ON HOW TO USE DRIFT TO PUSH THE SPECTRAL GAP OF A DIFFUSION ON S^2 TO INFINITY

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Abstract. We prove that on the sphere S^2 , one can find a sequence of divergence free vector fields b_n with the property that the spectral gap of the operators $A_{b_n} = \Delta + b_n \cdot \nabla$ goes to infinity. The proof uses some suitable adapted Faber-Krahn type inequality for functions which are in the kernel of the operator $b_n \cdot \nabla$. Questions of this type arise when trying to accelerate Markov Monte Carlo methods by adding convergence enhancing motion.

Introduction.

0.1. *Motivation:* Let M be a compact d -dimensional Riemannian manifold. A natural task is to study the behavior of the spectral gap ρ_b for the antisymmetric perturbations of the Laplacian of the form $A_b = \Delta + b \cdot \nabla$, where b is a divergence free vector field. Note that all diffusions generated by operators of the above form have the same invariant measure, which is the Riemannian volume on M . The spectral gap is an important indicator for understanding the speed at which the diffusion generated by A_b converges to its equilibrium distribution. For example it is used when one wants to measure the performance of Markov chain Monte Carlo algorithms. In [9] it was shown that the optimal algorithm is to be found in the class of non-reversible diffusions where the vector field b is non-zero and the generator A_b is not self-adjoint. In this context it is then important to understand how a change in the vector field b affects the convergence to equilibrium. In particular it is interesting to see if one can find vector fields which push the spectral gap of A_b over a prescribed value of arbitrary size. If the compact manifold

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has dimension larger or equal to three, then it is known that there exists on this manifold some smooth measure preserving mixing flows (see [11]). For vector fields b generating mixing flows the operator $b \cdot \nabla$ has no non-constant eigenfunctions, and it then follows that the spectral gap of A_{ab} goes to infinity as the real parameter a grows to infinity (see [4], [7]). Thus for dimensions larger than two one can find divergence free vector fields which yield a spectral gap of arbitrary size. However, those vector fields are not very explicit and are therefore not very practicable for application. Our goal is to find simple construction principles for the construction of vector fields with strong capacity to accelerate diffusivity. Note also that in the two dimensional case there are no mixing flows and one has to find some different line of arguments in this case. A result was recently published by Hwang and Pai which says that on N -torus the spectral gap can exceed any prescribed value after choice of a suitable vector field b (see [10]). Their result uses Fourier series arguments and cannot be extended to other two dimensional manifolds.

In this paper we study the case of the two dimensional sphere S^2 and construct explicit vector fields b in such a way that the operator A_b has a spectral gap which is larger than a prescribed size. The case of the sphere is quite special, since by the hairball theorem every vector field vanishes in at least one point. However, we will see that this is not an obstruction for the construction of accelerating vector fields. Our method is based on the possibility of finding vector fields with flow lines of arbitrary length on the sphere. This implies that the functions which are constant along those flow lines have large integrated square gradients. It then follows from a result from Franke, Hwang, Pai and Sheu (see [7]) that the spectral gap can be pushed to arbitrary size by varying the vector field.

0.2. *Formal definitions.* Let $S^2 = \{x \in \mathbb{R}^3; |x| = 1\}$ be the unit sphere in $\mathbb{R}^3$. We denote the area-measure on S^2 by ω_{S^2} . For $f \in C^\infty(S^2)$ we can define the operator

$$A_b f = \Delta f + b \cdot \nabla f,$$

where Δ is the Laplacian operator on S^2 and b is a divergence free vector field. The spectrum of this operator is located in the negative half space of the complex plane, and the associated eigenfunctions are complex valued. We therefore introduce the following Hilbert space of mean-zero complex-valued functions

$$H = \left\{ f = f_1 + i f_2; f_1, f_2 \in L^2(S^2, \omega_{S^2}) : \int f_1 d\omega_{S^2} = \int f_2 d\omega_{S^2} = 0 \right\}$$

with scalar product $\langle f, g \rangle = \int f \bar{g} d\omega_{S^2}$. Further, we define the Sobolev space of mean-zero functions

$$H^1 = \left\{ f = f_1 + i f_2 \in H; \int |\nabla f_1|^2 d\omega_{S^2} + \int |\nabla f_2|^2 d\omega_{S^2} < \infty \right\}$$

with scalar product $\langle f, g \rangle_1 = \int \nabla f \overline{\nabla g} d\omega_{S^2}$. The resulting norms of functions $f \in H$, resp. $f \in H^1$, will be denoted $\|f\|_{L^2}$, resp. $\|f\|_{H^1}$.

0.3. *Main result.* In this article we will study the problem of the spectral gap:

$$\rho_b = -\sup \left\{ \operatorname{Re}(z); z \in \operatorname{sp}(A_b) \setminus \{0\} \right\}.$$

Here $\text{sp}(A_b)$ denotes the spectrum of the operator A_b . We will prove the following theorem:

THEOREM 0.1. There exists a sequence of divergence free vector fields $(b_n)_{n \in \mathbb{N}}$ on S^2 such that one has

$$\lim_{n \rightarrow \infty} \rho_{b_n} = \infty.$$

The proof of this result is presented at the end of the paper.

1. Construction of the vector fields. In this section we will construct an explicit sequence of vector fields b_n with periodic flow lines on the sphere. The particularity of those vector fields will be that the length of the associated flow lines will grow to infinity as n becomes large. In order to do the construction we will need cylindrical coordinates.

1.1. *Cylindrical coordinates of S^2 .* In order to construct the vector fields on the sphere we use the following coordinates for the unit sphere S^2 :

$$F : \mathfrak{Z}^2 =]-1, 1[\times [0, 2\pi[\rightarrow S^2; (z, \theta) \mapsto (\sqrt{1-z^2} \sin \theta, \sqrt{1-z^2} \cos \theta, z).$$

This map is a bijection from $\mathfrak{Z}^2$ to $S^2 \setminus \{\mathbf{n}, \mathbf{s}\}$ where $\mathbf{n} = (0, 0, 1)$ and $\mathbf{s} = (0, 0, -1)$. We denote the surface measure on one of the cylinder $\mathfrak{Z}^2$ by $\omega_{\mathfrak{Z}^2}$. In fact, $\omega_{\mathfrak{Z}^2}$ is just the Lebesgue measure on $] -1, 1[\times [0, 2\pi[$. It is well known that the above identification F satisfies $\omega_{S^2} = \omega_{\mathfrak{Z}^2} \circ F^{-1}$. For a Riemannian manifold M we denote the arc length of a curve $\gamma : [a, b] \rightarrow M$ by

$$\ell_M(\gamma) = \int_{\gamma} d\ell_M.$$

We have the following local inequality between the length elements $d\ell_{S^2}$ and $d\ell_{\mathfrak{Z}^2}$:

$$d\ell_{S^2} = \sqrt{(dz)^2 + (1-z^2)(d\theta)^2} \geq \delta_z \sqrt{(dz)^2 + (d\theta)^2} = \delta_z d\ell_{\mathfrak{Z}^2}$$

with $\delta_z = \sqrt{1-z^2}$.

1.2. *Construction of the trajectories.* For all $n \in \mathbb{N}$, let us consider the $\mathcal{C}^1$ -function:

$$\gamma_n :]-1, 1[\times [0, 2\pi[\rightarrow]-1, 1[; (z, \theta) \mapsto \frac{(1-z^2)^2}{2} \sin(n\theta) + z^2 \text{sgn}(z).$$

(1) For $z \in]-1, 1[$ we note the graph of the function $\theta \mapsto \gamma_n(z, \theta)$ by

$$\Gamma_z^{(n)} = \left\{ (\theta, \gamma_n(z, \theta)); \theta \in [0, 2\pi[\right\}.$$

(2) We note the images of $\Gamma_x^{(n)}$ with respect to F on the sphere S^2 by

$$\Upsilon_z^{(n)} = \left\{ F(\gamma_n(z, \theta), \theta); \theta \in [0, 2\pi[\right\}.$$

REMARK 1.1. Since for all $\theta \in [0, 2\pi[$ and $z \in]-1, 1[\setminus \{0\}$ the derivative

$$\frac{d}{dz} \gamma_n(z, \theta) = (-2z + 2z^3) \sin(n\theta) + 2|z|$$

is positive it follows that the graphs Υ_z are disjoint for different values of z .

For $z \in]-1, 1[$ define $\widehat{D}_z^{(n)} = \{(\tilde{z}, \theta) \in \mathfrak{Z}^2; \tilde{z} > \gamma_n(z, \theta)\}$ and $\widehat{C}_z^{(n)} = F(\widehat{D}_z^{(n)})$. Since the map $F : \mathfrak{Z}^2 \rightarrow S^2$ is area preserving, we have $\omega_{\mathfrak{Z}^2}(\widehat{D}_z^{(n)}) = \omega_{S^2}(\widehat{C}_z^{(n)})$. Note also that those areas do not depend on $n \in \mathbb{N}$. Obviously $\partial \widehat{D}_z^{(n)} = \Gamma_z^{(n)}$ and $\partial \widehat{C}_z^{(n)} = \Upsilon_z^{(n)}$. For later use, we will need some control on the arc length of the boundary of $\widehat{C}_z^{(n)}$. This is provided by the following lemma.

LEMMA 1.2. For all $z \in]-1, 1[$ we have that $\ell_{S^2}(\Upsilon_z^{(n)}) \geq \delta_{\frac{1+z^4}{2}} \sqrt{\frac{16+n^2(1-z^2)^4}{3}}$.

Proof. For all $z \in]-1, 1[$, we calculate the infinitesimal arc length element $d\ell_{\mathfrak{Z}^2}$ on $\mathfrak{Z}^2$ of the graph $\Gamma_x^{(n)}$:

$$\begin{aligned} (d\ell_{\mathfrak{Z}^2}(\Gamma_z^{(n)}))^2 &= \left(\frac{(1-z^2)^2}{2} \right)^2 \left(\sin(n(\theta + d\theta)) - \sin(n\theta) \right)^2 + (d\theta)^2 \\ &= \left(1 + n^2 \left(\frac{(1-z^2)^2}{2} \right)^2 \cos^2(n\theta) \right) (d\theta)^2. \end{aligned}$$

So it follows that

$$\begin{aligned} \ell_{\mathfrak{Z}^2}(\Gamma_z^{(n)}) &= \int_0^{2\pi} \sqrt{1 + n^2 \left(\frac{(1-z^2)^2}{2} \right)^2 \cos^2(n\theta)} d\theta \\ &= 2n \int_0^{\frac{\pi}{n}} \frac{\sqrt{1 + n^2 \left(\frac{(1-z^2)^2}{2} \right)^2 \cos^2(n\theta)}}{\sqrt{1 - \cos^2(n\theta)}} \sin(n\theta) d\theta \\ &= 2 \int_{-1}^1 \sqrt{\frac{1 + n^2 \left(\frac{(1-z^2)^2}{2} \right)^2 u^2}{1 - u^2}} du \\ &\geq 4 \int_{\frac{1}{2}}^1 \sqrt{\frac{1 + n^2 \left(\frac{(1-z^2)^2}{2} \right)^2 \left(\frac{1}{2} \right)^2}{1 - \left(\frac{1}{2} \right)^2}} du \\ &= \sqrt{\frac{16 + n^2(1-z^2)^4}{3}}. \end{aligned}$$

Let's first assume that $z \geq 0$. Then the curve $\Upsilon_z^{(n)}$ always stays below the level $\frac{1+z^4}{2}$. Since we have $d\ell_{S^2} \geq \delta_z d\ell_{\mathfrak{Z}^2}$ for all points on the curve $\Upsilon_z^{(n)}$ the statement of the proposition follows for $z \geq 0$. By symmetry the statement also holds for $z \leq 0$. $\square$

REMARK 1.3. The area above the curve $\Gamma_z^{(n)}$ in $\mathfrak{Z}^2$ is equal to

$$\begin{aligned} 4\pi - \int_0^{2\pi} (\gamma_n(z, \theta) + 1) d\theta &= 4\pi - \int_0^{2\pi} \left(\frac{(1-z^2)^2}{2} \sin(n\theta) + z^2 \operatorname{sgn}(z) + 1 \right) d\theta \\ &= 2\pi(1 - z^2 \operatorname{sgn}(z)). \end{aligned}$$

If for a given $c \in [0, 4\pi]$ we define $z(c) = \sqrt{1 - \frac{c}{2\pi}} \operatorname{sgn}(1 - \frac{c}{2\pi})$, the curve $\Upsilon_{z(c)}^{(n)}$ encloses the volume c in the component containing the north pole $\mathbf{n}$.

We denote by $\Omega_c \subset S^2$ the ball of volume c around the north pole $\mathbf{n}$.

LEMMA 1.4. As $c \rightarrow 0$ one has $\ell_{S^2}(\Upsilon_{z(c)}) \sim \ell_{S^2}(\partial\Omega_c)$.

Proof. Since the area on S^2 which is located above the curve Υ_z equals the area located above the level $x_3 = z^2$, it is sufficient to prove that as $z \rightarrow 1$ one has

$$\ell_{S^2}(\Upsilon_z) \sim \ell_{S^2}(S^2 \cap \{x \in \mathbb{R}^3; x_3 = z^2\}) = 2\pi\sqrt{1-z^4}.$$

Note that the curve Υ_z never moves above the level $x_3 = \frac{1+z^4}{2}$ and below the level $x_3 = 2z^2 - \frac{1+z^4}{2}$ as it performs one rotation around the x_3 -axis. On one revolution it oscillates exactly n times between those two levels. It follows that $\ell_{S^2}(\Upsilon_z)$ has the lower bound $\ell_{S^2}(S^2 \cap \{x; x_3 = \frac{1+z^4}{2}\})$ and the upper bound $\ell_{S^2}(S^2 \cap \{x; x_3 = 2z^2 - \frac{1+z^4}{2}\}) + n(\arcsin \frac{1+z^4}{2} - \arcsin(2z^2 - \frac{1+z^4}{2}))$.

Clearly $\sqrt{1 - (\frac{1+z^4}{2})^2} \sim \sqrt{1-z^4}$ and $\sqrt{1 - (2z^2 - \frac{1+z^4}{2})^2} \sim \sqrt{1-z^4}$ when $z \rightarrow 1$. Thus $\ell_{S^2}(S^2 \cap \{x; x_3 = \frac{1+z^4}{2}\}) \sim \sqrt{1-z^4} \sim \ell_{S^2}(S^2 \cap \{x; x_3 = z^2 - \frac{1+z^4}{2}\})$. To finish the proof of the statement it is sufficient to show that

$$\lim_{z \rightarrow 1} \frac{\arcsin(\frac{1+z^4}{2}) - \arcsin(2z^2 - \frac{1+z^4}{2})}{\sqrt{1-z^4}} = 0.$$

Using L'Hôpital's rule, this is equivalent to

$$\frac{\frac{2z^2}{\sqrt{1 - (\frac{1+z^4}{2})^2}} - \frac{4z - 2z^3}{\sqrt{1 - (2z^2 - \frac{1+z^4}{2})^2}}}{\frac{2z^3}{\sqrt{1-z^4}}} \rightarrow 0 \quad \text{as } z \rightarrow 1.$$

This follows, since our previous considerations show that the two quotients

$$\sqrt{\frac{1-z^4}{1 - (\frac{1+z^4}{2})^2}} \quad \text{and} \quad \sqrt{\frac{1-z^4}{1 - (2z^2 - \frac{1+z^4}{2})^2}}$$

converge toward the same limit as $z \rightarrow 1$. □

LEMMA 1.5. The function

$$L_{b_n} :]0, 4\pi[\rightarrow \mathbb{R}; \quad c \mapsto \ell_{S^2}(\Upsilon_{z(c)}^{(n)})$$

satisfies that

$$\inf_{0 < c < 2\pi} \frac{L_{b_n}(c)}{c} \xrightarrow{n \rightarrow \infty} \infty.$$

Proof. We have by Lemma 1.2 that

$$\begin{aligned} L_{b_n}(c) &\geq \sqrt{1 - \left(\frac{1 + (z(c))^4}{2}\right)^2} \sqrt{\frac{16 + n^2(1 - (z(c))^2)^4}{3}} \\ &= \sqrt{\frac{c}{\pi} - \frac{c^2}{2\pi^2} + \frac{c^3}{8\pi^3} - \frac{c^4}{64\pi^4}} \sqrt{\frac{256\pi^4 + n^2c^4}{48\pi^4}}. \end{aligned}$$

We separate the task into two sub-problems as follows:

$$\inf_{0 < c < 2\pi} \frac{L_{b_n}(c)}{c} = \min \left(\inf_{0 < c \leq \frac{1}{\sqrt[n]{n}}} \frac{L_{b_n}(c)}{c}, \inf_{\frac{1}{\sqrt[n]{n}} < c < 2\pi} \frac{L_{b_n}(c)}{c} \right).$$

The first infimum satisfies

$$\inf_{0 < c \leq \frac{1}{\sqrt[4]{n}}} \frac{L_{b_n}(c)}{c} \geq \sqrt{\frac{\sqrt[4]{n}}{\pi} - \frac{1}{2\pi^2} + 0 - \frac{1}{64\pi^4\sqrt{n}}} \sqrt{\frac{256\pi^4 + 0}{48\pi^4}}.$$

This expression goes to infinity as $n \rightarrow \infty$. In order to treat the second infimum, we observe that the function

$$a(c) = \frac{1}{c\pi} - \frac{1}{2\pi^2} + \frac{c}{8\pi^3} - \frac{c^2}{64\pi^4}$$

can be written as a sum of two functions:

$$a_1(c) = \frac{1}{c\pi} - \frac{1}{2\pi^2} \quad \text{and} \quad a_2(c) = \frac{c}{8\pi^3} - \frac{c^2}{64\pi^4}.$$

Note that $a_1(c)$ is continuous and decreases strictly from infinity to zero on the interval $[0, 2\pi]$ and that a_2 is also continuous and increases strictly from zero to the value $\frac{1}{4\pi^2} - \frac{1}{16\pi^2}$ on the interval $[0, 2\pi]$. It follows that there is a lower bound $\delta > 0$ for the function $a(c)$ on the interval $[0, 2\pi]$. This implies

$$\inf_{\frac{1}{\sqrt[4]{n}} < c < 2\pi} \frac{L_{b_n}(c)}{c} \geq \sqrt{\delta} \sqrt{\frac{256\pi^4 + n}{48\pi^4}}.$$

Since this also diverges to infinity as $n \rightarrow \infty$ the statement follows. $\square$

1.3. *Construction of the vector field.* For a function $U : S^2 \rightarrow \mathbb{R}$ the gradient has the following representation in cylindrical coordinates:

$$\nabla U = (1 - z^2)\partial_z U \partial_z + \frac{1}{1 - z^2}\partial_\theta U \partial_\theta.$$

The vector field $\nabla^\perp U = \partial_\theta U \partial_z - \partial_z U \partial_\theta$ satisfies the properties $\operatorname{div}(\nabla^\perp U) = 0$ and $g(\nabla^\perp U, \nabla U) = 0$ where g is the Riemannian metric on S^2 . It follows from the monotonicity which was established in Remark 1.1 that there exists a function $\alpha_n :]-1, 1[\times [0, 2\pi[\rightarrow]-1, 1[$ which satisfies $\gamma_n(\alpha_n(z, \theta), \theta) = z$ for all $(z, \theta) \in]-1, 1[\times [0, 2\pi[$. Note that α_n is constant on every curve $\Upsilon_z^{(n)}$. Now, we consider the vector fields b_n^0 on S^2 defined by

$$b_n^0(z, \theta) = \nabla^\perp \alpha_n(z, \theta).$$

By construction b_n^0 is a divergence free vector field which generates a flow with trajectories $\Upsilon_z^{(n)}$.

Note that for every function $V \in \ker(b_n^0 \cdot \nabla) \cap \mathcal{C}^1(S^2)$ we have $\operatorname{div}(Vb_n^0) = 0$. The vector field $b_n = Vb_n^0$ has the same trajectories as b_n^0 . Those trajectories are the level sets of the function α_n which are closed sets. Let $(\phi_t)_{t \in \mathbb{R}}$ be the flow generated by b_n . For $(z, \theta) \in]-1, 1[\times [0, 2\pi[$ we denote by $\tau_{(z, \theta)}$ the return time for the flow $(\phi_t)_{t \geq 0}$ to the point (z, θ) . This means that we have

$$\phi_{\tau_{(z, \theta)}}(z, \theta) = (z, \theta).$$

LEMMA 1.6. For all $n \in \mathbb{N}$, there exists a function $V \in \ker(b_n^0 \cdot \nabla) \cap \mathcal{C}^1(S^2)$ such that the return time into the point $(z, 0)$ of the flow $(\phi_t)_{t \in \mathbb{R}}$ generated by the vector field $b_n = Vb_n^0$ is equal to $\sqrt{1 - z^2}/2$ for all $z \in]-1, 1[$.

Proof. We introduce the flows $(\phi_t)_{t \in \mathbb{R}}$ and $(\phi_t^0)_{t \in \mathbb{R}}$ generated by b_n and b_n^0 through the equations $\dot{\phi}_t^0 = b_n^0(\phi_t^0)$ and $\dot{\phi}_t = b_n(\phi_t)$. This implies the relation $\phi_t = \phi_{tV}^0$ for all $t \in \mathbb{R}$. If we denote by $\tau_{(z,\theta)}^0$ the return time of $(\phi_t^0)_{t \in \mathbb{R}}$, then we have that $\phi_{\tau_{(z,\theta)}^0}^0(z, \theta) = (z, \theta)$. From this we obtain that

$$\phi_{\tau_{z,\theta}^0/V(z,\theta)}(z, \theta) = \phi_{\tau_{(z,\theta)}^0}^0(z, \theta) = (z, \theta),$$

which means that $\tau_{(z,\theta)} = \frac{\tau_{(z,\theta)}^0}{V(z,\theta)}$ is the return time of $(\phi_t)_{t \in \mathbb{R}}$. As a consequence the function

$$V(z, \theta) = \frac{\tau_{(z,\theta)}^0}{\sqrt{1 - \alpha_n^2(z, \theta)/2}} \quad \forall z \in]-1, 1[, \theta \in [0, 2\pi[$$

satisfies the requirements of the statement. Note here that $z \mapsto \tau_{(z,\theta)}^0$ is continuously differentiable (see [8]) and also in $\ker(b_n \cdot \nabla)$. The result follows now since we have $\alpha_n(z, 0) = z$ for all $z \in]-1, 1[$. $\square$

From now on we will work with the sequence of vector fields $b_n = Vb_n^0$.

2. The rearrangement method for functions from the kernel. In this section, let b be one of the vectors fields constructed in the previous section. In order to keep notation simple we prefer to suppress the n in the notation concerning objects related to b_n ; i.e., we will simply write γ for γ_n , α for α_n , etc. Further, let $(\phi_t)_{t \in \mathbb{R}}$ be the flow generated by the vector field b .

2.1. *Some results concerning continuity along the trajectory.* The following weak eigenspace for the anti-symmetric operator $b \cdot \nabla$ will be important later:

$$H_\mu^1 = \left\{ f \in H^1; \int f(x) b \cdot \nabla \psi(x) dx = i\mu \int f(x) \psi(x) dx, \forall \psi \in C^1(S^2) \right\}.$$

Note that although those spaces depend on the vector field b , we opt to use a simplified notation for the sake of shorter symbols. The next proposition will help to restrict our considerations to functions which are continuous along the trajectory. It is based on the following lemma.

LEMMA 2.1. If $f \in H_\mu^1$, then for every $t \in \mathbb{R}$ there holds

$$f \circ \phi_{-t} = e^{i\mu t} f \quad \omega_{S^2}\text{-a.e.}$$

Proof. For $\psi \in C^1(S^2)$, we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \int_{S^2} f \circ \phi_{-t} \psi dx &= \frac{d}{dt} \Big|_{t=0} \int_{S^2} f \psi \circ \phi_t dx = \int_{S^2} f \frac{d}{dt} \Big|_{t=0} (\psi \circ \phi_t) dx \\ &= \int_{S^2} f b \cdot \nabla \psi dx = i\mu \int_{S^2} f(x) \psi(x) dx. \end{aligned}$$

Now, for $t = t_0$, we have

$$\frac{d}{dt} \Big|_{t=t_0} \int_{S^2} f \circ \phi_{-t} \psi dx = \frac{d}{dt} \Big|_{t=0} \int_{S^2} f \psi \circ \phi_{t_0} \circ \phi_t dx = i\mu \int_{S^2} f(x) \psi \circ \phi_{t_0}(x) dx.$$

In the last equality, we used that $\psi \circ \phi_{t_0}$ is in $\mathcal{C}^1(S^2)$. This implies that the function

$$u(t) = \int_{S^2} f \circ \phi_{-t}(x) \psi(x) dx$$

satisfies the differential equation

$$\frac{d}{dt} u(t) = i\mu u(t).$$

We obtain that

$$u(t) = e^{i\mu t} u(0) = e^{i\mu t} \int_{S^2} f(x) \psi(x) dx.$$

As a consequence, one has for all $\psi \in \mathcal{C}^1(S^2)$,

$$\int_{S^2} (f \circ \phi_{-t}(x) - e^{i\mu t} f(x)) \psi(x) dx = 0.$$

Finally, we have $f \circ \phi_{-t}(x) = e^{i\mu t} f(x)$ for ω_{S^2} -almost all $x \in S^2$. $\square$

Let $\xi_\epsilon : \mathbb{R} \mapsto [0, \infty[$ be a smooth function such that $\xi_\epsilon(s) = 0$ if $s \notin]-\epsilon, \epsilon[$ and $\int_{-\epsilon}^\epsilon \xi(s) ds = 1$. For $f \in H_\mu^1$ and $x \in S^2$, we consider $f_\epsilon(x) = \int_{-\epsilon}^\epsilon f \circ \phi_{-s} \xi_\epsilon(s) ds$.

PROPOSITION 2.2. For any $f \in H_\mu^1$ the function f_ϵ satisfies that $\|f_\epsilon - f\|_{H^1} \rightarrow 0$ as $\epsilon \rightarrow 0$. Moreover, for ω_{S^2} -almost all $x \in S^2$, we have

$$f_\epsilon \circ \phi_{-t}(x) = e^{i\mu t} f_\epsilon(x), \forall t \in \mathbb{R}.$$

Proof. We will first prove that $\|f_\epsilon - f\|_{H^1} \rightarrow 0$ as $\epsilon \rightarrow 0$. We have

$$\begin{aligned} \|\nabla(f_\epsilon - f)\|_{L^2}^2 &= \int_{S^2} |\nabla(f_\epsilon(x) - f(x))|^2 dx \\ &= \int_{S^2} \left| \nabla \left\{ \int_{-\epsilon}^\epsilon f \circ \phi_{-s}(x) \xi_\epsilon(s) ds \right\} \right|^2 dx + \int_{S^2} |\nabla f(x)|^2 dx \\ &\quad - 2 \int_{S^2} \nabla \left\{ \int_{-\epsilon}^\epsilon f \circ \phi_{-s}(x) \xi_\epsilon(s) ds \right\} \nabla f(x) dx \\ &= \int_{-\epsilon}^\epsilon \int_{-\epsilon}^\epsilon \xi_\epsilon(s) \xi_\epsilon(s') \int_{S^2} \nabla(f \circ \phi_{-s})(x) \nabla(f \circ \phi_{-s'})(x) dx ds ds' \\ &\quad + \int_{S^2} |\nabla f(x)|^2 dx - 2 \int_{-\epsilon}^\epsilon \xi_\epsilon(s) \int_{S^2} \nabla(f \circ \phi_{-s})(x) \nabla f(x) dx ds. \end{aligned}$$

Note that one can use the change of variable formula for weakly differentiable functions (see [12], p. 49) to see that $(s, s') \mapsto \int_{S^2} \nabla(f \circ \phi_{-s}) \nabla(f \circ \phi_{-s'}) dx$ and that $s \mapsto \int_{S^2} \nabla(f \circ \phi_{-s}) \nabla f dx$ are continuous. Moreover, $s \mapsto \xi_\epsilon(s)$ is a probability density concentrating around zero as $\epsilon \rightarrow 0$. So we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \|\nabla(f_\epsilon - f)\|_{L^2}^2 &= \int_{S^2} \nabla(f \circ \phi_0)(x) \nabla(f \circ \phi_0)(x) dx + \int_{S^2} |\nabla f(x)|^2 dx \\ &\quad - 2 \int_{S^2} \nabla(f \circ \phi_0)(x) \nabla f(x) dx \\ &= 0. \end{aligned}$$

It is easy to prove, with similar arguments, that $\|f_\epsilon - f\|_{L^2}$ goes to zero when ϵ goes to zero. This proves that $\|f_\epsilon - f\|_{H^1} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Now, we will prove that the function f_ϵ is in H_μ^1 . Indeed, let $\psi \in \mathcal{C}^1(S^2)$:

$$\begin{aligned}
 \int_{S^2} f_\epsilon(x) \mathbf{b} \cdot \nabla \psi(x) dx &= \int_{S^2} \int_{-\epsilon}^\epsilon f \circ \phi_{-s}(x) \xi_\epsilon(s) ds \mathbf{b} \cdot \nabla \psi(x) dx \\
 &= \int_{-\epsilon}^\epsilon \xi_\epsilon(s) \int_{S^2} f \circ \phi_{-s}(x) \mathbf{b} \cdot \nabla \psi(x) dx ds \\
 &= \int_{-\epsilon}^\epsilon \xi_\epsilon(s) \int_{S^2} f(x) \mathbf{b} \cdot \nabla (\psi \circ \phi_s)(x) dx ds \\
 &= \int_{-\epsilon}^\epsilon \xi_\epsilon(s) i\mu \int_{S^2} f(x) (\psi \circ \phi_s)(x) dx ds \\
 &= i\mu \int_{S^2} f_\epsilon(x) \psi(x) dx.
 \end{aligned}$$

In the third equality, we used that

$$\begin{aligned}
 \int_{S^2} f \circ \phi_{-s}(x) \mathbf{b} \cdot \nabla \psi(x) dx &= \left. \frac{d}{dt} \right|_{t=0} \int_{S^2} f \circ \phi_{-s}(x) \psi \circ \phi_t(x) dx \\
 &= \left. \frac{d}{dt} \right|_{t=0} \int_{S^2} f(x) \psi \circ \phi_s \circ \phi_t(x) dx \\
 &= \int_{S^2} f(x) \mathbf{b} \cdot \nabla (\psi \circ \phi_s)(x) dx.
 \end{aligned}$$

Therefore it follows from $f \in H_\mu^1$ and Lemma 2.1 that

$$f_\epsilon \circ \phi_{-t} = e^{i\mu t} f_\epsilon \quad \omega_{S^2}\text{-a.e.}$$

We will see that $t \mapsto f_\epsilon \circ \phi_t$ is continuous, and so finally we conclude from this that for ω_{S^2} almost all $x \in S^2$ holds:

$$f_\epsilon \circ \phi_t(x) = e^{i\mu t} f_\epsilon(x) \quad \forall t \in \mathbb{R}.$$

To finish the proof, let us prove that $t \mapsto f_\epsilon \circ \phi_t$ is continuous. We prove the right continuity and note that the left continuity can be obtained by similar arguments. Let $t > t_0$ and $x \in S^2$ be fixed, so we have

$$\begin{aligned}
 &\left| f_\epsilon \circ \phi_t(x) - f_\epsilon \circ \phi_{t_0}(x) \right| \\
 &= \left| \int_{-\epsilon}^\epsilon f \circ \phi_t \circ \phi_{-s}(x) \xi_\epsilon(s) ds - \int_{-\epsilon}^\epsilon f \circ \phi_{t_0} \circ \phi_{-s}(x) \xi_\epsilon(s) ds \right| \\
 &= \left| \int_{-\epsilon-t}^{\epsilon-t} f \circ \phi_{-s}(x) \xi_\epsilon(s+t) ds - \int_{-\epsilon-t_0}^{\epsilon-t_0} f \circ \phi_{-s}(x) \xi_\epsilon(s+t_0) ds \right| \\
 &\leq \int_{\epsilon-t}^{\epsilon-t_0} |\xi_\epsilon(s+t_0) f \circ \phi_{-s}(x)| ds + \int_{-\epsilon-t}^{-\epsilon-t_0} |\xi_\epsilon(s+t) f \circ \phi_{-s}(x)| ds \\
 &\quad + \int_{-\epsilon-t_0}^{\epsilon-t} |\xi_\epsilon(s+t) - \xi_\epsilon(s+t_0)| |f \circ \phi_{-s}(x)| ds.
 \end{aligned}$$

Let us now show that

$$\int_{-\epsilon-t}^{-\epsilon-t_0} |\xi_\epsilon(s+t)f \circ \phi_{-s}(x)| ds \longrightarrow 0 \quad \text{as } t \rightarrow t_0.$$

The set $\Upsilon_x = \{\phi_t(x); t \in \mathbb{R}\}$ is a one dimensional sub-manifold of S^2 . Note that the restriction operator

$$R_\Upsilon : H^1 \rightarrow L^2(\Upsilon_x); f \mapsto f|_{\Upsilon_x}$$

is continuous (see [1, p. 114]). Since Υ_x has finite arc length, the identity map from $L^2(\Upsilon_x)$ into $L^1(\Upsilon_x)$ is also continuous. So we have as a consequence that $|\xi_\epsilon(s+t)f \circ \phi_{-s}(x)|1_{[-\epsilon-t, -\epsilon-t_0]}$ is uniformly bounded by the integrable function $s \mapsto \|\xi_\epsilon\|_{\sup}|f \circ \phi_{-s}(x)|$. By the dominated convergence theorem, we conclude that

$$\lim_{t \rightarrow t_0} \int_{-\epsilon-t}^{-\epsilon-t_0} |\xi_\epsilon(s+t)f \circ \phi_{-s}(x)| ds = 0.$$

Similar arguments show that

$$\lim_{t \rightarrow t_0} \int_{\epsilon-t}^{\epsilon-t_0} |\xi_\epsilon(s+t_0)f \circ \phi_{-s}(x)| ds = 0$$

and

$$\lim_{t \rightarrow t_0} \int_{-\epsilon-t}^{\epsilon-t} |\xi_\epsilon(s+t) - \xi_\epsilon(s+t_0)| |f \circ \phi_{-s}(x)| ds = 0.$$

This proves the continuity. $\square$

REMARK 2.3. For $f \in H_0^1$ it follows from the previous proposition that f_ϵ is constant along the trajectories of the flow $(\phi_t)_{t \in \mathbb{R}}$. It follows from this that there exists some weakly differentiable function $f_\epsilon^0 :]-1, 1[\rightarrow \mathbb{R}$ with $f_\epsilon = f_\epsilon^{(0)} \circ \alpha$.

2.2. *The comparison manifold.* In order to obtain some control on the gradients of functions from H_0^1 , we will need some suitable comparison manifold, which is constructed through a deformation procedure from the unit sphere S^2 . The function

$$G_{S^2} : [-1, 1] \rightarrow \mathbb{R}_0^+ : r \mapsto \omega_{S^2}(S^2 \cap \{x_3 \geq r\})$$

is called the polar cap area function of S^2 . The polar cap area function is strictly decreasing on $[-1, 1]$. For $c \in [0, 4\pi]$ the open set

$$\Omega_c := S^2 \cap \{x_3 > G_{S^2}^{-1}(c)\}$$

is called the polar cap of volume c . The function

$$L_{S^2} : [0, 4\pi] \rightarrow \mathbb{R}_0^+ : c \mapsto \ell_{S^2}(\partial\Omega_c)$$

is called the latitude function of S^2 . For every vector field b constructed in the previous section every trajectory cuts the sphere into two connected parts, one containing the north pole $\mathbf{n}$ and the other one containing the south pole $\mathbf{s}$. There exists a unique trajectory $\Upsilon_{z(c)}$ with the property that the volume of the northern part equals c . We define $L_b(c) = \ell_{S^2}(\Upsilon_{z(c)})$.

PROPOSITION 2.4. There exists a Riemannian metric g^* on S^2 such that the resulting Riemannian manifold (M^*, g^*) satisfies the following four properties:

- (1) $\omega_{M^*}(M^*) = 4\pi$.

- (2) Elements of $SO(3)$ fixing the x_3 -axis act as isometry on M^* .
- (3) Reflection with respect to the x_1x_2 -plane acts as an isometry on M^* .
- (4) $\ell_{M^*}(\partial\Omega_c^*) = L_b(c)$ for all $c \in [0, 4\pi]$, where ℓ_{M^*} is the arc length measure on M^* and Ω_c^* is the polar cap with volume $\omega_{M^*}(\Omega_c^*) = c$ around $\mathbf{n}^* = (0, 0, 1)$.

REMARK 2.5. The metric g^* depends on the vector field $\mathbf{b}$ through the function L_b . We will call the Riemannian manifold (M^*, g^*) from Proposition 2.4 a comparison manifold for the vector field $\mathbf{b}$.

Proof. To construct the comparison manifold we follow the method given in [6]. We start with the canonical metric g on S^2 and transform it to a new metric g^* . Let Ω_c be the polar cap with volume c around the north pole. Its boundary $\partial\Omega_c = \{x \in S^2 \subset \mathbb{R}^3, x_3 = G_{S^2}^{-1}(c)\}$ is a smooth curve. For some point $x \in \partial\Omega_c$, we can decompose any tangential vector V into a part $V^\parallel$ tangent to $\partial\Omega_c$ and a part $V^\perp$ orthogonal to $\partial\Omega_c$. The metric g now satisfies, for any two tangential vectors V and W , that

$$g(V, W) = g(V^\perp, W^\perp) + g(V^\parallel, W^\parallel).$$

We define the new metric g^* through

$$g^*(V, W) = \left(\frac{L_b(c)}{\ell_{S^2}(\partial\Omega_c)} \right)^2 g(V^\parallel, W^\parallel) + \left(\frac{\ell_{S^2}(\partial\Omega_c)}{L_b(c)} \right)^2 g(V^\perp, W^\perp).$$

Note that it follows from Lemma 1.4 that this new metric behaves well in the poles. Finally with the same arguments as in [6], we can prove that (M^*, g^*) satisfies the four stated properties. $\square$

REMARK 2.6. Note that on M^* , we also have a north pole $\mathbf{n}^* = (0, 0, 1)$ and a south pole $\mathbf{s}^* = (0, 0, -1)$.

REMARK 2.7. One could also use the method proposed by Bérard (see [2, p. 86]) to prove the existence of (M^*, g^*) .

2.3. *Faber-Krahn type inequality.* In this paragraph, we associate to each function f from $L^1(S^2)$ a symmetrized function f^* in $L^1(M^*)$. We will see that once restricted to functions from H_0^1 , the rearrangement decreases the H^1 -norm, while it conserves the L^2 -norm. This is a version of the well known Faber-Krahn inequality.

For a measurable function $f \in H_0^1$, we define $\lambda(s) = \omega_{S^2}\{x; f(x) \geq s\}$ and $\mu(s) = \omega_{S^2}\{x; f(x) > s\}$. In addition, we define the function

$$h : [0, d_{M^*}(\mathbf{n}^*, \mathbf{s}^*)] \rightarrow [\inf f, \sup f]; \quad t \mapsto \sup\{s; \lambda(s) > \omega_{M^*}(B_t(\mathbf{n}^*))\}$$

where $B_t(\mathbf{n}^*)$ is the ball of radius t around $\mathbf{n}^*$ in (M^*, g^*) .

The function $f^*(x) = h(d_{M^*}(\mathbf{n}^*, x))$ is defined for $x \in M^*$; it will be called the polar rearrangement of f on M^* .

REMARK 2.8. The function f^* constructed above verifies, for all $s \in \mathbb{R}$, that

$$\omega_{S^2}\{x; f(x) \geq s\} = \omega_{M^*}\{x; f^*(x) \geq s\}.$$

From this follows that $\|f\|_{L^p} = \|f^*\|_{L^p}$.

REMARK 2.9. It follows from Remark 2.3 that $C^1(S^2) \cap \ker(\mathbf{b} \cdot \nabla)$ is dense in $H^1 \cap \ker(\mathbf{b} \cdot \nabla)$ with respect to the H^1 -norm.

The following result is a variant of the well known Faber-Krahn theorem for functions from $\ker(\mathbf{b} \cdot \nabla)$.

PROPOSITION 2.10. For all $f \in C^1(S^2) \cap \ker(b \cdot \nabla)$ we have that

$$\int_{S^2} |\nabla f|^2 d\omega_{S^2} \geq \int_{M^*} |\nabla f^*|^2 d\omega_{M^*}.$$

Proof. Our proof follows the classical method based on the co-area formula that can be found for example in [2] or in [3]. In the previous paragraph, we saw that functions from H_0^1 are constant on the sets $\Upsilon_{z(c)}$ which enclose the area c . By construction the boundaries of the polar caps Ω_c of area c on M^* have shorter arc length than $\Upsilon_{z(c)}$. Thus the level sets of f^* are shorter than the corresponding ones of f .

We first note that the rearrangement of a C^1 -function is not necessarily a C^1 -function again. However, the non-differentiability can only occur on level sets of f^* corresponding to critical values of f . One can use the Sard theorem to see that the set of points where f^* is not differentiable is a set of ω_{M^*} -measure zero. It follows from the co-area formula (see [5]) that

$$\int_{S^2} |\nabla f|^2 d\omega_{S^2} = \int_{\inf f}^{\sup f} \left(\int_{f^{-1}(t)} |\nabla f| d\ell_{S^2} \right) dt.$$

From the Cauchy-Schwarz inequality it follows that the right side of the above equation is larger than or equal to

$$\int_{\inf f}^{\sup f} \left(\ell_{S^2}(f^{-1}(t)) \right)^2 \left(\int_{f^{-1}(t)} \frac{1}{|\nabla f|} d\ell_{S^2} \right)^{-1} dt.$$

The same consideration and the additional fact that $|\nabla f^*|$ is constant along the level sets $(f^*)^{-1}(t)$ shows that

$$\int_{M^*} |\nabla f^*|^2 d\omega_{M^*} = \int_{\inf f}^{\sup f} \left(\ell_{M^*}((f^*)^{-1}(t)) \right)^2 \left(\int_{(f^*)^{-1}(t)} \frac{1}{|\nabla f^*|} d\ell_{M^*} \right)^{-1} dt.$$

Note that f^* is Lipschitz continuous, and by the co-area formula we have

$$\int_{(f^*)^{-1}(t)} \frac{1}{|\nabla f^*|} d\ell_{M^*} = -\frac{d}{dt} \omega_{M^*} \{x; f^*(x) \geq t\},$$

which, by the construction of f^* , equals

$$-\frac{d}{dt} \omega_{S^2} \{x; f(x) \geq t\} = \int_{f^{-1}(t)} \frac{1}{|\nabla f|} d\ell_{S^2}.$$

Since by construction of M^* and by the fact that the level sets $f^{-1}(t)$ coincide with some $\Upsilon_{z(c)}$ for a suitable choice of $c \in [0, 4\pi]$, it follows that

$$\ell_{S^2}(f^{-1}(t)) = \ell_{S^2}(\Upsilon_{z(c)}) \geq \ell_{M^*}(\partial\Omega_c^*) = \ell_{M^*}((f^*)^{-1}(t)).$$

This finishes the proof of our result. □

3. Proof of the main results. Our proof is based on the following remark, which contains a theorem which reduces the study of the asymptotics of the spectral gap of the non-self adjoint operators A_{ab} as $a \rightarrow \infty$ to some Rayleigh quotient.

REMARK 3.1. It was proved in [7] that for a divergence free vector field b one has

$$\lim_{|a| \rightarrow \infty} \rho_{ab} = \inf_{\mu \in \mathbb{R}} \inf \left\{ \frac{\int_{S^2} |\nabla f|^2 d\omega_{S^2}}{\int_{S^2} |f|^2 d\omega_{S^2}}; f \in H_\mu^1 \right\} \quad (3.1)$$

with

$$H_\mu^1 = \left\{ f \in H^1; \int_{S^2} f(x) b \cdot \nabla \psi(x) dx = i\mu \int_{S^2} f(x) \psi(x) dx, \forall \psi \in \mathcal{C}^1(S^2) \right\}.$$

In view of the fact presented in the previous remark it will be important to understand the spaces H_μ^1 . We have the following result.

PROPOSITION 3.2. The sequence of divergence free vector fields b_n constructed in Lemma 1.6 satisfies that $H_\mu^1 = \{0\}$ for all $\mu \neq 0$.

Proof. We saw that for all $f \in H_\mu^1$ there exists a family of functions $f_\epsilon \in H_\mu^1$ which converges toward f in $H^1(S^2)$ as $\epsilon \rightarrow 0$. Denote by $(S_t^{(n)})_{t \in \mathbb{R}}$ the group of isometries generated by the operator $b_n \cdot \nabla$. The fact that $f_\epsilon \in H_\mu^1$ implies that $S_t^{(n)} f_\epsilon = e^{i\mu t} f_\epsilon$. Let $(\phi_t)_{t \in \mathbb{R}}$ be the flow generated by the vector field b_n . Remember that as a consequence of Lemma 1.6 the return time of $(\phi_t)_{t \in \mathbb{R}}$ in the point $(z, 0)$ equals $\sqrt{1 - z^2}/2$ for all $z \in]-1, 1[$. So we obtain the following equality:

$$f_\epsilon(z, 0) = f_\epsilon \circ \phi_{-\sqrt{1-z^2}/2} = e^{i\mu\sqrt{1-z^2}/2} f_\epsilon(z, 0) \quad \forall z \in]-1, 1[.$$

So for all $z \in]-1, 1[$ one has either

$$\exists k \in \mathbb{Z} : \mu\sqrt{1-z^2}/2 = 2\pi k$$

or

$$f_\epsilon(z, 0) = 0.$$

The first possibility can only be true for a zero set of $z \in]-1, 1[$. Since the absolute value of f_ϵ is constant along the trajectories, we obtain from this that

$$f_\epsilon = 0 \quad \omega_{S^2}\text{-a.e.}$$

This implies $f(x) = 0$ for ω_{S^2} -almost all $x \in S^2$. □

We have to introduce an appropriate Cheeger constant for the space of rotationally invariant functions on M^* . For this denote by $\mathcal{S}^*$ the set of all closed subsets in M^* which are invariant with respect to rotations fixing the x_3 -axis:

$$C^* = \inf_{A \in \mathcal{S}^* : \omega_{M^*}(A) < 2\pi} \frac{\ell_{M^*}(\partial A)}{\omega_{M^*}(A)}.$$

Note that the constant C^* depends on the vector field b_n which was used to construct M^* . We will emphasize this dependence through the notation C_n^* in the following proposition.

PROPOSITION 3.3. We have $\lim_{n \rightarrow \infty} C_n^* = \infty$.

Proof. Let A be some closed set from $\mathcal{S}^*$ and suppose that it is not of the form $M^* \cap \{x \in \mathbb{R}^3; z_o \geq x_3 \geq z_u\}$ for a suitable $z_o \in \mathbb{R}$. Then there exist $z_r > z_l$ with

$$A = (A \cap \{x; x_3 \geq z_r\}) \cup (A \cap \{x; x_3 \leq z_l\}) =: A_1 \cup A_2.$$

For $i = 1, 2$ let $l_i := \ell_{M^*}(\partial A_i)$ and $c_i := \omega_{M^*}(A_i)$. It then follows that $l := \ell_{M^*}(A) = l_1 + l_2$ and $c := \omega_{M^*}(A) = c_1 + c_2$. Now suppose that $\frac{l_1}{c_1} \leq \frac{l_2}{c_2}$. It then follows that

$$\frac{l}{c} = \frac{l_1 + l_2}{c_1 + c_2} = \frac{l_1 + l_2}{\frac{c_1}{l_1} l_1 + \frac{c_2}{l_2} l_2} \geq \frac{l_1 + l_2}{\frac{c_1}{l_1} l_1 + \frac{c_1}{l_1} l_2} = \frac{l_1}{c_1}.$$

This means that in the definition of the constant C^* the minimization can be reduced to the set $\mathcal{B}^*$ of elements from $\mathcal{S}^*$ which have the particular form $M^* \cap \{x; z_1 \leq x_3 \leq z_2\}$. Now let $B = M^* \cap \{x; z_o \leq x_3 \leq z_u\}$ be a set from $\mathcal{B}^*$ with $\omega_{M^*}(B) \leq 2\pi$. It then follows for $B_n := M^* \cap \{x; x_3 \geq z_o\}$ and $B_s := M^* \cap \{x; z_u \geq x_3\}$ that either $\omega_{M^*}(B_n) > \pi$ or $\omega_{M^*}(B_s) > \pi$ holds. Without loss of generality we assume that $\omega_{M^*}(B_s) > \pi$. Then one has

$$\frac{\ell_{M^*}(\partial B)}{\omega_{M^*}(B)} \geq \frac{\ell_{M^*}(\partial B_s) + \ell_{M^*}(\partial B_n)}{2\pi} \geq \frac{\ell_{M^*}(\partial B_s)}{2\omega_{M^*}(B_s)}.$$

It then follows from the properties of the comparison manifold that

$$C^* \geq \frac{1}{2} \inf_{0 < c < 2\pi} \frac{\ell_{M^*}(\partial \Omega_c^*)}{c}.$$

The statement now follows from Lemma 1.5 and Proposition 2.4. $\square$

PROPOSITION 3.4. For all $f \in H^1(M^*)$ which are invariant by rotations through the x_3 -axis, we have that

$$\frac{\int_{M^*} |\nabla f|^2 d\omega_{M^*}}{\int_{M^*} |f|^2 d\omega_{M^*}} \geq \frac{C^*}{4}.$$

Proof. Note first that the minimizer of the left quotient exists and that this minimizer is an eigenfunction of the self-adjoint operator that one obtains after restricting the Laplace-Beltrami operator on (M^*, g^*) to the functions which are invariant by rotations fixing the x_3 -axis in $\mathbb{R}^3$.

Let f be such a minimizer. We define the two nodal domains $\Omega_1^* := \{f > 0\}$ and $\Omega_2^* := \{f < 0\}$. Obviously the restriction of f to Ω_i^* is a solution of the eigenvalue problem with Dirichlet boundary condition at $\partial\Omega_i$. Without loss of generality we assume $\omega_{M^*}(\Omega_1^*) \leq \omega_{M^*}(\Omega_2^*)$. Since f and $f|_{\Omega_1^*}$ are associated to the same eigenvalue, we have by the Rayleigh principle that

$$\frac{\int_{M^*} |\nabla f|^2 d\omega_{M^*}}{\int_{M^*} |f|^2 d\omega_{M^*}} = \frac{\int_{\Omega_1^*} |\nabla f|^2 d\omega_{M^*}}{\int_{\Omega_1^*} |f|^2 d\omega_{M^*}}. \quad (3.2)$$

It is then sufficient to prove that the right quotient is larger than $C^*/4$. Since we only take care of functions which are rotationally invariant, their level sets are in $\mathcal{S}^*$. Following Cheeger's arguments restricted to rotationally invariant functions one can prove that the quotient in (3.2) is bounded from below by $C^*/4$. (See Chavel [3] for more details.) $\square$

Proof of Theorem 0.1. Let $K > 0$ be large. By Proposition 3.3 the sequence of vector fields $(b_n)_{n \in \mathbb{N}}$ constructed in the first section satisfies that there exists $n \in \mathbb{N}$ such that

$$\inf \left\{ \frac{\int_{M_n^*} |\nabla f|^2 d\omega_{M_n^*}}{\int_{M_n^*} |f|^2 d\omega_{M_n^*}}; f \in H^1(M_n^*) \right\} \geq \frac{C_n^*}{4} \geq 2K,$$

where M_n^* denotes the comparison manifold, which is associated to b_n . Further, it follows from Proposition 2.10, together with Remark 2.9, that

$$\inf \left\{ \frac{\int_{S^2} |\nabla f|^2 d\omega_{S^2}}{\int_{S^2} |f|^2 d\omega_{S^2}}; f \in H_0^1 \right\} \geq 2K.$$

Then Remark 3.1 and Proposition 3.2 imply for the vector field b_n chosen above that

$$\lim_{a \rightarrow \infty} \rho_{ab_n} \geq 2K.$$

In conclusion, there exists a large constant $a \in \mathbb{R}$ such that $\rho_{ab_n} > K$. $\square$

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