QUASINEUTRAL LIMIT OF THE PRESSURELESS EULER-POISSON EQUATION FOR IONS

BY

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Abstract. In this paper, we consider the quasineutral limit of the Euler-Poisson equation for cold ions when the Debye length tends to zero. In the cold ion case the Euler-Poisson equation is pressureless and hence fails to be Friedrich symmetrizable, excluding the application of the PsDO energy estimates method of Grenier to obtain uniform estimates independent of ε . To overcome this difficulty, we use ε -weighted norms which combine energy estimates in different levels with weights depending on ε . Finally, that the quasineutral regimes are the compressible Euler equations is proven for well prepared initial data. As a natural extension, we also obtain the zero temperature limit of the Euler-Poisson equation.

1. Introduction. In this paper, we consider the Euler-Poisson equation for cold ions in plasma:

$$(EP) \quad \begin{cases} \partial_t n + \operatorname{div}(n\mathbf{u}) = 0, & (1.1a) \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \phi, & (1.1b) \\ \varepsilon \Delta \phi = e^\phi - n, & (1.1c) \end{cases}$$

where n is the density of the ions, $\mathbf{u} = (u_1, \dots, u_d)$ is the velocity field and ϕ is the electric potential at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^d$, $d \leq 3$. Here e^ϕ is the rescaled electron density by the famous Boltzmann relation and $\varepsilon \ll 1$ is a small parameter representing the squared scaled Debye length $\varepsilon = \lambda_D^2/L^2 = \epsilon_0 k_B T_i / N_i e^2 L^2$, where λ_D is the Debye length, L is the characteristic observation length, ϵ_0 is the vacuum permittivity, κ_B is the

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universal Boltzmann constant, T_i and N_i are respectively the average temperature and density of ions and e is the fundamental electric charge. For typical plasma applications, the Debye length is very small compared to the characteristic length of physical interest, and it is therefore necessary to consider the limiting system when $\varepsilon \rightarrow 0$. For more physical background on the Euler-Poisson equation or the ion-acoustic plasma, one may refer to [14].

Formally, by letting $\varepsilon \rightarrow 0$, we obtain from the third equation in (1.1) that $\phi = \ln n$, and hence the following compressible Euler system:

$$(EQ) \quad \begin{cases} \partial_t n + \operatorname{div}(n\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \ln n = 0. \end{cases} \quad (1.2)$$

This limit system (1.2) is a hyperbolic symmetrizable system whose classical result for the existence and uniqueness of sufficiently smooth solutions in a small time interval is available in [17]. The system (1.2) has to be supplemented by suitable initial conditions. We shall assume that the plasma is uniform and electrically neutral near infinity, i.e., $n \rightarrow n^\pm$ and $\mathbf{u} \rightarrow 0$ as $x \rightarrow \pm\infty$. More precisely, let $\tilde{n}$ be a smooth strictly positive function, constant outside $x \in [-1, +1]$, going to $n^\pm$ as $x \rightarrow \infty$. We assume that the initial conditions $(n_0^0, \mathbf{u}_0^0)$ satisfy

$$(n_0^0 - \tilde{n}) \in H^s(\mathbb{R}^d), \quad \mathbf{u}_0^0 \in \mathbf{H}^s(\mathbb{R}^d), \quad n_0^0 \geq \sigma > 0, \quad (1.3)$$

for some $s > 3/2$ and some constant $\sigma > 0$.

THEOREM 1.1. Let $(n_0^0, \mathbf{u}_0^0) \in H^{s'} \times H^{s'}$ be initial data with $s' > \frac{d}{2} + 1$ and satisfying (1.3). Then there exists $T > 0$, maximal time of existence and a solution $(n^0, \mathbf{u}^0)$ of (1.2) on $0 \leq t < T$ with initial data $(n_0^0, \mathbf{u}_0^0)$ such that for every $T' < T$,

$$(n^0 - \tilde{n}, \mathbf{u}^0) \in (C([0, T]; H^{s'}) \times C([0, T]; \mathbf{H}^{s'})) \cap (C^1([0, T]; H^{s'-1}) \times C^1([0, T]; \mathbf{H}^{s'-1}))$$

and T depends only on $\|(n_0^0 - \tilde{n}, \mathbf{u}_0^0)\|_{H^{s'} \times \mathbf{H}^{s'}}$.

REMARK 1.2. For the maximal existence time T , either $T = \infty$ in the case of global existence or $T < \infty$ and the solution blows up when $t \rightarrow T$:

$$\limsup_{t \rightarrow T} (\|\mathbf{u}^0\|_{L^\infty} + \|n^0\|_{L^\infty} + \|\frac{1}{n^0}\|_{L^\infty} + \int_0^t (\|\partial_x \mathbf{u}^0\|_{L^\infty} + \|\partial_x n^0\|_{L^\infty}) d\tau) = \infty. \quad (1.4)$$

Therefore, we will work on a time interval $[0, T']$ for $T' < T$ (but arbitrarily close to T) in order to insure $0 < \sigma' < n^0(t, x) < \sigma''$ for all (t, x) , for some constants $\sigma', \sigma'' > 0$. Here, σ' may approach 0 as T' goes to T .

Let us define $\phi^0 = \ln n^0$. The main result in this paper is the following.

THEOREM 1.3. Let $s' \in \mathbb{N}$ with $s' > [\frac{d}{2}] + 2$ be sufficiently large. Let $(n_0^0, \mathbf{u}_0^0) \in H^{s'} \times \mathbf{H}^{s'}$ and $(n^0, \mathbf{u}^0)$ be the solution of the limit system (1.2) on $[0, T]$ with initial data $(n_0^0, \mathbf{u}_0^0)$, given in Theorem 1.1. Then there exist solutions $(n^\varepsilon(t), \mathbf{u}^\varepsilon(t))$ of (1.1) with the same initial data on $[0, T^\varepsilon)$ with $\liminf_{\varepsilon \rightarrow 0} T^\varepsilon \geq T$. Moreover, for every $T' < T$ and for every ε small enough, $\varepsilon^{-1}(n^\varepsilon - n^0)$ and $\varepsilon^{-1}(\mathbf{u}^\varepsilon - \mathbf{u}^0)$ are bounded in $L^\infty([0, T']; H^s)$ and $L^\infty([0, T']; \mathbf{H}^s)$, respectively, for some $s < s'$.

This theorem will be proved in Section 2. Without essential difficulties, we can show that the same result holds on the torus $\mathbb{T}^d$ following the method in the present paper. The details are omitted.

Before proving this theorem, we make some points that stimulated our work in the present paper. The more general isothermal Euler-Poisson equation for ions has the following form:

$$\begin{cases} \partial_t n + \operatorname{div}(n\mathbf{u}) = 0, & (1.5a) \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{T_i}{n} \nabla n = -\nabla \phi, & (1.5b) \\ \varepsilon \Delta \phi = e^\phi - n, & (1.5c) \end{cases}$$

where $T_i > 0$ is the ion temperature. When $T_i = 0$ (compared with the electron temperature), this equation reduces to (1.1) for the cold ions. When $T_i > 0$, the pressure term introduces a smoothing effect that enables Cordier and Grenier [2] to prove the quasineutral limit as $\varepsilon \rightarrow 0$ by using the pseudodifferential energy estimates method of [6]. It is shown that, under suitable conditions, the solution of (1.5) converges to the following Euler equation as $\varepsilon \rightarrow 0$:

$$\begin{cases} \partial_t n + \operatorname{div}(n\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + (T_i + 1) \nabla \ln n = 0. \end{cases} \quad (1.6)$$

But their method cannot be applied to the case when $T_i = 0$, where such a smoothing effect vanishes. The main difference between (1.1) and (1.5) is that (1.5) has the pressure term $T_i \nabla \ln n$, which is crucial in proving the quasineutral result of the Euler-Poisson equation (1.5). With this term, the hyperbolic part of (1.5) is Friedrich symmetrizable and the general framework of pseudodifferential operator energy estimates methods of Grenier [6] can be applied. One may refer to [2] for more details of application of this method in treating the quasineutral limit of (1.5). But without the pressure term, as is the case in the present paper, the pseudodifferential energy method cannot apply and no quasineutral limit can be drawn without introducing new techniques, since the hyperbolic part (the equations (1.1a) and (1.1b)) of (1.1) is not symmetrizable.

Some historic results follow. For the Euler-Poisson equation (1.5), Guo and Pausader [8] constructed global smooth irrotational solutions with small amplitude for this equation with fixed $\varepsilon > 0$ and $T_i > 0$. Very recently, Guo and the first author of the present paper [9] derived the KdV equation from (1.5) for the full range of $T_i \geq 0$, which inspired the first author [19] to derive the Kadomtsev-Petviashvili II equation and the Zakharov-Kuznetsov equation (ZKE) via the Gardner-Morikawa type transformations. Lannes *et al.* independently studied the ZKE limit of the Euler-Poisson equation under long wavelength limit [15]. Very recently, Han-Kwan also studied the KdV and ZKE limit from the kinetic Vlasov-Poisson system under long wavelength limit. Guo *et al.* [7] made a breakthrough for the Euler-Maxwell two-fluid system in 3D and proved that irrotational, smooth and localized perturbations of a constant background with small amplitude lead to global smooth solutions. The long wavelength limit of the Euler-Poisson system in [15, 19] could be interpreted as a quasineutral limit of the Euler-Poisson equation to

the constant solution $(1, 0, 0)$. However, the quasineutral limit of the present paper concerning solutions $(n, \mathbf{u}, \phi)$ to a general compressible Euler equation (1.6) is new.

For the quasineutral limit of the Euler-Poisson equation, as pointed out above, Cordier and Grenier proved the quasineutral limit from the isothermal Euler-Poisson equation (1.5) to the Euler equation (1.6) in [2]. Wang [24] studied the quasineutral limit of the Euler-Poisson system with and without viscosity. We also would like to remark that Loeper [16] proved quasineutral limit results for the electron Euler-Poisson equation without pressure term and derived the incompressible Euler equation and hence is irrelevant to the present result. Han-Kwan [10] studied the quasineutral limit of the Vlasov-Poisson system with massless electrons, where the author derived the classical isothermal Euler equation for cold ions using the relative entropy method. Very recently, Gerard-Varet *et al.* studied the quasineutral limit of the system (1.5) in $\mathbb{R}_+^3$ and boundary layers were constructed. For numerical studies for the pressureless Euler-Poisson equation (1.1), the reader may refer to a recent paper of Degond *et al.* [3], which analyzes various schemes for the Euler-Poisson-Boltzmann equation. For more results on the quasineutral limit results of the Euler-Poisson equation and related models, one may refer to various recent papers and the references therein; see [1, 4, 5, 12, 18, 20–22, 24, 25] to list only a few. The difference between [20] and the present paper is that [20] briefly discusses the weak convergence of weak solutions, while the present paper considers the strong convergence of smooth solutions, which requires subtle estimates.

Now, we make a remark on Cordier and Grenier's results on the quasineutral limit for the Euler-Poisson equation for $T_i > 0$ fixed. The presence of the pressure term $T_i \nabla \ln n$ in (1.5) enables them to derive the Euler equation as Debye length goes to zero. But their estimates are not uniform in $T_i > 0$, because the smoothing effects of the pressure term vanish as $T_i \rightarrow 0$. One may refer to Guo-Pu [9] for a clear study of why these smoothing effects vanish. But by a similar method to the one we use to prove Theorem 1.3 in Section 2, we can indeed give uniform estimates in $T_i > 0$ and hence improve their results. This would yield the zero temperature limit as $T_i \rightarrow 0$ stated in Theorem 1.4 below. Let $(n^{\varepsilon, T_i}, \mathbf{u}^{\varepsilon, T_i}, \phi^{\varepsilon, T_i})$ be a solution of (1.5) and $(n^{0, T_i}, \mathbf{u}^{0, T_i}, \phi^{0, T_i})$ be a solution of (1.6) with the same initial data. We let

$$n^{\varepsilon, T_i} = n^{0, T_i} + \varepsilon n^{1, T_i}, \quad \mathbf{u}^{\varepsilon, T_i} = \mathbf{u}^{0, T_i} + \varepsilon \mathbf{u}^{1, T_i}, \quad \phi^{\varepsilon, T_i} = \phi^{0, T_i} + \varepsilon \phi^{1, T_i}.$$

Here ϕ^{ε, T_i} and n^{ε, T_i} satisfy the Poisson equation (1.5c), and indeed ϕ^{ε, T_i} can be solved via $\phi^{\varepsilon, T_i} = \phi^{\varepsilon, T_i}[n^{\varepsilon, T_i}]$ and $\phi^{0, T_i} = \ln n^{0, T_i}$. We will prove the following.

THEOREM 1.4. Let $(n^{0, T_i}, \mathbf{u}^{0, T_i}, \phi^{0, T_i})$ be a solution of (1.6) on $[0, T]$ with initial data $(n_0, \mathbf{u}_0) \in H^{s'}$ with s' sufficiently large. There exist solutions $(n^{\varepsilon, T_i}, \mathbf{u}^{\varepsilon, T_i}, \phi^{\varepsilon, T_i})$ of (1.5) with the same initial data on $[0, T^{\varepsilon, T_i}]$ with $\liminf_{\varepsilon, T_i \rightarrow 0} T^{\varepsilon, T_i} \geq T$. Moreover, for every $T' < T$ and for ε, T_i small enough, $(n^{1, T_i}, \mathbf{u}^{1, T_i})$ are bounded in $L^\infty([0, T']; H^s)$ for some $s < s'$.

By such a result, we can indeed derive the Euler equation (1.2) from (1.5) by first letting $\varepsilon \rightarrow 0$ and then letting $T_i \rightarrow 0$. The limit $T_i \rightarrow 0$ is usually known as the cold ion limit. Furthermore, $T_i > 0$ and $\varepsilon > 0$ don't depend on each other. They can go to zero independently.

The next section is devoted to the proof of Theorem 1.3. For this purpose, we write the solution of (1.1) as $n^\varepsilon = n^0 + \varepsilon n^1$ and $\mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1$ and consider the remainder system (R_ε) of n^1 and $\mathbf{u}^1$. The main idea is then to show that $(n^1, \mathbf{u}^1)$ is uniformly bounded in $H^s \times \mathbf{H}^s$ when $\varepsilon \rightarrow 0$. To overcome the difficulty of non-symmetrizability of (1.1), we introduce some triple norm $||| \cdot |||_{\varepsilon, s}$,

$$\begin{aligned} ||| \mathbf{u}^1 |||_{\varepsilon, s}^2 &= \| \mathbf{u}^1 \|_{H^s}^2 + \varepsilon \| \nabla \mathbf{u}^1 \|_{H^s}^2, \\ ||| \phi^1 |||_{\varepsilon, s}^2 &= \| \phi^1 \|_{H^s}^2 + \varepsilon \| \phi^1 \|_{H^s}^2 + \varepsilon^2 \| \Delta \phi^1 \|_{H^s}^2, \end{aligned} \quad (1.7)$$

and then show that $|||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}$ is uniformly bounded on some time interval independent of ε . The main novelty of the proof is then to combine the s -order energy estimates with the $(s+1)$ -order energy estimates with weights 1 and ε . By such a combination, we obtain some Gronwall type inequality for $|||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}$, which enables us to obtain uniform estimates independent of ε . The estimates of $\|n^1\|_{H^s}$ are obtained through some elliptic estimates from the Poisson equation (1.1c) in Section 2.2. This method was successfully employed in our previous paper to handle the long wavelength limit of the Euler-Poisson system to the KdV equation in [9] and could be potentially useful in treating the quasineutral limit for the pressureless electron Euler-Poisson equations.

Theorem 1.4 is proven in Section 3 in the same spirit as the proof of Theorem 1.3.

We introduce some notation. We let L^p denote the usual Lebesgue space of p -th integrable functions normed by $\|\cdot\|_{L^p}$. When $p = 2$, we usually use $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$. The Sobolev space H^s , $s \in \mathbb{Z}$, $s \geq 0$ is defined as $H^s(\mathbb{R}^d) = \{f(x) : \sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2 < \infty\}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex, $|\alpha| = \sum \alpha_i$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$. H^s is a Banach space with norm $\|f\|_{H^s} = (\sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2)^{1/2}$. For definiteness, we will restrict ourselves to the physical space dimensions $d \leq 3$ in this paper.

2. Proof of Theorem 1.3. The purpose of this section is to prove Theorem 1.3. Let $(n^\varepsilon, \mathbf{u}^\varepsilon, \phi^\varepsilon)$ satisfy the Euler-Poisson equation (1.1) and $(n^0, \mathbf{u}^0, \phi^0)$ be a sufficiently smooth solution of the Euler equation (1.2). We let

$$n^\varepsilon = n^0 + \varepsilon n^1, \quad \mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1, \quad \phi^\varepsilon = \phi^0 + \varepsilon \phi^1. \quad (2.1)$$

Here ϕ^ε and n^ε satisfy the Poisson equation (1.1c), and indeed ϕ^ε can be solved via $\phi^\varepsilon = \phi^\varepsilon[n^\varepsilon]$ and $\phi^0 = \ln n^0$. Then $(n^1, \mathbf{u}^1, \phi^1)$ satisfy the remainder system (R_ε) :

$$(R_\varepsilon) \quad \begin{cases} \partial_t n^1 + \nabla \cdot (n^0 \mathbf{u}^1 + \mathbf{u}^0 n^1) + \varepsilon \nabla \cdot (n^1 \mathbf{u}^1) = 0 & (2.2a) \\ \partial_t \mathbf{u}^1 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^1 + \mathbf{u}^1 \cdot \nabla \mathbf{u}^0 + \varepsilon \mathbf{u}^1 \cdot \nabla \mathbf{u}^1 = -\nabla \phi^1 & (2.2b) \\ -\varepsilon \Delta \phi^1 = \Delta \phi^0 + n^1 - n^0 \phi^1 + \sqrt{\varepsilon} R^1, & (2.2c) \end{cases}$$

where

$$R^1 = \varepsilon^{-3/2} (n^0 + \varepsilon n^0 \phi^1 - e^{\phi^0 + \varepsilon \phi^1}). \quad (2.3)$$

To prove Theorem 1.3, we need only to derive some uniform bound for the remainder equation (2.2). To slightly simplify the presentation, we assume that (2.2) has smooth solutions in a small time T_ε dependent on ε . Let $\tilde{C}$ be a constant to be determined later,

much larger than the bound of $\|(n_0^1, \mathbf{u}_0^1)\|_s$, such that on $[0, T_\varepsilon]$,

$$\sup_{[0, T_\varepsilon]} \|(n^1, \mathbf{u}^1, \phi^1)\|_{H^s} \leq \tilde{C}. \quad (2.4)$$

We will prove that $T_\varepsilon > T$ as $\varepsilon \rightarrow 0$ for some $T > 0$. Recalling the expressions for n and $\mathbf{u}$ in (2.1), we immediately know that there exists some $\varepsilon_1 = \varepsilon_1(\tilde{C}) > 0$ such that on $[0, T_\varepsilon]$,

$$\sigma'/2 < n^\varepsilon < 2\sigma'', \quad |\mathbf{u}^\varepsilon| \leq 1/2, \quad (2.5)$$

for all $0 < \varepsilon < \varepsilon_1$.

2.1. *Estimates for R_1 .* We first bound R^1 in terms of ϕ^1 . More precisely, we have the following.

LEMMA 2.1. Let $(n^0, \mathbf{u}^0, \phi^0)$ be a sufficiently smooth solution of (1.2) by Theorem 1.1. Then for the remainder term (2.3), we have on $[0, T_\varepsilon]$,

$$\begin{aligned} \|R^1\|_{H^k} &\leq C(\sqrt{\varepsilon}\tilde{C})\|\phi^1\|_{H^k} \quad \text{and} \\ \|\partial_t R^1\|_{H^k} &\leq C(\sqrt{\varepsilon}\tilde{C})(\|\phi^1\|_{H^k} + \|\partial_t \phi^1\|_{H^k}), \quad \forall k \geq 0. \end{aligned} \quad (2.6)$$

In particular, there exists some $\varepsilon_1 > 0$ and $C_1 = C(1)$ such that

$$\begin{aligned} \|R^1\|_{H^k} &\leq C_1\|\phi^1\|_{H^k} \quad \text{and} \\ \|\partial_t R^1\|_{H^k} &\leq C_1(\|\phi^1\|_{H^k} + \|\partial_t \phi^1\|_{H^k}), \quad \forall k \geq 0, \end{aligned} \quad (2.7)$$

for all $0 < \varepsilon < \varepsilon_1$ and $t \in [0, T_\varepsilon]$.

Proof. From the Taylor expansion in the integral form, we have

$$R^1 = \varepsilon^{1/2} e^{\phi^0} \int_0^1 e^{\theta \varepsilon \phi^1} (1 - \theta) d\theta (\phi^1)^2.$$

By taking L^2 norm, we have

$$\|R^1\| \leq \sqrt{\varepsilon} \|e^{\phi^0}\|_{L^\infty} e^{\varepsilon \|\phi^1\|_{L^\infty}} \|\phi^1\|_{L^2} \|\phi^1\|_{L^\infty}.$$

From the continuity assumption (2.4), we have $\|\phi^1\|_{L^\infty} \leq C\tilde{C}$ on $[0, T_\varepsilon]$ and hence

$$\|R^1\|_{L^2} \leq C(\sqrt{\varepsilon}\tilde{C})\|\phi^1\|_{L^2}, \quad \forall t \in [0, T_\varepsilon].$$

By applying ∂^α with $|\alpha| = k$, $k \geq 1$ integers, similar estimates yield

$$\|R^1\|_{H^k} \leq C(\sqrt{\varepsilon}\tilde{C})\|\phi^1\|_{H^k}.$$

Taking ∂_t to R^1 and then taking the H^k norm, we obtain

$$\|\partial_t R^1\|_{H^k} \leq C(\sqrt{\varepsilon}\tilde{C})(\|\phi^1\|_{H^k} + \|\partial_t \phi^1\|_{H^k}).$$

Finally, choosing $\varepsilon_1 = (1/\tilde{C})^2$ yields (2.7). □

2.2. *Elliptical estimates.* The following lemmas provide useful estimates between n^1 , $\mathbf{u}^1$ and ϕ^1 . These will be used widely in the uniform estimates in the next subsection.

LEMMA 2.2. Let $(n^1, \mathbf{u}^1, \phi^1)$ be a smooth solution for the remainder system (R_ε) and α a multiindex. There exist ε_1 and C such that for any $0 < \varepsilon < \varepsilon_1$ and any multiindices α with $|\alpha| = k \geq 0$, there hold

$$\begin{aligned} \|\partial_x^\alpha n^1\|^2 &\leq C + C\|\phi^1\|_{H^k}^2 + C\varepsilon^2\|\Delta\phi^1\|_{H^k}^2 \quad \text{and} \\ \|\partial_x^\alpha \phi^1\|^2 + \varepsilon\|\partial_x^\alpha \nabla\phi^1\| + \varepsilon^2\|\Delta\partial_x^\alpha \phi^1\|^2 &\leq C + C\|\partial_x^\alpha n^1\|^2 \end{aligned}$$

on the interval $[0, T_\varepsilon]$.

Proof. Taking the L^2 inner product of (2.2c) with ϕ^1 and then integrating by parts yield

$$\varepsilon\|\nabla\phi^1\|^2 + \int n^0|\phi^1|^2 = \int \phi^1\Delta\phi^0 + \int n^1\phi^1 + \sqrt{\varepsilon} \int \phi^1 R^1.$$

Hereafter, $\int = \int_{\mathbb{R}^d} \cdots dx$. As $n^0 > \sigma' > 0$ for $t \leq T' < T$, we obtain by Young's inequality

$$\varepsilon\|\nabla\phi^1\|^2 + \sigma'\|\phi^1\|^2 \leq \frac{\sigma'}{4}\|\phi^1\|^2 + \frac{4C}{\sigma'}(\|\Delta\phi^0\|^2 + \|n^1\|^2 + \varepsilon\|R^1\|^2).$$

From Lemma 2.1, there exists $\varepsilon_1 > 0$ such that $\|R^1\| \leq C_1\|\phi^1\|$. Then by choosing a new smaller ε_1 such that $\varepsilon_1 \leq \frac{\sigma'^2}{16CC_1}$, we then have for any $0 < \varepsilon < \varepsilon_1$ that

$$\varepsilon\|\nabla\phi^1\|^2 + \sigma'\|\phi^1\|^2 \leq \frac{C}{\sigma'}(1 + \|n^1\|^2). \quad (2.8)$$

Similarly, by taking the L^2 inner product of (2.2c) with $\varepsilon\Delta\phi^1$ and integrating by parts, we obtain that

$$\begin{aligned} \varepsilon^2\|\Delta\phi^1\|^2 + \varepsilon \int n^0|\nabla\phi^1|^2 &= -\varepsilon \int \Delta\phi^0\Delta\phi^1 + \varepsilon \int n^1\Delta\phi^1 \\ &\quad - \varepsilon \int \nabla n^0\phi^1\nabla\phi^1 + \varepsilon^{3/2} \int R^1\phi^1 \\ &\leq \frac{\varepsilon^2}{2}\|\Delta\phi^1\|^2 + 4(\|\Delta\phi^0\|^2 + \|n^1\|^2 + \varepsilon\|R^1\|^2 + \varepsilon\|\phi^1\|^2 + \varepsilon\|\nabla\phi^1\|^2), \end{aligned}$$

which yields for any $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1 > 0$ that

$$\varepsilon^2\|\Delta\phi^1\|^2 + \varepsilon\sigma'\|\nabla\phi^1\|^2 \leq C(1 + \|n^1\|^2), \quad (2.9)$$

as $n^0 > \sigma' > 0$ for $t \leq T' < T$, where the constant C depends on σ' . By combining (2.8) and (2.9) we easily obtain that for any $0 < \varepsilon < \varepsilon_1$,

$$\varepsilon^2\|\Delta\phi^1\|^2 + \varepsilon\sigma'\|\nabla\phi^1\|^2 + \|\phi^1\|^2 \leq C(1 + \|n^1\|^2),$$

for some constant C depending on σ' . On the other hand, by taking the L^2 norm of (2.2c), we obtain that for any $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} \|n^1\|^2 &\leq \varepsilon^2\|\Delta\phi^1\|^2 + \|n^0\|_{L^\infty}^2\|\phi^1\|^2 + \|\Delta\phi^0\|^2 + \varepsilon\|R^1\|^2 \\ &\leq C(1 + \|\phi^1\|^2 + \varepsilon^2\|\Delta\phi^1\|^2), \end{aligned}$$

thanks again to Lemma 2.1, where C depends on σ'' and C_1 . Therefore, we finish the proof when $k = 0$. Higher order estimates can be handled similarly, and we omit further details. $\square$

LEMMA 2.3. Let $(n^1, \mathbf{u}^1, \phi^1)$ be a smooth solution for the remainder system (R_ε) and α an integer. There exist ε_1 and C such that for any $0 < \varepsilon < \varepsilon_1$ and any multiindices α with $|\alpha| = k$, there holds

$$\|\partial^\alpha \partial_t n^1\|^2 \leq C(1 + \|\mathbf{u}^1\|_{\mathbf{H}^{k+1}}^2 + \|\phi^1\|_{H^{k+1}}^2 + \varepsilon^2 \|\Delta \phi^1\|_{H^{k+1}}^2) \quad (2.10)$$

on the time interval $[0, T_\varepsilon]$.

Proof. We take the L^2 norm of (2.2a) to obtain

$$\begin{aligned} \|\partial_t n^1\|^2 &\leq C(\|\mathbf{u}^1\|_{\mathbf{H}^1}^2 + \|n^1\|_{H^1}^2) + \varepsilon^2(\|n^1\|_{L^\infty}^2 \|\nabla \mathbf{u}^1\|^2 + \|\mathbf{u}^1\|_{L^\infty}^2 \|\nabla n^1\|^2) \\ &\leq C(1 + \varepsilon^2(\|n^1\|_{L^\infty}^2 + \|\mathbf{u}^1\|_{L^\infty}^2))(\|\mathbf{u}^1\|_{\mathbf{H}^1}^2 + \|n^1\|_{H^1}^2), \end{aligned}$$

for some constant C depending on $(n^0, \mathbf{u}^0)$. By the continuity assumption (2.4) and Lemma 2.2, we have

$$\begin{aligned} \|\partial_t n^1\|^2 &\leq C(1 + \varepsilon^2 \tilde{C})(1 + \|\mathbf{u}^1\|_{\mathbf{H}^1}^2 + \|\phi^1\|_{H^1}^2 + \varepsilon^2 \|\Delta \phi^1\|_{H^1}^2), \\ &\leq C(1 + \|\mathbf{u}^1\|_{\mathbf{H}^1}^2 + \|\phi^1\|_{H^1}^2 + \varepsilon^2 \|\Delta \phi^1\|_{H^1}^2) \end{aligned}$$

for any $0 < \varepsilon < \varepsilon_1$, for some $\varepsilon_1 > 0$.

Higher order inequalities are proved similarly. Taking ∂^α with $|\alpha| = k \geq 1$ to the equation (2.2a) and then taking the L^2 norm, we obtain

$$\begin{aligned} \|\partial^\alpha \partial_t n^1\|^2 &\leq C(\|\mathbf{u}^1\|_{\mathbf{H}^{k+1}}^2 + \|n^1\|_{H^{k+1}}^2) + \varepsilon^2(\|n^1\|_{L^\infty}^2 \|\partial^\alpha \nabla \mathbf{u}^1\|^2 + \|\mathbf{u}^1\|_{L^\infty}^2 \|\partial^\alpha \nabla n^1\|^2) \\ &\leq C(1 + \varepsilon^2(\|n^1\|_{L^\infty}^2 + \|\mathbf{u}^1\|_{L^\infty}^2))(\|\mathbf{u}^1\|_{\mathbf{H}^{k+1}}^2 + \|n^1\|_{H^{k+1}}^2) \\ &\leq C(1 + \|\mathbf{u}^1\|_{\mathbf{H}^{k+1}}^2 + \|\phi^1\|_{H^{k+1}}^2 + \varepsilon^2 \|\Delta \phi^1\|_{H^{k+1}}^2), \end{aligned}$$

for any $0 < \varepsilon < \varepsilon_1$, for some $\varepsilon_1 > 0$, where we have used the multiplicative estimates in Lemma A.1. $\square$

LEMMA 2.4. Let $(n^1, \mathbf{u}^1, \phi^1)$ be a smooth solution for the remainder system (R_ε) and α an integer. There exist ε_1 and C such that for any $0 < \varepsilon < \varepsilon_1$ and any multiindices α with $|\alpha| = k$,

$$\|\partial^\alpha \partial_t \phi^1\|^2 + \varepsilon \|\partial^\alpha \nabla \partial_t \phi^1\|^2 + \varepsilon^2 \|\partial^\alpha \Delta \partial_t \phi^1\|^2 \leq C(1 + \|\partial^\alpha \partial_t n^1\|^2 + \|\phi^1\|_{H^k}^2)$$

on the time interval $[0, T_\varepsilon]$.

Proof. Taking ∂_t of (2.2c), we obtain

$$-\varepsilon \Delta \partial_t \phi^1 = \Delta \partial_t \phi^0 + \partial_t n^1 - \partial_t(n^0 \phi^1) + \sqrt{\varepsilon} \partial_t R^1.$$

Taking the L^2 inner product with $\partial_t \phi^1$ and then integrating by parts, we obtain

$$\varepsilon \|\nabla \partial_t \phi^1\|^2 + \int n^0 |\partial_t \phi^1|^2 = \int \partial_t \phi^1 (\Delta \partial_t \phi^0 - \partial_t n^0 \phi^1 + \partial_t n^1 + \sqrt{\varepsilon} \partial_t R^1).$$

As $n^0 > \sigma' > 0$ for $t \leq T' < T$, we have by Hölder inequality

$$\begin{aligned} \varepsilon \|\nabla \partial_t \phi^1\|^2 + \sigma' \|\partial_t \phi^1\|^2 &\leq \frac{\sigma'}{4} \|\partial_t \phi^1\|^2 \\ &\quad + \frac{4}{\sigma'} (\|\Delta \partial_t \phi^0\|^2 + \|\partial_t n^0\|_{L^\infty}^2 \|\phi^1\|^2 + \|\partial_t n^1\|^2 + \varepsilon \|\partial_t R^1\|^2) \\ &\leq \frac{\sigma'}{4} \|\partial_t \phi^1\|^2 + \frac{4}{\sigma'} (C + C \|\phi^1\|^2 + \|\partial_t n^1\|^2 + \varepsilon \|\partial_t R^1\|^2). \end{aligned}$$

By choosing a small $\varepsilon_1 > 0$ such that $16C_1\varepsilon_1 \leq \sigma'^2$, we then have for any $0 < \varepsilon < \varepsilon_1$ that

$$\varepsilon \|\nabla \partial_t \phi^1\|^2 + \sigma' \|\partial_t \phi^1\|^2 \leq \frac{C}{\sigma'} (1 + \|\phi^1\|^2 + \|\partial_t n^1\|^2),$$

thanks to Lemma 2.1. Similarly, by taking the inner product with $\varepsilon \Delta \partial_t \phi^1$, we obtain

$$\varepsilon^2 \|\Delta \partial_t \phi^1\|^2 + \sigma' \varepsilon \|\partial_t \phi^1\|^2 \leq \frac{C}{\sigma'} (1 + \|\phi^1\|^2 + \|\partial_t n^1\|^2).$$

Adding them together, we obtain that for any $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1 > 0$,

$$\|\partial_t \phi^1\|^2 + \varepsilon \|\nabla \partial_t \phi^1\|^2 + \varepsilon^2 \|\Delta \partial_t \phi^1\|^2 \leq C(1 + \|\partial_t n^1\|_{H^k}^2 + \|\phi^1\|_{H^k}^2),$$

for some constant C depending on σ' . Higher order estimates can be treated similarly, and we obtain for any α with $|\alpha| = k$ that

$$\|\partial_t \partial^\alpha \phi^1\|^2 + \varepsilon \|\partial_t \partial^\alpha \nabla \phi^1\|^2 + \varepsilon^2 \|\partial_t \partial^\alpha \Delta \phi^1\|^2 \leq C(1 + \|\partial_t n^1\|_{H^k}^2 + \|\phi^1\|_{H^k}^2),$$

for some constant C depending on σ' . □

By recalling Lemma 2.3, we have the following.

COROLLARY 2.5. Let $(n^1, \mathbf{u}^1, \phi^1)$ be a smooth solution for the remainder system (R_ε) and α an integer. There exist ε_1 and C such that for any $0 < \varepsilon < \varepsilon_1$ and any multiindices α with $|\alpha| = k$,

$$\begin{aligned} \|\partial^\alpha \partial_t \phi^1\|^2 + \varepsilon \|\partial^\alpha \nabla \partial_t \phi^1\|^2 + \varepsilon^2 \|\partial^\alpha \Delta \partial_t \phi^1\|^2 \\ \leq C(1 + \|\mathbf{u}^1\|_{\mathbf{H}^{k+1}}^2 + \|\phi^1\|_{H^{k+1}}^2 + \varepsilon^2 \|\Delta \phi^1\|_{H^{k+1}}^2) \end{aligned}$$

on the time interval $[0, T_\varepsilon]$.

As a direct consequence of Lemma 2.1, Lemma 2.4 and 2.3, we also have:

COROLLARY 2.6. Let $(n^1, \mathbf{u}^1, \phi^1)$ be a smooth solution for the remainder system (R_ε) and α an integer. There exist ε_1 and C such that

$$\|\partial_t R^1\|_{H^k} \leq C(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, k+1}),$$

for any $0 < \varepsilon < \varepsilon_1$ and any multiindices α with $|\alpha| = k$.

2.3. Estimates of the s order. In this subsection, we give several estimates of the s order. However, the H^s -norm of the solutions depends on the H^{s+1} -norm and hence cannot be closed until the next subsection. The main result in this subsection is Proposition 2.10. In the following, $\gamma \geq 0$ will always denote a multiindex with $|\gamma| = s$.

LEMMA 2.7. Let $\gamma \geq 0$ be a multiindex with $|\gamma| = s$ and $(n^1, \mathbf{u}^1, \phi^1)$ a smooth solution for the system (2.2). There exist $\varepsilon_1 > 0$ and $C > 0$ such that

$$\begin{aligned} & \left\{ \frac{1}{2} \frac{d}{dt} \|\partial^\gamma \mathbf{u}^1\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^0 + \varepsilon n^1} |\partial^\gamma \phi^1|^2 + \frac{\varepsilon}{2} \frac{d}{dt} \int \frac{1}{n^0 + \varepsilon n^1} |\partial^\gamma \nabla \phi^1|^2 \right\} \\ & \leq C(1 + \varepsilon^3 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 5}^3)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, s \vee 3}^2), \end{aligned} \quad (2.11)$$

for any $0 < \varepsilon < \varepsilon_1$, where $s \vee 3 = \max\{s, 3\}$.

Proof. Let γ be a multiindex with $|\gamma| = s \geq 0$. Taking ∂^γ to (2.2b), we obtain

$$\partial_t \partial^\gamma \mathbf{u}^1 + \partial^\gamma (\mathbf{u}^0 \cdot \nabla \mathbf{u}^1) + \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0) + \varepsilon \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^1) = -\partial^\gamma \nabla \phi^1.$$

Taking the L^2 inner product with $\partial^\gamma \mathbf{u}^1$, we obtain

$$\begin{aligned} \int \partial_t \partial^\gamma \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 &= - \int \partial^\gamma \nabla \phi^1 \partial^\gamma \mathbf{u}^1 - \varepsilon \int \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^1) \partial^\gamma \mathbf{u}^1 \\ &\quad - \int \partial^\gamma (\mathbf{u}^0 \cdot \nabla \mathbf{u}^1) \partial^\gamma \mathbf{u}^1 - \int \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0) \partial^\gamma \mathbf{u}^1 \\ &=: I + II + III + IV. \end{aligned} \quad (2.12)$$

- *Estimate of the fourth term IV.*

The term IV can be bounded by

$$\begin{aligned} IV &\leq C \|\partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0)\|_{L^2} \|\partial^\gamma \mathbf{u}^1\|_{L^2} \\ &\leq C(\|\mathbf{u}^1\|_{H^s} \|\nabla \mathbf{u}^0\|_{L^\infty} + \|\nabla \mathbf{u}^0\|_{H^s} \|\mathbf{u}^1\|_{L^\infty}) \|\mathbf{u}^1\|_{H^s} \\ &\leq C(\|\mathbf{u}^1\|_{H^s}^2 + \|\mathbf{u}^1\|_{H^2}^2), \end{aligned} \quad (2.13)$$

where we have used the commutator estimates (A.1), the Sobolev embedding $H^2 \hookrightarrow L^\infty$ when $d \leq 3$ and the fact that $(n^0, \mathbf{u}^0)$ is a known smooth solution of the Euler equation (1.2) by Theorem 1.1.

- *Estimate of the third term III.*

By integration by parts, the third term III can be rewritten as

$$\begin{aligned} III &= - \int \mathbf{u}^0 \cdot \nabla \partial^\gamma \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 - \int [\partial^\gamma, \mathbf{u}^0] \cdot \nabla \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 \\ &= \frac{1}{2} \int \nabla \cdot \mathbf{u}^0 |\partial^\gamma \mathbf{u}^1|^2 - \int [\partial^\gamma, \mathbf{u}^0] \cdot \nabla \mathbf{u}^1 \partial^\gamma \mathbf{u}^1. \end{aligned} \quad (2.14)$$

By using commutator estimates (A.1), we obtain

$$\begin{aligned} \left| \int [\partial^\gamma, \mathbf{u}^0] \cdot \nabla \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 \right| &\leq C \|[\partial^\gamma, \mathbf{u}^0] \cdot \nabla \mathbf{u}^1\|_{L^2} \|\partial^\gamma \mathbf{u}^1\|_{L^2} \\ &\leq C(\|\nabla \mathbf{u}^0\|_{L^\infty} \|\nabla \mathbf{u}^1\|_{H^{s-1}} + \|\mathbf{u}^0\|_{H^s} \|\nabla \mathbf{u}^1\|_{L^\infty}) \|\partial^\gamma \mathbf{u}^1\|_{L^2} \\ &\leq C(\|\mathbf{u}^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^s}^2), \end{aligned}$$

where we have used the Hölder inequality and the Sobolev embedding $H^2 \hookrightarrow L^\infty$. This yields the estimate

$$III \leq C(\|\mathbf{u}^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^s}^2), \quad (2.15)$$

since the first term on the RHS of (2.14) is bounded by $C\|\mathbf{u}^1\|_{H^s}^2$.

- *Estimate of the second term II.*

Similar to the estimate of III, we have by integration by parts that

$$\begin{aligned}
 II &= \frac{\varepsilon}{2} \int \nabla \cdot \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 - \varepsilon \int [\partial^\gamma, \mathbf{u}^1] \cdot \nabla \mathbf{u}^1 \partial^\gamma \mathbf{u}^1 \\
 &\leq C\varepsilon \|\nabla \cdot \mathbf{u}^1\|_{L^\infty} \|\partial^\gamma \mathbf{u}^1\|_{L^2}^2 \\
 &\quad + C\varepsilon (\|\nabla \mathbf{u}^1\|_{L^\infty} \|\nabla \mathbf{u}^1\|_{H^{s-1}} + \|\nabla \mathbf{u}^1\|_{L^\infty} \|\mathbf{u}^1\|_{H^s}) \|\partial^\gamma \mathbf{u}^1\|_{L^2} \\
 &\leq C\varepsilon \|\mathbf{u}^1\|_{H^3} \|\mathbf{u}^1\|_{H^s}^2.
 \end{aligned} \tag{2.16}$$

- *Estimate of the first term I.*

By integration by parts, the term I in (2.12) is rewritten as

$$I = \int \partial^\gamma \phi^1 \partial^\gamma \nabla \cdot \mathbf{u}^1.$$

To handle this term, we note that from the remainder equation (2.2a),

$$\begin{aligned}
 (n^0 + \varepsilon n^1) \partial^\gamma \nabla \cdot \mathbf{u}^1 + [\partial^\gamma, n^0 + \varepsilon n^1] \nabla \cdot \mathbf{u}^1 + \partial_t \partial^\gamma n^1 \\
 + \partial^\gamma ((\mathbf{u}^0 + \varepsilon \mathbf{u}^1) \cdot \nabla n^1 + \mathbf{u}^1 \cdot \nabla n^0) + \partial^\gamma (n^1 \nabla \cdot \mathbf{u}^0) = 0.
 \end{aligned} \tag{2.17}$$

Inserting this into I , we obtain

$$\begin{aligned}
 I &= - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial_t \partial^\gamma n^1 - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma ((\mathbf{u}^0 + \varepsilon \mathbf{u}^1) \cdot \nabla n^1) \\
 &\quad - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} [\partial^\gamma, n^0 + \varepsilon n^1] \nabla \cdot \mathbf{u}^1 - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (\mathbf{u}^1 \cdot \nabla n^0) \\
 &\quad - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (n^1 \cdot \nabla \mathbf{u}^0) =: \sum_{i=1}^5 I_i.
 \end{aligned} \tag{2.18}$$

In the following, we estimate $I_3 \sim I_5$ while leaving the estimates of I_1 and I_2 to the next lemmas.

For I_3 , we have

$$\begin{aligned}
 I_3 &\leq C \|\partial^\gamma \phi^1\|_{L^2} (\|\nabla (n^0 + \varepsilon n^1)\|_{L^\infty} \|\nabla \cdot \mathbf{u}^1\|_{H^{s-1}} + \|n^0 + \varepsilon n^1\|_{H^s} \|\nabla \cdot \mathbf{u}^1\|_{L^\infty}) \\
 &\leq C \|\phi^1\|_{H^s} (\|\mathbf{u}^1\|_{H^s} + \|\mathbf{u}^1\|_{H^3}) + C\varepsilon (\|n^1\|_{H^3} + \|\mathbf{u}^1\|_{H^3}) \|\phi^1\|_{H^s} (\|n^1\|_{H^s} + \|\mathbf{u}^1\|_{H^s}) \\
 &\leq C(1 + \varepsilon \|(n^1, \mathbf{u}^1)\|_{H^3}) \|(n^1, \mathbf{u}^1, \phi^1)\|_{H^s}^2 + C\|\mathbf{u}^1\|_{H^3}^2,
 \end{aligned}$$

where we have used the Hölder inequality, commutator estimates (A.1) and the fact that n^0 and $n^0 + \varepsilon n^1$ are bounded from above and below by positive numbers when $\varepsilon < \varepsilon_1$ is small enough in (2.5).

For I_4 , directly applying the Hölder inequality and Lemma A.1 yields

$$\begin{aligned}
 I_{24} &\leq C \|\phi^1\|_{H^s} (\|\mathbf{u}^1\|_{L^\infty} \|\nabla n^0\|_{H^s} + \|\mathbf{u}^1\|_{H^s} \|\nabla n^0\|_{L^\infty}) \\
 &\leq C \|\phi^1\|_{H^s}^2 + C \|\mathbf{u}^1\|_{H^s}^2 + C \|\mathbf{u}^1\|_{H^2}^2.
 \end{aligned}$$

Similarly, I_5 can be bounded by

$$I_{25} \leq C \|\phi^1\|_{H^s}^2 + C \|n^1\|_{H^s}^2 + C \|n^1\|_{H^2}^2.$$

Summarizing, we have that

$$I \leq I_1 + I_2 + C\|(n^1, \mathbf{u}^1)\|_{H^3}^2 + C(1 + \varepsilon^2\|(n^1, \mathbf{u}^1)\|_{H^3}^2)\|(n^1, \mathbf{u}^1, \phi^1)\|_{H^s}^2. \quad (2.19)$$

To end the proof of Lemma 2.7, we need to get suitable estimates for I_1 and I_2 . However, this is not straightforward, and to make it easier to read, we leave the proof to the next two lemmas. $\square$

LEMMA 2.8. The term of I_2 in (2.18) is bounded by

$$I_{21} \leq C(1 + \varepsilon^3\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,4}^3)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,s\vee 3}^2), \quad (2.20)$$

for some constant $C > 0$ and for all $0 < \varepsilon < \varepsilon_1$.

Proof. First, we observe that I_2 in (2.18) can be decomposed into

$$I_2 = - \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \nabla \partial^\gamma n^1 - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} [\partial^\gamma, \mathbf{u}^0 + \varepsilon \mathbf{u}^1] \cdot \nabla n^1 =: I_{21} + I_{22}.$$

By commutator estimate (A.1),

$$\begin{aligned} I_{22} &\leq C \|\partial^\gamma \phi^1\| (\|\nabla \mathbf{u}^0 + \varepsilon \nabla \mathbf{u}^1\|_{L^\infty} \|\nabla n^1\|_{H^{s-1}} + \|\mathbf{u}^0 + \varepsilon \mathbf{u}^1\|_{H^s} \|\nabla n^1\|_{L^\infty}) \\ &\leq C \|\phi^1\|_{H^s} (\|n^1\|_{H^s} + \|n^1\|_{H^3}) + C\varepsilon (\|(n^1, \mathbf{u}^1)\|_{H^3}) \|\phi^1\|_{H^s} (\|(n^1, \mathbf{u}^1)\|_{H^s}) \\ &\leq C \|n^1\|_{H^3}^2 + C(1 + \varepsilon \|(n^1, \mathbf{u}^1)\|_{H^3}) \|(n^1, \mathbf{u}^1, \phi^1)\|_{H^s}^2. \end{aligned} \quad (2.21)$$

To treat I_{21} , we first note that from the remainder equation (2.2c),

$$\partial^\gamma \nabla n^1 = \nabla \partial^\gamma (n^0 \phi^1) - \varepsilon \partial^\gamma \nabla \Delta \phi^1 - \partial^\gamma \nabla \Delta \phi^0 - \sqrt{\varepsilon} \partial^\gamma \nabla R^1. \quad (2.22)$$

Hence I_{21} is accordingly divided into

$$\begin{aligned} I_{21} &= - \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \nabla \partial^\gamma (n^0 \phi^1) + \varepsilon \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \partial^\gamma \nabla \Delta \phi^1 \\ &\quad + \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \partial^\gamma \nabla \Delta \phi^0 + \sqrt{\varepsilon} \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \partial^\gamma \nabla R^1 \\ &=: \sum_{i=1}^4 I_{21i}. \end{aligned} \quad (2.23)$$

For the first term I_{211} , we have

$$\begin{aligned} I_{211} &= - \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot n^0 \nabla \partial^\gamma \phi^1 - \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot [\partial^\gamma, n^0] \nabla \phi^1 \\ &= \frac{1}{2} \int \nabla \cdot \left(\frac{n^0(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right) |\partial^\gamma \phi^1|^2 - \int \frac{\partial^\gamma \phi^1(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot [\partial^\gamma, n^0] \nabla \phi^1. \end{aligned}$$

By direct computation and Sobolev embedding, we have

$$\|\nabla \cdot \left(\frac{n^0(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right)\|_{L^\infty} \leq C + C\varepsilon^2 (\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2),$$

which yields

$$\begin{aligned} I_{211} &\leq C\|\phi^1\|_{H^s}^2 + C\varepsilon^2(\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2)\|\phi^1\|_{H^s}^2 + C\|\partial^\gamma \phi^1\|_{L^2}^2 \\ &\quad \cdot (1 + \varepsilon\|\mathbf{u}^1\|_{L^\infty})(\|\nabla n^0\|_{L^\infty}\|\nabla \phi^1\|_{H^{s-1}} + \|n^0\|_{H^s}\|\nabla \phi^1\|_{L^\infty}) \\ &\leq C(1 + \varepsilon^2(\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2))\|\phi^1\|_{H^s}^2 + C\|\phi^1\|_{H^3}^2, \end{aligned} \quad (2.24)$$

thanks to the commutator estimates (A.1). For I_{212} , by integration by parts, we obtain

$$\begin{aligned} I_{212} &= \varepsilon \int \frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1) \partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \cdot \partial^\gamma \nabla \Delta \phi^1 \\ &= -\varepsilon \int \frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \partial^\gamma \nabla \phi^1 \cdot \partial^\gamma \nabla \nabla \phi^1 - \varepsilon \int \nabla \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \cdot \partial^\gamma \nabla \nabla \phi^1 \\ &= \frac{3\varepsilon}{2} \int \nabla \cdot \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right) |\partial^\gamma \nabla \phi^1|^2 + \varepsilon \int \nabla^2 \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \partial^\gamma \nabla \phi^1. \end{aligned}$$

By direct computation and the Sobolev embedding $H^2 \hookrightarrow L^\infty$, we have

$$\begin{aligned} \|\nabla \cdot \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right)\|_{L^\infty} &\leq C + C\varepsilon^2(\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2), \\ \|\nabla^2 \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right)\|_{L^\infty} &\leq C + C\varepsilon^3(\|n^1\|_{H^4}^3 + \|\mathbf{u}^1\|_{H^4}^3), \end{aligned}$$

which yield

$$I_{212} \leq C(1 + \varepsilon^3(\|(n^1, \mathbf{u}^1)\|_{H^4}^3))(\|\phi^1\|_{H^s}^2 + \varepsilon\|\nabla \phi^1\|_{H^s}^2). \quad (2.25)$$

For I_{213} , by Hölder inequality, we obtain

$$I_{213} \leq C(1 + \|\mathbf{u}^1\|_{L^2}^2 + \|\phi^1\|_{H^s}^2). \quad (2.26)$$

For I_{214} , by integration by parts,

$$I_{214} = -\sqrt{\varepsilon} \int \nabla \cdot \left(\frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \partial^\gamma R^1 - \sqrt{\varepsilon} \int \frac{(\mathbf{u}^0 + \varepsilon \mathbf{u}^1)}{n^0 + \varepsilon n^1} \cdot \nabla \partial^\gamma \phi^1 \partial^\gamma R^1.$$

By using Lemma 2.1, we obtain

$$\begin{aligned} I_{214} &\leq C(1 + \varepsilon^2\|(n^1, \mathbf{u}^1)\|_{H^3}^2)\|\phi^1\|_{H^s}^2 \\ &\quad + C(1 + \varepsilon\|\mathbf{u}^1\|_{L^\infty})(\|\phi^1\|_{H^s}^2 + \varepsilon\|\nabla \phi^1\|_{H^s}^2) \\ &\leq C(1 + \varepsilon\|(\mathbf{u}^1, n^1)\|_{H^3})(\|\phi^1\|_{H^s}^2 + \varepsilon\|\nabla \phi^1\|_{H^s}^2). \end{aligned} \quad (2.27)$$

By (2.23) and putting (2.24)-(2.27) together, we obtain

$$\begin{aligned} I_{21} &\leq C + C\|(n^1, \mathbf{u}^1, \phi^1)\|_{H^3}^2 \\ &\quad + C(1 + \varepsilon^3\|(n^1, \mathbf{u}^1)\|_{H^4}^3)(\|(n^1, \mathbf{u}^1, \phi^1)\|_{H^s}^2 + \varepsilon\|\nabla \phi^1\|_{H^s}^2) \\ &\leq C(1 + \varepsilon^3\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,4}^3)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,s\vee 3}^2), \end{aligned} \quad (2.28)$$

where we have used the definition of the norm $\|\cdot\|_{\varepsilon,s}$ in (1.7) and Lemma 2.2 to replace the norms of n^1 with the norms of ϕ^1 . By putting (2.21) and (2.28) together and using Lemma 2.2, we obtain (2.20). This ends the proof of Lemma 2.8. $\square$

LEMMA 2.9. For I_1 in (2.18), we have the estimates

$$\begin{aligned} I_1 \leq & -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \phi^1|^2 - \frac{\varepsilon}{2} \frac{d}{dt} \int \frac{1}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 \\ & + C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,5}^3)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,sv3}^2). \end{aligned}$$

Proof. From the remainder equation (2.2c), we obtain

$$\partial^\gamma \partial_t n^1 = \partial^\gamma \partial_t (n^0 \phi^1) - \varepsilon \partial^\gamma \partial_t \Delta \phi^1 - \partial^\gamma \partial_t \Delta \phi^0 - \sqrt{\varepsilon} \partial^\gamma \partial_t R^1. \quad (2.29)$$

In this way, we can divide I_1 in (2.18) into the following:

$$\begin{aligned} I_1 = & \varepsilon \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t \Delta \phi^1 - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t (n^0 \phi^1) \\ & - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t \Delta \phi^0 - \sqrt{\varepsilon} \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t R^1 = \sum_{i=1}^4 I_{1i}. \end{aligned} \quad (2.30)$$

In the following, we treat the RHS terms of (2.30) one by one.

- *Estimate of I_{12} .*

For the term I_{12} , we have

$$\begin{aligned} I_{12} = & - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (\phi^1 \partial_t n^0) - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (n^0 \partial_t \phi^1) \\ = & - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (\phi^1 \partial_t n^0) - \int \frac{n^0 \partial^\gamma \phi^1}{n^0 + \varepsilon n^1} \partial_t \partial^\gamma \phi^1 - \int \frac{\partial^\gamma \phi^1}{n^0 + \varepsilon n^1} [\partial^\gamma, n^0] \partial_t \phi^1 \\ = & : \sum_{i=1}^3 I_{12i}. \end{aligned}$$

For the first term I_{121} , since n^0 is known and is assumed to be smooth in Theorem 1.1, we have

$$I_{121} \leq C \|\phi^1\|_{H^s}^2,$$

thanks to the multiplicative estimate in Lemma A.1. For the second term I_{122} , by integration by parts and Lemma 2.3, we have

$$\begin{aligned} I_{122} = & -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^0 + \varepsilon n^1} |\partial^\gamma \phi^1|^2 + \frac{1}{2} \int \partial_t \left(\frac{n^0}{n^0 + \varepsilon n^1} \right) |\partial^\gamma \phi^1|^2 \\ \leq & -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^0 + \varepsilon n^1} |\partial^\gamma \phi^1|^2 + C(1 + \varepsilon \|\partial_t n^1\|_{L^\infty}) \|\partial^\gamma \phi^1\|_{L^2}^2 \\ \leq & -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^0 + \varepsilon n^1} |\partial^\gamma \phi^1|^2 + C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2) \|\phi^1\|_{H^s}^2, \end{aligned}$$

where we have used the Sobolev embedding $H^2 \hookrightarrow L^\infty$ for $d \leq 3$. For the third term I_{123} , by commutator estimates in Lemma A.1 and Lemma 2.3, we have

$$\begin{aligned} I_{123} \leq & C \|\partial^\gamma \phi^1\|_{L^2} (\|\nabla n^0\|_{L^\infty} \|\partial_t \phi^1\|_{H^{s-1}} + \|n^0\|_{H^s} \|\partial_t \phi^1\|_{L^\infty}) \\ \leq & C(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s}^2 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2). \end{aligned}$$

Summarizing, we have

$$I_{12} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^0 + \varepsilon n^1} |\partial^\gamma \phi^1|^2 + C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s}^2 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2). \quad (2.31)$$

- *Estimate of I_{13} .*

For the term I_{13} , since ϕ^0 is known and smooth, it is easy to obtain

$$I_{13} \leq C(1 + \|\phi^1\|_{H^s}^2). \quad (2.32)$$

- *Estimate of I_{14} .*

For the term I_{14} , by integration by parts, we have

$$I_{14} = \sqrt{\varepsilon} \int \frac{\partial^{\gamma+\gamma_1} \phi^1}{n^0 + \varepsilon n^1} \partial^{\gamma-\gamma_1} \partial_t R^1 + \sqrt{\varepsilon} \int \partial^{\gamma_1} \left(\frac{1}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \partial^{\gamma-\gamma_1} \partial_t R^1,$$

where $\gamma_1 \leq \gamma$ is a multiindex with $|\gamma_1| = 1$. By Lemma 2.2, Corollary 2.6 and the definition of the triple norm $||| \cdot |||_{\varepsilon,s}$, we have the bound

$$\begin{aligned} I_{14} &\leq C\sqrt{\varepsilon} \|\partial^{\gamma+\gamma_1} \phi^1\|_{L^2} \|\partial_t R^1\|_{H^{s-1}} + C(1 + \varepsilon \|\partial^{\gamma_1} n^1\|_{L^\infty}) \|\partial^\gamma \phi^1\|_{L^2} \|\partial_t R^1\|_{H^{s-1}} \\ &\leq C \|\partial_t R^1\|_{H^{s-1}}^2 + C\varepsilon \|\nabla \phi^1\|_{H^s}^2 + C(1 + \varepsilon^2 \|n^1\|_{H^3}^2) \|\phi^1\|_{H^s}^2 \\ &\leq C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s}^2). \end{aligned} \quad (2.33)$$

- *Estimate of I_{11} .*

We next deal with the term I_{11} in (2.30). By integration by parts, we have

$$\begin{aligned} I_{11} &= -\varepsilon \int \frac{1}{n^0 + \varepsilon n^1} \partial^\gamma \nabla \phi^1 \cdot \partial_t \partial^\gamma \nabla \phi^1 - \varepsilon \int \nabla \left(\frac{1}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \cdot \partial_t \partial^\gamma \nabla \phi^1 \\ &= -\frac{\varepsilon}{2} \frac{d}{dt} \int \frac{1}{n^0 + \varepsilon n^1} |\partial^\gamma \nabla \phi^1|^2 + \frac{\varepsilon}{2} \int \partial_t \left(\frac{1}{n^0 + \varepsilon n^1} \right) |\partial^\gamma \nabla \phi^1|^2 \\ &\quad - \varepsilon \int \nabla \left(\frac{1}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \cdot \partial_t \partial^\gamma \nabla \phi^1 =: I_{111} + I_{112} + I_{113}. \end{aligned}$$

Since from Sobolev embedding and Lemma 2.3, we have

$$\|\partial_t \left(\frac{n^0}{n^0 + \varepsilon n^1} \right)\|_{L^\infty} \leq C(1 + \varepsilon \|\partial_t n^1\|_{L^\infty}) \leq C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2),$$

it is immediate that

$$I_{112} \leq C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2)(\varepsilon \|\partial^\gamma \nabla \phi^1\|^2).$$

For the term I_{113} , we have by integration by parts,

$$\begin{aligned} I_{113} &= \varepsilon \int \Delta \left(\frac{1}{n^0 + \varepsilon n^1} \right) \partial^\gamma \phi^1 \partial_t \partial^\gamma \phi^1 + \varepsilon \int \nabla \left(\frac{1}{n^0 + \varepsilon n^1} \right) \partial^\gamma \nabla \phi^1 \partial_t \partial^\gamma \phi^1 \\ &= -\varepsilon \int \partial^{\gamma_1} \Delta \left(\frac{1}{n^\varepsilon} \right) \partial^\gamma \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1 - \varepsilon \int \Delta \left(\frac{1}{n^\varepsilon} \right) \partial^{\gamma+\gamma_1} \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1 \\ &\quad - \varepsilon \int \partial^{\gamma_1} \nabla \left(\frac{1}{n^\varepsilon} \right) \nabla \partial^\gamma \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1 - \varepsilon \int \nabla \left(\frac{1}{n^\varepsilon} \right) \nabla \partial^{\gamma+\gamma_1} \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1, \end{aligned} \quad (2.34)$$

where $\gamma_1 \leq \gamma$ is a multiindex with $|\gamma_1| = 1$. By direct computation, Hölder inequality and Sobolev embedding $H^2 \hookrightarrow L^\infty$, it is easy to obtain

$$\begin{aligned} \|\partial^\alpha(\frac{1}{n^\varepsilon})\|_{L^\infty} &\leq C(1 + \varepsilon^{|\alpha|} \|n^1\|_{H^{2+|\alpha|}}^{|\alpha|}) \\ &\leq C(1 + \varepsilon^{|\alpha|} |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 2+|\alpha|}^{|\alpha|}) \end{aligned} \quad (2.35)$$

for any smooth function n^0 and any multiindex α , thanks to Lemma 2.2. On the other hand, from Lemma 2.4 and Lemma 2.3, we obtain

$$\begin{aligned} \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2}^2 &\leq C(1 + \|\phi^1\|_{H^{s-1}}^2 + \|\partial_t n^1\|_{H^{s-1}}^2) \\ &\leq C(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2). \end{aligned} \quad (2.36)$$

Since the order of the derivatives on $n^0/(n^0 + \varepsilon n^1)$ in (2.34) does not exceed 3, by using Hölder inequality, (2.35) and (2.36), I_{113} can be bounded by

$$\begin{aligned} I_{113} &\leq C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3) \cdot \varepsilon(\|\partial^\gamma \phi^1\|_{L^2} + \|\partial^{\gamma+\gamma_1} \phi^1\|_{L^2} \\ &\quad + \|\nabla \partial^\gamma \phi^1\|_{L^2} + \|\nabla \partial^{\gamma+\gamma_1} \phi^1\|_{L^2}) \cdot \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2} \\ &\leq C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3) \\ &\quad \times (\|\partial^\gamma \phi^1\|_{L^2}^2 + \varepsilon \|\partial^\gamma \nabla \phi^1\|_{L^2}^2 + \varepsilon^2 \|\partial^\gamma \Delta \phi^1\|_{L^2}^2 + \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2}^2) \\ &\leq C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2), \end{aligned}$$

where we have used the definition of $||| \cdot |||_{\varepsilon, s}$ in (1.7) and the L^2 boundedness of the Riesz operator [23]. To be more precise, there exists some constant $C > 0$ such that $\|\partial_i \partial_j \phi^1\| \leq C \|\Delta \phi^1\|$ since $\partial_i \partial_j = -R_i R_j \Delta$, where R_i is the i^{th} Riesz operator. In particular, we have $\|\nabla \partial^{\gamma+\gamma_1} \phi^1\| \leq C \|\partial^\gamma \Delta \phi^1\|$.

Summarizing, we have

$$I_{11} = -\frac{\varepsilon}{2} \frac{d}{dt} \int \frac{1}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 + C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2). \quad (2.37)$$

By (2.30) and the estimates of (2.31), (2.32), (2.33) and (2.37), we have

$$\begin{aligned} I_1 &\leq -\frac{1}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \phi^1|^2 - \frac{\varepsilon}{2} \frac{d}{dt} \int \frac{1}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 \\ &\quad + C(1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s \vee 3}^2). \end{aligned}$$

This ends the proof of Lemma 2.9. $\square$

Now, we can end the proof of Lemma 2.7.

End of proof of Lemma 2.7. The proof of Lemma 2.7 is closed by (2.12), (2.13), (2.15), (2.16), (2.19) and Lemma 2.8 and 2.9. $\square$

PROPOSITION 2.10. Let $s \geq 0$ be a positive integer and $(n^1, \mathbf{u}^1, \phi^1)$ a smooth solution for the system (2.2). There exists $\varepsilon_1 > 0$ and $C, C' > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, there holds

$$\begin{aligned} &\|\mathbf{u}^1(t)\|_{H^s}^2 + \|\phi^1(t)\|_{H^s}^2 + \varepsilon \|\nabla \phi^1(t)\|_{H^s}^2 \\ &\leq C'(\|\mathbf{u}^1(0)\|_{H^s}^2 + \|\phi^1(0)\|_{H^s}^2 + \varepsilon \|\nabla \phi^1(0)\|_{H^s}^2) \\ &\quad + C \int_0^t (1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s \vee 3}^2) d\tau, \end{aligned} \quad (2.38)$$

where C' depends only on σ' and σ'' .

Proof. This is shown by integrating (2.11) over $[0, t]$ and summing them up for $|\gamma| \leq s$, and then using $\sigma' < n^0 < \sigma''$ and $\frac{\sigma'}{2} < n^\varepsilon < 2\sigma''$ for any $t \in [0, T_\varepsilon]$ in (1.4) and (2.5) for $0 < \varepsilon < \varepsilon_1$. $\square$

However, this Gronwall inequality is not closed since the right hand side of (2.38) depends on $\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2$, which does not appear on the left hand side. This will be treated in the next subsection.

2.4. *Weighted $s+1$ order estimates.* In the following, we also let γ be a multiindex with $|\gamma| = s$. The main result in this subsection is Proposition 2.14.

LEMMA 2.11. Let $s \geq 0$ be a positive integer and $(n^1, \mathbf{u}^1, \phi^1)$ a smooth solution for the system (2.2). There exists $\varepsilon_1 > 0$ and $C, C' > 0$ such that

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} \int |\partial^\gamma \nabla \mathbf{u}^1|^2 &\leq -\frac{\varepsilon}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 - \frac{\varepsilon^2}{2} \frac{d}{dt} \int \frac{1}{n^\varepsilon} |\partial^\gamma \Delta \phi^1|^2 \\ &\quad + C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,4}^2)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s\sqrt{3}}^2), \end{aligned} \quad (2.39)$$

for any $0 < \varepsilon < \varepsilon_1$.

Proof. Let γ be a multiindex with $|\gamma| = s \geq 0$. Taking ∂^γ in the second equation of (2.2), we obtain

$$\partial_t \partial^\gamma \mathbf{u}^1 + \partial^\gamma (\mathbf{u}^0 \cdot \nabla \mathbf{u}^1) + \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0) + \varepsilon \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^1) = -\partial^\gamma \nabla \phi^1.$$

Taking the L^2 inner product with $-\varepsilon \partial^\gamma \Delta \mathbf{u}^1$, we obtain

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} \int |\partial^\gamma \nabla \mathbf{u}^1|^2 &= -\varepsilon \int \partial_t \partial^\gamma \mathbf{u}^1 \partial^\gamma \Delta \mathbf{u}^1 \\ &= \varepsilon \int \partial^\gamma \nabla \phi^1 \partial^\gamma \Delta \mathbf{u}^1 + \varepsilon^2 \int \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^1) \partial^\gamma \Delta \mathbf{u}^1 \\ &\quad + \varepsilon \int \partial^\gamma (\mathbf{u}^0 \cdot \nabla \mathbf{u}^1) \partial^\gamma \Delta \mathbf{u}^1 + \varepsilon \int \partial^\gamma (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0) \partial^\gamma \Delta \mathbf{u}^1 \\ &=: I^\varepsilon + II^\varepsilon + III^\varepsilon + IV^\varepsilon. \end{aligned} \quad (2.40)$$

• *Estimate of IV^ε .*

The term IV^ε can be bounded by

$$\begin{aligned} IV^\varepsilon &= -\varepsilon \int \partial^\gamma \nabla (\mathbf{u}^1 \cdot \nabla \mathbf{u}^0) \partial^\gamma \nabla \mathbf{u}^1 \\ &= -\varepsilon \int (\partial^\gamma (\nabla \mathbf{u}^1 \cdot \nabla \mathbf{u}^0) + \partial^\gamma (\mathbf{u}^1 \cdot \nabla^2 \mathbf{u}^0)) \partial^\gamma \nabla \mathbf{u}^1 \\ &\leq C\varepsilon (\|\partial^\gamma (\nabla \mathbf{u}^1 \cdot \nabla \mathbf{u}^0)\|_{L^2}^2 + \|\partial^\gamma (\mathbf{u}^1 \cdot \nabla^2 \mathbf{u}^0)\|_{L^2}^2) + C\varepsilon \|\partial^\gamma \nabla \mathbf{u}^1\|_{L^2}^2 \\ &\leq C(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2 + \|\mathbf{u}^1\|_{H^3}^2), \end{aligned} \quad (2.41)$$

where we have used the commutator estimates (A.1), the Sobolev embedding $H^2 \hookrightarrow L^\infty$ when $d \leq 3$ and the fact that n^0 and $\mathbf{u}^0$ are known smooth solutions of the Euler equation (1.2) by Theorem 1.1.

• *Estimate of III^ε .*

By integration by parts, the third term III^ε can be rewritten as

$$\begin{aligned}
III^\varepsilon &= \varepsilon \int \mathbf{u}^0 \cdot \nabla \partial^\gamma \mathbf{u}^1 \partial^\gamma \Delta \mathbf{u}^1 + \varepsilon \int [\partial^\gamma, \mathbf{u}^0] \cdot \nabla \mathbf{u}^1 \partial^\gamma \Delta \mathbf{u}^1 \\
&= \varepsilon \int \sum_{i,j,k} u_i^0 \partial_i \partial^\gamma u_k^1 \partial^\gamma \partial_j \partial_j u_k^1 + \varepsilon \int \sum_{\beta=1}^{\gamma} C_\gamma^\beta \partial^\beta \mathbf{u}^0 \cdot \partial^{\gamma-\beta} \nabla \mathbf{u}^1 \cdot \partial^\gamma \Delta \mathbf{u}^1 \\
&= -2\varepsilon \int \sum_{i,j,k} \partial_j u_i^0 \partial_i \partial^\gamma u_k^1 \partial^\gamma \partial_j u_k^1 + \varepsilon \int \sum_{i,j,k} \partial_i u_i^0 \partial^\gamma \partial_j u_k^1 \partial^\gamma \partial_j u_k^1 \\
&\quad - \varepsilon \int \sum_{\beta=1}^{\gamma} C_\gamma^\beta \partial^\beta \mathbf{u}^0 \cdot \partial^{\gamma-\beta} \nabla^2 \mathbf{u}^1 \cdot \partial^\gamma \nabla \mathbf{u}^1 - \varepsilon \int \sum_{\beta=1}^{\gamma} C_\gamma^\beta \partial^\beta \nabla \mathbf{u}^0 \cdot \partial^{\gamma-\beta} \nabla \mathbf{u}^1 \cdot \partial^\gamma \nabla \mathbf{u}^1 \\
&=: III_1^\varepsilon + III_2^\varepsilon + III_3^\varepsilon + III_4^\varepsilon,
\end{aligned}$$

where $\mathbf{u}^0 = (u_1^0, \dots, u_d^0)$ and $\mathbf{u}^1 = (u_1^1, \dots, u_d^1)$ and we have used integration by parts twice in the third equality. For III_1^ε , III_2^ε and III_3^ε , we easily obtain

$$|III_1^\varepsilon, III_2^\varepsilon, III_3^\varepsilon| \leq C\varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2.$$

For III_4^ε , by Hölder inequality, we obtain

$$|III_4^\varepsilon| \leq C\varepsilon \|\mathbf{u}^1\|_{H^s} \|\nabla \mathbf{u}^1\|_{H^s} \leq C(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2).$$

Summing them up, we obtain

$$|III^\varepsilon| \leq C(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2). \quad (2.42)$$

• *Estimate of II^ε .*

For the second term II^ε , by integration by parts, we have

$$\begin{aligned}
II^\varepsilon &= \varepsilon^2 \int \mathbf{u}^1 \cdot \partial^\gamma \nabla \mathbf{u}^1 \partial^\gamma \Delta \mathbf{u}^1 + \varepsilon^2 \int [\partial^\gamma, \mathbf{u}^1] \cdot \nabla \mathbf{u}^1 \partial^\gamma \Delta \mathbf{u}^1 \\
&= -\varepsilon^2 \int \sum_{i,j,k} \partial_j u_i^1 \partial^\gamma \partial_i u_k^1 \partial^\gamma \partial_j u_k^1 + \frac{\varepsilon^2}{2} \int \sum_{i,j,k} \partial_i u_i^1 \partial^\gamma \partial_j u_k^1 \partial^\gamma \partial_j u_k^1 \\
&\quad - \varepsilon^2 \int [\partial^\gamma, \nabla \mathbf{u}^1] \cdot \nabla \mathbf{u}^1 \cdot \partial^\gamma \nabla \mathbf{u}^1 - \varepsilon^2 \int [\partial^\gamma, \mathbf{u}^1] \cdot \nabla^2 \mathbf{u}^1 \cdot \partial^\gamma \nabla \mathbf{u}^1 =: \sum_{i=1}^4 II_i^\varepsilon.
\end{aligned}$$

By Hölder inequality, II_1^ε and II_2^ε are bounded by

$$\begin{aligned}
|II_1^\varepsilon, II_2^\varepsilon| &\leq C(\varepsilon \|\nabla \mathbf{u}^1\|_{L^\infty})(\varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2) \\
&\leq C(\varepsilon \|\mathbf{u}^1\|_{H^3})(\varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2).
\end{aligned}$$

By commutator estimates,

$$\begin{aligned}
\|[\partial^\gamma, \nabla \mathbf{u}^1] \cdot \nabla \mathbf{u}^1\|_{L^2} &\leq C(\|\nabla^2 \mathbf{u}^1\|_{L^\infty} \|\nabla \mathbf{u}^1\|_{H^{s-1}} + \|\nabla \mathbf{u}^1\|_{H^s} \|\nabla \mathbf{u}^1\|_{L^\infty}) \\
&\leq C\|\mathbf{u}^1\|_{H^4}(\|\mathbf{u}^1\|_{H^s} + \|\nabla \mathbf{u}^1\|_{H^s}).
\end{aligned}$$

Hence

$$|II_3^\varepsilon| \leq C\varepsilon \|\mathbf{u}^1\|_{H^4}(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^1\|_{H^s}^2).$$

Similarly, by commutator estimates

$$\begin{aligned} |II_4^\varepsilon| &\leq C\varepsilon^2(\|\nabla \mathbf{u}^1\|_{L^\infty}\|\nabla^2 \mathbf{u}^1\|_{H^{s-1}} + \|\mathbf{u}^1\|_{H^s}\|\nabla^2 \mathbf{u}^1\|_{L^\infty})\|\nabla \mathbf{u}^1\|_{H^s} \\ &\leq C\varepsilon\|\mathbf{u}^1\|_{H^4}(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon\|\nabla \mathbf{u}^1\|_{H^s}^2). \end{aligned}$$

Summing them up, we obtain

$$|II^\varepsilon| \leq C\varepsilon\|\mathbf{u}^1\|_{H^4}(\|\mathbf{u}^1\|_{H^s}^2 + \varepsilon\|\nabla \mathbf{u}^1\|_{H^s}^2). \quad (2.43)$$

• *Estimate of I^ε .*

In the following, we treat the first term I^ε in (2.40). By integration by parts thrice, we obtain

$$I^\varepsilon = -\varepsilon \int \partial^\gamma \Delta \phi^1 \partial^\gamma \nabla \cdot \mathbf{u}^1.$$

By using (2.17), we obtain

$$\begin{aligned} I^\varepsilon &= \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \partial_t \partial^\gamma n^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \partial^\gamma (\mathbf{u}^\varepsilon \cdot \nabla n^1) \\ &\quad + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} [\partial^\gamma, n^\varepsilon] \nabla \cdot \mathbf{u}^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \partial^\gamma (\mathbf{u}^1 \cdot \nabla n^0) \\ &\quad + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \partial^\gamma (n^1 \cdot \nabla \mathbf{u}^0) =: \sum_{i=1}^5 I_i^\varepsilon. \end{aligned} \quad (2.44)$$

In the following, we estimate $I_3^\varepsilon \sim I_5^\varepsilon$ while leaving the estimates of I_1^ε and I_2^ε to the next two lemmas. For I_3^ε , we have

$$\begin{aligned} I_3^\varepsilon &\leq C\varepsilon\|\partial^\gamma \Delta \phi^1\|_{L^2}(\|\nabla(n^0 + \varepsilon n^1)\|_{L^\infty}\|\nabla \cdot \mathbf{u}^1\|_{H^{s-1}} + \|n^0 + \varepsilon n^1\|_{H^s}\|\nabla \cdot \mathbf{u}^1\|_{L^\infty}) \\ &\leq C\varepsilon^2\|\Delta \phi^1\|_{H^s}^2 + \|\mathbf{u}^1\|_{H^s}^2 + \|\mathbf{u}^1\|_{H^3}^2 + C\varepsilon^2(\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2)(\|n^1\|_{H^s}^2 + \|\mathbf{u}^1\|_{H^s}^2), \end{aligned}$$

where we have used the commutator estimates (A.1) and the fact that $n^0 + \varepsilon n^1$ are bounded from above and below by positive numbers when $\varepsilon < \varepsilon_1$ is small enough in (2.5). Recalling Lemma 2.2 and the definition of the triple norm (1.7), we obtain

$$I_3^\varepsilon \leq C(1 + \varepsilon^2|||(\mathbf{u}^1, \phi)|||_{\varepsilon,3}^2)|||(\mathbf{u}^1, \phi)|||_{\varepsilon,s}^2 + C\|\mathbf{u}^1\|_{H^3}^2.$$

For I_4^ε , we have

$$\begin{aligned} I_4^\varepsilon &\leq C\varepsilon\|\Delta \phi^1\|_{H^s}(\|\mathbf{u}^1\|_{L^\infty}\|\nabla n^0\|_{H^s} + \|\mathbf{u}^1\|_{H^s}\|\nabla n^0\|_{L^\infty}) \\ &\leq C\varepsilon^2\|\Delta \phi^1\|_{H^s}^2 + C\|\mathbf{u}^1\|_{H^s}^2 + C\|\mathbf{u}^1\|_{H^2}^2. \end{aligned}$$

Similarly, I_5^ε can be bounded by

$$\begin{aligned} I_5^\varepsilon &\leq C\varepsilon\|\Delta \phi^1\|_{H^s}^2 + C\|n^1\|_{H^s}^2 + C\|n^1\|_{H^2}^2 \\ &\leq C(1 + |||\phi|||_{\varepsilon,s}^2) + C\|n^1\|_{H^2}^2, \end{aligned}$$

thanks to Lemma 2.2. Summarizing, we obtain from (2.44)

$$\begin{aligned} I^\varepsilon &\leq I_1^\varepsilon + I_2^\varepsilon + C\|\mathbf{u}^1\|_{H^3}^2 + C\|n^1\|_{H^2}^2 \\ &\quad + C(1 + \varepsilon^2|||(\mathbf{u}^1, \phi)|||_{\varepsilon,3}^2)(1 + |||(\mathbf{u}^1, \phi)|||_{\varepsilon,s}^2), \end{aligned} \quad (2.45)$$

where I_1^ε and I_2^ε will be treated in the next two lemmas. $\square$

LEMMA 2.12. The term I_2^ε in (2.44) can be estimated as

$$I_2^\varepsilon \leq C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,3}^2)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,s}^2) + \|\phi^1\|_{\varepsilon,3}^2,$$

for all $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1 > 0$.

Proof. Recall that I_2^ε is given by (2.44). By integration by parts, it can be rewritten as

$$I_2^\varepsilon = \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla \partial^\gamma n^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} [\partial^\gamma, \mathbf{u}^\varepsilon] \cdot \nabla n^1 =: I_{21}^\varepsilon + I_{22}^\varepsilon.$$

• *Estimate of I_{22}^ε .*

For the second term I_{22}^ε , we obtain

$$\begin{aligned} I_{22}^\varepsilon &\leq C\varepsilon \|\partial^\gamma \Delta \phi^1\|_{L^2} \|[\partial^\gamma, \mathbf{u}^\varepsilon] \cdot \nabla n^1\|_{L^2} \\ &\leq C\varepsilon \|\partial^\gamma \Delta \phi^1\|_{L^2} (\|\nabla \mathbf{u}^\varepsilon\|_{L^\infty} \|\nabla n^1\|_{H^{s-1}} + \|\mathbf{u}^\varepsilon\|_{H^s} \|\nabla n^1\|_{L^\infty}) \\ &\leq C\varepsilon \|\Delta \phi^1\|_{H^s} (\|n^1\|_{H^s} + \|n^1\|_{H^3} + \varepsilon \|\mathbf{u}^1\|_{H^3} \|n^1\|_{H^s} + \varepsilon \|n^1\|_{H^3} \|\mathbf{u}^1\|_{H^s}) \\ &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C(\|n^1\|_{H^s}^2 + \|n^1\|_{H^3}^2 + \varepsilon^2 \|(\mathbf{u}^1, n^1)\|_{H^3}^2 (\|(n^1, \mathbf{u}^1)\|_{H^s}^2)) \\ &\leq C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,3}^2)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,s}^2) + \|\phi^1\|_{\varepsilon,3}^2, \end{aligned} \quad (2.46)$$

thanks to the commutator estimates (A.1) in the second step, Lemma 2.2 in the last step and the definition of the triple norm in (1.7).

• *Estimate of I_{21}^ε .*

In the rest of this lemma, we focus on the treatment of I_{21}^ε . Recalling the remainder equation (2.2c) (see also (2.22)), I_{21}^ε can be divided into

$$\begin{aligned} I_{21}^\varepsilon &= \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla \partial^\gamma (n^0 \phi^1) - \varepsilon^2 \int \frac{\partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \partial^\gamma \nabla \Delta \phi^1 \\ &\quad - \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \partial^\gamma \nabla \Delta \phi^0 - \sqrt{\varepsilon} \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \partial^\gamma \nabla R^1 =: \sum_{i=1}^4 I_{21i}^\varepsilon. \end{aligned} \quad (2.47)$$

In the following, we estimate the four terms on the RHS of (2.47). For the first term I_{211}^ε , we have

$$\begin{aligned} I_{211}^\varepsilon &= \int \frac{\varepsilon n^0 \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla \partial^\gamma \phi^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot [\partial^\gamma, n^0] \nabla \phi^1 \\ &= \frac{\varepsilon}{2} \int \nabla \cdot \left(\frac{n^0 \mathbf{u}^\varepsilon}{n^\varepsilon} \right) |\partial^\gamma \nabla \phi^1|^2 - \varepsilon \int \sum_{i,j} \partial_i \left(\frac{n^0 u_j^\varepsilon}{n^\varepsilon} \right) \partial^\gamma \partial_i \phi^1 \partial^\gamma \partial_j \phi^1 \\ &\quad + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^\varepsilon} \mathbf{u}^\varepsilon \cdot [\partial^\gamma, n^0] \nabla \phi^1 =: I_{2111}^\varepsilon + I_{2112}^\varepsilon + I_{2113}^\varepsilon, \end{aligned} \quad (2.48)$$

where $u_j^\varepsilon = (\mathbf{u}^\varepsilon)_j = u_j^0 + \varepsilon u_j^1$. By using Sobolev embedding $H^2 \hookrightarrow L^\infty$,

$$\|\nabla \left(\frac{n^0 \mathbf{u}^\varepsilon}{n^\varepsilon} \right)\|_{L^\infty} \leq C + C\varepsilon^2 (\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2),$$

which yields for the first two terms on the RHS of (2.48)

$$\begin{aligned} |I_{2111}^\varepsilon, I_{2112}^\varepsilon| &\leq C\varepsilon(1 + \varepsilon^2(\|n^1\|_{H^3}^2 + \|\mathbf{u}^1\|_{H^3}^2))\|\nabla\phi^1\|_{H^s}^2 \\ &\leq C(1 + \varepsilon^2\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,3}^2)\|\phi^1\|_{\varepsilon,s}^2. \end{aligned}$$

For the term I_{2113}^ε , by the commutator estimates (A.1), we have

$$\begin{aligned} \|[\partial^\gamma, n^0]\nabla\phi^1\|_{L^2} &\leq C(\|\nabla n^0\|_{L^\infty}\|\nabla\phi^1\|_{H^{s-1}} + \|n^0\|_{H^s}\|\nabla\phi^1\|_{L^\infty}) \\ &\leq C(\|\phi^1\|_{H^3} + \|\phi^1\|_{H^s}), \end{aligned}$$

which implies that by Hölder inequality

$$\begin{aligned} I_{2113}^\varepsilon &\leq C\left\|\frac{1}{n^0 + \varepsilon n^1}\right\|_{L^\infty}\|\varepsilon\partial^\gamma\Delta\phi^1(\mathbf{u}^0 + \varepsilon\mathbf{u}^1)\|_{L^2}\|[\partial^\gamma, n^0]\nabla\phi^1\|_{L^2} \\ &\leq C(\|\phi^1\|_{H^3}^2 + \|\phi^1\|_{H^s}^2) + C(1 + \varepsilon^2\|\mathbf{u}^1\|_{L^\infty}^2)(\varepsilon^2\|\partial^\gamma\Delta\phi^1\|_{L^2}^2) \\ &\leq C(1 + \varepsilon^2\|\mathbf{u}^1\|_{H^2}^2)\|\phi^1\|_{\varepsilon,s}^2 + C\|\phi^1\|_{H^3}^2, \end{aligned}$$

thanks to (2.5), the Sobolev embedding $H^2 \hookrightarrow L^\infty$ and the definition of the triple norm (1.7).

For the first term I_{212} , by integration by parts and Lemma 2.2, we have

$$\begin{aligned} |I_{212}^\varepsilon| &= \left|\frac{\varepsilon^2}{2} \int \nabla \cdot \left(\frac{\mathbf{u}^0 + \varepsilon\mathbf{u}^1}{n^0 + \varepsilon n^1}\right) |\partial^\gamma\Delta\phi^1|^2\right| \\ &\leq C(1 + \varepsilon^2\|(n^1, \mathbf{u}^1)\|_{H^3}^2) \cdot \varepsilon^2\|\partial^\gamma\Delta\phi^1\|_{L^2}^2 \\ &\leq C(1 + \varepsilon^2\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,3}^2)\|\phi^1\|_{\varepsilon,s}^2. \end{aligned}$$

For the third term I_{213}^ε , by Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} |I_{213}^\varepsilon| &\leq \left\|\frac{\varepsilon\partial^\gamma\Delta\phi^1(\mathbf{u}^0 + \varepsilon\mathbf{u}^1)}{n^0 + \varepsilon n^1}\right\|_{L^2}\|\partial^\gamma\nabla\Delta\phi^0\|_{L^2} \\ &\leq \|\partial^\gamma\nabla\Delta\phi^0\|_{L^2}^2 + C(1 + \varepsilon^2\|\mathbf{u}^1\|_{L^\infty}^2)(\varepsilon^2\|\partial^\gamma\Delta\phi^1\|_{L^2}^2) \\ &\leq C + C(1 + \varepsilon^2\|\mathbf{u}^1\|_{H^2}^2)\|\phi^1\|_{\varepsilon,s}^2. \end{aligned}$$

For the fourth term I_{214}^ε , we have by Hölder inequality,

$$\begin{aligned} I_{214}^\varepsilon &\leq C\|\sqrt{\varepsilon}\partial^\gamma\nabla R^1\|_{L^2}^2 + C(1 + \varepsilon^2\|\mathbf{u}^1\|_{L^\infty}^2)(\varepsilon^2\|\partial^\gamma\Delta\phi^1\|_{H^s}^2) \\ &\leq C\varepsilon(\|\nabla\phi^1\|_{H^s}^2 + \|\phi^1\|^2) + C(1 + \varepsilon^2\|\mathbf{u}^1\|_{H^2}^2)(\varepsilon^2\|\partial^\gamma\Delta\phi^1\|_{L^2}^2) \\ &\leq C(1 + \varepsilon^2\|\mathbf{u}^1\|_{H^2}^2)\|\phi^1\|_{\varepsilon,s}^2, \end{aligned}$$

where we have used Lemma 2.1 with $|\alpha| = s + 1$ there and the fact that $\|\phi^1\|_{H^{s+1}} \approx \|\nabla\phi^1\|_{H^s} + \|\phi^1\|_{H^s}$. Summarizing, we obtain

$$I_{21}^\varepsilon \leq C + C\|\phi^1\|_{H^3}^2 + C(1 + \varepsilon^2\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,3}^2)\|\phi^1\|_{\varepsilon,s}^2. \quad (2.49)$$

Putting (2.46) and (2.49) together, we end the proof of Lemma 2.12. $\square$

LEMMA 2.13. The term I_1^ε in (2.44) is bounded by

$$\begin{aligned} I_1^\varepsilon &\leq -\frac{\varepsilon}{2} \frac{d}{dt} \int \left(\frac{n^0}{n^\varepsilon}\right) |\partial^\gamma\nabla\phi^1|^2 - \frac{\varepsilon^2}{2} \frac{d}{dt} \int \frac{1}{n^\varepsilon} |\partial^\gamma\Delta\phi^1|^2 \\ &\quad + C(1 + \varepsilon^2\|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,4}^2)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon,s\vee 3}^2), \end{aligned} \quad (2.50)$$

for all $0 < \varepsilon < \varepsilon_1$, for some $\varepsilon_1 > 0$.

Proof. Recall that I_1^ε is given in (2.44). From the remainder equation (2.2c) (see also (2.29)), I_1^ε can be divided into the following:

$$\begin{aligned} I_1^\varepsilon = & -\varepsilon \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t \Delta \phi^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t (n^0 \phi^1) \\ & + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t \Delta \phi^0 + \sqrt{\varepsilon} \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma \partial_t R^1 =: \sum_{i=1}^4 I_{1i}^\varepsilon. \end{aligned} \quad (2.51)$$

In the following, we will estimate the RHS terms one by one.

• *Estimate of I_{11}^ε .*

For the first term I_{11}^ε , we have

$$I_{11}^\varepsilon = -\frac{\varepsilon^2}{2} \frac{d}{dt} \int \frac{1}{n^0 + \varepsilon n^1} |\partial^\gamma \Delta \phi^1|^2 + \frac{\varepsilon^2}{2} \int \partial_t \left(\frac{1}{n^0 + \varepsilon n^1} \right) |\partial^\gamma \Delta \phi^1|^2.$$

Using Lemma 2.3 and Sobolev embedding, we have

$$\|\partial_t \left(\frac{1}{n^0 + \varepsilon n^1} \right)\|_{L^\infty} \leq C(1 + \varepsilon \|\partial_t n^1\|_{L^\infty}) \leq C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 3}^2),$$

which yields that

$$I_{11}^\varepsilon \leq -\frac{\varepsilon^2}{2} \frac{d}{dt} \int \frac{1}{n^0 + \varepsilon n^1} |\partial^\gamma \Delta \phi^1|^2 + C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 3}^2) \|\phi^1\|_{\varepsilon, s}^2. \quad (2.52)$$

• *Estimate of I_{12}^ε .*

For the second term I_{12}^ε , we have

$$\begin{aligned} I_{12}^\varepsilon &= \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (\partial_t n^0 \phi^1) + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (n^0 \partial_t \phi^1) \\ &= \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} \partial^\gamma (\partial_t n^0 \phi^1) + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} n^0 \partial_t \partial^\gamma \phi^1 + \int \frac{\varepsilon \partial^\gamma \Delta \phi^1}{n^0 + \varepsilon n^1} [\partial^\gamma, n^0] \partial_t \phi^1 \\ &=: \sum_{i=1}^3 I_{12i}^\varepsilon. \end{aligned}$$

For the first term I_{121}^ε , by the multiplicative estimates (A.1), we have

$$\begin{aligned} \|\partial^\gamma (\partial_t n^0 \phi^1)\|_{L^2} &\leq C(\|\partial_t n^0\|_{L^\infty} \|\phi^1\|_{H^s} + \|\partial_t n^0\|_{H^s} \|\phi^1\|_{L^\infty}) \\ &\leq C(\|\phi^1\|_{H^3} + \|\phi^1\|_{H^s}), \end{aligned}$$

which yields

$$\begin{aligned} I_{121}^\varepsilon &\leq C(\|\phi^1\|_{H^3} + \|\phi^1\|_{H^s}) \|\varepsilon \Delta \phi^1\|_{H^s} \\ &\leq C(\|\phi^1\|_{H^3}^2 + \|(\phi^1)\|_{\varepsilon, s}^2). \end{aligned}$$

For the second term I_{122}^ε , we have by integration by parts,

$$\begin{aligned} I_{122}^\varepsilon &= - \int \nabla \left(\frac{n^0}{n^0 + \varepsilon n^1} \right) \varepsilon \partial^\gamma \nabla \phi^1 \partial_t \partial^\gamma \phi^1 - \int \left(\frac{n^0}{n^0 + \varepsilon n^1} \right) \varepsilon \partial^\gamma \nabla \phi^1 \partial_t \nabla \partial^\gamma \phi^1 \\ &= \int \nabla \left(\frac{n^0}{n^\varepsilon} \right) \varepsilon \partial^{\gamma+\gamma_1} \nabla \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1 + \int \nabla \partial^{\gamma_1} \left(\frac{n^0}{n^\varepsilon} \right) \varepsilon \partial^\gamma \nabla \phi^1 \partial_t \partial^{\gamma-\gamma_1} \phi^1 \\ &\quad - \frac{\varepsilon}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 + \frac{\varepsilon}{2} \int \partial_t \left(\frac{n^0}{n^\varepsilon} \right) |\partial^\gamma \nabla \phi^1|^2 =: I_{122i}^\varepsilon \end{aligned}$$

where $\gamma_1 \leq \gamma$ is a multiindex with $|\gamma_1| = 1$. By Lemma 2.4 and Lemma 2.3, we have

$$\begin{aligned} \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2}^2 &\leq C(\|\partial_t \partial^{\gamma-\gamma_1} n^1\|^2 + \|\phi^1\|_{H^{s-1}}^2) \\ &\leq C(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, s}^2), \end{aligned}$$

where $\gamma_1 \leq \gamma$, $|\gamma_1| = 1$ and hence $|\gamma - \gamma_1| = s - 1$. On the other hand, by direct computation, we have

$$\begin{aligned} \|\nabla \left(\frac{n^0}{n^\varepsilon} \right)\|_{L^\infty} &\leq C(1 + \varepsilon \|\nabla n^1\|_{L^\infty}) \leq C(1 + \varepsilon^2 \|(\phi^1)\|_{\varepsilon, 3}^2), \\ \|\nabla \partial^{\gamma_1} \left(\frac{n^0}{n^\varepsilon} \right)\|_{L^\infty} &\leq C(1 + \varepsilon^2 \|\nabla n^1\|_{L^\infty}^2 + \varepsilon \|\nabla \partial^{\gamma_1} \phi^1\|_{L^\infty}) \\ &\leq C(1 + \varepsilon^2 \|(\phi^1)\|_{\varepsilon, 4}^2), \end{aligned}$$

where we have used Lemma 2.2 and Sobolev embedding. Hence, by Hölder inequality, we obtain

$$\begin{aligned} I_{122i}^\varepsilon, I_{122}^\varepsilon &\leq C \|\nabla \left(\frac{n^0}{n^\varepsilon} \right)\|_{L^\infty} \|\varepsilon \partial^{\gamma+\gamma_1} \nabla \phi^1\|_{L^2} \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2} \\ &\quad + C \|\nabla \partial^{\gamma_1} \left(\frac{n^0}{n^\varepsilon} \right)\|_{L^\infty} \|\varepsilon \partial^\gamma \nabla \phi^1\|_{L^2} \|\partial_t \partial^{\gamma-\gamma_1} \phi^1\|_{L^2} \\ &\leq C(1 + \varepsilon^2 \|(\phi^1)\|_{\varepsilon, 4}^2)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, s}^2), \end{aligned}$$

where we have used the boundedness of the Riesz operator. Similarly,

$$\|\partial_t \left(\frac{n^0}{n^\varepsilon} \right)\|_{L^\infty} \leq C(1 + \varepsilon^2 \|\partial_t n^1\|_{L^\infty}^2) \leq C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 3}^2),$$

which yields that

$$I_{1224}^\varepsilon \leq C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 3}^2) \|(\phi^1)\|_{\varepsilon, s}^2.$$

Therefore, I_{122}^ε is bounded by

$$I_{122}^\varepsilon \leq -\frac{\varepsilon}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 + C(1 + \varepsilon^2 \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 4}^2)(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, s}^2).$$

For the third term I_{123}^ε , we have

$$\begin{aligned} I_{123}^\varepsilon &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C\|[\partial^\gamma, n^0] \partial_t \phi^1\|_{L^2}^2 \\ &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C(\|\nabla n^0\|_{L^\infty}^2 \|\partial_t \phi^1\|_{H^{s-1}}^2 + \|\partial_t \phi^1\|_{L^\infty}^2 \|n^0\|_{H^s}^2) \\ &\leq C(1 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, s}^2 + \|(\mathbf{u}^1, \phi^1)\|_{\varepsilon, 3}^2), \end{aligned}$$

thanks to Lemma A.1 in the second inequality and Lemma 2.4 and Lemma 2.3 in the last inequality. In summary, I_{12}^ε can be bounded by

$$\begin{aligned} I_{12}^\varepsilon &\leq -\frac{\varepsilon}{2} \frac{d}{dt} \int \frac{n^0}{n^\varepsilon} |\partial^\gamma \nabla \phi^1|^2 \\ &\quad + C(1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,4}^2)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s}^2 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,3}^2). \end{aligned} \quad (2.53)$$

• *Estimate of I_{13}^ε .*

For the third term I_{13}^ε , it is straightforward that

$$I_{13}^\varepsilon \leq C(1 + \varepsilon^2 \|\Delta \phi^1\|_{H^s}^2). \quad (2.54)$$

• *Estimate of I_{14}^ε .*

For the fourth term I_{14}^ε , we have

$$\begin{aligned} |I_{14}^\varepsilon| &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C\varepsilon \|\partial_t R^1\|_{H^s}^2 \\ &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C_1(\|\phi^1\|_{H^s}^2 + \varepsilon \|\partial_t \phi^1\|_{H^s}^2) \\ &\leq C\varepsilon^2 \|\Delta \phi^1\|_{H^s}^2 + C_1(\|\phi^1\|_{H^s}^2 + \|\partial_t n^1\|_{H^{s-1}}^2) \\ &\leq C(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s}^2), \end{aligned} \quad (2.55)$$

where we have used Hölder inequality in the first inequality, Lemma 2.1 in the second inequality, Lemma 2.4 in the third inequality and Lemma 2.3 in the last inequality. Here, we also have used the fact that $\|\partial_t \phi^1\|_{H^s} \approx \|\partial_t \phi^1\|_{H^{s-1}} + \|\partial_t \partial^\alpha \nabla \phi^1\|_{L^2}$ with $|\gamma| = s-1$ and $\|\phi^1\|_{H^{s-1}} \leq \|\phi^1\|_{H^s}$ for all integers $s > 0$.

By (2.51), using (2.52), (2.53), (2.54) and (2.55), we obtain the estimate (2.50) for I^ε . $\square$

Now, we can end the proof of Lemma 2.11.

End of proof of Lemma 2.11. By using (2.40), the estimates of (2.41), (2.42), (2.43) and (2.45), and Lemmas 2.12 and 2.13, we close the proof of Lemma 2.11. $\square$

Summarizing these lemmas, we obtain the following.

PROPOSITION 2.14. Let $s \geq 0$ be a positive integer and $(n^1, \mathbf{u}^1, \phi^1)$ a smooth solution for the system (2.2). There exists $\varepsilon_1 > 0$ and $C, C' > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, there holds

$$\begin{aligned} &\varepsilon \|\nabla \mathbf{u}^1(t)\|_{H^s}^2 + \varepsilon \|\nabla \phi^1(t)\|_{H^s}^2 + \varepsilon^2 \|\Delta \phi^1(t)\|_{H^s}^2 \\ &\leq C'(\varepsilon \|\nabla \mathbf{u}^1(0)\|_{H^s}^2 + \varepsilon \|\nabla \phi^1(0)\|_{H^s}^2 + \varepsilon^2 \|\Delta \phi^1(0)\|_{H^s}^2) \\ &\quad + C \int_0^t (1 + \varepsilon^2 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,4}^2)(1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon,s\vee 3}^2) d\tau, \end{aligned} \quad (2.56)$$

where C' depends only on σ' and σ'' .

Proof. This is shown by integrating (2.39) over $[0, t]$ and summing them up for $|\gamma| \leq s$, and then using $\sigma' < n^0 < \sigma''$ and $\sigma'/2 < n^\varepsilon < 2\sigma''$ for any $t \in [0, T_\varepsilon]$ in (1.4) and (2.5) for $0 < \varepsilon < \varepsilon_1$. $\square$

2.5. End of proof of Theorem 1.3. Now, we are in a good position to end the proof of Theorem 1.3.

Proof. Let $s \geq 7$ be an integer. By Proposition 2.10 and Proposition 2.14 and recalling the definition of the norm (1.7), we obtain the Gronwall type inequality

$$\begin{aligned} |||(\mathbf{u}^1, \phi^1)(t)|||_{\varepsilon, s}^2 &\leq C' C_\varepsilon(0) \\ &+ C \int_0^t (1 + \varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3) (1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2) d\tau, \end{aligned} \quad (2.57)$$

where $C_\varepsilon(0) = |||(\mathbf{u}^1, \phi^1)(0)|||_{\varepsilon, s}^2$ and we have used the fact that $||| \cdot |||_{\varepsilon, 3} \leq ||| \cdot |||_{\varepsilon, s}$ for $s \geq 3$. From (2.4), there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, $\varepsilon^3 |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, 5}^3 \leq 1$, and hence (2.57) yields

$$|||(\mathbf{u}^1, \phi^1)(t)|||_{\varepsilon, s}^2 \leq C_2 C_\varepsilon(0) + C_2 \int_0^t (1 + |||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2) d\tau, \quad (2.58)$$

where $C_2 = \max\{C', 2C\}$. On the other hand, from Lemma 2.2, there exists constant $C_3 > 1$ such that

$$\|n^1(t)\|_{H^s}^2 \leq C_3(1 + |||\phi^1|||_{\varepsilon, s}^2). \quad (2.59)$$

Let $C_0 = \sup_{0 < \varepsilon < 1} C_\varepsilon(0)$. We let $\tilde{C}$ in (2.4) satisfy $\tilde{C} \geq 2C_3(1 + C_2 C_0)e^{C_2 T_\varepsilon}$. Then from (2.58)

$$|||(\mathbf{u}^1, \phi^1)|||_{\varepsilon, s}^2 \leq (1 + C_2 C_0)e^{C_2 T_\varepsilon} \leq \tilde{C},$$

and hence from (2.59)

$$\|n^1(t)\|_{H^s}^2 \leq C_3(1 + (1 + C_2 C_0)e^{C_2 T_\varepsilon}) \leq \tilde{C}.$$

Then by the continuity principle, it is standard to get the uniform in ε estimates for $|||(n^1, \mathbf{u}^1, \phi^1)|||_{\varepsilon, s}$. In particular, for every $T' < T$, $\varepsilon^{-1}(n^\varepsilon - n^0)$ and $\varepsilon^{-1}(\mathbf{u}^\varepsilon - \mathbf{u}^0)$ are bounded in $L^\infty([0, T']; H^s)$ and $L^\infty([0, T']; \mathbf{H}^s)$, respectively, uniformly in ε for ε small enough for some $s < s'$. $\square$

3. Proof of Theorem 1.4. This section is devoted to the proof of Theorem 1.4. Let $(n^{\varepsilon, T_i}, \mathbf{u}^{\varepsilon, T_i}, \phi^{\varepsilon, T_i})$ be a solution of (1.5) and $(n^{0, T_i}, \mathbf{u}^{0, T_i}, \phi^{0, T_i})$ be a solution of (1.6) with the same initial data. Let

$$n^{\varepsilon, T_i} = n^{0, T_i} + \varepsilon n^{1, T_i}, \quad \mathbf{u}^{\varepsilon, T_i} = \mathbf{u}^{0, T_i} + \varepsilon \mathbf{u}^{1, T_i}, \quad \phi^{\varepsilon, T_i} = \phi^{0, T_i} + \varepsilon \phi^{1, T_i}. \quad (3.1)$$

Then $(n^{1, T_i}, \mathbf{u}^{1, T_i}, \phi^{1, T_i})$ satisfy the remainder system:

$$(R_\varepsilon) \quad \begin{cases} \partial_t n^{1, T_i} + \nabla \cdot (n^{0, T_i} \mathbf{u}^{1, T_i} + \mathbf{u}^{0, T_i} n^{1, T_i}) + \varepsilon \nabla \cdot (n^{1, T_i} \mathbf{u}^{1, T_i}) = 0 & (3.2a) \\ \partial_t \mathbf{u}^{1, T_i} + \mathbf{u}^{0, T_i} \cdot \nabla \mathbf{u}^{1, T_i} + \mathbf{u}^{1, T_i} \cdot \nabla \mathbf{u}^{0, T_i} + \varepsilon \mathbf{u}^{1, T_i} \cdot \nabla \mathbf{u}^{1, T_i} \\ \quad + \frac{T_i \nabla n^{1, T_i}}{n^{0, T_i} + \varepsilon n^{1, T_i}} - \frac{T_i n^{1, T_i} \nabla n^{0, T_i}}{n^{0, T_i} (n^{0, T_i} + \varepsilon n^{1, T_i})} = -\nabla \phi^{1, T_i} & (3.2b) \\ -\varepsilon \Delta \phi^{1, T_i} = \Delta \phi^{0, T_i} + n^{1, T_i} - n^{0, T_i} \phi^{1, T_i} + \sqrt{\varepsilon} R^1, & (3.2c) \end{cases}$$

where R^1 is given in (2.3).

We need only show estimates uniform in ε and T_i for $(n^{1, T_i}, \mathbf{u}^{1, T_i})$. We also let $\tilde{C}$ be a constant to be determined later, much larger than the bound of $\|(n_0, \mathbf{u}_0)\|_{H^s}$, such that on $[0, T^{\varepsilon, T_i}]$:

$$\sup_{[0, T^{\varepsilon, T_i}]} \|(n^{1, T_i}, \mathbf{u}^{1, T_i}, \phi^{1, T_i})\|_{H^s} \leq \tilde{C}. \quad (3.3)$$

From the expression (1), there exists some $\varepsilon_1 = \varepsilon_1(\tilde{C}) > 0$ and $T_{i1} > 0$ such that $\sigma'/2 < n^{\varepsilon, T_i} < 2\sigma'$ and $|\mathbf{u}^{\varepsilon, T_i}| \leq 1/2$ on $[0, T^{\varepsilon, T_i}]$ for all $0 < \varepsilon < \varepsilon_1$ and $0 < T_i < T_{i1}$.

Let γ be a multiindex with $|\gamma| = s$. By taking ∂^γ to (3.2b), and then taking the L^2 inner product with $\partial^\gamma \mathbf{u}^{1, T_i}$, we obtain an equality similar to (2.12) but with the following two more terms on the RHS:

$$V + VI := -T_i \int \partial^\gamma \mathbf{u}^{1, T_i} \partial^\gamma \left(\frac{\nabla n^{1, T_i}}{n^{0, T_i} + \varepsilon n^{1, T_i}} \right) + T_i \int \partial^\gamma \mathbf{u}^{1, T_i} \partial^\gamma \left(\frac{\nabla n^{0, T_i}}{n^{0, T_i}} \frac{n^1}{n^{0, T_i} + \varepsilon n^{1, T_i}} \right). \quad (3.4)$$

• *Estimate of the term VI.*

For this term, we obtain by using the Leibnitz formula and multiplicative estimates

$$VI \leq T_i C(\varepsilon \tilde{C}) \|(\partial^\gamma \mathbf{u}^{1, T_i}, \partial^\gamma n^{1, T_i})\|^2,$$

where $\tilde{C}$ is given in (2.4).

• *Estimate of the term V.*

For this term, we rewrite

$$\begin{aligned} V &= -T_i \int \frac{\partial^\gamma \mathbf{u}^{1, T_i} \cdot \nabla \partial^\gamma n^{1, T_i}}{n^{0, T_i} + \varepsilon n^{1, T_i}} - T_i \int \partial^\gamma \mathbf{u}^{1, T_i} \cdot [\partial^\gamma, \frac{1}{n^{0, T_i} + \varepsilon n^{1, T_i}}] \nabla n^{1, T_i} \\ &= T_i \int \frac{\partial^\gamma \nabla \cdot \mathbf{u}^{1, T_i} \partial^\gamma n^{1, T_i}}{n^{0, T_i} + \varepsilon n^{1, T_i}} + T_i \int \partial^\gamma \mathbf{u}^{1, T_i} \partial^\gamma n^{1, T_i} \cdot \nabla \left(\frac{1}{n^{0, T_i} + \varepsilon n^{1, T_i}} \right) \\ &\quad - T_i \int \partial^\gamma \mathbf{u}^{1, T_i} \cdot [\partial^\gamma, \frac{1}{n^{0, T_i} + \varepsilon n^{1, T_i}}] \nabla n^{1, T_i} =: V_1 + V_2 + V_3. \end{aligned}$$

For V_2 and V_3 , we obtain by using commutator estimates (A.1) that

$$V_2 + V_3 \leq T_i C(\varepsilon \tilde{C}) \|(\partial^\gamma \mathbf{u}^{1, T_i}, \partial^\gamma n^{1, T_i})\|^2.$$

Recalling decomposition (2.17), we decompose V_1 into

$$\begin{aligned} V_1 &= -T_i \int \frac{\partial_t \partial^\gamma n^{1, T_i} \partial^\gamma n^{1, T_i}}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2} - T_i \int \frac{[\partial^\gamma, n^{0, T_i} + \varepsilon n^{1, T_i}] \nabla \cdot \mathbf{u}^{1, T_i} \partial^\gamma n^{1, T_i}}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2} \\ &\quad - T_i \int \frac{\partial^\gamma ((\mathbf{u}^{0, T_i} + \varepsilon \mathbf{u}^{1, T_i}) \cdot \nabla n^{1, T_i}) \partial^\gamma n^{1, T_i}}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2} - T_i \int \frac{\partial^\gamma (\mathbf{u}^{1, T_i} \cdot \nabla n^{0, T_i}) \partial^\gamma n^{1, T_i}}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2} \\ &\quad - T_i \int \frac{\partial^\gamma (n^{1, T_i} \nabla \cdot \mathbf{u}^{0, T_i}) \partial^\gamma n^{1, T_i}}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2} =: V_{11} + \dots + V_{15}. \end{aligned}$$

The term V_{11} can be estimated as

$$V_{11} = -\frac{T_i}{2} \frac{d}{dt} \int \frac{|\partial^\gamma n^{1, T_i}|^2}{(n^{0, T_i} + \varepsilon n^{1, T_i})^2}.$$

The estimate of term V_{12} is similar to I_3 in (2.18), and the estimate of term V_{13} is similar to II in (2.12). After integration by parts and commutator estimates, they can be estimated as

$$V_{12} + V_{13} \leq T_i C(\varepsilon \tilde{C}) \|(n^{1, T_i}, \mathbf{u}^{1, T_i})\|_{H^s}^2.$$

The estimates of the last two terms V_{14} and V_{15} are direct since the derivative on n^{1, T_i} and $\mathbf{u}^{1, T_i}$ is no greater than $|\gamma|$ and we have

$$V_{14} + V_{15} \leq T_i C(\varepsilon \tilde{C}) \|(n^{1, T_i}, \mathbf{u}^{1, T_i})\|_{H^s}^2.$$

These estimates lead to the following:

$$V + VI \leq -\frac{T_i}{2} \frac{d}{dt} \int \frac{|\partial^\gamma n^{1,T_i}|^2}{(n^{0,T_i} + \varepsilon n^{1,T_i})^2} + T_i C(\varepsilon \tilde{C}) \|(n^{1,T_i}, \mathbf{u}^{1,T_i})\|_{H^s}^2. \quad (3.5)$$

Therefore, similar to Proposition 2.10, there exist $\varepsilon_1 > 0$ and $T_{i1} > 0$ and constants $C', C > 0$ such that

$$\begin{aligned} & T_i \|n^{1,T_i}(t)\|_{H^s}^2 + \|\mathbf{u}^{1,T_i}(t)\|_{H^s}^2 + \|\phi^{1,T_i}(t)\|_{H^s}^2 + \varepsilon \|\nabla \phi^{1,T_i}(t)\|_{H^s}^2 \\ & \leq C'(T_i \|n^{1,T_i}(0)\|_{H^s}^2 + \|\mathbf{u}^{1,T_i}(0)\|_{H^s}^2 + \|\phi^{1,T_i}(0)\|_{H^s}^2 + \varepsilon \|\nabla \phi^{1,T_i}(0)\|_{H^s}^2) \\ & \quad + C \int_0^t T_i C(\varepsilon \tilde{C}) \|(n^{1,T_i}, \mathbf{u}^{1,T_i})\|_{H^s}^2 \\ & \quad + (1 + \varepsilon^3 \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,5}^3) (1 + \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon, s \vee 3}^2) d\tau, \end{aligned} \quad (3.6)$$

where C' depends only on σ' and σ'' .

Likewise, at the weighted $(s+1)$ -order, there holds

$$\begin{aligned} & T_i \varepsilon \|\nabla n^{1,T_i}(t)\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^{1,T_i}(t)\|_{H^s}^2 + \varepsilon \|\nabla \phi^{1,T_i}(t)\|_{H^s}^2 + \varepsilon^2 \|\Delta \phi^{1,T_i}(t)\|_{H^s}^2 \\ & \leq C'(T_i \varepsilon \|\nabla n^{1,T_i}(0)\|_{H^s}^2 + \varepsilon \|\nabla \mathbf{u}^{1,T_i}(0)\|_{H^s}^2 + \varepsilon \|\nabla \phi^{1,T_i}(0)\|_{H^s}^2 + \varepsilon^2 \|\Delta \phi^{1,T_i}(0)\|_{H^s}^2) \\ & \quad + C \int_0^t T_i \varepsilon C(\varepsilon \tilde{C}) \|(n^{1,T_i}, \mathbf{u}^{1,T_i})\|_{H^{s+1}}^2 \\ & \quad + (1 + \varepsilon^2 \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,4}^2) (1 + \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon, s \vee 3}^2) d\tau, \end{aligned} \quad (3.7)$$

where C' depends only on σ' and σ'' .

Now let $\|n^{1,T_i}\|_{\varepsilon,s}^2 = \|n^{1,T_i}\|_{H^s}^2 + \varepsilon \|\nabla n^{1,T_i}\|_{H^s}^2$. Putting (3.6) and (3.7) together, we then obtain the following Gronwall type inequality:

$$\begin{aligned} & \|(\sqrt{T_i} n^{1,T_i}, \mathbf{u}^{1,T_i}, \phi^{1,T_i})(t)\|_{\varepsilon,s}^2 \leq C' C_\varepsilon(0) \\ & \quad + \int_0^t C(\varepsilon \tilde{C}) (1 + \varepsilon^3 \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,5}^3) (1 + \|(\sqrt{T_i} n^{1,T_i}, \mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,s}^2) d\tau, \end{aligned} \quad (3.8)$$

where $C_\varepsilon(0) = \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})(0)\|_{\varepsilon,s}^2$ and $s \vee 3 = s$ for $s \geq 3$. From (3.3), there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, $\varepsilon^3 \|(\mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,5}^3 \leq 1$ and $C(\varepsilon \tilde{C}) \leq C(1)$, and hence (3.8) yields

$$\begin{aligned} & \|(\sqrt{T_i} n^{1,T_i}, \mathbf{u}^{1,T_i}, \phi^{1,T_i})(t)\|_{\varepsilon,s}^2 \leq C_2 C_\varepsilon(0) \\ & \quad + C_2 \int_0^t (1 + \|(\sqrt{T_i} n^{1,T_i}, \mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{\varepsilon,s}^2) d\tau, \end{aligned}$$

where $C_2 = \max\{C', 2C(1)\}$. Then invoking Lemma 2.2 and following the proof of Theorem 1.3 in Section 2.5, we obtain the estimates uniform both in ε and T_i for $\|(n^{1,T_i}, \mathbf{u}^{1,T_i}, \phi^{1,T_i})\|_{H^s}$. In deed, we obtain that $\|n^{1,T_i}\|_{H^s}^2 + T_i \varepsilon \|\nabla n^{1,T_i}\|_{H^s}^2$ is uniformly bounded. In particular, for every $T' < T$, $\varepsilon^{-1}(n^\varepsilon - n^0)$ and $\varepsilon^{-1}(\mathbf{u}^\varepsilon - \mathbf{u}^0)$ are bounded in $L^\infty([0, T']; H^s)$ and $L^\infty([0, T']; \mathbf{H}^s)$, respectively, uniformly in ε and T_i for ε small enough for some $s < s'$. This completes the proof.

Appendix A. Commutator estimates. For the reader's convenience, we give two important inequalities that are widely used throughout this paper [13, Lemma X1 and Lemma X4].

LEMMA A.1. Let α be any multiindex with $|\alpha| = k$ and $p \in (1, \infty)$. Then there exists some constant $C > 0$ such that

$$\begin{aligned} \|\partial_x^\alpha(fg)\|_{L^p} &\leq C\{\|f\|_{L^{p_1}}\|g\|_{\dot{H}^{k,p_2}} + \|f\|_{\dot{H}^{k,p_3}}\|g\|_{L^{p_4}}\}, \\ \|[\partial_x^\alpha, f]g\|_{L^p} &\leq C\{\|\nabla f\|_{L^{p_1}}\|g\|_{\dot{H}^{k-1,p_2}} + \|f\|_{\dot{H}^{k,p_3}}\|g\|_{L^{p_4}}\}, \end{aligned} \quad (\text{A.1})$$

where $f, g \in \mathcal{S}$, the Schwartz class and $p_2, p_3 \in (1, +\infty)$ are such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

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