{1+2\alpha}{1+\alpha}}\Big]=V. \qquad (3.12)$$

If we define

$$\xi=J^{\frac{2(1+\alpha)}{1+2\alpha}},\quad H(\xi)=\xi^{\frac{1+2\alpha}{1+\alpha}},\quad G(\xi)=\frac{1+\alpha}{\alpha}Lq_b^{1+\alpha}\xi^{\frac{1}{2(1+\alpha)}},$$

the equation (3.12) can be rewritten as

$$\frac{\alpha}{(1+2\alpha)q_b^{1+2\alpha}}\Big[H(\xi+G(\xi))-H(\xi)\Big]=V.$$

This equation in ξ has for every value of $\alpha\geq 1$, $q_b>0$, $L>0$ and $V>0$ one and only one solution as a consequence of the following elementary lemma.

Lemma 3.1. Let $H(\xi)\in C^2((0,\infty))\cup C^0([0,\infty))$, $H'(\xi)>0$, $H''(\xi)>0$ for $\xi>0$, $G(\xi)\in C^1((0,\infty))\cup C^0([0,\infty))$, $G(\xi)>0$, $G'(\xi)>0$ for $\xi>0$. Then the function $F(\xi)=H(\xi+G(\xi))-H(\xi)$ is strictly increasing.

Proof. Simply note that $F'(\xi)=H'(\xi+G(\xi))(1+G'(\xi))-H'(\xi)\geq H'(\xi)G'(\xi)$. □

Since the right hand side of (3.12) tends to infinity when J tends to infinity, we conclude that problem (3.1)-(3.5) has one and only one solution if $f(q)=q^\alpha$. Taking the principal part of the left hand side of (3.12) with respect to $J^{\frac{1}{1+\alpha}}$ we have

$$\frac{\alpha}{1+\alpha}\Big(\frac{1+\alpha}{\alpha}\Big)L^{\frac{1+2\alpha}{1+\alpha}}J^{\frac{1}{1+\alpha}}+o\big(J^{\frac{1}{1+\alpha}}\big)=V.$$

Thus, defining

$$K=\frac{\alpha}{1+2\alpha}\Big(\frac{1+\alpha}{\alpha}\Big)^{\frac{1+2\alpha}{1+\alpha}}$$

we have, within the limit of the above approximation,

$$L^{\frac{1+2\alpha}{1+\alpha}}KJ^{\frac{1}{1+\alpha}}=V.$$

Hence

$$J=BV^{1+\alpha},\ B=\frac{\Big(\frac{1+2\alpha}{\alpha}\Big)^{1+\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+2\alpha}}{L^{1+2\alpha}},$$

which is compatible with (1.3) and with the result of [13] since $1+\alpha\geq 2$.

We now consider problem (3.1)-(3.5) when

$$f(q)=\frac{1}{1+q^\alpha},\quad \alpha\geq 1. \qquad (3.13)$$

This will give an example of nonuniqueness and of nonexistence. Let $\alpha>1$. We have

$$g(y)=\Big(\frac{1-y}{y}\Big)^{1/\alpha}$$

and, by (3.8) and (3.9),

$$\int_{J(1+q_b^\alpha)}^{E}\Big(\frac{t-J}{J}\Big)^{-1/\alpha}=x,$$

i.e.

$$\frac{J\alpha}{\alpha-1}\Big[\Big(\frac{E-J}{J}\Big)^{\frac{\alpha-1}{\alpha}}-q_b^{\alpha-1}\Big]=x.$$

Solving with respect to E we obtain

$$E = J + \frac{\gamma}{J^{\alpha-1}}(x+\beta)^{\frac{\alpha}{\alpha-1}}$$

where

$$\beta = q_e^{\alpha-1}, \quad \gamma = \left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

From (3.10) we have, in this case,

$$\int_0^L \left[J + \frac{\gamma}{J^{\alpha-1}}(x+\beta)^{\frac{\alpha}{\alpha-1}}\right]dx = V \tag{3.14}$$

or

$$\mathcal{F}(J) = V$$

where

$$\mathcal{F}(J) = JL + MJ^{\frac{1}{1-\alpha}} \quad \text{and} \quad M = \frac{\gamma(\alpha-1)}{2\alpha-1}\left[(L+\beta)^{\frac{2\alpha-1}{\alpha-1}} - \beta^{\frac{2\alpha-1}{\alpha-1}}\right] > 0. \tag{3.15}$$

Since $\frac{1}{1-\alpha} < 0$ by (3.13) we obtain for $\mathcal{F}(J)$ the graph of Figure 1. We conclude that if

$$0 < V < \mathcal{F}(J_c) \text{ where } J_c = \left(\frac{L\alpha - L}{M}\right)^{\frac{1-\alpha}{\alpha}} > 0$$

problem (3.1)-(3.5) (when $f(q) = \frac{1}{1+q^{\alpha}}$, $\alpha > 1$) has no solution. If

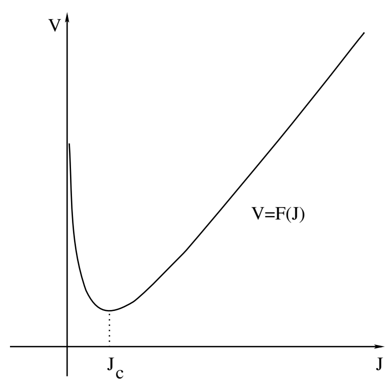

FIG. 1

$$\mathcal{F}(J_c) < V$$

the same problem has exactly two solutions. The case $\alpha = 1$ is better treated separately, for from (3.14) and (3.15) we have the linear equation

$$J\varphi'' - \varphi' - J = 0. \tag{3.16}$$

Taking into account the boundary conditions (3.2), (3.3) and (3.5) we obtain

$$\mathcal{F}_1(J) = V \tag{3.17}$$

where now

$$\mathcal{F}_1(J) = q_b J^2\left(e^{\frac{L}{J}} - 1\right) + JL.$$

It is easily seen that for every $q_b > 0$, $L > 0$ there exists $\bar{V} > 0$ such that if $0 \leq V < \bar{V}$ the equation (3.17), and therefore problem (3.1)-(3.5), has no solution. If $V > \bar{V}$ the equation (3.17), and therefore problem (3.1)-(3.5), has one and only one solution.

4. The problem with other boundary conditions. The system of partial differential equations

$$-\Delta\varphi = q \quad \text{in } \Omega, \tag{4.1}$$

$$-\kappa\Delta q - \nabla\cdot(f(q)\nabla) = 0 \quad \text{in } \Omega \tag{4.2}$$

can, in principle, be supplemented also with boundary conditions of the Dirichlet-Neumann type. Let Γ, the boundary of Ω, consist of two regular surfaces Γ_1 and Γ_2 such that $\Gamma_1 \subset \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_2 \neq \emptyset$. We could add to (4.1) and (4.2) e.g. the conditions

$$\varphi = \varphi_b,\ q = q_b \text{ on } \Gamma_2,\ \frac{d\varphi}{dn} = 0,\ \frac{dq}{dn} = 0 \text{ on } \Gamma_1. \tag{4.3}$$

Using the Schauder estimates for elliptic second-order equations with mixed boundary conditions, a theorem of existence of a weak solution for problem (4.1), (4.2), (4.3) can be proved.

In this last section we prefer to propose a nonstandard formulation for the problem without diffusion. Let the potential φ be given on Γ_1 and Γ_2,

$$\varphi = V \text{ on } \Gamma_1, \quad \varphi = 0 \text{ on } \Gamma_2, \tag{4.4}$$

where $V > 0$ is a constant. We assume $\mathbf{J}\cdot\mathbf{n}$ to be assigned on Γ_1, where $\mathbf{n}$ is the unit vector normal to Γ_1 pointing outward with respect to Ω. Recalling that $\mathbf{J} = -f(-\Delta\varphi)\nabla\varphi$ we obtain

$$\nabla\cdot(f(-\Delta\varphi)\nabla\varphi) = 0 \text{ in } \Omega \tag{4.5}$$

and

$$-f(-\Delta\varphi)\frac{d\varphi}{dn} = K \text{ on } \Gamma_1. \tag{4.6}$$

We show in the next example that, in certain cases, this problem has a unique solution. Let $f(q) = q$, $\Omega = \{(\rho,\theta),\ 1 < \rho < R,\ 0 \leq \theta < 2\pi\}$, $\Gamma_1 = \{(1,\theta),\ 0 \leq \theta < 2\pi\}$, $\Gamma_2 = \{(R,\theta),\ 0 \leq \theta < 2\pi\}$, $R > 1,\ K > 0$. If we search for a solution depending only on ρ, equation (4.5) takes the form

$$\frac{1}{\rho}\frac{d}{d\rho}\Big(\rho\Big(\frac{1}{\rho}\frac{d}{d\rho}\Big(\rho\frac{d\varphi}{d\rho}\Big)\Big)\frac{d\varphi}{d\rho}\Big) = 0. \tag{4.7}$$

The conditions (4.4) become

$$\varphi(1) = V,\ V > 0, \tag{4.8}$$

$$\varphi(R) = 0. \tag{4.9}$$

Moreover (4.6) gives

$$\frac{d}{d\rho}\Big(\rho\frac{d\varphi}{d\rho}\Big)\frac{d\varphi}{d\rho}\Big|_{\rho=1} = K. \tag{4.10}$$

From (4.10) and (4.7) we have

$$\frac{d}{d\rho}\Big(\rho\frac{d\varphi}{d\rho}\Big)\frac{d\varphi}{d\rho} = K \text{ for } 1 < \rho < R. \tag{4.11}$$

The problem (4.11), (4.8) and (4.9) can be solved elementarily. In fact, from (4.11) we have

$$\Big(\frac{d\varphi}{d\rho}\Big)^2+\frac{1}{2}\rho\frac{d}{d\rho}\Big(\frac{d\varphi}{d\rho}\Big)^2=K \tag{4.12}$$

or, if we define $z=\left(\frac{d\varphi}{d\rho}\right)^2$,

$$z+\frac{\rho}{2}\frac{dz}{d\rho}=K. \tag{4.13}$$

This gives

$$z(\rho)=K+\frac{\xi}{\rho^2} \tag{4.14}$$

and

$$\frac{d\varphi}{d\rho}=-\sqrt{\frac{K\rho^2+\xi}{\rho^2}}. \tag{4.15}$$

Integrating and taking into account (4.9) and (4.8) we obtain for the determination of the constant of integration ξ the equation

$$h(\xi;K,R)=V, \tag{4.16}$$

where

$$h(\xi;K,R)=\sqrt{KR^2+\xi}-\sqrt{\xi}\ln\frac{2(\xi+\sqrt{\xi}\sqrt{KR^2+\xi})}{R}-\sqrt{K+\xi}+\sqrt{\xi}\ln 2(\xi+\sqrt{\xi}\sqrt{K+\xi}). \tag{4.17}$$

For every fixed $R>1$ and $K>0$ the function $h(\xi,K,R)$ is strictly increasing in $0<\xi<\infty$. Moreover,

$$\lim_{\xi\to 0^+}h(\xi,K,R)=\sqrt{K}(R-1), \tag{4.18}$$

$$\lim_{\xi\to\infty}h(\xi,K,R)=\infty. \tag{4.19}$$

We conclude that if

$$0<V<\sqrt{K}(R-1)$$

the problem (4.7)-(4.10) has no solution. If

$$\sqrt{K}(R-1)\le V<\infty$$

the problem (4.7)-(4.10) has one and only one solution. The one-dimensional analogue of the problem is easily stated with $\Omega=(0,L)$. We find a similar result of existence and of nonexistence. If

$$0<V<\frac{2^{\frac{3}{2}}L^{\frac{3}{2}}K^{\frac{1}{2}}}{3}$$

the problem has no solution. If

$$\frac{2^{\frac{3}{2}}L^{\frac{3}{2}}K^{\frac{1}{2}}}{3}\le V<\infty$$

the problem has one and only one solution. These examples suggest that the problem may have solutions also in other doubly connected plane domains.

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