QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXV, NUMBER 3
SEPTEMBER 2017, PAGES 433–503
http://dx.doi.org/10.1090/qam/1461
Article electronically published on February 2, 2017

GREEN'S FUNCTION AND POINTWISE CONVERGENCE FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this paper we introduce a program to construct the Green's function for the linearized compressible Navier-Stokes equations in several space dimensions. This program contains three components, a procedure to isolate global singularities in the Green's function for a multi-spatial-dimensional problem, a long wave-short wave decomposition for the Green's function and an energy method together with Sobolev inequalities. These three components together split the Green's function into singular and regular parts with the singular part given explicitly and the regular part bounded by exponentially sharp pointwise estimates. The exponentially sharp singular-regular description of the Green's function together with Duhamel's principle and results of Matsumura-Nishida on L^∞ decay yield through a bootstrap procedure an exponentially sharp space-time pointwise description of solutions of the full compressible Navier-Stokes equations in $\mathbb{R}^n (n = 2, 3)$.

Contents

Received November 20, 2016.
2010 *Mathematics Subject Classification.* Primary 35Q30, 65M80.
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1. Introduction. The compressible isentropic Navier-Stokes equations (NS),

$$\begin{cases}(\partial_t\rho + \nabla\cdot(\rho\vec{\boldsymbol{u}}))(\vec{\boldsymbol{x}},t) = 0,\\ (\partial_t(\rho\vec{\boldsymbol{u}}) + \nabla\cdot(\rho\vec{\boldsymbol{u}}\otimes\vec{\boldsymbol{u}}) + \nabla p(\rho) - \mu\Delta\vec{\boldsymbol{u}} - \zeta\nabla(\nabla\cdot\vec{\boldsymbol{u}}))(\vec{\boldsymbol{x}},t) = 0,\end{cases} \tag{1.1}$$

for $(\vec{\boldsymbol{x}},t) \in \mathbb{R}^n \times \mathbb{R}_+$ is a fundamental mathematical model for viscous isentropic gas flow, where ρ stands for density, $p(\rho) = \rho^\gamma$, $\gamma \geq 1$, is for gas pressure; μ and ζ stand for dissipation constants. This nonlinear problem has been extensively studied in [6,7, 11–13,15,18,24–26,30,35].

It is important to have a precise qualitative and quantitative mathematical description for the solutions of NS because of its practical importance in fluid mechanics and physics. In the classical result of Matsumura-Nishida [24], it was shown that small perturbations around a constant state for NS equations converge to the constant state in the H_2-norm space by use of spectral properties of the semi-group of a linear system and the classical energy method; see also [11–13,25,26,30]. Kawashima [12,13] and later Shizuta-Kawashima [31] generalized the method in [24] via an approach characterized by a stability condition called the Shizuta-Kawashima condition for hyperbolic-parabolic systems. This general approach has been applied to various problems such as relaxation models, etc. (see [1,5,32] and the references therein). The Sobolev norm analysis also applies to the time-asymptotic stability problems for both a viscous shock profile and boundary layer problem undergoing planar wave perturbations; cf. [3,10,14,16,27].

The Green's function for NS was investigated in [6,7,15,18]. In [6], Hoff-Zumbrun derived the asymptotic behavior of solutions by analysis involving the Fourier multiplier and Paley-Wiener theory. In [7], they studied a unique "effective artificial viscosity" system which approximates the behavior of the compressible Navier-Stokes equations as shown in their previous work [6]. They separated the system into a hyperbolic system and a parabolic one. Based on this separation, entries of the Green's function could be represented as convolutions of the wave operator and heat kernel and a scaling argument and integration by parts yielded the pointwise estimates. In [18] and [15], the isentropic case and the NS system with the additional energy balance equation were studied respectively. The authors developed the contour integral method for the inverse Fourier transform in the 1-D case and used the micro-local analysis to achieve algebraic wave structure for the long wave components in the multi-D case. They also adopted micro-local analysis for short wave components to give a description about the singularities of the Green's function. Those works are sufficient for nonlinear stability problems around constant states. However, they are not sharp enough to bound the dispersive wave motion so that one might fail to understand the interaction between the interior dispersive wave motions and the viscous shock front. Thus, as a preparation for the treatment of dispersive behavior along the shock front for the nonplanar wave perturbation problem for a viscous shock profile, it is important to obtain uniformly exponentially sharp pointwise structure of the Green's function. This is the main purpose of this paper. The program introduced in this paper for construction of the Green's function with exponentially sharp estimates is called **Program A**.

This program involves three sequential steps mixed with Fourier, complex, and real analysis, and introduces a new ingredient so that the singularities in the Green's function of NS equations can be removed completely for all time $t > 0$. The other ingredients of this program (which were first used in [20]) are a combination of spectral analysis and energy estimates. This program is rather universal in the sense that it should work for the Navier-Stokes equations with an energy balance equation. For the sake of simplicity, we restrict the calculation to the case of isentropic flow.

In summary there are three steps in **Program A:**

Step 1. Singularities removal.

Step 2. Long wave-short wave decomposition.

Step 3. Weighted energy estimates.

The basic principle of this program is due to Liu-Yu [19–21], for the purpose of resolution of the coupling of δ-functions and the collision kernel of the Boltzmann equation. This coupling was first realized in [35], and resolved by extracting approximated singularities. However, the resolution in [35] is neither sufficient for the Boltzmann equation nor NS in multi-D, since there are no known equations to approximate the singularities. The principle of resolution of this coupling by Liu-Yu in [19–21] is to identify the mechanism of the singularities and its interaction with regular wave structures for the analysis of the Boltzmann equation. The mechanism for NS is completely different from the Boltzmann equation. Here we follow the idea in [19–21] and split singularities before the long wave-short wave decomposition and modify the original spirit of [35] via the use of a more generalized approach to remove singularities by introducing "singular support functions" based on Taylor expansions. Unlike the studies for the Boltzmann equation, the "singular support functions" are not constructed from the equations directly. They are based on an asymptotic analysis of the spectra of the linearized problem. In this sense, it seems that such a singularity removal step should work in other situations. This is Step 1. The other two steps are similar to those in [19–21].

After the precise structures of the Green's function is obtained, we use the procedure used in [35] to analyze the solutions under a priori estimates given in [13, 24, 25] and obtain the time asymptotic pointwise convergence to the constant state.

We outline the basics in this paper:

Without loss of generality, one considers the unit constant state $(\rho, \vec{\boldsymbol{u}}) = (1, \vec{0})$. The linearized equations around this state are:

$$\begin{cases} \partial_t \rho + \nabla \cdot \vec{\boldsymbol{u}} = 0, \\ \partial_t \vec{\boldsymbol{u}} + \gamma \nabla \rho - \mu \Delta \vec{\boldsymbol{u}} - \zeta \nabla (\nabla \cdot \vec{\boldsymbol{u}}) = 0. \end{cases} \tag{1.2}$$

By rescaling space-time variables, the system (1.2) can be transformed into the form:

$$\begin{cases} \partial_t \rho + \nabla \cdot \vec{\boldsymbol{u}} = 0, \\ \partial_t \vec{\boldsymbol{u}} + \nabla \rho - \mu' \Delta \vec{\boldsymbol{u}} - \zeta' \nabla (\nabla \cdot \vec{\boldsymbol{u}}) = 0. \end{cases} \tag{1.3}$$

For convenience, we still use (ν, ζ) to denote (ν', ζ') for the rest of this paper.

The Green's function $U(\vec{\boldsymbol{x}}, t) = (U_{ij}(\vec{\boldsymbol{x}}, t))_{(n+1)\times(n+1)}$ of (1.3) is the solution of an initial value problem for (1.3) with a matrix-valued initial data:

$$\begin{cases} \left(\partial_t + \begin{pmatrix} 0 & \nabla^t_{\vec{\boldsymbol{x}}} \\ \nabla_{\vec{\boldsymbol{x}}} & 0 \end{pmatrix} - \mu \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \Delta - \zeta \begin{pmatrix} 0 & 0 \\ 0 & \nabla_{\vec{\boldsymbol{x}}} \nabla^T_{\vec{\boldsymbol{x}}} \end{pmatrix} \right) U = 0, \\ U(\vec{\boldsymbol{x}}, 0) = \delta_n(\vec{\boldsymbol{x}}) \mathbb{I}, \end{cases} \tag{1.4}$$

where $\mathbb{I}$ is an $(n+1) \times (n+1)$ identity matrix; $\delta_n(\vec{\boldsymbol{x}})$ is an n-dimensional δ-function; Δ is the Laplacian in $\vec{\boldsymbol{x}} \in \mathbb{R}^n$; $\mathbb{F}[U](\vec{\eta}, t)$ is the Fourier transform of the solution of (1.4)

$$\mathbb{F}[U](\vec{\eta}, t) \equiv \int_{\mathbb{R}^n} e^{-i\vec{\eta}\cdot\vec{\boldsymbol{x}}} U(\vec{\boldsymbol{x}}, t) d\vec{\eta},$$

with the Fourier variable $\vec{\eta} = (\eta^1, \cdots, \eta^n)^t \in \mathbb{R}^n$; and $e^{\boldsymbol{A}(\vec{\eta})t}$ is the semi-group

$$\mathbb{F}[U](\vec{\eta}, t) = \exp(-\boldsymbol{A}(\vec{\eta})t), \quad \boldsymbol{A}(\vec{\eta}) \equiv \begin{pmatrix} 0 & i\vec{\eta}^t \\ i\vec{\eta} & \mu|\vec{\eta}|^2 \boldsymbol{I} + \mathbb{T} \end{pmatrix},$$

where $\boldsymbol{I}$ is an $n \times n$ identity matrix and the stress tensor

$$\mathbb{T}(\vec{\eta}) = (\mathbb{T}_{kl}(\vec{\eta}))_{n\times n}, \ \mathbb{T}_{kl}(\vec{\eta}) \equiv \zeta \eta^k \eta^l,$$

with $\vec{\eta} \equiv (\eta^1, \eta^2, \cdots, \eta^n)^t$. The semi-group $\exp(-A(\vec{\eta})t)$ is, [6],

$$\exp(-A(\vec{\eta})t) = \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -i\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \vec{\eta}^t \\ -i\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \vec{\eta} & e^{-\mu|\vec{\eta}|^2 t} \boldsymbol{I} + \left(\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-\mu|\vec{\eta}|^2 t} \right) \frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2} \end{pmatrix} \tag{1.5}$$

where

$$\lambda_\pm \equiv -\frac{\mu + \zeta}{2} |\vec{\eta}|^2 \pm \frac{1}{2} \sqrt{(\mu + \zeta)^2 |\vec{\eta}|^4 - 4|\vec{\eta}|^2}.$$

The Green's function $U(\vec{\boldsymbol{x}}, t)$ can be represented by

$$U(\vec{\boldsymbol{x}}, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}} e^{-A(\vec{\eta})t} d\vec{\eta}.$$

This function $U(\vec{x}, t)$ contains singularities for all $t > 0$, due to the degeneracy of λ_+ at $\vec{\eta} = \infty$ i.e. $\lambda_+ \to -\nu^{-1}$ as $\vec{\eta} \to \infty$ so that the limit $\lim_{|\vec{\eta}|\to\infty} e^{-A(\vec{\eta})t} \neq 0$. The nondecaying component $e^{\lambda_+ t}$ at $\vec{\eta} = \infty$ causes a δ-function in the Green's function and the factor $\lambda_+/(\lambda_- - \lambda_+)$ cancels the parabolic singularity in $e^{\lambda_- t}$ when t is small. This δ-function singularity was first realized in [35] for a 1-D problem. In this paper, Step 1 is designed to remove such singularities caused by the degeneracy of the eigenvalues of the matrix $A(\vec{\eta})$ at $\vec{\eta}=\infty$ for any finite spatial dimensional problem properly. This step starts from establishing the relationship between $e^{\lambda_\pm t}$ and the initial singularity $\delta_n(\vec{\boldsymbol{x}})$ at $t=0$. From the representation (1.5) for the matrix $\exp(-A(\vec{\eta})t)$, one needs to remove the singularities in the functions $H_0(\vec{\boldsymbol{x}},t)$ and $H_1(\vec{\boldsymbol{x}},t)$,

$$\begin{cases} H_0(\vec{\boldsymbol{x}},t)\equiv\mathbb{F}^{-1}[(\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t})/(\lambda_+-\lambda_-)],\\ H_1(\vec{\boldsymbol{x}},t)\equiv\mathbb{F}^{-1}[(e^{\lambda_+t}-e^{\lambda_-t})/(\lambda_+-\lambda_-)],\end{cases}\tag{1.6}$$

which relate to the entries of the Green's function $U(\vec{\boldsymbol{x}},t)=(U_{ij}(\vec{\boldsymbol{x}},t))_{(n+1)\times(n+1)}$:

$$\begin{cases} U_{11}(\vec{\boldsymbol{x}},t)=H_0(\vec{\boldsymbol{x}},t),\\ U_{1,j+1}(\vec{\boldsymbol{x}},t)=U_{j+1,1}(\vec{\boldsymbol{x}},t)\equiv\partial_{x^j}H_1(\vec{\boldsymbol{x}},t)\text{ for } j=1,\cdots,n.\end{cases}$$

Here, the expressions in (1.6) in terms of the Fourier variable $\vec{\eta}$ give the complete information on the singularities inherited from the initial data $\delta_n(\vec{\boldsymbol{x}})\mathbb{I}$. Hence we see the approximate spectra $\lambda^*_\pm(\vec{\eta})$ (given by Definition 1.1 below) to $\lambda_\pm(\vec{\eta})$ at $\vec{\eta}=\infty$ is *local analytic around the real axis* in the Fourier variable $\vec{\eta}$ so that one can assert that the singularities of $H_l(\vec{\boldsymbol{x}},t)$ are contained in the *singular support functions* $H^*_l(\vec{\boldsymbol{x}},t)$, $l=0,1$,

$$\begin{cases} H^*_0(\vec{\boldsymbol{x}},t)\equiv\mathbb{F}^{-1}[(\lambda^*_+e^{\lambda^*_-t}-\lambda^*_-e^{\lambda^*_+t})/(\lambda^*_+-\lambda^*_-)],\\ H^*_1(\vec{\boldsymbol{x}},t)\equiv\mathbb{F}^{-1}[(e^{\lambda^*_+t}-e^{\lambda^*_-t})/(\lambda^*_+-\lambda^*_-)].\end{cases}$$

Here, the notion *"singular support function H^*_l for H_l"* means that $H_l-H^*_l$ is a $C^k(\mathbb{R}^n\times\mathbb{R}_+)$ up to some integer $k\geq 1$.

DEFINITION 1.1 (Local analyticity around the real axis). A complex-valued function $f(\vec{\eta})$ is locally analytic around the real axis if and only if there exists $\delta_0>0$ such that $f(\vec{\eta})$ are analytic in the region $|Im(\eta^k)|\leq\delta_0$ for all $k=1,\cdots,n$.

The *singular support functions* H^*_0 and H^*_1 can be constructed in terms of $\delta_n(\vec{\boldsymbol{x}})$ functions, Yukawa potential $\mathbb{Y}_n$, and $j^*_l(\vec{\boldsymbol{x}},t)$, which are functions with exponentially sharp pointwise estimates described by the following theorem.

THEOREM 1.2. Let $n=2,3$ be the spatial dimension. There exist distribution functions $H^*_0(\vec{\boldsymbol{x}},t)$, $j^*_0(\vec{\boldsymbol{x}},t)$, $H^*_1(\vec{\boldsymbol{x}},t)$, $j^*_1(\vec{\boldsymbol{x}},t)$ and constants $C_n>0$ satisfying

$$|j^*_0(\vec{\boldsymbol{x}},t)|,\ |j^*_1(\vec{\boldsymbol{x}},t)|\leq C_nL_n(t)e^{-(|\vec{\boldsymbol{x}}|+t)/C_n}\text{ for } t>0,\tag{1.7}$$

$$\begin{aligned}&\left|H^*_0(\vec{\boldsymbol{x}},t)-j^*_0(\vec{\boldsymbol{x}},t)-e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}})-(\nu^{-3}t+\nu^{-2})e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}})\right|\\ &\qquad\leq C_ne^{-(|\vec{\boldsymbol{x}}|+t)/C_0},\\ &\left|H^*_1(\vec{\boldsymbol{x}},t)-j^*_1(\vec{\boldsymbol{x}},t)-\frac{1}{\nu}e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}})\right|\leq C_ne^{-(|\vec{\boldsymbol{x}}|+t)/C_0}.\end{aligned}\tag{1.8}$$

Furthermore, the singular support functions $H_0^*(\vec{\boldsymbol{x}},t)$ and $H_1^*(\vec{\boldsymbol{x}},t)$ satisfy

$$|(\partial_t^2-(1+\nu\partial_t)\Delta)H_k^*|\le C_n e^{-(|\vec{\boldsymbol{x}}|+t)/C_n} \text{ for } k=0,1, \tag{1.9}$$
$$\mathbb{F}[(\partial_t^2-(1+\nu\partial_t)\Delta)H_0^*](\vec{\eta},t)=O(1)/(1+|\vec{\eta}|^2)^4, \tag{1.10}$$
$$\mathbb{F}[(\partial_t^2-(1+\nu\partial_t)\Delta)H_1^*](\vec{\eta},t)=O(1)/(1+|\vec{\eta}|^2)^5, \tag{1.11}$$

where $\nu=\mu+\zeta$. Here, $\mathbb{Y}_n(\vec{\boldsymbol{x}})$ is the n-dimensional Yukawa potential with unit mass (i.e.,$\mathbb{Y}_n\equiv(\Delta_n+1)^{-1}\delta_n(\vec{\boldsymbol{x}})$):

$$\begin{cases}\mathbb{Y}_1(x)=\dfrac{e^{-|x|}}{2\sqrt{2\pi}},\\ \mathbb{Y}_2(\vec{\boldsymbol{x}})=\dfrac{1}{2\pi}BesselK_0(|\vec{\boldsymbol{x}}|),\\ \mathbb{Y}_3(\vec{\boldsymbol{x}})=-\dfrac{e^{-|\vec{\boldsymbol{x}}|}}{4\pi|\vec{\boldsymbol{x}}|},\end{cases} \tag{1.12}$$

where $BesselK_0(x)$ is the modified Bessel function of the second kind with degree 0; and the function $L_n(t)$ is defined by

$$L_n(t)\equiv\begin{cases}1,\ n=1,\\ \log(t),\ n=2,\\ t^{-\frac{1}{2}},\ n=3.\end{cases}$$

A fundamental fact given by (1.7) and (1.8) is that the **parabolic singularity** at $t=0$ is not present in $U_{11}(\vec{\boldsymbol{x}},t)(\equiv H_0(\vec{\boldsymbol{x}},t))$ for $0<t\ll 1$. Here, a parabolic singularity means that the singular structure can be bounded from above and below by heat kernel type singularities $C_+e^{-|\vec{\boldsymbol{x}}|^2/(C_+t)}/t^{n/2}$ and $e^{-C_-|\vec{\boldsymbol{x}}|^2/t}/(C_-t^{n/2})$. This illustrates the difference between NS and parabolic systems.

The properties (1.9), (1.10), and (1.11) assure that $H_0^*(\vec{\boldsymbol{x}},t)$ and $H_1^*(\vec{\boldsymbol{x}},t)$ completely absorb the singularities and the truncation errors $(\partial_t^2-(1+(\mu+\zeta)\partial_t)\Delta)H_k^*$ for $k=0,1$ decay exponentially in the space-time domain and possess sufficient regularities. Then, one can apply Step 2 and Step 3 to $H_l(\vec{\boldsymbol{x}},t)-H_l^*(\vec{\boldsymbol{x}},t)$ to yield the global pointwise structures of $H_l(\vec{\boldsymbol{x}},t)-H_l^*(\vec{\boldsymbol{x}},t)$ for $l=0,1$.

Theorem 1.3 (Main Theorem I). There exists C_0 such that for $t>0$ and $0\le|\alpha|\le 3$,

$$\begin{aligned}&|\partial_{\vec{\boldsymbol{x}}}^{\alpha}(H_0(\vec{\boldsymbol{x}},t)-H_0^*(\vec{\boldsymbol{x}},t))|,|\partial_{\vec{\boldsymbol{x}}}^{\alpha}\partial_{x^k}(H_1(\vec{\boldsymbol{x}},t)-H_1^*(\vec{\boldsymbol{x}},t))|\\ &\quad\le O(1)\left(\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^{\frac{n+1+|\alpha|}{2}}}+e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}\right)\\ &\quad\quad+O(1)\frac{1}{1+t^{\frac{|\alpha|}{2}}}\begin{cases}\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^{5/4}}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}+e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} \text{ for } n=2,\\ \\ 0 \text{ for } n=3,\end{cases}\end{aligned}$$

where

$$\mathbf{H}(\tau)=\begin{cases}0 \text{ for } \tau<0,\\ 1 \text{ for } \tau\ge 0.\end{cases}$$

After obtaining the global pointwise structures of H_0 and H_1, one needs to separate the mechanisms due to the heat dissipation, d'Alembert wave propagation in the long wave, and the singularities in H_1^* together to obtain the pointwise structure of $\mathbb{F}^{-1}\left[((\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t})/(\lambda_+-\lambda_-)-e^{-\mu|\vec{\eta}|^2t})\vec{\eta}\vec{\eta}^t/|\vec{\eta}|^2\right]$ in the space-time domain. One rearranges

$$\begin{aligned}\mathbb{F}^{-1}\left[\frac{\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-\mu|\vec{\eta}|^2t}}{|\vec{\eta}|^2}\vec{\eta}\vec{\eta}^t\right] &= \mathbb{J}+\mathbb{F}^{-1}\left[\frac{e^{-\nu|\vec{\eta}|^2t}-e^{-\mu|\vec{\eta}|^2t}}{|\vec{\eta}|^2}\vec{\eta}\vec{\eta}^t\right],\\ \mathbb{J} &\equiv \mathbb{F}^{-1}\left[\frac{\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-\nu|\vec{\eta}|^2t}}{|\vec{\eta}|^2}\vec{\eta}\vec{\eta}^t\right];\end{aligned}$$

and uses H_1 to represent $\mathbb{J}=(\mathbb{J}_{kl})_{n\times n}$ by

$$\mathbb{J}_{kl}(\vec{\boldsymbol{x}},t)=-\partial_{x^k}\partial_{x^l}H_1(\vec{\boldsymbol{x}},t)\underset{(\vec{\boldsymbol{x}},t)}{*}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\nu t}}}{(4\pi\nu t)^{\frac{n}{2}}}.$$

The singular support matrix $\mathbb{J}^*=(\mathbb{J}^*_{kl})_{n\times n}$ for $\mathbb{J}$ is defined by

$$\mathbb{J}^*_{kl}(\vec{\boldsymbol{x}},t)=-\partial_{x^k}\partial_{x^l}\int_0^t H_1^*(\vec{\boldsymbol{x}},\tau)\underset{\vec{\boldsymbol{x}}}{*}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\nu(t-\tau)}}}{(4\pi\nu(t-\tau))^{\frac{n}{2}}}d\tau.$$

Subtract $\mathbb{J}^*$ from $\mathbb{F}^{-1}\left[\frac{\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-\mu|\vec{\eta}|^2t}}{|\vec{\eta}|^2}\vec{\eta}\vec{\eta}^t\right]$, and one can apply Step 2 and Step 3 to yield the following theorem:

THEOREM 1.4 (Main Theorem II). There exists $C_0>0$ such that

$$\begin{aligned}&\left|\mathbb{F}^{-1}\left[\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-\mu|\vec{\eta}|^2t}\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2}\right](\vec{\boldsymbol{x}},t)\right|\le C_0\left(\frac{e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}}{|\vec{\boldsymbol{x}}|^{n-1}}+\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C_0t}}}{t^{n/2}}\right)\\ &+C_0\begin{cases}\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^{5/4}} & \text{for } n=2,\\ \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}})+\sqrt{t})^2}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^2} & \text{for } n=3.\end{cases}\end{aligned} \tag{1.13}$$

For $|\alpha|\le 3$,

$$\begin{aligned}&\left|\partial^\alpha_{\vec{\boldsymbol{x}}}\left(\mathbb{F}^{-1}\left[\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-\mu|\vec{\eta}|^2t}\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2}\right]-\mathbb{J}^*\right)(\vec{\boldsymbol{x}},t)\right|\le C_0\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C_0t}}}{t^{(n+|\alpha|)/2}}\\ &+C_0\frac{1}{1+t^{\frac{|\alpha|}{2}}}\begin{cases}\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^{5/4}} & \text{for } n=2,\\ \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}})+\sqrt{t})^2}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{1+t^2} & \text{for } n=3,\end{cases}\end{aligned} \tag{1.14}$$

and

$$|\mathbb{J}^*_{kl}(\vec{\boldsymbol{x}},t)| \le C_0 \frac{e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}}{|\vec{\boldsymbol{x}}|^{n-1}}. \tag{1.15}$$

Theorems 1.3 and 1.4 give the pointwise structures of the Green's function $U(\vec{\boldsymbol{x}},t)$ for the spatial dimension $n=2$ and 3 and one can decompose the Green's function $U(\vec{\boldsymbol{x}},t)$ into singular and nonsingular parts:

$$U(\vec{\boldsymbol{x}},t) = U^*(\vec{\boldsymbol{x}},t) + U^R(\vec{\boldsymbol{x}},t), \tag{1.16}$$

where

$$U^*(\vec{\boldsymbol{x}},t) \equiv \begin{pmatrix} H_0^*(\vec{\boldsymbol{x}},t) & (\nabla_{\vec{\boldsymbol{x}}} H_1^*(\vec{\boldsymbol{x}},t))^T \\ \nabla_{\vec{\boldsymbol{x}}} H_1^*(\vec{\boldsymbol{x}},t) & \mathbb{J}^*(\vec{\boldsymbol{x}},t) \end{pmatrix},$$

and the matrix $U^R(\vec{\boldsymbol{x}},t)$ satisfies that for $0 \le |\alpha| \le 3$ and some constant $C_0 > 0$,

$$\begin{aligned}
&|\partial^\alpha_{\vec{\boldsymbol{x}}} U^R(\vec{\boldsymbol{x}},t)| = O(1)\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C_0 t}}}{t^{(n+|\alpha|)/2}} + O(1)e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} \\
&+\frac{O(1)}{1+t^{|\alpha|/2}} \begin{cases} \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}} + \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}} \text{ for } n=2, \\ \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} + \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|)+\sqrt{t})^2} + \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{(1+t)^2} \text{ for } n=3. \end{cases}
\end{aligned} \tag{1.17}$$

With the structures (1.16) and (1.17) of Green's function, one can treat the nonlinear wave couplings. Since NS is quasi-linear, one needs a priori estimates for the closure of the nonlinearity. We quote the following theorem for a priori L^∞ bounds for the high order derivatives of the solution.

Theorem 1.5 (Matsurmura-Nishida [24, 25]). Let $s \in \mathbb{N}$ and

$$E_s \equiv \int_{\mathbb{R}^n} \left(|\rho-1| + |\rho-1|^2 + |\vec{\boldsymbol{u}}| + |\vec{\boldsymbol{u}}|^2 + \sum_{1\le|\alpha|\le s} |\partial^\alpha_{\vec{\boldsymbol{x}}}\rho|^2 + |\partial^\alpha_{\vec{\boldsymbol{x}}}\vec{\boldsymbol{u}}|^2 \right)(\vec{\boldsymbol{x}},0)d\vec{\boldsymbol{x}}.$$

Suppose that $s \in \mathbb{N}$ is sufficiently large. Then, there exist $\varepsilon_0 > 0$ and $N(n,s) \ge 2$ such that when $E_s < \varepsilon_0$ the solution $(\rho, \vec{\boldsymbol{u}})$ of (1.1) exists for $t \in (0,\infty]$ and

$$\|\vec{\boldsymbol{u}}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|\rho(\cdot,t)-1\|_{L^\infty(\mathbb{R}^n)} \le O(1)t^{-n/2}\sqrt{E_s},$$
$$\|\partial^\alpha_{\vec{\boldsymbol{x}}}\vec{\boldsymbol{u}}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|\partial^\alpha_{\vec{\boldsymbol{x}}}\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le O(1)t^{-(n+|\alpha|)/2}\sqrt{E_s} \text{ for } |\alpha| = 1,2,\cdots,N(n,s).$$

Let $V(\vec{\boldsymbol{x}},t)$ be the perturbation of $(\rho, \rho\vec{\boldsymbol{u}})$ around $(1,0)$ for the system (1.1):

$$V(\vec{\boldsymbol{x}},t) = (V^0, V^1, \cdots, V^n)^t(\vec{\boldsymbol{x}},t) \equiv (\rho-1, \rho\vec{\boldsymbol{u}}^t)^t(\vec{\boldsymbol{x}},t),$$

and the perturbation $V(\vec{\boldsymbol{x}},t)$ satisfies

$$\left(\partial_t + \begin{pmatrix} 0 & \nabla^t_{\vec{\boldsymbol{x}}} \\ \nabla_{\vec{\boldsymbol{x}}} & 0 \end{pmatrix} - \mu \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \Delta - \zeta \begin{pmatrix} 0 & 0 \\ 0 & \nabla_{\vec{\boldsymbol{x}}} \nabla^T_{\vec{\boldsymbol{x}}} \end{pmatrix}\right) V = \begin{pmatrix} 0 \\ \nabla\cdot\nabla S_1 + \partial_{x^1} J_1 - \nabla_{\vec{\boldsymbol{x}}}\cdot m^1 \vec{\boldsymbol{m}} \\ \nabla\cdot\nabla S_2 + \partial_{x^2} J_2 - \nabla_{\vec{\boldsymbol{x}}}\cdot m^2 \vec{\boldsymbol{m}} \\ \vdots \\ \nabla\cdot\nabla S_n + \partial_{x^n} J_n - \nabla_{\vec{\boldsymbol{x}}}\cdot m^n \vec{\boldsymbol{m}} \end{pmatrix}, \tag{1.18}$$

where

$$\begin{cases} (m^1, m^2, \cdots, m^n)^t \equiv \vec{\boldsymbol{m}} \equiv \rho \vec{\boldsymbol{u}}, \\ S_i \equiv \mu(\frac{1}{1+V^0} - 1) m^i, \\ J_i \equiv -(p(\rho) - p(1) - p'(1)V^0) + \zeta \sum_{k=1}^n \left(\frac{m^i}{1+V_0} \partial_{x^k} \left(\frac{m^k}{1+V_0} \right) - m^i \partial_{x^k} m^k \right). \end{cases}$$

One can rewrite (1.18) in the following two forms:

$$\left(\partial_t + \begin{pmatrix} 0 & \nabla^t_{\vec{\boldsymbol{x}}} \\ \nabla_{\vec{\boldsymbol{x}}} & 0 \end{pmatrix} - \mu \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \Delta - \zeta \begin{pmatrix} 0 & 0 \\ 0 & \nabla_{\vec{\boldsymbol{x}}} \nabla^T_{\vec{\boldsymbol{x}}} \end{pmatrix}\right) V = \mathscr{S}^*,$$

$$\left(\partial_t + \begin{pmatrix} 0 & \nabla^t_{\vec{\boldsymbol{x}}} \\ \nabla_{\vec{\boldsymbol{x}}} & 0 \end{pmatrix} - \mu \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \Delta - \zeta \begin{pmatrix} 0 & 0 \\ 0 & \nabla_{\vec{\boldsymbol{x}}} \nabla^T_{\vec{\boldsymbol{x}}} \end{pmatrix}\right) V = \sum_{k=1} \partial_{x^k} \mathscr{S}^R_k,$$

where

$$\mathscr{S}^*(\vec{\boldsymbol{x}},t) \equiv \sum_{k=1}^n \partial_{x^k} \begin{pmatrix} 0 \\ m^k m^1 \\ m^k m^2 \\ \vdots \\ m^k m^n \end{pmatrix} + \partial_{x^k} \begin{pmatrix} 0 \\ J_1 \delta^k_1 \\ J_2 \delta^k_2 \\ \vdots \\ J_n \delta^k_n \end{pmatrix} + \Delta \begin{pmatrix} 0 \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix}, \tag{1.19}$$

$$\mathscr{S}^R_k(\vec{\boldsymbol{x}},t) \equiv -\begin{pmatrix} 0 \\ m^k m^1 \\ m^k m^2 \\ \vdots \\ m^k m^n \end{pmatrix} + \partial_{x^k} \begin{pmatrix} 0 \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix} - \begin{pmatrix} 0 \\ J_1 \delta^k_1 \\ J_2 \delta^k_2 \\ \vdots \\ J_n \delta^k_n \end{pmatrix}.$$

By Duhamel's principle and the splitting of the Green's function $U(\vec{\boldsymbol{x}},t) = U^*(\vec{\boldsymbol{x}},t) + U^R(\vec{\boldsymbol{x}},t)$, one has the representation of the solution $V(\vec{\boldsymbol{x}},t)$ as follows:

$$\begin{aligned} V(\vec{\boldsymbol{x}},t) &= U(\vec{\boldsymbol{x}},t) \underset{\vec{\boldsymbol{x}}}{*} V(\vec{\boldsymbol{x}},0) + (U^*(\vec{\boldsymbol{x}},t) + U^R(\vec{\boldsymbol{x}},t)) \underset{(\vec{\boldsymbol{x}},t)}{*} \sum_{k=1}^n \partial_{x^k} \mathscr{S}^R_l(\vec{\boldsymbol{x}},t) \\ &= U(\vec{\boldsymbol{x}},t) \underset{\vec{\boldsymbol{x}}}{*} V(\vec{\boldsymbol{x}},0) + U^*(\vec{\boldsymbol{x}},t) \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{S}^*(\vec{\boldsymbol{x}},t) - \sum_{k=1}^n \partial_{x^k} U^R(\vec{\boldsymbol{x}},t) \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{S}^R_k(\vec{\boldsymbol{x}},t). \end{aligned} \tag{1.20}$$

This representation gives a bootstrapping procedure to obtain a precise pointwise estimate.

By both (1.17) and Theorem 1.3, the initial data gives the structure of $U(\vec{\boldsymbol{x}},t) \underset{(\vec{\boldsymbol{x}},t)}{*} V(\vec{\boldsymbol{x}},0)$, for $|\alpha|\le 2$:

$$
\begin{cases}
|\partial^\alpha_{\vec{\boldsymbol{x}}} U(\vec{\boldsymbol{x}},t) \underset{\vec{\boldsymbol{x}}}{*} V(\vec{\boldsymbol{x}},0)| \le \mathscr{C}_0 \dfrac{\varepsilon}{1+t^{\frac{|\alpha|}{2}}} \boldsymbol{A}_n(\vec{\boldsymbol{x}},t), \\
\boldsymbol{A}_2(\vec{\boldsymbol{x}},t) \equiv \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}} + \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C(1+t)}}}{(1+t)^{5/4}}, \\
\boldsymbol{A}_3(\vec{\boldsymbol{x}},t) \equiv \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} + \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C(1+t)}}}{(1+t)^2},
\end{cases}
\tag{1.21}
$$

which gives a fairly good upper bound of $V(\vec{\boldsymbol{x}},t)$ for the nonlinear closure up to second derivatives. From this structure one forms the ansatz assumption for $\partial^\alpha_{\vec{\boldsymbol{x}}} V(\vec{\boldsymbol{x}},t)$:

$$
|\partial^\alpha_{\vec{\boldsymbol{x}}} V(\vec{\boldsymbol{x}},t)| \le 2\varepsilon\mathscr{C}_0 \left(1+t^{\frac{|\alpha|}{2}}\right)^{-1} \boldsymbol{A}_n(\vec{\boldsymbol{x}},t) \text{ for } n=2,3 \text{ and } 0\le|\alpha|\le 2.
$$

To justify this assumption, one simply differentiates the representation (1.20):

$$
\begin{aligned}
\partial^\alpha_{\vec{\boldsymbol{x}}} V(\vec{\boldsymbol{x}},t) &= \partial^\alpha_{\vec{\boldsymbol{x}}} U(\vec{\boldsymbol{x}},t) \underset{\vec{\boldsymbol{x}}}{*} V(\vec{\boldsymbol{x}},0) \\
&\quad + U^*(\vec{\boldsymbol{x}},t) \underset{(\vec{\boldsymbol{x}},t)}{*} \partial^\alpha_{\vec{\boldsymbol{x}}} \mathscr{S}^*(\vec{\boldsymbol{x}},t) - \sum_{k=1}^{n} (\partial^\alpha_{\vec{\boldsymbol{x}}} \partial_{x^k} U^R(\vec{\boldsymbol{x}},t)) \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{S}^R_k(\vec{\boldsymbol{x}},t).
\end{aligned}
\tag{1.22}
$$

The $\partial^\alpha_{\vec{\boldsymbol{x}}}$ derivatives of nonlinear terms $\mathscr{S}^*$ and $\mathscr{S}^R_k$ with $|\alpha|\le 2$ in (1.22) can be written as a sum of products of a lower order derivative term with order less than 2 and a higher order derivative term. This together with a priori assumption, and Theorem 1.5 yields a sufficiently strong pointwise decaying structure of $\partial^\alpha_{\vec{\boldsymbol{x}}}\mathscr{S}^*$ for the representation in (1.22), though $\partial^\alpha_{\vec{\boldsymbol{x}}}\mathscr{S}^*$ contains factors with derivative up to the order $|\alpha|+2$ and one does have pointwise structure for such factors. With computational lemmas for nonlinear wave couplings given in Appendix C, one has the pointwise structure of the nonlinear wave propagations.

THEOREM 1.6 (Main Theorem). Suppose that both $s\in\mathbb{N}$ and $1/\varepsilon>0$ is sufficiently large; and

$$
\int_{\mathbb{R}^n} e^{|\vec{\boldsymbol{x}}|} \sum_{0\le|\alpha|\le s} |\partial^\alpha_{\vec{\boldsymbol{x}}} V(\vec{\boldsymbol{x}},0)|^2 d\vec{\boldsymbol{x}} \le \varepsilon^2. \tag{1.23}
$$

Then, the solution $V(\vec{\boldsymbol{x}},t)$ of (1.18) satisfies

$$
|V(\vec{\boldsymbol{x}},t)| = O(1)\varepsilon \boldsymbol{A}_n(\vec{\boldsymbol{x}},t) \text{ for } n=2,3,
$$

where $\boldsymbol{A}_n(\vec{\boldsymbol{x}},t)$ is given by (1.21).

In Section 2, we prepare the complex analysis tool for the inverse Fourier transform, the representation of the solutions of d'Alembert wave equation in 2-D and 3-D, the parabolic waves, and basic properties of the Yukawa potentials, which are used to construct the singular support functions.

In Section 3, we perform Step 1 to construct the singular support functions H_l^* for H_l in terms of δ-function and the Yukawa potentials.

In Section 4, we perform Step 2 and Step 3 to obtain the global pointwise estimates of $H_l - H_l^*$ in the space-time domain.

In Section 5, we use the singular support function H_1^* to construct the singular support matrix for $\mathbb{F}^{-1}[\eta^k\eta^l(e^{\lambda_- t} - e^{\lambda_+ t})/|\vec{\eta}|^2]$. The key issue in this section is to yield the pointwise space-time structure in $\mathbb{F}^{-1}[\eta^k\eta^l(e^{\lambda_- t} - e^{\lambda_+ t})/|\vec{\eta}|^2]$ with the presence of the operator $\partial_{x^k}\partial_{x^l}\Delta^{-1}(=\mathbb{F}^{-1}[\eta^k\eta^l/|\vec{\eta}|^2])$.

In Section 6, we use the singular and nonsingular decomposition of the Green's function $U(\vec{\boldsymbol{x}},t)$ and Duhamel's principle to bootstrap the result of [24] into pointwise ansatz structure.

In the appendix, we give the proofs of computational lemmas; and we also give the lemmas for computing the nonlinear wave couplings and the proofs of those lemmas.

2. Preliminaries.

In this section we prepare some basic analytic tools for further development in the paper.

Inverse Fourier transform.

One denotes

$$\mathscr{D}_\delta \equiv \{\vec{\eta} \in \mathbb{C}^n \mid |Im(\eta^k)| < \delta \text{ for } k = 1, \cdots, n\}. \tag{2.1}$$

LEMMA 2.1. Suppose a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\mathbb{F}[f](\vec{\eta})$, $\vec{\eta} = (\eta^1, \cdots, \eta^n)$ is analytic in $\mathscr{D}_{\nu_0}$ and satisfies

$$|\mathbb{F}[f](\vec{\eta})| < \frac{E_1}{(1+|\vec{\eta}|)^{n+1}} \text{ for } |Im(\eta^j)| < \nu_0 \text{ and } j = 1, \cdots, n.$$

Then, the function $f(x)$ satisfies

$$|f(x)| \le E_1 e^{-\nu_0|x|/C},$$

for any positive constant $C > 1$.

Proof. By both inverse Fourier transform and analyticity property together, one has that

$$\begin{aligned}
|f(\vec{\boldsymbol{x}})| &= \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}}\mathbb{F}[f](\vec{\eta})d\vec{\eta}\right| = \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{i(\vec{\eta}+i\vec{\eta}_0)\cdot\vec{\boldsymbol{x}}}\mathbb{F}[f](\vec{\eta}+i\vec{\eta}_0)d(\vec{\eta}+i\vec{\eta}_0)\right| \\
&= \frac{1}{(2\pi)^n}e^{-\vec{\eta}_0\cdot\vec{\boldsymbol{x}}}\left|\int_{\mathbb{R}^n} e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}}\mathbb{F}[f](\vec{\eta}+i\vec{\eta}_0)d\vec{\eta}\right| \\
&\le \frac{1}{(2\pi)^n}e^{-\vec{\eta}_0\cdot\vec{\boldsymbol{x}}}\int_{\mathbb{R}^n}\left|\frac{E_1}{(1+|\vec{\eta}+i\vec{\eta}_0|)^{n+1}}\right|d\vec{\eta} \le E_1 e^{-\vec{\eta}_0\cdot\vec{\boldsymbol{x}}}
\end{aligned}$$

for all $\vec{\eta}_0 \in (-\nu_0,\nu_0)^n \subset \mathbb{R}^n$. This concludes the lemma. □

Fourier transform of the fundamental solution.

The Fourier transform of the problem (1.4) is a system of ODE:

$$\begin{cases}(\partial_t + A(\vec{\eta}))\mathbb{F}[U] = \vec{0}, \\ \mathbb{F}[U](\vec{\eta},0) = \mathbb{I},\end{cases}$$

where

$$\begin{cases} A(\vec{\eta}) \equiv \begin{pmatrix} 0 & i\vec{\eta}^t \\ i\vec{\eta} & \mu|\vec{\eta}|^2\boldsymbol{I} + \mathbb{T} \end{pmatrix}_{(n+1)\times(n+1)}, \\ \boldsymbol{I} \equiv (\delta_k^l)_{n\times n}, \\ (\mathbb{T}_{kl}(\vec{\eta}))_{n\times n} \equiv \left(\zeta\eta^k\eta^l\right)_{n\times n}. \end{cases}$$

The solution $\mathbb{F}[U](\vec{\eta},t)$ is the semi-group $e^{-A(\vec{\eta})t}$. By a straight computation, $e^{-A(\vec{\eta})t}$ is, [6]:

$$\mathbb{F}[U](\vec{\eta},t) = \begin{pmatrix} \frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-} & -i\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\vec{\eta}^t \\ -i\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\vec{\eta} & e^{-\mu|\vec{\eta}|^2t}\boldsymbol{I} + \left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} - e^{-\mu|\vec{\eta}|^2t}\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2} \end{pmatrix} \tag{2.2}$$

where

$$\lambda_\pm \equiv -\frac{\nu}{2}|\vec{\eta}|^2 \pm \frac{1}{2}\sqrt{\nu^2|\vec{\eta}|^4 - 4|\vec{\eta}|^2}, \tag{2.3}$$

with

$$\nu \equiv \mu + \zeta,$$

and $\boldsymbol{I}$ is an $n \times n$ identity matrix.

Solutions of the d'Alembert wave equation in $\mathbb{R}^n$.

Let $W(\vec{\boldsymbol{x}},t)$ be the solution of the d'Alembert wave equation for $\vec{\boldsymbol{x}} \in \mathbb{R}^n$:

$$\begin{cases} (\partial_t^2 - \Delta)W(\vec{\boldsymbol{x}},t) = 0 \text{ for } (\vec{\boldsymbol{x}},t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ W(\vec{\boldsymbol{x}},0) = W_1(\vec{\boldsymbol{x}}), \\ \partial_t W(\vec{\boldsymbol{x}},0) = W_0(\vec{\boldsymbol{x}}). \end{cases}$$

The representation of the solution $W(\vec{\boldsymbol{x}},t)$ in terms of the sine transform and the cosine transform is as follows:

$$\begin{cases} \mathbb{F}[W](\vec{\eta},t) = \mathbb{F}[\mathbb{S}_n](\vec{\eta},t)\mathbb{F}[W_0](\vec{\eta}) + \mathbb{F}[\mathbb{C}_n](\vec{\eta},t)\mathbb{F}[W_1](\vec{\eta}), \\ \mathbb{F}[\mathbb{S}_n](\vec{\eta},t) \equiv \dfrac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}, \\ \mathbb{F}[\mathbb{C}_n](\vec{\eta},t) \equiv \cos(|\vec{\eta}|t). \end{cases}$$

Kirchhoff formula.

When the space dimension $n = 3$, the solution $W(\vec{\boldsymbol{x}},t)$ of the d'Alembert wave equation can be represented by the 2-dimensional surface integral and the Kirchhoff formula together:

$$W(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi t}\int_{|\vec{\boldsymbol{y}}-\vec{\boldsymbol{x}}|=t} W_0(\vec{\boldsymbol{y}})dA_{\vec{\boldsymbol{y}}} + \partial_t\left(\frac{1}{4\pi t}\int_{|\vec{\boldsymbol{y}}-\vec{\boldsymbol{x}}|=t} W_1(\vec{\boldsymbol{y}})dA_{\vec{\boldsymbol{y}}}\right),$$

where $A_{\vec{\boldsymbol{y}}}$ is the 2-dimensional area element of the surface integral.

The above representation of $W(\vec{\boldsymbol{x}},t)$ yields the kernel function of the sine transform:

$$\mathbb{S}_3(\vec{\boldsymbol{x}},t) = \frac{1}{4\pi t}\delta(|\vec{\boldsymbol{x}}| - t). \tag{2.4}$$

This formula and the Hadamard's method of descendent yield the kernel function of the sine transform for the 2-D case:

$$\mathbb{S}_2(\vec{\boldsymbol{x}},t)=\begin{cases}\dfrac{1}{2\pi\sqrt{t^2-|\vec{\boldsymbol{x}}|^2}} \text{ for } |\vec{\boldsymbol{x}}|<t,\\ 0 \text{ for } |\vec{\boldsymbol{x}}|>t.\end{cases} \tag{2.5}$$

Large time behavior of a parabolic wave.

LEMMA 2.2 (Parabolic wave, [20]). For a given function $h(z)$ analytic in z around $z=0$ with the property $h(0)=0$, there exist $\varepsilon_0>0$ and $D_0>0$ such that for $\vec{\boldsymbol{x}}\in\mathbb{R}^n$ with $|\vec{\boldsymbol{x}}|<4(\sigma+1)$

$$\left|\int_{|\vec{\eta}|<\varepsilon_0}\vec{\eta}^\alpha\frac{\sin(|\vec{\eta}|h(|\vec{\eta}|^2\sigma))}{|\vec{\eta}|}e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}-\frac{\nu}{2}|\vec{\eta}|^2\sigma}d\vec{\eta}\right|,\ \left|\int_{|\vec{\eta}|<\varepsilon_0}\vec{\eta}^\alpha\cos(|\vec{\eta}|h(|\vec{\eta}|^2\sigma))e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}-\frac{\nu}{2}|\vec{\eta}|^2\sigma}d\vec{\eta}\right|$$
$$\le D_0\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{D_0(\sigma+1)}}}{(1+\sigma)^{(n+|\alpha|)/2}} \text{ for } |\alpha|\le n/2. \tag{2.6}$$

Here, $\vec{\eta}^\alpha\equiv\eta_1^{\alpha_1}\cdots\eta_n^{\alpha_n}$ and $|\alpha|\equiv\sum_{i=1}^n\alpha_i$.

The proof of this lemma will be given in Appendix A.

Singularity of the Yukawa potential.

LEMMA 2.3. For $n=2,3$ there exists C_0 such that the Yukawa potential functions $\mathbb{Y}_n(\vec{\boldsymbol{x}})$ given in (1.12) satisfy

$$\begin{cases}\dfrac{(\big|\log|\vec{\boldsymbol{x}}|\big|+1)}{4\pi C_0}\le|\mathbb{Y}_2(\vec{\boldsymbol{x}})|\le C_0\dfrac{(\big|\log|\vec{\boldsymbol{x}}|\big|+1)}{\pi} \text{ for } |\vec{\boldsymbol{x}}|<1,\\ |\mathbb{Y}_n(\vec{\boldsymbol{x}})|\le C_0e^{-|\vec{\boldsymbol{x}}|/2} \text{ for } |\vec{\boldsymbol{x}}|\ge1,\ n=2,3,\\ |\nabla\mathbb{Y}_2(\vec{\boldsymbol{x}})|\le C_0\dfrac{e^{-|\vec{\boldsymbol{x}}|/2}}{|\vec{\boldsymbol{x}}|},\\ |\nabla\mathbb{Y}_3(\vec{\boldsymbol{x}})|\le C_0\dfrac{e^{-|\vec{\boldsymbol{x}}|/2}}{|\vec{\boldsymbol{x}}|^2}.\end{cases}$$

REMARK 2.4. This lemma assures that both $\mathbb{Y}_n$ and $\nabla\mathbb{Y}_n$ contain an integrable singularity at $\vec{\boldsymbol{x}}=0$ for $n=2,3$. ∎

3. Singularity removal (Proof of Theorem 1.2).

3.1. *Spectral approximation.* In [35], for a 1-dimensional compressible NS equation, the fundamental solution contains singularities, which are δ-functions. It is due to the fact that the factor $e^{\lambda_+(\vec{\eta})t}$ does not decay to zero as $\vec{\eta}\to\infty$, where $\lambda_\pm(\vec{\eta})$ are the spectra of the linear Navier-Stokes equations given in (2.3). In this section, we present an effective and universal method to remove the singularities without compromising global space-time structure, i.e. to construct sharper singular support functions.

The Taylor's expansions of $\lambda_\pm(\vec{\eta})$ at $|\vec{\eta}| = \infty$ are

$$\lambda_+ = -\nu^{-1} - \nu^{-3}|\vec{\eta}|^{-2} - 2\nu^{-5}|\vec{\eta}|^{-4} - 5\nu^{-7}|\vec{\eta}|^{-6} - 14\nu^{-9}|\vec{\eta}|^{-8} - 42\nu^{-11}|\vec{\eta}|^{-10} + O(1)|\vec{\eta}|^{-12}, \tag{3.1}$$

$$\lambda_- = -\lambda_+ - \nu|\vec{\eta}|^2, \tag{3.2}$$

where

$$\nu = \mu + \zeta.$$

By the expansions (3.1) and (3.2), one has that the $(1,1)$ entry of the semi-group $\mathbb{F}[U]$ in (2.2) satisfies

$$\lim_{\vec{\eta}\to\infty} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \lim_{\vec{\eta}\to\infty} \frac{\lambda_- e^{\lambda_+ t}}{\lambda_- - \lambda_+} = e^{-\nu^{-1}t} \neq 0.$$

This nondecaying property results in that the inverse Fourier transform of $\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}$ in (2.2) contains singularities in spatial variables. In order to realize the singularities, one starts to approximate $\lambda_\pm$ at $\vec{\eta} = \infty$ by introducing $\lambda_\pm^*$:

$$\begin{cases} \lambda_+^*(\vec{\eta}) = -\nu^{-1} - \dfrac{\nu^{-3}}{1+|\vec{\eta}|^2} - \dfrac{\nu^{-3}+2\nu^{-5}}{(1+|\vec{\eta}|^2)^2} - \dfrac{\nu^{-3}+4\nu^{-5}+5\nu^{-7}}{(1+|\vec{\eta}|^2)^3} \\ \qquad\qquad - \dfrac{\nu^{-3}+6\nu^{-5}+15\nu^{-7}+14\nu^{-9}}{(1+|\vec{\eta}|^2)^4}, \\ \lambda_-^*(\vec{\eta}) = -\nu|\vec{\eta}|^2 + \nu^{-1} + \dfrac{\nu^{-3}}{1+|\vec{\eta}|^2} + \dfrac{\nu^{-3}+2\nu^{-5}}{(1+|\vec{\eta}|^2)^2} + \dfrac{\nu^{-3}+4\nu^{-5}+5\nu^{-7}}{(1+|\vec{\eta}|^2)^3} \\ \qquad\qquad + \dfrac{\nu^{-3}+6\nu^{-5}+15\nu^{-7}+14\nu^{-9}}{(1+|\vec{\eta}|^2)^4} - \dfrac{J_0}{(1+|\vec{\eta}|^2)^5}. \end{cases} \tag{3.3}$$

Here, J_0 is a chosen sufficiently large positive number so that there exists $J_0^* \in (0, \nu^{-1})$ such that the spectral information satisfies

$$\sup_{\vec{\eta}\in\mathscr{D}_{1/2}} Re(\lambda_\pm^*(\vec{\eta})) < -J_0^*, \tag{3.4}$$

and

$$\inf_{\vec{\eta}\in\mathscr{D}_{1/2}} |\lambda_-^*(\vec{\eta}) - \lambda_+^*(\vec{\eta})| > 0. \tag{3.5}$$

The approximated spectrum λ_+^* (λ_-^*) is up to degree 10 approximation of λ_+ (λ_-) for $\vec{\eta} \to \infty$, i.e.

$$\sup_{\vec{\eta}\in\mathscr{D}_{1/2}} |\vec{\eta}|^{10}(\lambda_\pm(\vec{\eta}) - \lambda_\pm^*(\vec{\eta})) < \infty,$$

where $\mathscr{D}_{1/2}$ is defined by (2.1). This degree 10 approximation yields the following lemma:

LEMMA 3.1. The approximated spectra $\lambda_\pm^*$ given in (3.3) satisfy

$$\left| \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*} \right| \le O(1)(1+|\vec{\eta}|)^{-10} \quad \text{for } \vec{\eta} \in \mathbb{R}^n. \tag{3.6}$$

This lemma yields that for $n \le 3$,

$$\left\| \mathbb{F}^{-1}\left[\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}\right](\cdot, t)\right\|_{L^\infty(\mathbb{R}^n)} = O(1).$$

This estimate asserts that $\mathbb{F}^{-1}\left[\frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}\right]$ contains the singular part of $\mathbb{F}^{-1}\left[\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right]$.

It remains to analyze the structures of $\mathbb{F}^{-1}[\frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}]$ and $\mathbb{F}^{-1}[\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}]$ in the space-time domain.

3.2. *A differential equation associated with* $\mathbb{F}^{-1}[e^{\lambda_\pm^* t}]$. The system (1.3) can be reduced to a scalar equation. One applies ∂_t and $\nabla\cdot$ to the first equation and the second equation in (1.3) respectively to yield another two equations. These two new equations and the conservation laws yield that

$$\begin{cases} \rho_t = -\nabla \cdot \vec{\boldsymbol{u}}, \\ \partial_t^2 \rho + \nabla \cdot \vec{\boldsymbol{u}}_t = 0, \\ \nabla \cdot \vec{\boldsymbol{u}}_t + \Delta\rho - \nu\Delta(\nabla \cdot \vec{\boldsymbol{u}}) = 0, \end{cases}$$

which results in

$$\partial_t^2 \rho - \Delta\rho - \nu\partial_t \Delta\rho = 0. \tag{3.7}$$

The Fourier transform of (3.7) is

$$\left(\partial_t^2 + |\vec{\eta}|^2(1 + \nu\partial_t)\right)\mathbb{F}[\rho] = 0.$$

For convenience, we denote

$$\begin{cases} H_0(\vec{\boldsymbol{x}}, t) \equiv \mathbb{F}^{-1}\left[\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right], \\ H_0^*(\vec{\boldsymbol{x}}, t) \equiv \mathbb{F}^{-1}\left[\frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}\right], \end{cases} \qquad \begin{cases} H_1(\vec{\boldsymbol{x}}, t) \equiv \mathbb{F}^{-1}\left[\dfrac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right], \\ H_1^*(\vec{\boldsymbol{x}}, t) \equiv \mathbb{F}^{-1}\left[\dfrac{e^{\lambda_+^* t} - e^{\lambda_-^* t}}{\lambda_+^* - \lambda_-^*}\right], \end{cases}$$

and $H_k^*(\mathbf{1}ym\vec{b}olx, t)$ are introduced as the *singular support function* for $H_k(\vec{\boldsymbol{x}}, t)$ for $k = 0, 1$. The functions $H_k(\vec{\boldsymbol{x}}, t)$ are the solutions of (3.7) corresponding to different initial conditions:

$$\begin{cases} (\partial_t^2 - \Delta - \nu\partial_t\Delta)H_0(\vec{\boldsymbol{x}}, t) = 0 \text{ for } (\vec{\boldsymbol{x}}, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ H_0(\vec{\boldsymbol{x}}, 0) = \delta_n(\vec{\boldsymbol{x}}), \\ \partial_t H_0(\vec{\boldsymbol{x}}, 0) = 0, \end{cases}$$

$$\begin{cases} (\partial_t^2 - \triangle - \nu\partial_t\triangle)H_1(\vec{\boldsymbol{x}}, t) = 0 \text{ for } (\vec{\boldsymbol{x}}, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ H_1(\vec{\boldsymbol{x}}, 0) = 0, \\ \partial_t H_1(\vec{\boldsymbol{x}}, 0) = \delta_n(\vec{\boldsymbol{x}}), \end{cases} \tag{3.8}$$

where $\delta_n(\vec{\boldsymbol{x}})$ is the n-dimensional delta function.

Denote

$$\begin{cases}\mathscr{E}(\vec{\boldsymbol{x}},t)\equiv-(\partial_t^2-\Delta-\nu\partial_t\Delta)H_0^*(\vec{\boldsymbol{x}},t),\\ h^*(\vec{\boldsymbol{x}},t)\equiv H_0(\vec{\boldsymbol{x}},t)-H_0^*(\vec{\boldsymbol{x}},t).\end{cases}$$
$$\begin{cases}\mathscr{F}(\vec{\boldsymbol{x}},t)\equiv-\left(\partial_t^2-\triangle-\nu\partial_t\triangle\right)H_1^*(\vec{\boldsymbol{x}},t),\\ m^*(\vec{\boldsymbol{x}},t)\equiv H_1(\vec{\boldsymbol{x}},t)-H_1^*(\vec{\boldsymbol{x}},t).\end{cases}$$

The functions $\mathscr{E}(\vec{\boldsymbol{x}},t)$ and $\mathscr{F}(\vec{\boldsymbol{x}},t)$ can be realized as the truncation errors of the singular support functions for $H_0^*(\vec{\boldsymbol{x}},t)$ and $H_1^*(\vec{\boldsymbol{x}},t)$:

$$\begin{cases}(\partial_t^2-\Delta-\nu\partial_t\Delta)H_0^*(\vec{\boldsymbol{x}},t)=-\mathscr{E}(\vec{\boldsymbol{x}},t)\text{ for }(\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}_+,\\ H_0^*(\vec{\boldsymbol{x}},0)=\delta_n(\vec{\boldsymbol{x}}),\\ \partial_tH_0^*(\vec{\boldsymbol{x}},0)=0.\end{cases}$$

$$\begin{cases}(\partial_t^2-\Delta-\nu\partial_t\Delta)H_1^*(\vec{\boldsymbol{x}},t)=-\mathscr{F}(\vec{\boldsymbol{x}},t)\text{ for }(\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}_+,\\ H_1^*(\vec{\boldsymbol{x}},0)=0,\\ \partial_tH_1^*(\vec{\boldsymbol{x}},0)=\delta_n(\vec{\boldsymbol{x}}).\end{cases}$$

The error functions $h^*(\vec{\boldsymbol{x}},t)$ and $m^*(\vec{\boldsymbol{x}},t)$ satisfy the equations

$$\begin{aligned}&\begin{cases}(\partial_t^2-\Delta-\nu\partial_t\Delta)h^*(\vec{\boldsymbol{x}},t)-\mathscr{E}(\vec{\boldsymbol{x}},t)=0\text{ for }(\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}_+,\\ h^*(\vec{\boldsymbol{x}},0)=0,\\ \partial_th^*(\vec{\boldsymbol{x}},0)=0.\end{cases}\\ &\begin{cases}(\partial_t^2-\Delta-\nu\partial_t\Delta)m^*(\vec{\boldsymbol{x}},t)-\mathscr{F}(\vec{\boldsymbol{x}},t)=0\text{ for }(\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}_+,\\ m^*(\vec{\boldsymbol{x}},0)=0,\\ \partial_tm^*(\vec{\boldsymbol{x}},0)=0.\end{cases}\end{aligned}\tag{3.9}$$

One needs to access both structures of $\mathscr{E}(\vec{\boldsymbol{x}},t)$ and $\mathscr{F}(\vec{\boldsymbol{x}},t)$ in the space-time domain as well as the transform domain. The Fourier transforms $\mathbb{F}[\mathscr{E}](\vec{\eta},t)$ and $\mathbb{F}[\mathscr{F}](\vec{\eta},t)$ satisfy

$$\begin{aligned}\mathbb{F}[\mathscr{E}](\vec{\eta},t)&=-(\partial_t^2+|\vec{\eta}|^2(1+\nu\partial_t))\frac{\lambda_+^*e^{\lambda_-^*t}-\lambda_-^*e^{\lambda_+^*t}}{\lambda_+^*-\lambda_-^*}\\ &=\frac{-\left((\lambda_-^*)^2+|\vec{\eta}|^2(1+\nu\lambda_-^*)\right)\lambda_+^*e^{\lambda_-^*t}-\left((\lambda_+^*)^2+|\vec{\eta}|^2(1+\nu\lambda_+^*)\right)\lambda_-^*e^{\lambda_+^*t}}{\lambda_+^*-\lambda_-^*},\end{aligned}\tag{3.10}$$

$$\begin{aligned}\mathbb{F}[\mathscr{F}](\vec{\eta},t)&=-(\partial_t^2+|\vec{\eta}|^2(1+\nu\partial_t))\frac{e^{\lambda_+^*t}-e^{\lambda_-^*t}}{\lambda_+^*-\lambda_-^*}\\ &=\frac{-\left((\lambda_+^*)^2+|\vec{\eta}|^2(1+\nu\lambda_+^*)\right)e^{\lambda_+^*t}-\left((\lambda_-^*)^2+|\vec{\eta}|^2(1+\nu\lambda_-^*)\right)e^{\lambda_-^*t}}{\lambda_+^*-\lambda_-^*}.\end{aligned}$$

3.3. *Truncation errors and singular support functions.*

3.3.1. *Truncation errors.* To access the truncation error $\mathbb{F}[\mathscr{E}](\vec{\eta},t)$, one uses the expansions (3.3) and (3.10) to obtain the following lemma:

LEMMA 3.2. The approximated spectra $\lambda_\pm^*$ satisfy that there exists $C_0>0$ such that

$$\left|\frac{((\lambda_-^*)^2+|\vec{\eta}|^2(1+\nu\lambda_-^*))}{\lambda_+^*-\lambda_-^*}\right|,\ \left|\frac{((\lambda_+^*)^2+|\vec{\eta}|^2(1+\nu\lambda_+^*))}{\lambda_+^*-\lambda_-^*}\right|\le\frac{C_0}{(1+|\vec{\eta}|^2)^5}.$$

By Lemma 3.2, (3.10), and (3.4) one has the following lemma:

LEMMA 3.3. The Fourier transforms $\mathbb{F}[\mathscr{E}](\vec{\eta},t)$ and $\mathbb{F}[\mathscr{F}](\vec{\eta},t)$ of truncation error functions are analytic in $\mathscr{D}_{1/2}$ and satisfy

$$\begin{cases}|\mathbb{F}[\mathscr{E}](\vec{\eta},t)|=O(1)\dfrac{e^{-J_0^*t}}{(1+|\vec{\eta}|)^8},\\ |\mathbb{F}[\mathscr{F}](\vec{\eta},t)|=O(1)\dfrac{e^{-J_0^*t}}{(1+|\vec{\eta}|)^{10}},\end{cases}$$

for $\vec{\eta}\in\mathscr{D}_{1/2}$.

With Lemma 3.3 and Lemma 2.1 one has the following lemma for truncation errors:

LEMMA 3.4. For $n=2,3$, there exists $C_0>0$ such that

$$|\mathscr{E}(\vec{\boldsymbol{x}},t)|,|\mathscr{F}(\vec{\boldsymbol{x}},t)|\le C_0e^{-(|\vec{\boldsymbol{x}}|+t)/C_0},$$

$$\int_{\mathbb{R}^n}e^{|\vec{\boldsymbol{x}}|/C_0}|\partial_{\vec{\boldsymbol{x}}}^\alpha\mathscr{E}(\vec{\boldsymbol{x}},t)|^2d\vec{\boldsymbol{x}}\le O(1)e^{-t/C_0}\text{ for }|\alpha|<8-\frac{n}{2},\tag{3.11}$$

$$\int_{\mathbb{R}^n}e^{|\vec{\boldsymbol{x}}|/C_0}|\partial_{\vec{\boldsymbol{x}}}^\alpha\mathscr{F}(\vec{\boldsymbol{x}},t)|^2d\vec{\boldsymbol{x}}\le O(1)e^{-t/C_0}\text{ for }|\alpha|<10-\frac{n}{2}.$$

REMARK 3.5. This lemma asserts that there is no singularity in $\mathscr{E}(\vec{\boldsymbol{x}},t)$ and $\mathscr{F}(\vec{\boldsymbol{x}},t)$; and the truncation errors decay exponentially fast in the space-time domain. ∎

3.3.2. *Singular support functions.* One breaks $\frac{\lambda_+^*e^{\lambda_-^*t}-\lambda_-^*e^{\lambda_+^*t}}{\lambda_+^*-\lambda_-^*}$ as follows:

$$\frac{\lambda_+^*e^{\lambda_-^*t}-\lambda_-^*e^{\lambda_+^*t}}{\lambda_+^*-\lambda_-^*}=\left(1-\frac{\lambda_+^*}{\lambda_+^*-\lambda_-^*}\right)e^{\lambda_+^*t}+\frac{\lambda_+^*e^{\lambda_-^*t}}{\lambda_+^*-\lambda_-^*}.$$

Apply the representation of λ_+^* in (3.3) to yield that

$$\begin{aligned}e^{\lambda_+^*t}&=e^{-\nu^{-1}t}e^{-\frac{\nu^{-3}}{1+|\vec{\eta}|^2}t-\frac{\nu^{-3}+2\nu^{-5}}{(1+|\vec{\eta}|^2)^2}t-\frac{\nu^{-3}+4\nu^{-5}+5\nu^{-7}}{(1+|\vec{\eta}|^2)^3}t-\frac{\nu^{-3}+6\nu^{-5}+15\nu^{-7}+14\nu^{-9}}{(1+|\vec{\eta}|^2)^4}t}\\ &=e^{-\nu^{-1}t}-te^{-\nu^{-1}t}\frac{\nu^{-3}}{1+|\vec{\eta}|^2}-te^{-\nu^{-1}t}\frac{(\nu^{-3}+2\nu^{-5}-\frac{1}{2}\nu^{-6}t)}{(1+|\vec{\eta}|^2)^2}+\mathscr{O}(\vec{\eta},t)\,.\end{aligned}\tag{3.12}$$

Here, the function $\mathscr{O}(\vec{\eta},t)$ is analytic in $\vec{\eta}\in\mathscr{D}_{1/2}$ and satisfies

$$|\mathscr{O}(\vec{\eta},t)|=O(1)\frac{e^{-\nu^{-1}t/2}}{(1+|\vec{\eta}|^2)^3}\text{ for }\vec{\eta}\in\mathscr{D}_{1/2}.\tag{3.13}$$

We denote $\mathbb{Y}_n(\vec{\boldsymbol{x}})$ as the Yukawa potential in n-dimensional space with a unit mass:

$$\mathbb{Y}_n(\vec{\boldsymbol{x}}) \equiv \mathbb{F}^{-1}\left[\frac{1}{1+|\vec{\eta}|^2}\right](\vec{\boldsymbol{x}}) \text{ for } \vec{\boldsymbol{x}} \in \mathbb{R}^n.$$

They have explicit forms for $n = 1, 2, 3$, etc.:

$$\begin{cases} \mathbb{Y}_1(x) = \dfrac{e^{-|x|}}{2\sqrt{2\pi}}, \\ \mathbb{Y}_2(\vec{\boldsymbol{x}}) = \dfrac{1}{2\pi} BesselK_0(|\vec{\boldsymbol{x}}|), \\ \mathbb{Y}_3(\vec{\boldsymbol{x}}) = -\dfrac{e^{-|\vec{\boldsymbol{x}}|}}{4\pi|\vec{\boldsymbol{x}}|}, \end{cases}$$

where $BesselK_0(x)$ is the modified Bessel function of the second kind of degree 0.

Apply Lemma 2.1 to (3.13) and (3.12) concludes that there exists $C_0 > 0$ such that

$$\left|\mathbb{F}^{-1}[e^{\lambda_+^* t}](\vec{\boldsymbol{x}}, t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \nu^{-3}te^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}})\right| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}. \tag{3.14}$$

Due to (3.5), the denominator $\lambda_+^* - \lambda_-^*$ of the function $\frac{\lambda_+^*(e^{\lambda_-^* t} - e^{\lambda_+^* t})}{\lambda_+^* - \lambda_-^*}$ is nonzero for $\vec{\eta} \in \mathscr{D}_{1/2}$ and the function is analytic in $\mathscr{D}_{1/2}$. This analytic property, the following decaying properties:

$$\begin{cases} \dfrac{\lambda_+^*}{\lambda_+^* - \lambda_-^*} = \dfrac{-\nu^{-2}}{1+|\vec{\eta}|^2} + \dfrac{-\nu^{-2} - 3\nu^{-4}}{(1+|\vec{\eta}|^2)^2} + O(|\vec{\eta}|^{-6}) \text{ as } |\vec{\eta}| \to \infty, \\ \dfrac{1}{\lambda_+^* - \lambda_-^*} = \dfrac{\nu^{-1}}{1+|\vec{\eta}|^2} + \dfrac{\nu^{-1} + 3\nu^{-3}}{(1+|\vec{\eta}|^2)^2} + O(|\vec{\eta}|^{-6}) \text{ as } |\vec{\eta}| \to \infty, \end{cases}$$

and (3.12) together result in the following asymptotic behaviors:

$$\begin{cases} \dfrac{\lambda_+^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*} = \dfrac{-\nu^{-2}e^{-\nu^{-1}t}}{1+|\vec{\eta}|^2} + \mathscr{O}_+(\vec{\eta}, t), \\ \dfrac{e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*} = \dfrac{\nu^{-1}e^{-\nu^{-1}t}}{1+|\vec{\eta}|^2} + \tilde{\mathscr{O}}_+(\vec{\eta}, t), \end{cases} \tag{3.15}$$

with the functions $\mathscr{O}_+(\vec{\eta}, t)$ and $\tilde{\mathscr{O}}_+(\vec{\eta}, t)$ analytic in $\vec{\eta} \in \mathscr{D}_{1/2}$ and with properties similar to (3.13):

$$|\mathscr{O}_+(\vec{\eta}, t)|,\ |\tilde{\mathscr{O}}_+(\vec{\eta}, t)| \ \le C_0 \frac{e^{-t/C_0}}{(1+|\vec{\eta}|^4)}$$

for $C_0 > 0$. It together with Lemma 2.1 yields that there exists $C_0 > 0$ such that

$$|\mathbb{F}^{-1}[\mathscr{O}_+](\vec{\boldsymbol{x}}, t)|,\ |\mathbb{F}^{-1}[\tilde{\mathscr{O}}_+](\vec{\boldsymbol{x}}, t)| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} \text{ for } n = 2, 3.$$

The function $\mathbb{F}^{-1}[\lambda_+^* e^{\lambda_-^* t}/(\lambda_-^* - \lambda_+^*)]$ does not contain singularities in $\vec{\boldsymbol{x}}$ variable due to its asymptotic when $|\vec{\eta}| \to \infty$ for $t > 0$:

$$\left|\lambda_+^* e^{\lambda_-^* t}/(\lambda_-^* - \lambda_+^*)\right| \le K_0 \frac{e^{-t|\vec{\eta}|^2/K_0 - J_0^* t}}{1+|\vec{\eta}|^2}$$

for some $K_0 > 0$. One has that there exist $C_0, C_1 > 0$ such that for $\varepsilon \in (-1/2, 1/2)$

$$\begin{aligned}\int_{\substack{Im(\eta^k)=\varepsilon\\1\le k\le n}} \left|\frac{\lambda_+^* e^{\lambda_-^* t}}{\lambda_+^*-\lambda_-^*}\right| d\vec{\eta} &\le C_1\int_{\mathbb{R}^n}\frac{e^{-|\vec{\eta}|^2 t/C_1 - J_0^* t}}{(1+|\vec{\eta}|)^2}d\vec{\eta} = C_1\Gamma(n)\int_0^\infty \frac{e^{-r^2t/C_1-J_0^*t}}{(1+r)^2}r^{n-1}dr\\ &\le C_0 e^{-t/C_0}L_n(t),\end{aligned} \tag{3.16}$$

where

$$L_n(t) \equiv \begin{cases} 1, \ n=1,\\ \log(t), \ n=2,\\ t^{-\frac{1}{2}}, \ n=3.\end{cases} \tag{3.17}$$

Thus, (3.16) together with Lemma 2.1 results in that there exists $C_0 > 0$ such that

$$|j_0^*(\vec{\boldsymbol{x}}, t)| \le C_0 e^{-|\vec{\boldsymbol{x}}|/C_0 - \nu^{-1}t/2} L_n(t). \tag{3.18}$$

Here, one denotes

$$\begin{cases} j_0^*(\vec{\boldsymbol{x}}, t) \equiv \mathbb{F}^{-1}[\lambda_+^* e^{\lambda_-^* t}/(\lambda_+^* - \lambda_-^*)],\\ j_1^*(\vec{\boldsymbol{x}}, t) \equiv -\mathbb{F}^{-1}[e^{\lambda_-^* t}/(\lambda_+^* - \lambda_-^*)],\end{cases}$$

and thus obtains the estimate for H_0^* by combining the estimates (3.14), (3.15) and (3.18):

$$\left|H_0^*(\vec{\boldsymbol{x}}, t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \left(\nu^{-3}t + \nu^{-2}\right) e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}}) - j_0^*(\vec{\boldsymbol{x}}, t)\right| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}. \tag{3.19}$$

Similar to (3.18), since

$$|\mathbb{F}[j_1^*](\vec{\eta}, t)| \le C e^{-t|\vec{\eta}|^2/C - J_0^* t}/(1+|\vec{\eta}|^2),$$

there exists $C_0 > 0$ such that for $t > 0$

$$|j_1^*(\vec{\boldsymbol{x}}, t)| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} L_n(t). \tag{3.20}$$

Combining (3.15) and (3.20), one has

$$\left|H_1^*(\vec{\boldsymbol{x}}, t) - \nu^{-1}e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}}) - j_1^*(\vec{\boldsymbol{x}}, t)\right| = O(1)e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}. \tag{3.21}$$

Thus, we have finished the proof of Theorem 1.2.

4. Long wave-short wave decomposition (Proof of Theorem 1.3). We introduce a long wave-short wave decomposition

$$\begin{cases} f(\vec{\boldsymbol{x}}, t) = f_L(\vec{\boldsymbol{x}}, t) + f_S(\vec{\boldsymbol{x}}, t),\\ \mathbb{F}[f_L] = \Lambda\left(\frac{|\vec{\eta}|}{\varepsilon_0}\right)\mathbb{F}[f],\\ \mathbb{F}[f_S] = \left(1 - \Lambda\left(\frac{|\vec{\eta}|}{\varepsilon_0}\right)\right)\mathbb{F}[f],\end{cases}$$

with the parameter $\varepsilon_0 \ll 1$, where

$$\Lambda(r) \equiv \mathbf{H}(1 - |r|).$$

4.1. *Splitting of hyperbolic and parabolic waves in long wave component.* For $|\vec{\eta}| \le \varepsilon_0 \ll 1$, one substitutes (2.3) into $\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}$ to yield that

$$\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu|\vec{\eta}|^2 \sin\left(|\vec{\eta}|\sqrt{1-\nu^2|\vec{\eta}|^2/4}\; t\right)}{2|\vec{\eta}|\sqrt{1-\nu^2|\vec{\eta}|^2/4}} + \cos\left(|\vec{\eta}|\sqrt{1-\nu^2|\vec{\eta}|^2/4}\; t\right)\right).$$

Then, one expands

$$\sqrt{1-\nu^2|\vec{\eta}|^2/4} = 1 + h(|\vec{\eta}|^2),$$

where $h(z)$ is a locally analytic function around $z = 0$ with the property $h(0) = 0$ so that

$$\begin{aligned}\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\Bigg(|\vec{\eta}|\sin(|\vec{\eta}|t)\left(\frac{\nu\cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{|\vec{\eta}|}\right)\\ + \cos(|\vec{\eta}|t)\left(\frac{\nu|\vec{\eta}|\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)\right)\Bigg). \quad (4.1)\end{aligned}$$

Here, since the functions $\cos(x)$ and $\sin(x)/x$ are even functions, every term in (4.1) is locally analytic in $\vec{\eta}$ around $\vec{\eta} = 0$. Thus, we have the following identity:

$$\begin{aligned}H_{0L}(\vec{\boldsymbol{x}},t) &\equiv \int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\Lambda(|\vec{\eta}|/\varepsilon_0)d\vec{\eta}\\
&= \int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\left(\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu\cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{|\vec{\eta}|}\right)|\vec{\eta}|^2\right)\\
&\qquad\cdot\left(\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right)d\vec{\eta}\\
&+ \int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\left(\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu|\vec{\eta}|\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)\right)\right)(\cos(|\vec{\eta}|t))\,d\vec{\eta}\\
&= \left(\int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu\cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{|\vec{\eta}|}\right)|\vec{\eta}|^2 d\vec{\eta}\right)\\
&\qquad\underset{\vec{\boldsymbol{x}}}{*}\left(\mathbb{F}^{-1}\left[\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right]\right)\\
&+ \left(\int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu|\vec{\eta}|\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)\right)d\vec{\eta}\right)\\
&\qquad\underset{\vec{\boldsymbol{x}}}{*}\left(\mathbb{F}^{-1}\left[\cos(|\vec{\eta}|t)\right]\right)\\
&= \mathbb{D}_1 \underset{\vec{\boldsymbol{x}}}{*} \mathbb{S}_n + \mathbb{D}_2 \underset{\vec{\boldsymbol{x}}}{*} \mathbb{C}_n, \quad (4.2)\end{aligned}$$

where $\mathbb{S}_n$ and $\mathbb{C}_n$ are the hyperbolic waves, the sine and cosine transforms (the sine transforms are given explicitly in (2.5) and (2.4) for $n = 2, 3$ respectively); and

$$\begin{cases} \mathbb{D}_1(\vec{x}, t) \equiv \int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\eta}}\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu\cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{|\vec{\eta}|}\right)|\vec{\eta}|^2 d\vec{\eta}, \\ \mathbb{D}_2(\vec{x}, t) \equiv \int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\eta}}\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\frac{\nu|\vec{\eta}|\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)}{1+h(|\vec{\eta}|^2)} + \cos\left(|\vec{\eta}|h(|\vec{\eta}|^2)t\right)\right)d\vec{\eta} \end{cases}$$

are the parabolic waves. The notion $\underset{\vec{x}}{*}$ denotes the convolution in the space variable $\vec{x}$.

4.2. *Long wave component in a finite Mach region $|\vec{x}| < 3t$ for H_0.*

LEMMA 4.1. For space dimension $n = 2, 3$ and for $|\vec{x}| < 3t$, there exists $C_0 > 1$ such that

$$|\partial_{\vec{x}}^{\alpha} h_L^*(\vec{x}, t)|,\ |\partial_{\vec{x}}^{\alpha} H_{0L}(\vec{x}, t)| = O(1)\left(1 + t^{\frac{1+|\alpha|}{2}}\right)^{-1}\mathbb{K}_n(\vec{x}, t; C_0), \tag{4.3}$$

where

$$\mathbb{K}_2(\vec{x}, t; C_0) = \frac{\mathbf{H}(t - |\vec{x}|)}{1 + \sqrt{(t + |\vec{x}|)(t - |\vec{x}| + \sqrt{t})}} + \frac{e^{-\frac{(|\vec{x}|-t)^2}{4C_0(t+1)}}}{(1+t)^{\frac{3}{4}}},$$
$$\mathbb{K}_3(\vec{x}, t; C_0) = \frac{e^{-\frac{(|\vec{x}|-t)^2}{4C_0(t+1)}}}{(1+t)^{\frac{3}{2}}}.$$

Proof. By (2.6), there exists $C_0 > 0$ such that for $|\vec{x}| < 4t$

$$\begin{aligned} \left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_1(\vec{x}, t)\right| &\le C_0(1+t)^{-(n+2+|\alpha|)/2}e^{-\frac{|\vec{x}|^2}{C_0(t+1)}}, \\ \left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_2(\vec{x}, t)\right| &\le C_0(1+t)^{-(n+|\alpha|)/2}e^{-\frac{|\vec{x}|^2}{C_0(t+1)}}. \end{aligned} \tag{4.4}$$

By the fact that $\mathbb{S}_n(\vec{x}, t) = \mathbb{C}_n(\vec{x}, t) = 0$ for $|\vec{x}| > t$, one has the following estimates from (4.4) for $|\vec{x}| < 3t$ for the cases $n = 2$ and $n = 3$.

CASE. $n = 2$ and $|\vec{x}| < 3t$. From (4.4) and Lemma B.1, one has that

$$\begin{aligned} \left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_1 \underset{\vec{x}}{*} \mathbb{S}_2(\vec{x}, t)\right| &= \left|\int_{\mathbb{R}^2}\mathbb{S}_2(\vec{x} - \vec{x}_*, t)\partial_{\vec{x}_*}^{\alpha}\mathbb{D}_1(\vec{x}_*, t)d\vec{x}_*\right| \\ &= \frac{1}{2\pi}\left|\int_{|\vec{x}-\vec{x}_*|<t}\frac{1}{\sqrt{t^2 - |\vec{x} - \vec{x}_*|^2}}\partial_{\vec{x}_*}^{\alpha}\mathbb{D}_1(\vec{x}_*, t)d\vec{x}_*\right| \\ &= O(1)\int_{|\vec{x}-\vec{x}_*|<t}\frac{1}{\sqrt{t^2 - |\vec{x} - \vec{x}_*|^2}}\frac{e^{-\frac{|\vec{x}_*|^2}{C_0(t+1)}}}{(1+t)^{2+|\alpha|/2}}d\vec{x}_* \\ &= O(1)\frac{1}{1 + t^{|\alpha|/2}}\left(\frac{\mathbf{H}(t - |\vec{x}|)}{1 + t^{3/2}(t - |\vec{x}| + \sqrt{t})^{1/2}} + \frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{7/4}}\right), \end{aligned}$$

and

$$\left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_2 \underset{\vec{x}}{*} \mathbb{C}_2(\vec{x},t)\right| = \left|\int_{\mathbb{R}^2} \mathbb{C}_2(\vec{x}-\vec{x}_*,t)\partial_{\vec{x}_*}^{\alpha}\mathbb{D}_2(\vec{x}_*,t)d\vec{x}_*\right|$$

$$= O(1)\left|\int\limits_{|\vec{x}-\vec{x}_*|<t} \frac{1}{\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}}\left|\partial_{\vec{x}_*}^{\alpha}\nabla_{\vec{x}}\mathbb{D}_2(\vec{x}_*,t)\right| d\vec{x}_*\right|$$

$$+ O(1)\frac{1}{1+t}\left|\int\limits_{|\vec{x}-\vec{x}_*|<t} \frac{1}{\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}}\left|\partial_{\vec{x}_*}^{\alpha}\mathbb{D}_2(\vec{x}_*,t)\right| d\vec{x}_*\right|$$

$$= O(1)\int\limits_{|\vec{x}-\vec{x}_*|<t} \frac{1}{\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}}\frac{e^{-\frac{|\vec{x}_*|^2}{C_0(t+1)}}}{(1+t)^{(3+|\alpha|)/2}}d\vec{x}_*$$

$$= O(1)\frac{1}{1+t^{|\alpha|/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}}\right).$$

Case ($n=3$ and $|\vec{x}|<3t$). Similarly, from (4.4) and Lemma B.1 one has that

$$\left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_1 \underset{\vec{x}}{*} \mathbb{S}_3(\vec{x},t)\right| = \left|\int_{\mathbb{R}^3} \mathbb{S}_3(\vec{x}-\vec{x}_*,t)\partial_{\vec{x}}^{\alpha}\mathbb{D}_1(\vec{x}_*,t)d\vec{x}_*\right|$$

$$= O(1)\int\limits_{|\vec{x}-\vec{x}_*|=t} \frac{e^{-\frac{|\vec{x}_*|^2}{C_0(1+t)}}}{t(1+t)^{(5+|\alpha|)/2}}d\vec{x}_* = O(1)\frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{(5+|\alpha|)/2}},$$

$$\left|\partial_{\vec{x}}^{\alpha}\mathbb{D}_2 \underset{\vec{x}}{*} \mathbb{C}_3(\vec{x},t)\right| = \left|\int_{\mathbb{R}^3} \mathbb{C}_3(\vec{x}-\vec{x}_*,t)\partial_{\vec{x}}^{\alpha}\mathbb{D}_2(\vec{x}_*,t)d\vec{x}_*\right|$$

$$= O(1)\int\limits_{|\vec{x}-\vec{x}_*|=t} \frac{e^{-\frac{|\vec{x}_*|^2}{C_0(1+t)}}}{t(1+t)^{2+|\alpha|/2}}d\vec{x}_* = O(1)\frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{2+|\alpha|/2}}.$$

The above cases conclude (4.3) for $\partial_{\vec{x}}^{\alpha}H_L(\vec{x},t)$.

The choices of λ_+^* and λ_-^* in (3.3) and (3.4) yield

$$|\partial_{\vec{x}}^{\alpha}H_{0L}^*(\vec{x},t)| \le \int\limits_{|\vec{\eta}|<\varepsilon_0} |\vec{\eta}|^{|\alpha|}\left|\mathbb{F}[H_0^*](\vec{\eta},t)\right| d\vec{\eta} = O(1)\varepsilon_0^{n+|\alpha|}e^{-J_0^*t}. \tag{4.5}$$

The identity $H_0(\vec{x},t) = h^*(\vec{x},t) + H_0^*(\vec{x},t)$, and the above estimates result in that for $|\vec{x}|<3t$,

$$|\partial_{\vec{x}}^{\alpha}h_L^*(\vec{x},t)| = O(1)\frac{1}{1+t^{(1+|\alpha|)/2}}\mathbb{K}_n(\vec{x},t) \text{ for } n=2,3. \tag{4.6}$$

The lemma follows. □

4.3. *Short wave component in a finite Mach region* $|\vec{x}|<3t$. In Lemma 4.1 and (4.5), we have the long wave components $H_{0L}(\vec{x},t)$, $h_L^*(\vec{x},t)$ and $H_{0L}^*(\vec{x},t)$ for $|\vec{x}|<3t$. Next one considers the short wave components $h_S^*(\vec{x},t)$, H_{0S}^*, $H_{0S}(\vec{x},t)$, and the whole wave $H_0(\vec{x},t)$ for the finite Mach number region $|\vec{x}|\le 3t$.

By (3.19), (4.5) and $H_{0S}^* + H_{0L}^* = H_0^*$, one has that for $|\vec{\boldsymbol{x}}| < 3t$

$$\begin{aligned}&\left|H_{0S}^*(\vec{\boldsymbol{x}},t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \left(\nu^{-3}t + \nu^{-2}\right)e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}})\right| \le |H_{0L}^*(\vec{\boldsymbol{x}},t)| + |j_0^*(\vec{\boldsymbol{x}},t)| \quad (4.7)\\ &\qquad + \left|H_0^*(\vec{\boldsymbol{x}},t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \left(\nu^{-3}t + \nu^{-2}\right)e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}}) - j_0^*(\vec{\boldsymbol{x}},t)\right|\\ &= O(1)e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}L_n(t).\end{aligned}$$

This estimate, (4.5), and (4.3) give the short wave component $H_{0S}^*(\vec{\boldsymbol{x}},t)$ in the region $|\vec{\boldsymbol{x}}| < 3t$.

By the property of $\lambda_\pm$ one has that for $|\eta| > \varepsilon_0$ there exists $\varepsilon_1 > 0$ such that

$$(1 - \Lambda(|\vec{\eta}|/\varepsilon_0))\left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right) = O(1)e^{-\varepsilon_1 t}.$$

It follows together with (3.6) that there exists $\varepsilon_2 > 0$ such that for $n \le 3$ and $0 \le |\alpha| \le 6$,

$$\begin{aligned}&|\partial_{\vec{\boldsymbol{x}}}^\alpha h_S^*(\vec{\boldsymbol{x}},t)|\\ &\quad = \left|\int_{\mathbb{R}^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}(1 - \Lambda(|\vec{\eta}|/\varepsilon_0))\,|\vec{\eta}|^{|\alpha|}\left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+^* e^{\lambda_-^* t} - \lambda_-^* e^{\lambda_+^* t}}{\lambda_+^* - \lambda_-^*}\right)d\vec{\eta}\right|\\ &\quad = O(1)e^{-\varepsilon_2 t}.\end{aligned} \tag{4.8}$$

From (4.6) and (4.8), one has the pointwise estimates of $h^*(\vec{\boldsymbol{x}},t)$ in the region $|\vec{\boldsymbol{x}}| < 3t$ by $h^*(\vec{\boldsymbol{x}},t) = h_L^*(\vec{\boldsymbol{x}},t) + h_S^*(\vec{\boldsymbol{x}},t)$; and one has the pointwise estimates of $H_0(\vec{\boldsymbol{x}},t)$ for $|\vec{\boldsymbol{x}}| < 3t$ by $H_0(\vec{\boldsymbol{x}},t) = H_{0L}(\vec{\boldsymbol{x}},t) + H_{0S}^*(\vec{\boldsymbol{x}},t) + h_{0S}^*(\vec{\boldsymbol{x}},t)$ with $H_{0L}(\vec{\boldsymbol{x}},t)$ given in Lemma 4.1, $H_S^*(\vec{\boldsymbol{x}},t)$ in (4.7) and $h_S^*(\vec{\boldsymbol{x}},t)$ in (4.8) to conclude the following lemma:

LEMMA 4.2. For $n = 2, 3$, $0 \le |\alpha| \le 6$ and $|\vec{\boldsymbol{x}}| < 3t$,

$$|\partial_{\vec{\boldsymbol{x}}}^\alpha h^*(\vec{\boldsymbol{x}},t)| = O(1)e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} + O(1)\frac{1}{1 + t^{(1+|\alpha|)/2}}\mathbb{K}_n(\vec{\boldsymbol{x}},t), \tag{4.9}$$

$$\begin{aligned}&\left|H_0(\vec{\boldsymbol{x}},t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \left(\nu^{-3}t + \nu^{-2}\right)e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}})\right|\\ &\qquad = O(1)e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}L_n(t) + O(1)\frac{1}{\sqrt{t+1}}\mathbb{K}_n(\vec{\boldsymbol{x}},t).\end{aligned}$$

4.4. *Energy estimate for obtaining wave structure in* $|\vec{\boldsymbol{x}}| > 3t$. It remains to construct the wave structure in the region $|\vec{\boldsymbol{x}}| > 3t$. It is sufficient to study $h^*(\vec{\boldsymbol{x}},t)$ outside the finite Mach region since the global structure of $H_0^*(\vec{\boldsymbol{x}},t)$ is clear by (3.19). Since the error function $h^*(\vec{\boldsymbol{x}},t)$ contains no singularity, the energy estimate for (3.9) can be applied to yield exponentially sharp estimates.

Outside the finite Mach region, we apply the weighted energy method to (3.9). The weight function is chosen to be

$$w(\vec{\boldsymbol{x}},t) \equiv e^{(|\vec{\boldsymbol{x}}|-Mt)/C_0} \text{ with } C_0 \gg 1,\ \frac{3}{2} \le M < 2,$$

with

$$w_t = -\frac{M}{C_0}w, \quad \nabla w = \frac{\vec{\boldsymbol{x}}}{C_0|\vec{\boldsymbol{x}}|}w.$$

Consider

$$\int_{\mathbb{R}^n} w(\vec{x},t)h_t^*(\vec{x},t)\left((\partial_t^2-\nabla\cdot\nabla-\nu\partial_t\nabla\cdot\nabla)h^*(\vec{x},t)-\mathscr{E}(\vec{x},t)\right)d\vec{x}=0, \tag{4.10}$$

and the divergent theorem ensures that the following estimate holds under the assumption $1/C_0 \ll \nu$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\partial_t h^*|^2+|\nabla h^*|^2\right)wd\vec{x} \qquad (4.11)\\
&\quad=-\frac{M}{2C_0}\int_{\mathbb{R}^n}\left(|\partial_t h^*|^2+|\nabla h^*|^2\right)wd\vec{x}-\nu\int_{\mathbb{R}^n}|\nabla\partial_t h^*|^2wd\vec{x}\\
&\quad-\frac{1}{C_0}\int_{\mathbb{R}^n}\partial_t h^*\frac{\vec{x}}{|\vec{x}|}w\cdot(\nabla h^*+\nu\nabla\partial_t h^*)d\vec{x}+\int_{\mathbb{R}^n}\partial_t h^*\mathscr{E}wd\vec{x}\\
&\quad\le-\frac{M-1}{4C_0}\int_{\mathbb{R}^n}\left(|\partial_t h^*|^2+|\nabla h^*|^2\right)wd\vec{x}-\frac{\nu}{2}\int_{\mathbb{R}^n}|\nabla\partial_t h^*|^2wd\vec{x}+C_0^{3/2}\int_{\mathbb{R}^n}\mathscr{E}^2wd\vec{x}.
\end{aligned}$$

From (3.11) and (4.11), there exists $C_*>0$ such that

$$\int_{\mathbb{R}^n}\left(|\partial_t h^*|^2+|\nabla h^*|^2\right)wd\vec{x}\le C_*e^{-t/C_*}.$$

One can also get similar estimates for higher derivatives:

$$\int_{\mathbb{R}^n}\left(|\partial_t\partial_{\vec{x}}^{\alpha}h^*|^2+|\nabla\partial_{\vec{x}}^{\alpha}h^*|^2\right)wd\vec{x}\le C_*e^{-t/C_*}\text{ for }|\alpha|<8-\frac{n}{2}. \tag{4.12}$$

Let $M=\frac{3}{2}$ and from (4.12) together with the Sobolev inequality for $1\le n\le 3$, there exists $C_0>0$ such that for $0\le|\alpha|\le 4$,

$$\sup_{(\vec{x},t)\in\mathbb{R}^n\times\mathbb{R}^+}e^{(|\vec{x}|-3t/2)/C_*}\left(|\partial_t\partial_{\vec{x}}^{\alpha}h^*(\vec{x},t)|+|\nabla\partial_{\vec{x}}^{\alpha}h^*(\vec{x},t)|\right)\le C_0e^{-t/C_0}. \tag{4.13}$$

For $|\vec{x}|>3t$,

$$|\vec{x}|-3t/2=\frac{3|\vec{x}|}{8}+\left(\frac{5|\vec{x}|}{8}-3t/2\right)>\frac{3|\vec{x}|}{8}+\left(\frac{15t}{8}-3t/2\right)=\frac{3|\vec{x}|+3t}{8}.$$

Thus, there exist positive constants C_0 and C_1 such that for $|\vec{x}|>3t$ and $0\le|\alpha|\le 4$,

$$\left(|\partial_t\partial_{\vec{x}}^{\alpha}h^*(\vec{x},t)|+|\nabla\partial_{\vec{x}}^{\alpha}h^*(\vec{x},t)|\right)\le C_1e^{-(|\vec{x}|+t)/C_1}, \tag{4.14}$$

and

$$|h^*(\vec{x},t)|\le\int_0^\infty\left|\nabla h^*\left(\vec{x}+s\frac{\vec{x}}{|\vec{x}|},t\right)\right|ds\le C_0e^{-|\vec{x}|+t/C_0}. \tag{4.15}$$

The estimates (4.14) and (4.15) together with (4.9) yield that there exist positive constants C_0 and D_0 such that for $0\le|\alpha|\le 4$ and $\vec{x}\in\mathbb{R}^n(n=2,3)$,

$$|\partial_{\vec{x}}^{\alpha}h^*(\vec{x},t)|=O(1)e^{-(|x|+t)/C_0}+O(1)\frac{1}{1+t^{(1+|\alpha|)/2}}\mathbb{K}_n(\vec{x},t;D_0).$$

The estimate with (3.19) concludes the theorem for $H_0(\vec{x},t)$; and $H_1(\vec{x},t)$ follows by a similar argument.

THEOREM 4.3. For $n = 2, 3$, there exist positive constants C_0 and D_0 such that for all $\vec{\boldsymbol{x}} \in \mathbb{R}^n$,

$$\begin{aligned}&\left|H_0(\vec{\boldsymbol{x}},t) - e^{-\nu^{-1}t}\delta_n(\vec{\boldsymbol{x}}) - \left(\nu^{-3}t+\nu^{-2}\right)e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}}) - j_0^*(\vec{\boldsymbol{x}},t)\right| \\ &\qquad\qquad \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} + \frac{C_0}{\sqrt{t+1}}\mathbb{K}_n(\vec{\boldsymbol{x}},t;D_0), \\ &\left|H_1(\vec{\boldsymbol{x}},t) - \nu^{-1}e^{-\nu^{-1}t}\mathbb{Y}_n(\vec{\boldsymbol{x}}) - j_1^*(\vec{\boldsymbol{x}},t)\right| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} + C_0\mathbb{K}_n(\vec{\boldsymbol{x}},t;D_0),\end{aligned}$$

and

$$|j_k^*(\vec{\boldsymbol{x}},t)| \le C_0 L_n(t) e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} \text{ for } k=0,1.$$

Furthermore, for $0 \le |\alpha| \le 4$,

$$\left|\partial_{\vec{\boldsymbol{x}}}^\alpha\big(H_0(\vec{\boldsymbol{x}},t) - H_0^*(\vec{\boldsymbol{x}},t)\big)\right| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} + \frac{C_0}{1+t^{(1+|\alpha|)/2}}\mathbb{K}_n(\vec{\boldsymbol{x}},t;D_0),$$

$$\left|\partial_{\vec{\boldsymbol{x}}}^\alpha\partial_{x^k}\big(H_1(\vec{\boldsymbol{x}},t) - H_1^*(\vec{\boldsymbol{x}},t)\big)\right| \le C_0 e^{-(|\vec{\boldsymbol{x}}|+t)/C_0} + \frac{C_0}{1+t^{(1+|\alpha|)/2}}\mathbb{K}_n(\vec{\boldsymbol{x}},t;D_0).$$

This theorem concludes Theorem 1.3.

5. Global wave structure of auxiliary function (Proof of Theorem 1.4).

It remains to invert the space-time structure of the symbols $\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} - e^{-\mu|\vec{\eta}|^2t}\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2}$ by the rearrangement

$$\begin{aligned}&\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} - e^{-\mu|\vec{\eta}|^2t}\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2} \\ &\quad = \left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} - e^{-\nu|\vec{\eta}|^2t} + \left(e^{-\nu|\vec{\eta}|^2t} - e^{-\mu|\vec{\eta}|^2t}\right)\right)\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2}\end{aligned}$$

and a differential equation approach. Here, for the sake of the differential equation approach, we replace the factor $e^{-\mu|\vec{\eta}|^2t}$ in the expression by $e^{-\nu|\vec{\eta}|^2t}$; and introduce the function $\mathbb{U}(\vec{\boldsymbol{x}},t)$ as follows:

$$\mathbb{U}(\vec{\boldsymbol{x}},t) \equiv \mathbb{F}^{-1}\left[\frac{1}{|\vec{\eta}|^2}\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} - e^{-\nu|\vec{\eta}|^2t}\right)\right], \tag{5.1}$$

which solves the problem

$$\begin{cases}(\partial_t^2 - \triangle - \nu\partial_t\triangle)\mathbb{U}(\vec{\boldsymbol{x}},t) = -\mathscr{H}_n(\vec{\boldsymbol{x}},t) \text{ for } (\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}^+, \\ \mathbb{U}(\vec{\boldsymbol{x}},0) = \partial_t\mathbb{U}(\vec{\boldsymbol{x}},0) = 0,\end{cases} \tag{5.2}$$

where

$$\mathscr{H}_n(\vec{\boldsymbol{x}},t) \equiv \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\nu t}}}{(4\pi t)^{n/2}}.$$

Then, one has that

$$\mathbb{U}(\vec{\boldsymbol{x}},t) = -H_1(\vec{\boldsymbol{x}},t) \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{H}_n(\vec{\boldsymbol{x}},t). \tag{5.3}$$

In this section, we will mainly discuss the pointwise structure for $\partial_{x^k}\partial_{x^l}\mathbb{U}(\vec{\boldsymbol{x}},t)$.

The remaining symbol $\frac{\vec{\eta}\vec{\eta}^t}{|\vec{\eta}|^2}\left(e^{-\nu|\vec{\eta}|^2 t}-e^{-\mu|\vec{\eta}|^2 t}\right)$ can be estimated by the following lemma:

LEMMA 5.1. For given $\nu \geq \mu > 0$, there exists $C_0 > 0$ such that for $t > 0$ and $k, l = 1, \cdots, n$,

$$\left|\mathbb{F}^{-1}\left[\eta^k\eta^l(e^{-\nu|\vec{\eta}|^2 t}-e^{-\mu|\vec{\eta}|^2 t})/|\vec{\eta}|^2\right](\vec{\boldsymbol{x}},t)\right| \leq C_0|\nu-\mu|(\nu/\mu)^{n/2}\mathscr{H}_n(\vec{\boldsymbol{x}}/C_0,\nu t).$$

Proof. The identity

$$\eta^k\eta^l(e^{-\mu|\vec{\eta}|^2 t}-e^{-\nu|\vec{\eta}|^2 t})/|\vec{\eta}|^2 = \int_{\mu t}^{\nu t}\eta^k\eta^l e^{-\tau|\vec{\eta}|^2}d\tau,$$

yields

$$\begin{aligned}
&\left|\mathbb{F}^{-1}\left[\eta^k\eta^l(e^{-\mu|\vec{\eta}|^2 t}-e^{-\nu|\vec{\eta}|^2 t})/|\vec{\eta}|^2\right](\vec{\boldsymbol{x}},t)\right| \\
&= \left|\int_{\mu t}^{\nu t}\partial_{x^k}\partial_{x^l}\mathscr{H}_n(\vec{\boldsymbol{x}},\tau)d\tau\right| \\
&\leq \int_{\mu t}^{\nu t}|\partial_{x^k}\partial_{x^l}\mathscr{H}_n(\vec{\boldsymbol{x}},\tau)|d\tau \\
&\leq \int_{\mu t}^{\nu t}\frac{C_1}{\tau}\mathscr{H}_n(\vec{\boldsymbol{x}}/C_1,\tau)d\tau \leq C_0|\nu-\mu|(\nu/\mu)^{n/2}\mathscr{H}_n(\vec{\boldsymbol{x}}/C_0,\nu t).
\end{aligned}$$

□

5.1. *Two representations for* $\mathbb{F}[\mathbb{U}]$. By substituting $\lambda_\pm$ given in (2.3) into (5.1), one has that

$$\begin{aligned}
\mathbb{F}[\mathbb{U}](i\vec{\eta},t) &= \frac{1}{|\vec{\eta}|^2} \qquad (5.4)\\
&\cdot\left(-e^{-\nu|\vec{\eta}|^2 t}+e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\right.\\
&\left.\left(\frac{\left(-\frac{\nu|\vec{\eta}|^2}{2}+\frac{\sqrt{\nu^2|\vec{\eta}|^4-4|\vec{\eta}|^2}}{2}\right)e^{\frac{\sqrt{\nu^2|\vec{\eta}|^4-4|\vec{\eta}|^2}}{2}t}-\left(-\frac{\nu|\vec{\eta}|^2}{2}-\frac{\sqrt{\nu^2|\vec{\eta}|^4-4|\vec{\eta}|^2}}{2}\right)e^{-\frac{\sqrt{|\vec{\eta}|^4-4|\vec{\eta}|^2}}{2}t}}{\sqrt{\nu^2|\vec{\eta}|^4-4|\vec{\eta}|^2}}\right)\right)\\
&= -\frac{e^{-\nu|\vec{\eta}|^2 t}}{|\vec{\eta}|^2}+\frac{e^{-\frac{\nu|\vec{\eta}|^2}{2}t}}{|\vec{\eta}|^2}\cos\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}t\right)-\frac{\nu}{2}e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\frac{\sin\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}t\right)}{|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}}.
\end{aligned}$$

It is used for the pointwise structure in $|\vec{\boldsymbol{x}}| < t - \sqrt{t}$. One also needs to rewrite the symbol as follows to resolve the symbol $1/|\vec{\eta}|^2$ for $\mathbb{U}(\vec{\boldsymbol{x}}, t)$ in the region $|\vec{\boldsymbol{x}}| \in (t - \sqrt{t}, 4t)$:

$$\begin{aligned}\mathbb{F}[\mathbb{U}](i\vec{\eta}, t) = & -\frac{e^{-\nu|\vec{\eta}|^2 t} - e^{-\frac{\nu|\vec{\eta}|^2 t}{2}}}{|\vec{\eta}|^2} + \frac{e^{-\frac{\nu|\vec{\eta}|^2}{2}t}}{|\vec{\eta}|^2}\left(\cos\left(|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}t\right) - 1\right) \\ & - \frac{\nu}{2} e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \frac{\sin\left(|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}t\right)}{|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}} \\ = & -\frac{e^{-\nu|\vec{\eta}|^2 t} - e^{-\frac{\nu|\vec{\eta}|^2 t}{2}}}{|\vec{\eta}|^2} - e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(1 - \frac{\nu^2|\vec{\eta}|^2}{4}\right)\left(\int_0^t \frac{\sin\left(|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}s\right)}{|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}} ds\right) \\ & - \frac{\nu}{2} e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \frac{\sin\left(|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}t\right)}{|\vec{\eta}|\sqrt{1 - \frac{\nu^2|\vec{\eta}|^2}{4}}}.\end{aligned} \tag{5.5}$$

5.2. *Wave structure inside the cone* $|\vec{\boldsymbol{x}}| < t - \sqrt{t}$. One starts from the lemmas for inverting the symbol $\eta^k \eta^l e^{-a|\vec{\eta}|^2 t}/|\vec{\eta}|^2$ with a positive constant a.

Lemma 5.2. There exists $C_0 > 0$ such that

$$|\mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-a|\vec{\eta}|^2 t}/|\vec{\eta}|^2](\vec{\boldsymbol{x}}, t) \le C_0 \frac{1}{(|\vec{\boldsymbol{x}}| + \sqrt{t})^n}.$$

Proof. One has the identity

$$\begin{aligned}\left|\mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-a|\vec{\eta}|^2 t}/|\vec{\eta}|^2](\vec{\boldsymbol{x}}, t)\right| &= O(1)\left|\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l \int_t^\infty e^{-a|\vec{\eta}|^2\tau} d\tau\right]\right| \\ &= O(1)\left|\int_t^\infty \partial_{x^k}\partial_{x^l} \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4a\tau}}}{(4\pi\tau)^{n/2}} d\tau\right|\end{aligned}$$

$$\le C_1 \int_t^\infty \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C_1\tau}}}{\tau^{(n+2)/2}} d\tau = O(1)\int_t^\infty \frac{1}{(|\vec{\boldsymbol{x}}| + \sqrt{\tau})^{n+2}} d\tau = O(1)\frac{1}{(|\vec{\boldsymbol{x}}| + \sqrt{t})^n}.$$

□

The Kirchhoff's formula together with Lemma 5.2 yields the estimate for $\mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-|\vec{\eta}|^2 t}\cos(|\vec{\eta}|t)/|\vec{\eta}|^2]$:

Lemma 5.3. For $|\vec{\boldsymbol{x}}| < t - \sqrt{t}$,

$$\begin{aligned}&|\mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)e^{-|\vec{\eta}|^2 t}\eta^k\eta^l\cos(|\vec{\eta}|t)/|\vec{\eta}|^2](\vec{\boldsymbol{x}}, t)| \\ &\quad = O(1)\begin{cases}\displaystyle\int_{|x - x_*| < t} \frac{1}{\sqrt{t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*|}\,\sqrt{t}\,(|\vec{\boldsymbol{x}}_*| + \sqrt{t})^3} d\vec{\boldsymbol{x}}_* & \text{for } n = 2, \\ \displaystyle\frac{1}{(|\vec{\boldsymbol{x}}| + \sqrt{t})(t - |\vec{\boldsymbol{x}}| + \sqrt{t})^2} & \text{for } n = 3.\end{cases}\end{aligned}$$

Proof.

CASE ($n = 3$). By the Kirchhoff's formula in 3-D,

$$\left|\frac{1}{(2\pi)^3}\int_{\vec{\eta}\in\mathbb{R}^3} e^{i\vec{\eta}\cdot\vec{x}}\cos(|\vec{\eta}|t)\mathbb{F}[g](\vec{\eta})d\vec{\eta}\right| = \left|\partial_r\left(\frac{1}{4\pi r}\int_{\substack{\vec{x}_*\in\mathbb{R}^3\\|\vec{x}_*|=r}} g(\vec{x}-\vec{x}_*)dA_{\vec{x}_*}\right)\bigg|_{r=t}\right|$$
$$= O(1)\left(\frac{1}{t^2}\int_{\substack{\vec{x}_*\in\mathbb{R}^3\\|\vec{x}-\vec{x}_*|=t}}|g(\vec{x}_*)|dA_{\vec{x}_*}+\frac{1}{t}\int_{\substack{\vec{x}_*\in\mathbb{R}^3\\|\vec{x}-\vec{x}_*|=t}}|\nabla g(\vec{x}_*)|dA_{\vec{x}_*}\right), \quad (5.6)$$

and by substituting $g = \mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-|\vec{\eta}|^2 t}/|\vec{\eta}|^2]$ into (5.6) together with the estimates

$$\begin{cases}|g(\vec{x},t)| = O(1)\left|\int_t^\infty \partial_{x^k}\partial_{x^l}\dfrac{e^{-\frac{|\vec{x}|^2}{\tau}}}{(4\pi\tau)^{3/2}}d\tau\right| \le \dfrac{C_0}{(|\vec{x}|+\sqrt{t})^3},\\ |\nabla g(\vec{x},t)| \le \dfrac{C_0}{(|\vec{x}|+\sqrt{t})^4},\end{cases} \quad (5.7)$$

one has that for $\sqrt{t} \le |\vec{x}| < t-\sqrt{t}$,

$$\left|\mathbb{F}^{-1}[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-|\vec{\eta}|^2 t}cos(|\vec{\eta}|t)/|\vec{\eta}|^2](\vec{x},t)\right|$$
$$= O(1)\left(\int_{|\vec{x}_*-\vec{x}|=t}\left(\frac{1}{t^2|\vec{x}_*|^3}+\frac{1}{t|\vec{x}_*|^4}\right)dA_{\vec{x}_*}\right).$$

Let θ be the angle between $\vec{x}$ and $\vec{x}_*-\vec{x}$ and $r = |\vec{x}_*| = \sqrt{t^2+|\vec{x}|^2-2t|\vec{x}|\cos\theta}$. Thus,

$$\begin{cases}\cos\theta = \dfrac{t^2+|\vec{x}|^2-r^2}{2t|\vec{x}|},\\ \sin\theta d\theta = \dfrac{r}{2t|\vec{x}|}dr.\end{cases}$$

Then,

$$\int_{|\vec{x}_*-\vec{x}|=t}\left(\frac{1}{t^2|\vec{x}_*|^3}+\frac{1}{t|\vec{x}_*|^4}\right)dA_{\vec{x}_*}$$
$$= 4\pi\int_0^\pi\left(\frac{1}{(t^2+|\vec{x}|^2-2t|\vec{x}|\cos\theta)^{3/2}}+\frac{t}{(t^2+|\vec{x}|^2-2t|\vec{x}|\cos\theta)^2}\right)\sin\theta d\theta$$
$$= 4\pi\int_{t-|\vec{x}|}^{t+|\vec{x}|}\left(\frac{1}{r^3}+\frac{t}{r^4}\right)\frac{r}{t|\vec{x}|}dr = O(1)\left(\frac{1}{t|\vec{x}|(t-|\vec{x}|)}+\frac{1}{|\vec{x}|(t-|\vec{x}|)^2}\right)$$
$$= \frac{O(1)}{(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2}.$$

When $|\vec{\boldsymbol{x}}|<\sqrt{t}$, the computation is straightforward and we omit the details. The lemma for the case $n=3$ follows.

CASE ($n=2$). In this case we use (5.6) together with Hadamard's descending method to extend a 2-D problem into a 3-D problem. For any $\vec{\boldsymbol{X}}\in\mathbb{R}^3$ and $\vec{\boldsymbol{x}}\in\mathbb{R}^2$ with $\pi_2\vec{\boldsymbol{X}}=\vec{\boldsymbol{x}}$, we set

$$g(\vec{\boldsymbol{X}})\equiv\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\int_t^\infty\eta^k\eta^l e^{-|\vec{\eta}|^2\tau}d\tau\right](\vec{\boldsymbol{x}})=O(1)\int_t^\infty\partial_{x^k}\partial_{x^l}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\tau}}}{4\pi\tau}d\tau,$$

where π_2 is the projection operator from $\mathbb{R}^3$ to the x-y plane. Similar to (5.7), there exists $C_0>0$ such that

$$\begin{cases}|g(\vec{\boldsymbol{x}},t)|\le\dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^2},\\ |\nabla g(\vec{\boldsymbol{x}},t)|\le\dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}.\end{cases}$$

Thus, there exists $C>0$ such that for $|\vec{\boldsymbol{x}}|<t-\sqrt{t}$

$$\begin{aligned}&\left|\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-|\vec{\eta}|^2t}\cos(|\vec{\eta}|t)/|\vec{\eta}|^2\right]\right|(\vec{\boldsymbol{x}})\\
&=\frac{O(1)}{t^2}\int\limits_{\substack{\vec{\boldsymbol{X}}_*\in\mathbb{R}^3\\ |\vec{\boldsymbol{X}}-\vec{\boldsymbol{X}}_*|=t}}\int_t^\infty\frac{e^{-\frac{|\vec{\boldsymbol{x}}_*|^2}{C\tau}}}{\tau^2}d\tau dA_{\vec{\boldsymbol{X}}_*}+\frac{O(1)}{t}\int\limits_{\substack{\vec{\boldsymbol{X}}_*\in\mathbb{R}^3\\ |\vec{\boldsymbol{X}}-\vec{\boldsymbol{X}}_*|=t}}\int_t^\infty\frac{e^{-\frac{|\vec{\boldsymbol{x}}_*|^2}{C\tau}}}{\tau^{5/2}}d\tau dA_{\vec{\boldsymbol{X}}_*}\\
&=\frac{O(1)}{t^2}\int\limits_{\substack{\vec{\boldsymbol{X}}_*\in\mathbb{R}^3\\ |\vec{\boldsymbol{X}}-\vec{\boldsymbol{X}}_*|=t}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^2}dA_{\vec{\boldsymbol{X}}_*}+\frac{O(1)}{t}\int\limits_{\substack{\vec{\boldsymbol{X}}_*\in\mathbb{R}^3\\ |\vec{\boldsymbol{X}}-\vec{\boldsymbol{X}}_*|=t}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}dA_{\vec{\boldsymbol{X}}_*}\\
&=\frac{O(1)}{t}\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{\sqrt{t^2-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|^2}\,(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^2}d\vec{\boldsymbol{x}}_*\\
&\quad+O(1)\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{\sqrt{t^2-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|^2}\,(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*\\
&=O(1)\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}\,(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*.\end{aligned}$$

□

LEMMA 5.4. For $n=2$ and $|\vec{\boldsymbol{x}}|<t-\sqrt{t}$, there exists $C_0>0$ such that

$$\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}\,(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*\le\frac{C_0}{t\sqrt{t-|\vec{\boldsymbol{x}}|}}.$$

The proof is given in the appendix.

In Lemma 5.3 and Lemma 5.4 we have obtained the pointwise structure of $\mathbb{F}^{-1}[\eta^k\eta^l e^{-\nu|\vec{\eta}|^2 t}\cos(|\vec{\eta}|t)/|\vec{\eta}|^2]$ in the region $|\vec{\boldsymbol{x}}| < t-\sqrt{t}$. It remains to estimate the structure of the inverse of Q_{kl}, defined below, in the region $|\vec{\boldsymbol{x}}| < t-\sqrt{t}$:

$$Q_{kl}(\vec{\eta},t) \equiv \Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l \left(\frac{e^{-\frac{\nu|\vec{\eta}|^2}{2}t}}{|\vec{\eta}|^2} \left(\cos\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}t \right) - \cos(|\vec{\eta}|t) \right) \right.$$
$$\left. -\frac{\nu}{2}e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \left(\frac{\sin\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}t \right)}{|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}} - \frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|} \right) \right).$$

By a similar decomposition to that in (4.2), one has that

$$\eta^k\eta^l \frac{e^{-\frac{\nu|\vec{\eta}|^2}{2}t}}{|\vec{\eta}|^2} \left(\cos\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}t \right) - \cos(|\vec{\eta}|t) \right)$$
$$= \eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \frac{\left(\cos\left(|\vec{\eta}| \left(\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}} - 1 \right) t \right) - 1 \right)}{|\vec{\eta}|^2} \cos(|\vec{\eta}|t)$$
$$- \eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \frac{\sin\left(|\vec{\eta}| \left(\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}} - 1 \right) t \right)}{|\vec{\eta}|} \frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}.$$

One can apply Lemma 5.2 to define the parabolic wave

$$g(\vec{\boldsymbol{x}},t) \equiv \mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t} \left(\cos\left(|\vec{\eta}| \left(\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}} - 1 \right) t \right) - 1 \right) /|\vec{\eta}|^2 \right]$$

and it satisfies that for $|\vec{\boldsymbol{x}}| < 4t$,

Case $n=3$

$$\begin{cases} |g(\vec{\boldsymbol{x}},t)| \le \dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^5}, \\ |\nabla g(\vec{\boldsymbol{x}},t)| \le \dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^6}. \end{cases}$$

Case $n=2$

$$\begin{cases} |g(\vec{\boldsymbol{x}},t)| \le \dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^4}, \\ |\nabla g(\vec{\boldsymbol{x}},t)| \le \dfrac{C_0}{(|\vec{\boldsymbol{x}}|+\sqrt{t})^5}. \end{cases}$$

Then, one substitutes them into the constructions in Lemma 5.3 and Lemma 5.4 to result in that for $|\vec{\boldsymbol{x}}| < t - \sqrt{t}$,

Case $n = 3$

$$\mathbb{F}^{-1}\left[\frac{\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\cos\left(|\vec{\eta}|\left(\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}-1\right)t\right)-1\right)\cos(|\vec{\eta}|t)}{|\vec{\eta}|^2}\right](\vec{\boldsymbol{x}},t)$$
$$= \frac{O(1)}{t(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}.$$

Case $n = 2$

$$\mathbb{F}^{-1}\left[\frac{\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\cos\left(|\vec{\eta}|\left(\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}-1\right)t\right)-1\right)\cos(|\vec{\eta}|t)}{|\vec{\eta}|^2}\right](\vec{\boldsymbol{x}},t)$$
$$= O(1)\frac{1}{t^2\sqrt{t-|\vec{\boldsymbol{x}}|}}.$$

The rest of the parts in Q_{kl} contain no singularity for $|\eta| = 0$ and can be treated similarly to (4.2) and we conclude the pointwise structures of $\mathbb{F}^{-1}[Q_{kl}]$:

LEMMA 5.5. For $|\vec{\boldsymbol{x}}| < t - \sqrt{t}$,

$$\mathbb{F}^{-1}\left[Q_{kl}\right](\vec{\boldsymbol{x}},t) = O(1)\begin{cases}\dfrac{1}{t^2\sqrt{t-|\vec{\boldsymbol{x}}|}} & \text{for } n=2,\\[2ex] \dfrac{1}{t(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2} & \text{for } n=3.\end{cases}$$

5.3. *Structure in the region* $|\vec{\boldsymbol{x}}| \in (t-\sqrt{t}, 4t)$. Next, one continues to establish sharper estimates for $\mathbb{U}(\vec{\boldsymbol{x}},t)$ in a region outside the cone, $|\vec{\boldsymbol{x}}| \in (t-\sqrt{t}, 4t)$ by the representation (5.5). This approach is based on the rearrangement of the hyperbolic waves in contrast to the rearrangement of the parabolic waves in Lemma 5.2.

From the representation (5.5), we have the following basic lemma.

LEMMA 5.6. There exists $C_0 > 0$ such that for $|\vec{\boldsymbol{x}}| \in (t-\sqrt{t}, 4t)$

$$\left|\int_{|\vec{\eta}|<\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}-\nu|\vec{\eta}|^2t/2}\eta^k\eta^l\int_0^t\frac{\sin(|\vec{\eta}|\tau)}{|\vec{\eta}|}d\tau d\vec{\eta}\right| \le C_0\begin{cases}\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}} & \text{for } n=2,\\[2ex] \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^2} & \text{for } n=3.\end{cases} \tag{5.8}$$

Proof. When $n=2$, one has that

$$\begin{aligned}&\int\limits_{|\vec{\eta}|<\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}-\nu|\vec{\eta}|^2t/2}\eta^k\eta^l\int_0^t\frac{\sin(|\vec{\eta}|\tau)}{|\vec{\eta}|}d\tau d\vec{\eta}\\
&\qquad= O(1)(1+t)^{-2}\int_0^t\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<\tau}\frac{e^{-|\vec{\boldsymbol{x}}_*|^2/(C_0(t+1))}}{\sqrt{\tau^2-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|^2}}d\vec{\boldsymbol{x}}_*d\tau\\
&\qquad= O(1)(1+t)^{-1}\int_0^t e^{-(|\vec{\boldsymbol{x}}|-t+t-\tau)^2/(C_0t)}\tau^{-3/4}d\tau = O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}}.\end{aligned}$$

For $n=3$,

$$\begin{aligned}\int\limits_{|\vec{\eta}|<\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}-\nu|\vec{\eta}|^2t/2}\eta^k\eta^l\int_0^t\frac{\sin(|\vec{\eta}|\tau)}{|\vec{\eta}|}d\tau d\vec{\eta} &= O(1)(1+t)^{-3/2}\int_0^t\frac{e^{-(|\vec{\boldsymbol{x}}|-\tau)^2/(C_0(t+1))}}{(\tau+1)}d\tau\\
&= O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2)}{C_0(t+1)}}}{(1+t)^2}.\end{aligned}$$

The above cases conclude (5.8) and we finish the proof. □

Similar to the procedure to obtain the estimate of $\mathbb{F}^{-1}[Q_{kl}]$ in Lemma 5.5, it follows for $|\vec{\boldsymbol{x}}|\in(t-\sqrt{t},4t)$,

$$\begin{aligned}&\left|\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\int_0^t\frac{\sin\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}s\right)}{|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}}ds\right)\right]\right|\\
&\le\left|\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\int_0^t\frac{\sin(|\vec{\eta}|s)}{|\vec{\eta}|}ds\right]\right|\\
&\quad+\left|\mathbb{F}^{-1}\left[\Lambda(|\vec{\eta}|/\varepsilon_0)\eta^k\eta^l e^{-\frac{\nu|\vec{\eta}|^2}{2}t}\left(\int_0^t\left(\frac{\sin\left(|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}s\right)}{|\vec{\eta}|\sqrt{1-\frac{\nu^2|\vec{\eta}|^2}{4}}}-\frac{\sin(|\vec{\eta}|s)}{|\vec{\eta}|}\right)ds\right)\right]\right|\\
&\qquad= O(1)\begin{cases}\dfrac{1}{(1+t)^{5/4}}e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0t}} & \text{for } n=2,\\[2mm] \dfrac{1}{(1+t)^2}e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0t}} & \text{for } n=3.\end{cases}\end{aligned}\tag{5.9}$$

Finally, we conclude a theorem for the long wave components $\partial_{x^k}\partial_{x^l}\mathbb{U}_L(\vec{\boldsymbol{x}},t)$ from Lemmas 5.1 - 5.6 and (5.9):

THEOREM 5.7. There exists $C_0>0$ such that for $|\vec{\boldsymbol{x}}|<4t$,

$$\begin{aligned}&\left|\partial_{x^k}\partial_{x^l}\mathbb{U}_L(\vec{\boldsymbol{x}},t)\right|\\&=O(1)\begin{cases}\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}} \text{ for } n=2,\\ \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{2}} \text{ for } n=3.\end{cases}\end{aligned} \tag{5.10}$$

$$\begin{aligned}&\left|\nabla\partial_{x^k}\partial_{x^l}\mathbb{U}_L(\vec{\boldsymbol{x}},t)\right|\\&=O(1)\begin{cases}\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+\sqrt{t}(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t^{3/2}\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{7/4}} \text{ for } n=2,\\ \dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+\sqrt{t}(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t^{1/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/2}} \text{ for } n=3.\end{cases}\end{aligned} \tag{5.11}$$

REMARK 5.8. The calculation for obtaining the estimate (5.11) is almost identical to that for (5.10) except that there is an extra operator ∇. This causes an extra factor $i\vec{\eta}$ in the symbol for $\mathbb{F}[\mathbb{U}_L]$. Acting on $e^{-\nu|\vec{\eta}|^2t}$, this factor produces one extra decay rate $t^{-1/2}$. The estimate (5.11) follows. The computation is omitted. ∎

5.4. *Short wave component in a finite Mach region* $|\vec{\boldsymbol{x}}|<4t$. To obtain the short wave component in the finite Mach region $|\vec{\boldsymbol{x}}|<4t$, we need to use the solution $H_1(\vec{\boldsymbol{x}},t)$ of (3.8). By (5.3), one has that

$$\eta^k\eta^l\mathbb{F}[\mathbb{U}](i\eta,t)=-\mathbb{F}[\partial_{x^k}\partial_{x^l}\mathbb{U}](i\eta,t)=\eta^k\eta^l\int_0^t\frac{e^{\lambda_+(t-\tau)}-e^{\lambda_-(t-\tau)}}{\lambda_+-\lambda_-}e^{-\nu|\vec{\eta}|^2\tau}d\tau. \tag{5.12}$$

One breaks $\partial_{x^k}\partial_{x^l}\mathbb{U}$ as follows:

$$\partial_{x^k}\partial_{x^l}\mathbb{U}=\partial_{x^k}\partial_{x^l}(H_1^*+m^*)\underset{(\vec{\boldsymbol{x}},t)}{*}\mathscr{K}_n. \tag{5.13}$$

We introduce the short wave components as follows:

$$\begin{cases} M_{S0}^{kl}(\vec{\boldsymbol{x}},t) \\ \quad = \frac{1}{(2\pi)^n}\int_0^{t/2}\int\limits_{|\vec{\eta}|>\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\frac{e^{\lambda_+^*(t-\tau)}-e^{\lambda_-^*(t-\tau)}}{\lambda_+^*-\lambda_-^*}\eta^k\eta^l e^{-\nu|\vec{\eta}|^2\tau}d\vec{\eta}d\tau, \\ M_{S1}^{kl}(\vec{\boldsymbol{x}},t) \\ \quad = \frac{1}{(2\pi)^n}\int_{t/2}^{t}\int\limits_{|\vec{\eta}|>\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\frac{e^{\lambda_+^*(t-\tau)}-e^{\lambda_-^*(t-\tau)}}{\lambda_+^*-\lambda_-^*}\eta^k\eta^l e^{-\nu|\vec{\eta}|^2\tau}d\vec{\eta}d\tau, \\ m_S^{kl}(\vec{\boldsymbol{x}},t) \\ \quad = \frac{1}{(2\pi)^n}\int\limits_{|\vec{\eta}|>\varepsilon_0} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\left(\frac{e^{\lambda_+(t-\tau)}-e^{\lambda_-(t-\tau)}}{\lambda_+-\lambda_-}-\frac{e^{\lambda_+^*(t-\tau)}-e^{\lambda_-^*(t-\tau)}}{\lambda_+^*-\lambda_-^*}\right)\eta^k\eta^l e^{-\nu|\vec{\eta}|^2\tau}d\vec{\eta}d\tau. \end{cases}$$

By the expansions of $\lambda_\pm$ for $\vec{\eta}\to\infty$, (3.1), one has that

$$\begin{aligned} &\left|\frac{e^{\lambda_+(t-\tau)}-e^{\lambda_-(t-\tau)}}{\lambda_+-\lambda_-}-\frac{e^{\lambda_+^*(t-\tau)}-e^{\lambda_-^*(t-\tau)}}{\lambda_+^*-\lambda_-^*}\right| \\ &\qquad\le C_*\left(\frac{e^{-(t-\tau)/C_*}}{1+|\vec{\eta}|^{10}}+\frac{e^{-(t-\tau)/C_*}\frac{t-\tau}{1+|\vec{\eta}|^{10}}}{1+|\vec{\eta}|^2}\right)\ \text{for } |\vec{\eta}|>1. \end{aligned}$$

Thus there exists $C_0>0$ such that short wave component m_S^{kl} satisfies that for $0\le|\alpha|\le 6$ and $n\le 3$,

$$\begin{aligned} \left|\partial_{\vec{\boldsymbol{x}}}^\alpha m_S^{kl}(\vec{\boldsymbol{x}},t)\right| &\le C_0\int_0^t\int\limits_{|\vec{\eta}|>\varepsilon_0}|\vec{\eta}|^{|\alpha|}\left(\frac{e^{-(t-\tau)/C_*}}{1+|\vec{\eta}|^{10}}+\frac{e^{-(t-\tau)/C_*}\frac{t-\tau}{1+|\vec{\eta}|^{10}}}{1+|\vec{\eta}|^2}\right)e^{-\nu|\vec{\eta}|^2\tau}d\vec{\eta}d\tau \\ &\le C_0e^{-t/C_0}. \end{aligned}$$

The component M_{S1}^{kl} satisfies

$$|M_{S1}^{kl}(\vec{\boldsymbol{x}},t)|=O(1)\int_{t/2}^t\int\limits_{|\vec{\eta}|>\varepsilon_0}e^{-(t-\tau)/C_0}e^{-\nu|\vec{\eta}|^2\tau/2}d\vec{\eta}d\tau=O(1)t^{-(n-2)/2}e^{-t/C_0}. \quad (5.14)$$

For the component $M_{S0}^{kl}(\vec{\boldsymbol{x}},t)$, one rearranges it as

$$\begin{aligned} |M_{S0}^{kl}(\vec{\boldsymbol{x}},t)| &= \left|\int_0^{t/2}\left(\frac{1}{(2\pi)^n}\int\limits_{|\vec{\eta}|>\varepsilon_0} i\eta^k e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\frac{e^{\lambda_+*(t-\tau)}-e^{\lambda_-*(t-\tau)}}{\lambda_+^*-\lambda_-^*}d\vec{\eta}\right)\underset{\vec{\boldsymbol{x}}}{*}\partial_{x^l}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\nu\tau}}}{(4\pi\tau)^{n/2}}d\tau\right| \\ &= \left|\left(\int_1^{t/2}+\int_0^1\right)\left(\partial_{x^k}H_1^*(\vec{\boldsymbol{x}},t-\tau)-\frac{1}{(2\pi)^n}\int\limits_{|\vec{\eta}|<\varepsilon_0} i\eta^k e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}\frac{e^{\lambda_+*(t-\tau)}-e^{\lambda_-*(t-\tau)}}{\lambda_+^*-\lambda_-^*}d\vec{\eta}\right)\right. \\ &\qquad\qquad \left.\underset{\vec{\boldsymbol{x}}}{*}\partial_{x^l}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4\nu\tau}}}{(4\pi\tau)^{n/2}}d\tau\right| \\ &= O(1)e^{-(t+|\vec{\boldsymbol{x}}|)/C_0}\left(\frac{1}{|\vec{\boldsymbol{x}}|^{n-1}}+\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C_0(1+t)}}}{(1+t)^{(n+1)/2}}\right)+O(1)e^{-t/C_0}\ \text{for } n=2,3. \quad (5.15) \end{aligned}$$

From (5.12), (5.13), (5.14) and (5.15), one concludes the following theorem:

THEOREM 5.9. For a given $\varepsilon_0 > 0$ there exists $C_0 > 0$ such that for $|\vec{\boldsymbol{x}}| < 4t$, and $n = 2, 3$,

$$\left|\partial_{x_k}\partial_{x_l}\mathbb{U}_S(\vec{\boldsymbol{x}},t)\right| \leq C_0\frac{e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}}{|\vec{\boldsymbol{x}}|^{n-1}}.$$

REMARK 5.10. The short wave component $\partial_{x_k}\partial_{x_l}\mathbb{U}_S$ contains an integrable singularity bounded by $e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}/|\vec{\boldsymbol{x}}|^{n-1}$. ∎

5.5. *Outside cone.* Theorem 5.7 and Theorem 5.9 do not give the structure at $|\vec{\boldsymbol{x}}| = \infty$. One needs to use (5.2) to resolve the structure in the region $|\vec{\boldsymbol{x}}| > 4t$.

The following theorem is for the singular part of $\partial_{x^k}\partial_{x^l}\left(H_1^* \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{K}_n\right)(\vec{\boldsymbol{x}},t)$ and can be obtained by a straightforward computation based on (3.21) and we omit the details:

THEOREM 5.11. There exists C_0 such that

$$\left|\partial_{x^k}\partial_{x^l}\left(H_1^* \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{K}_n\right)(\vec{\boldsymbol{x}},t)\right| \leq C_0\frac{e^{-(|\vec{\boldsymbol{x}}|+t)/C_0}}{|\vec{\boldsymbol{x}}|^{n-1}} \text{ for } n = 2, 3.$$

Taking $h_{kl}(\vec{\boldsymbol{x}},t) = \partial_{x^k}\partial_{x^l}\left(m^* \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{K}_n\right)$, and from (3.9), one has the equation

$$\begin{cases}(\partial_t^2 - \Delta - \nu\Delta\partial_t)h_{kl} = \partial_{x^k}\partial_{x^l}\left(\mathscr{F} \underset{(\vec{\boldsymbol{x}},t)}{*} \mathscr{K}_n\right),\\ h_{kl}(\vec{\boldsymbol{x}},0) = \partial_t h_{kl}(\vec{\boldsymbol{x}},0) = 0.\end{cases}$$

Similar to (4.10), one has the following estimates for $n \leq 3$:

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(|\partial_t\partial_{\vec{\boldsymbol{x}}}^{\alpha}h_{kl}|^2 + |\nabla\partial_{\vec{\boldsymbol{x}}}^{\alpha}h_{kl}|^2\right)wd\vec{\boldsymbol{x}} \leq C_* e^{-t/C_*} \text{ for } |\alpha| < 8 - \frac{n}{2},\\ &\sup_{(\vec{\boldsymbol{x}},t)\in\mathbb{R}^n\times\mathbb{R}_+} e^{(|\vec{\boldsymbol{x}}|-3t/2)/C_*}\left(|\partial_t h_{kl}(\vec{\boldsymbol{x}},t)| + |\nabla h_{kl}(\vec{\boldsymbol{x}},t)|\right) \leq C_0 e^{-t/C_0}.\end{aligned} \tag{5.16}$$

Lemma 5.1, Theorem 5.7, Theorem 5.9, Theorem 5.11 and (5.16) conclude (1.13) in Theorem 1.4. The estimates (1.14) for high order derivatives can be obtained by a similar procedure and the details are omitted.

6. Duhamel's principle and pointwise convergence (Proof of Theorem 1.6). For the solution $V(\vec{x},t)$ of (1.18), we have the representation given by Duhamel's principle (1.20). This representation gives that

$$\partial_{\vec{x}}^{\alpha} V(\vec{x},t) = U^*(\vec{x},t) \underset{\vec{x}}{*} \partial_{\vec{x}}^{\alpha} V(\vec{x},0) + \partial_{\vec{x}}^{\alpha} U^R(\vec{x},t) \underset{\vec{x}}{*} V(\vec{x},0) + U^*(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t) + \sum_{k=1}^{n} \partial_{x^k} U^R(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}_k^R(\vec{x},t). \quad (6.1)$$

In (6.1), the assumption of the initial data (1.23), the decomposition (1.15), (1.16), (1.17), and Main Theorem I for H_k^* with $k = 0, 1$, yield that there exists $\mathscr{K}_0 > 0$ such that for $|\alpha| \le 2$

$$\left| \partial_{\vec{x}}^{\alpha} U^R(\vec{x},t) \underset{\vec{x}}{*} V(\vec{x},0) \right| \le \mathscr{K}_0 \varepsilon (1 + t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t), \quad (6.2)$$

$$\left| \partial_{\vec{x}}^{\alpha} U^*(\vec{x},t) \underset{\vec{x}}{*} V(\vec{x},0) \right| = \left| U^*(\vec{x},t) \underset{\vec{x}}{*} \partial_{\vec{x}}^{\alpha} V(\vec{x},0) \right| \le \mathscr{K}_0 \varepsilon e^{-(|\vec{x}|+t)/C_0}. \quad (6.3)$$

Here, $\boldsymbol{A}_n(\vec{x},t)$ is defined by (1.21). In the representation (6.1), to close the ansatz for $\partial_{\vec{x}}^{\alpha} V(\vec{x},t)$, the term $U^*(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}^*$ requires the information on derivatives of the solution $V(\vec{x},t)$ up to order $|\alpha| + 2$. By the explicit form for the nonlinear term $\mathscr{S}^*$ in (1.19), it is possible to use the ansatz of $\partial_{\vec{x}}^{\beta_0} V$ for $|\beta_0| \le (|\alpha|/2 + 1)$ together with uniform time decay rates of $\partial_{\vec{x}}^{\beta_1} V$ for $|\beta_1| \in (|\alpha|/2 + 1, |\alpha| + 2)$ to obtain the ansatz of $\partial_{\vec{x}}^{\alpha} V$. Thus, for the closure of the nonlinearity, it imposes a condition:

$$|\alpha| \ge (|\alpha|/2 + 1),$$

to ensure that one can obtain the higher order ansatz by the lower order one. It results in the condition $|\alpha| \ge 2$.

By (6.2) and (6.3), one makes the following a priori assumption on the solution $V(\vec{x},t)$ for all $\vec{x} \in \mathbb{R}^n$ and $t > 0$:

Ansatz assumption.

For any α with $|\alpha| \le 2$, there is $\mathscr{K}_0 > 0$ such that

$$|\partial_{\vec{x}}^{\alpha} V(\vec{x},t)| \le 4\mathscr{K}_0 \varepsilon (1 + t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t). \quad (6.4)$$

Justification of the ansatz assumption (6.4)**.**

This ansatz assumption is used for evaluating $\partial_{\vec{x}}^{\alpha} \mathscr{S}^*$ and $\partial_{\vec{x}}^{\alpha} \mathscr{S}_k^R$ in (6.1) with $|\alpha| \le 2$. When $\partial_{\vec{x}}^{\beta} V$ with $|\beta| = 3, 4$, involved in $\mathscr{S}^*$ and $\mathscr{S}_k^R$, one applies Theorem 1.5 for $\partial_{\vec{x}}^{\beta} V$.

When $|\beta| \le 2$, one applies the ansatz assumption (6.4) for $\partial_{\vec{\boldsymbol{x}}}^{\beta} V$. This yields that for $0 \le |\alpha| \le 1$,

$$
\begin{aligned}
&\left|\partial_{\vec{\boldsymbol{x}}}^{\alpha} \mathscr{S}_k^R(\vec{\boldsymbol{x}},t)\right| = O(1)\mathscr{K}_0^2\varepsilon^2 e^{-2(|\vec{\boldsymbol{x}}|+t)/C_0} \\
&+\frac{O(1)\mathscr{K}_0^2\varepsilon^2}{1+t^{|\alpha|/2}}
\begin{cases}
\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^4}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t^2(t-|\vec{\boldsymbol{x}}|+\sqrt{t})}+\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/2}} & \text{for } n=2, \\
\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^6}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^4} \\
\qquad +\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^4} & \text{for } n=3,
\end{cases}
\end{aligned}
\tag{6.5}
$$

and for $2 \le |\alpha| \le 3$,

$$
\begin{aligned}
&\left|\partial_{\vec{\boldsymbol{x}}}^{\alpha} \mathscr{S}_k^R(\vec{\boldsymbol{x}},t)\right| = O(1)\mathscr{K}_0^2\varepsilon^2 e^{-2(|\vec{\boldsymbol{x}}|+t)/C_0} \\
&+O(1)\mathscr{K}_0^2\varepsilon^2
\begin{cases}
\dfrac{1}{1+t^{|\alpha|/2}}\left(\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^4}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t^2(t-|\vec{\boldsymbol{x}}|+\sqrt{t})}+\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/2}}\right) \\
+\dfrac{1}{1+t^{(|\alpha|+3)/2}}\left(\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}\right. \\
\qquad\left.+\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}}\right) & \text{for } n=2, \\
\dfrac{1}{1+t^{|\alpha|/2}}\left(\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^6}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^4}\right. \\
\qquad\left.+\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^4}\right) \\
+\dfrac{1}{1+t^{(|\alpha|+4)/2}}\left(\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}+\dfrac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}\right. \\
\qquad\left.+\dfrac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(t+1)}}}{(1+t)^2}\right) & \text{for } n=3,
\end{cases}
\end{aligned}
\tag{6.6}
$$

for $|\alpha| = 0$,

$$
\begin{aligned}
|\partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t)| &= O(1)\mathscr{K}_0^2\varepsilon^2 e^{-2(|\vec{x}|+t)/C_0} \\
&+ O(1)\mathscr{K}_0^2\varepsilon^2
\begin{cases}
\frac{1}{1+t^{(1+|\alpha|)/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^4} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+t^2(t-|\vec{x}|+\sqrt{t})} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{(1+|\alpha|+3)/2}}\right) & \text{for } n=2, \\
\frac{1}{1+t^{|\alpha|/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^6} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2(t-|\vec{x}|+\sqrt{t})^4} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^4}\right) & \text{for } n=3,
\end{cases}
\end{aligned}
\tag{6.7}
$$

for $1 \le |\alpha| \le 2$,

$$
\begin{aligned}
|\partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t)| &= O(1)\mathscr{K}_0^2\varepsilon^2 e^{-2(|\vec{x}|+t)/C_0} \\
&+ O(1)\mathscr{K}_0^2\varepsilon^2
\begin{cases}
\frac{1}{1+t^{(1+|\alpha|)/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^4} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+t^2(t-|\vec{x}|+\sqrt{t})} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{(|\alpha|+3)/2}}\right) \\
+\frac{1}{1+t^{(|\alpha|+4)/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{5/4}}\right) & \text{for } n=2, \\
\frac{1}{1+t^{(1+|\alpha|)/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^6} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2(t-|\vec{x}|+\sqrt{t})^4} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^4}\right) \\
+\frac{1}{1+t^{(|\alpha|+5)/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^3} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2} \right. \\
\qquad \left. + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^2}\right) & \text{for } n=3.
\end{cases}
\end{aligned}
\tag{6.8}
$$

To complete the ansatz assumption, one needs to verify that

$$\left| U^*(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t) + \sum_{k=1}^{n} \partial_{x^k} U^R(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}_k^R(\vec{x},t) \right| \le 2\mathscr{K}_0 \varepsilon (1+t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t)$$

for $|\alpha| \le 2$ with $\mathscr{S}^*$ and $\mathscr{S}_k^R$ satisfying (6.5), (6.6), (6.7), and (6.8).

By the space-time exponential decaying structures in H_0^*, $\nabla_\varepsilon H_1^*$, and $\mathbb{Q}^*$, and exponential decaying structures of $\partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t)$ in the region $|\vec{x}| > t$ with $|\alpha| \le 2$, thus one has that

$$\left| U^*(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}^*(\vec{x},t) \right| = O(1)\mathscr{K}_0^2 \varepsilon^2 (1+t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t) \ll \mathscr{K}_0 \varepsilon (1+t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t)$$

if $\varepsilon \ll 1$.

It remains to show that for $\varepsilon \ll 1$,

$$\left| \sum_{k=1}^{n} \partial_{x^k} U^R(\vec{x},t) \underset{(\vec{x},t)}{*} \partial_{\vec{x}}^{\alpha} \mathscr{S}_k^R(\vec{x},t) \right| \ll \mathscr{K}_0 \varepsilon (1+t^{|\alpha|/2})^{-1} \boldsymbol{A}_n(\vec{x},t). \tag{6.9}$$

Justification of the ansatz assumption for $n = 2$.

For $|\alpha| = 0$, one has

$$\mathscr{S}_k^R(\vec{x},t) = O(1)\mathscr{K}_0^2 \varepsilon^2 \left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t}|)^4} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+t^2(t-|\vec{x}|+\sqrt{t})} \right) + \frac{\mathbf{H}(|\vec{x}|-t) e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}}.$$

By (C.1) with $k = 2$, (C.4), (C.8) and (C.11) with $n = 2$, one has that for $|\vec{x}| < t$,

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{x}_*|<\tau \\ |\vec{x}-\vec{x}_*|<t-\tau}} \left|\partial_{x^k} U^R(\vec{x}-\vec{x}_*, t-\tau)\right| \left|\mathscr{S}_k^R(\vec{x}_*,\tau)\right| d\vec{x}_* d\tau \\
&\quad = O(1)\mathscr{K}_0^2 \varepsilon^2 \int_0^t \int_{\substack{|\vec{x}_*|<\tau \\ |\vec{x}-\vec{x}_*|<t-\tau}} |\partial_{x^k} U^R(\vec{x}-\vec{x}_*, t-\tau)| \left(\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau}|)^4} \right. \\
&\qquad \left. + \frac{1}{1+\tau^2(\tau-|\vec{x}_*|+\sqrt{\tau})} \right) d\vec{x}_* d\tau \\
&\quad = O(1)\mathscr{K}_0^2 \varepsilon^2 \int_0^t \int_{\substack{|\vec{x}_*|<\tau \\ |\vec{x}-\vec{x}_*|<t-\tau}} \frac{1}{1+\sqrt{t-\tau}} \left(\frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2} \right. \\
&\qquad \left. + \frac{1}{1+(t-\tau)\sqrt{t-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau}}} \right) \\
&\qquad \left(\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau}|)^4} + \frac{1}{1+\tau^2(\tau-|\vec{x}_*|+\sqrt{\tau})} \right) d\vec{x}_* d\tau \\
&\quad = O(1)\mathscr{K}_0^2 \varepsilon^2 \left(\frac{1}{1+(|\vec{x}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} \right).
\end{aligned} \tag{6.10}$$

One also has the following estimates for $|\vec{\boldsymbol{x}}|<t$ from (C.15), (C.18) with $n=2$ and (C.21):

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|>\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau}} \left|\partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau)\right| \left|\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)\right| d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1)\mathscr{K}_0^2\varepsilon^2 \int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|>\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau}} \frac{1}{1+\sqrt{t-\tau}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \right. \\
&\qquad\qquad \left. + \frac{1}{1+(t-\tau)\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}} \right) \\
&\qquad\qquad \left(\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_0(t-\tau+1)}}}{1+\tau^{5/2}} + e^{-(|\vec{\boldsymbol{x}}_*|+\tau)/C_0} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1)\mathscr{K}_0^2\varepsilon^2 \left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}} \right),
\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau}} \left|\partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau)\right| \left|\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)\right| d\vec{\boldsymbol{x}}_* d\tau \\
&= O(1)\mathscr{K}_0^2\varepsilon^2 \int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau}} \frac{1}{1+\sqrt{t-\tau}} \left(\frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-t+\tau)^2}{C_0(t-\tau+1)}}}{(1+t-\tau)^{5/4}} + e^{-C_0(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+t-\tau)} \right) \\
&\left(\frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} + \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^4} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&= O(1)\mathscr{K}_0^2\varepsilon^2 \left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}} \right).
\end{aligned} \tag{6.12}$$

Thus, (6.10), (6.11), and (6.12) justify (6.9) for $|\alpha|=0$ for $|\vec{\boldsymbol{x}}|<t$.

For $1\le|\alpha|\le 2$, integration by parts yields

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^n} \partial_{\vec{\boldsymbol{x}}}^\alpha \partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau) \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau) d\vec{\boldsymbol{x}}_* d\tau \right| \\
&\le \left| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \partial_{\vec{\boldsymbol{x}}}^\alpha \partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau) \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau) d\vec{\boldsymbol{x}}_* d\tau \right| \\
&\quad + \left| \int_{\frac{t}{2}}^t \int_{\mathbb{R}^n} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau) \partial_{\vec{\boldsymbol{x}}}^\alpha \partial_{x^k} \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau) d\vec{\boldsymbol{x}}_* d\tau \right|.
\end{aligned} \tag{6.13}$$

By (C.2), (C.5), (C.9), and(C.12) together with $l = |\alpha|$, one has that for $|\vec{\boldsymbol{x}}| < t$,

$$\begin{aligned}
&\left| \int_0^{\frac{t}{2}} \int\limits_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} \partial_{\vec{\boldsymbol{x}}}^{\alpha}\partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*, t-\tau)\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)d\vec{\boldsymbol{x}}_* d\tau \right| \\
&\quad = O(1)\mathscr{K}_0^2\varepsilon^2 \int_0^{\frac{t}{2}} \int\limits_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} \frac{1}{1+(t-\tau)^{\frac{|\alpha|+1}{2}}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \right. \\
&\qquad \left. + \frac{1}{1+(t-\tau)\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}} \right) \\
&\qquad \left(\frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} + \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^4} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1)\frac{\mathscr{K}_0^2\varepsilon^2}{1+t^{\frac{|\alpha|}{2}}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}} \right).
\end{aligned}$$

By (C.3), (C.6), (C.10), (C.13) together with $l = |\alpha|$, and when $|\alpha| = 2$ by (C.7), (C.14) with $l = |\alpha|$ one has that for $|\vec{\boldsymbol{x}}| < t$,

$$\begin{aligned}
&\left| \int_{\frac{t}{2}}^{t} \int\limits_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*, t-\tau)\partial_{\vec{\boldsymbol{x}}}^{\alpha}\partial_{x^k}\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)d\vec{\boldsymbol{x}}_* d\tau \right| \\
&\quad = O(1)\mathscr{K}_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} \int\limits_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} \frac{1}{1+\tau^{\frac{|\alpha|+1}{2}}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \right. \\
&\qquad \left. + \frac{1}{1+(t-\tau)\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}} \right) \\
&\qquad \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^4} + \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1)\frac{\mathscr{K}_0^2\varepsilon^2}{1+t^{\frac{|\alpha|}{2}}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}} \right).
\end{aligned}$$

The other parts can also be justified by (C.16), (C.19), (C.22) and (C.17), (C.20), (C.23) for $\tau \in (0, t/2)$ and $\tau \in (t/2, t)$ respectively.

The justification for $|\vec{\boldsymbol{x}}| > t$ is similar by using Lemmas C.5 - C.7.

Justification of the ansatz assumption for $n = 3$.

For $|\alpha| = 0$, one has

$$\mathscr{S}_k^R(\vec{\boldsymbol{x}}, t) = O(1)\varepsilon^2 \mathscr{K}_0^2 \left(\frac{\mathbf{H}(t - |\vec{\boldsymbol{x}}|)}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t}|)^6} + \frac{\mathbf{H}(t - |\vec{\boldsymbol{x}}|)}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t})^2 (t - |\vec{\boldsymbol{x}}| + \sqrt{t})^4} \right.$$
$$\left. + \frac{\mathbf{H}(|\vec{\boldsymbol{x}}| - t) e^{-\frac{(|\vec{\boldsymbol{x}}| - t)^2}{C_0(1+t)}}}{1 + t^2} \right)$$

so that by (C.11) with $n = 3$, (C.24), (C.27) and (C.31), one has that for $|\vec{\boldsymbol{x}}| < t$,

$$\begin{aligned}
&int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*| < \tau \\ |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| < t - \tau}} \left| \partial_{x^k} U^R(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*, t - \tau) \right| \left| \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*, \tau) \right| d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1) \mathscr{K}_0^2 \varepsilon^2 \int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*| < \tau \\ |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| < t - \tau}} \frac{1}{1 + \sqrt{t - \tau}} \left(\frac{1}{1 + (|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})^3} \right. \\
&\qquad \left. + \frac{1}{1 + (|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})(t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})^2} \right) \\
&\qquad \left(\frac{1}{1 + (|\vec{\boldsymbol{x}}_*| + \sqrt{\tau}|)^6} + \frac{1}{1 + (|\vec{\boldsymbol{x}}_*| + \sqrt{\tau})^2 (\tau - |\vec{\boldsymbol{x}}_*| + \sqrt{\tau})^4} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1) \mathscr{K}_0^2 \varepsilon^2 \left(\frac{1}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t})^3} + \frac{1}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t})(t - |\vec{\boldsymbol{x}}| + \sqrt{t})^2} \right).
\end{aligned} \tag{6.14}$$

One also has the following estimates for $|\vec{\boldsymbol{x}}| < t$ from (C.18) with $n = 3$, (C.34) and (C.37):

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*| > \tau \\ |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| < t - \tau}} \left| \partial_{x^k} U^R(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*, t - \tau) \right| \left| \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*, \tau) \right| d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1) \mathscr{K}_0^2 \varepsilon^2 \int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*| > \tau \\ |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| < t - \tau}} \frac{1}{1 + \sqrt{t - \tau}} \left(\frac{1}{1 + (|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})^3} \right. \\
&\qquad \left. + \frac{1}{1 + (|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})(t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| + \sqrt{t - \tau})^2} \right) \\
&\qquad \left(\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*| - \tau)^2}{C_0(t - \tau + 1)}}}{1 + \tau^4} + e^{-(|\vec{\boldsymbol{x}}_*| + \tau)/C_0} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1) \mathscr{K}_0^2 \varepsilon^2 \left(\frac{1}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t})^3} + \frac{1}{1 + (|\vec{\boldsymbol{x}}| + \sqrt{t})(t - |\vec{\boldsymbol{x}}| + \sqrt{t})^2} \right),
\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau}} \left|\partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau)\right| \left|\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)\right| d\vec{\boldsymbol{x}}_* d\tau \\
&\quad = O(1)\mathscr{K}_0^2\varepsilon^2 \int_0^t \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau}} \frac{1}{1+\sqrt{t-\tau}} \left(\frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-t+\tau)^2}{C_0(t-\tau+1)}}}{(1+t-\tau)^2} + e^{-C_0(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+t-\tau)} \right) \\
&\qquad \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^6} + \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1)\mathscr{K}_0^2\varepsilon^2 \left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} + \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2} \right). \qquad (6.16)
\end{aligned}$$

Thus, (6.14), (6.15), and (6.16) justify (6.9) for $|\alpha|=0$ and $|\vec{\boldsymbol{x}}|<t$.

For $1\le|\alpha|\le 2$, one also uses (6.13). By (C.12), (C.25), (C.28), and(C.32) together with $l=|\alpha|$, one has that for $|\vec{\boldsymbol{x}}|<t$,

$$\begin{aligned}
&\left| \int_0^{\frac{t}{2}} \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} \partial_{\vec{\boldsymbol{x}}}^\alpha \partial_{x^k} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*,t-\tau) \mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau) d\vec{\boldsymbol{x}}_* d\tau \right| \\
&\qquad = O(1)\mathscr{K}_0^2\varepsilon^2 \int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau\\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau}} \frac{1}{1+(t-\tau)^{(1+|\alpha|)/2}} \\
&\quad \left(\frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^3} + \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \right) \\
&\qquad \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^6} + \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} \right) d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1)\mathscr{K}_0^2\varepsilon^2 \left(\frac{1}{1+t^{|\alpha|/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} + \frac{1}{1+t^{|\alpha|/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2} \right).
\end{aligned}$$

By (C.13), (C.14), (C.26), (C.29), (C.30) and (C.33) together with $l = |\alpha|$, one has that for $|\vec{\boldsymbol{x}}| < t$,

$$\left| \int_{\frac{t}{2}}^{t} \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}|<t-\tau}} U^R(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*, t-\tau)\partial_{\vec{\boldsymbol{x}}}^{\alpha}\partial_{x^k}\mathscr{S}_k^R(\vec{\boldsymbol{x}}_*,\tau)d\vec{\boldsymbol{x}}_* d\tau \right|$$

$$= O(1)\mathscr{K}_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} \int_{\substack{|\vec{\boldsymbol{x}}_*|<\tau \\ |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau}} \frac{1}{1+(t-\tau)^{1/2}}$$

$$\left(\frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^3} + \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\right)$$

$$\left(\frac{1}{1+\tau^{|\alpha|/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau}|)^6} + \frac{1}{1+\tau^{|\alpha|/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4}\right) d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1)\mathscr{K}_0^2\varepsilon^2\left(\frac{1}{1+t^{|\alpha|/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} + \frac{1}{1+t^{|\alpha|/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}\right).$$

The other parts can also be justified by (C.19), (C.35), (C.38) and (C.20), (C.36), (C.39) for $\tau \in (0, t/2)$ and $\tau \in (t/2, t)$ respectively.

The justification for $|\vec{\boldsymbol{x}}| > t$ is similar by using Lemma C.6 and Lemmas C.11-C.12.

Appendix A. Asymptotic structure of parabolic waves (Proof of Lemma 2.2). We only prove (2.6) for $\int_{|\vec{\eta}|<\varepsilon_0} \vec{\eta}^{\alpha}\frac{\sin(|\vec{\eta}|h(|\vec{\eta}|^2 t))}{|\vec{\eta}|}e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}-\frac{\nu}{2}|\vec{\eta}|^2 t}d\vec{\eta}$. The proof for the estimate for $\left|\int_{|\vec{\eta}|<\varepsilon_0}\vec{\eta}^{\alpha}\cos(|\vec{\eta}|h(|\vec{\eta}|^2 t))e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}}-\frac{\nu}{2}|\vec{\eta}|^2 t}d\vec{\eta}\right|$ will follow by the same procedure.

One decomposes

$$\{|\vec{\eta}|<\varepsilon_0\} = \left[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}\right]^n \cup \left(\{|\vec{\eta}|<\varepsilon_0\}\setminus\left[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}\right]^n\right);$$

and there exists $\varepsilon_1 > 0$

$$\left| \int_{\left(\{|\vec{\eta}|<\varepsilon_0\}\setminus\left[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}\right]^n\right)} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma}\frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|}\vec{\eta}^{\alpha}d\vec{\eta} \right|$$

$$= O(1)e^{-\frac{\nu\varepsilon_0^2}{18n}\sigma}\int_{|\vec{\eta}|<\varepsilon_0}|\vec{\eta}|^{|\alpha|}d\vec{\eta} = O(1)e^{-\varepsilon_1\sigma}\varepsilon_0^{(n+|\alpha|)}.$$

To estimate the iterated integral, $\int_{\left[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}\right]^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}}e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma}\frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|}\vec{\eta}^{\alpha}d\vec{\eta}$, we introduce a sequence of paths in the complex domain $\mathbb{C}$, Γ^k, $k = 1, \cdots, n$, for a given $\vec{\boldsymbol{x}}$,

$(|\vec{\boldsymbol{x}}| < 4(\sigma+1))$,

$$\Gamma_k = \left\{ -\frac{\varepsilon_0}{3\sqrt{n}} + iz | z \in (0, \delta_0 \frac{x^k}{\nu\sigma}) \right\} \cup \left\{ i\delta_0 \frac{x^k}{\nu\sigma} + z | z \in (-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}) \right\}$$
$$\cup \left\{ \frac{\varepsilon_0}{3\sqrt{n}} + iz | z \in (\delta_0 \frac{x^k}{\nu\sigma}, 0) \right\},$$

where $0 < \delta_0 \ll 1$. Since $e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}} e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \cdots (\eta^n)^{\alpha_n}$ is an analytic function in each η^k, the iterated integral path integral satisfies

$$\begin{aligned}
&\int_{\left[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}\right]^n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}} e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} \vec{\eta}^{\alpha} d\vec{\eta} \\
&= \int_{-\frac{\varepsilon_0}{3\sqrt{n}}}^{\frac{\varepsilon_0}{3\sqrt{n}}} \cdots \int_{-\frac{\varepsilon_0}{3\sqrt{n}}}^{\frac{\varepsilon_0}{3\sqrt{n}}} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}} e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \cdots (\eta^n)^{\alpha_n} d\eta^1 \cdots d\eta^n \\
&= \int_{\Gamma_1} \cdots \int_{\Gamma_n} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}} e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \cdots (\eta^n)^{\alpha_n} d\eta^1 \cdots d\eta^n \\
&= \int_{[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}]^n + i\delta_0(\frac{x^1}{\nu\sigma}, \frac{x^2}{\nu\sigma}, \cdots, \frac{x^n}{\nu\sigma})} e^{i\vec{\boldsymbol{x}}\cdot\vec{\eta}} e^{-\frac{\nu|\vec{\eta}|^2}{2}\sigma} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \\
&\qquad \cdots (\eta^n)^{\alpha_n} d\vec{\eta} + O(1) e^{-\varepsilon_1\sigma/2} \varepsilon_0^{(n+|\alpha|)} \\
&= \int_{[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}]^n + i\delta_0(\frac{x^1}{\nu\sigma}, \frac{x^2}{\nu\sigma}, \cdots, \frac{x^n}{\nu\sigma})} e^{-\frac{|\vec{\boldsymbol{x}}|^2}{2\nu\sigma}} e^{-\frac{\sigma\nu}{2}\left((\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\cdot(\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\right)} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \\
&\qquad \cdots (\eta^n)^{\alpha_n} d\vec{\eta} + O(1) e^{-\varepsilon_1\sigma/2} \varepsilon_0^{(n+|\alpha|)}. \qquad (A.1)
\end{aligned}$$

For $\vec{\eta} \in [-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}]^n + i\delta_0(\frac{x^1}{\nu\sigma}, \frac{x^2}{\nu\sigma}, \cdots, \frac{x^n}{\nu\sigma})$, one has the estimate

$$\begin{aligned}
&\left| e^{-\frac{|\vec{\boldsymbol{x}}|^2}{2\nu\sigma}} e^{-\frac{\sigma\nu}{2}\left((\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\cdot(\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\right)} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} \right| |\vec{\eta}|^{|\alpha|} \\
&= O(1) e^{-(1-(1-\delta_0)^2)\frac{|\vec{x}|^2}{2\nu\sigma} - \frac{\nu\sigma|\vec{\eta}|^2}{2} + O(1)(|\vec{\eta}|^3 + \left|\frac{\vec{x}}{\sigma}\right|^3)\sigma} |\vec{\eta}|^{|\alpha|}
\end{aligned} \qquad (A.2)$$

with $0 < \delta_0 \ll 1$. Substitute (A.2) into (A.1) and one has

$$\begin{aligned}
&\left| \int_{[-\frac{\varepsilon_0}{3\sqrt{n}}, \frac{\varepsilon_0}{3\sqrt{n}}]^n + i\delta_0(\frac{x^1}{\nu\sigma}, \frac{x^2}{\nu\sigma}, \cdots, \frac{x^n}{\nu\sigma})} e^{-\frac{|\vec{\boldsymbol{x}}|^2}{2\nu\sigma}} e^{-\frac{\sigma\nu}{2}\left((\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\cdot(\vec{\eta} - i\frac{\vec{\boldsymbol{x}}}{\nu\sigma})\right)} \frac{\sin\left(|\vec{\eta}|h(|\vec{\eta}|^2)\sigma\right)}{|\vec{\eta}|} (\eta^1)^{\alpha_1} \right. \\
&\qquad \left. \cdots (\eta^n)^{\alpha_n} d\vec{\eta} \right| \\
&= O(1) e^{-\frac{|\vec{x}|^2}{4\nu\sigma}} \min\left(\sigma^{-(n+|\alpha|)/2)}, \varepsilon_0^{n+|\alpha|}\right).
\end{aligned}$$

It together with (A.1) yields the lemma for $\int_{|\vec{\eta}|<\varepsilon_0} \vec{\eta}^{\alpha} \frac{\sin(|\vec{\eta}|h(|\vec{\eta}|^2 t))}{|\vec{\eta}|} e^{i\vec{\eta}\cdot\vec{\boldsymbol{x}} - \frac{\nu}{2}|\vec{\eta}|^2 t} d\vec{\eta}$.

Appendix B. Computational Lemmas I.

LEMMA B.1. For $|\vec{x}| \le 3t$, there exists $C_0 > 0$ such that

CASE ($n = 2$).

$$\int_{|\vec{x}-\vec{x}_*|<t} \frac{e^{-\frac{|\vec{x}_*|^2}{4(t+1)}}}{(1+t)^k\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}} d\vec{x}_* \le C_0 \left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+(1+t)^{k-1/2}(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{k-1/4}} \right),$$

CASE ($n = 3$).

$$\int_{|\vec{x}-\vec{x}_*|=t} \frac{e^{-\frac{|\vec{x}_*|^2}{4(t+1)}}}{(1+t)^k} d\vec{x}_* \le C_0 \frac{e^{-\frac{(|\vec{x}|-t)^2}{C_0(t+1)}}}{(1+t)^{k-1}}.$$

Proof. First, one considers the case $n = 2$:

CASE ($|\vec{x}| < t - 2\sqrt{t}$).

One has

$$(1+t)^{-k} \int_{t-\sqrt{t}<|\vec{x}-\vec{x}_*|<t} \frac{e^{-\frac{|\vec{x}_*|^2}{4(t+1)}}}{\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}} d\vec{x}_* = O(1)(1+t)^{-k} \int_{\mathbb{R}} \frac{t^{1/4} e^{-\frac{(t-|\vec{x}|)^2+y_*^2}{32(t+1)}}}{\sqrt{t}} dy_*$$
$$= O(1) \frac{e^{-\frac{(t-|\vec{x}|)^2}{C_0(t+1)}}}{(1+t)^{k-1/4}},$$

and

$$(1+t)^{-k} \int_{|\vec{x}-\vec{x}_*|<t-\sqrt{t}} \frac{e^{-\frac{|\vec{x}_*|^2}{C_0(t+1)}}}{\sqrt{t^2-|\vec{x}-\vec{x}_*|^2}} d\vec{x}_*$$
$$= O(1)(1+t)^{-k} \left(\int_{|\vec{x}-\vec{x}_*|<\frac{t+|\vec{x}|}{2}} + \int_{\frac{t+|\vec{x}|}{2}<|\vec{x}-\vec{x}_*|<t-\sqrt{t}} \right) \frac{e^{-\frac{|\vec{x}_*|^2}{C_0(t+1)}}}{\sqrt{t}\sqrt{t-|\vec{x}-\vec{x}_*|}} d\vec{x}_*$$
$$= O(1)(1+t)^{-k} \left(\int_{|\vec{x}-\vec{x}_*|<\frac{t+|\vec{x}|}{2}} \frac{e^{-\frac{|\vec{x}_*|^2}{C_0(t+1)}}}{\sqrt{t}\sqrt{t-|\vec{x}|}} d\vec{x}_* + \int_{\frac{t+|\vec{x}|}{2}<|\vec{x}-\vec{x}_*|<t-\sqrt{t}} \frac{e^{-\frac{(t-|\vec{x}|)^2}{C_0(t+1)}}}{t^{3/4}} d\vec{x}_* \right)$$
$$= O(1) \left(\frac{(1+t)^{-k+1/2}}{\sqrt{t-|\vec{x}|}} + \frac{e^{-\frac{(t-|\vec{x}|)^2}{C_0(t+1)}}}{(1+t)^{k-1/4}} \right).$$

CASE $(t-2\sqrt{t}<|\vec{\boldsymbol{x}}|<t+\sqrt{t})$.

$$\begin{aligned}&\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{e^{-\frac{|\vec{\boldsymbol{x}}_*|^2}{4(t+1)}}}{(1+t)^k\sqrt{t^2-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|^2}}d\vec{\boldsymbol{x}}_*\\&\le C_0(1+t)^{-k}\left(\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\sqrt{t}}\frac{e^{-\frac{|\vec{\boldsymbol{x}}_*|^2}{C_0(t+1)}}}{\sqrt{t}t^{1/4}}d\vec{\boldsymbol{x}}_*+\int\limits_{t-\sqrt{t}<|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{e^{-\frac{y_*^2}{C_0(t+1)}}}{\sqrt{t}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}d\vec{\boldsymbol{x}}_*\right)\\&=O(1)(1+t)^{-k+1/4}.\end{aligned}$$

CASE $(t+\sqrt{t}<|\vec{\boldsymbol{x}}|<3t)$.

When $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t$, then $|\vec{\boldsymbol{x}}_*|\ge|\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>|\vec{\boldsymbol{x}}|-t>0$. Thus,

$$(1+t)^{-k}\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{e^{-\frac{|t-\vec{\boldsymbol{x}}|^2+y_*^2}{4(1+t)}}}{\sqrt{t}}d\vec{\boldsymbol{x}}_*\le C_0\frac{e^{-\frac{(t-|\vec{\boldsymbol{x}}|)^2}{C_0(t+1)}}}{(1+t)^{k-1/4}}.$$

The above cases conclude the lemma for $n=2$. The calculation for the case $n=3$ is similar and we omit the details here. □

Proof of Lemma 5.4. The region $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t$ is divided into two parts: $\{|\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}\}$ and $\{|\vec{\boldsymbol{x}}_*|>\frac{t-|\vec{\boldsymbol{x}}|}{4}\}$.

$$\begin{aligned}&\int\limits_{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t}\frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*\\&=\left(\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t,\\|\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}}}+\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t,\\|\vec{\boldsymbol{x}}_*|>\frac{t-|\vec{\boldsymbol{x}}|}{4}}}\right)\frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*.\end{aligned}$$

When $|\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}$, due to the triangle inequality, one has that

$$t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|\ge t-|\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}_*|>t-\frac{t+3|\vec{\boldsymbol{x}}|}{4}=\frac{3(t-|\vec{\boldsymbol{x}}|)}{4}.$$

When $|\vec{\boldsymbol{x}}_*|>\frac{t-|\vec{\boldsymbol{x}}|}{4}$, denote $\vec{\boldsymbol{x}}_*=(\boldsymbol{x}_*,\boldsymbol{y}_*)$ and one has

$$|\vec{\boldsymbol{x}}_*|\ge\frac{\sqrt{2}}{2}(|\boldsymbol{x}_*|+|\boldsymbol{y}_*|),$$

and

$$|\vec{\boldsymbol{x}}_*|\ge\frac{\sqrt{2}}{10}(|\boldsymbol{x}_*|+|\boldsymbol{y}_*|+t-|\vec{\boldsymbol{x}}|).$$

Thus,

$$\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t,\\ |\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}}} \frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_* < \frac{C}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}|}}\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t,\\ |\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}}} \frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_* \leq \frac{C}{t\sqrt{t-|\vec{\boldsymbol{x}}|}}. \tag{B.1}$$

$$\begin{aligned}
&\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t,\\ |\vec{\boldsymbol{x}}_*|>\frac{t-|\vec{\boldsymbol{x}}|}{4}}} \frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{1}{(|\vec{\boldsymbol{x}}_*|+\sqrt{t})^3}d\vec{\boldsymbol{x}}_*\\
&\qquad = O(1)\int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\sqrt{t},\\ |\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}}} \frac{1}{t^{1/2+1/4}}\frac{1}{(|\boldsymbol{x}_*|+|\boldsymbol{y}_*|+t-|\vec{\boldsymbol{x}}|+\sqrt{t})^3}d\boldsymbol{x}_*d\boldsymbol{y}_*\\
&\qquad + O(1)\int\limits_{\substack{t-\sqrt{t}<|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t\\ |\vec{\boldsymbol{x}}_*|<\frac{t-|\vec{\boldsymbol{x}}|}{4}}} \frac{1}{t^{1/2}\sqrt{t-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|}}\frac{1}{(|\boldsymbol{x}_*|+|\boldsymbol{y}_*|+t-|\vec{\boldsymbol{x}}|+\sqrt{t})^3}d\boldsymbol{x}_*d\boldsymbol{y}_*\\
&= O(1)t^{-3/4}\int_{-t}^{t}\frac{1}{(|\boldsymbol{y}_*|+t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}d\boldsymbol{y}_* + O(1)t^{-1/4}\int_{-t}^{t}\frac{1}{(|\boldsymbol{y}_*|+t-|\vec{\boldsymbol{x}}|+\sqrt{t})^3}d\boldsymbol{y}_*\\
&= O(1)\frac{1}{t^{3/4}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})}.
\end{aligned} \tag{B.2}$$

Combining (B.1) and (B.2), one finishes the proof. □

Appendix C. Computational Lemmas II: Coupling waves. In this section we prepare computational lemmas for analyzing nonlinear wave couplings through a fundamental solution and Duhamel's principle. The proofs of the lemmas will be presented in Appendix D.

Lemma C.1. For $\vec{\boldsymbol{x}} \in \mathbb{R}^2$, it is satisfied with $|\vec{\boldsymbol{x}}| < t$ and $k, l = 1, 2$,

$$\int_0^t \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{\frac{k+1}{2}}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{5/2}}d\vec{\boldsymbol{x}}_*d\tau = \frac{O(1)}{1+t^{k/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}; \tag{C.1}$$

$$\int_0^{t/2}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)^{\frac{3}{2}+\frac{l}{2}}(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{5/2}}d\vec{x}_*d\tau$$
$$=\frac{O(1)}{1+t^{1+\frac{l}{2}}(t-|\vec{x}|+\sqrt{t})^{1/2}};\quad \text{(C.2)}$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{(l+6)/2}}d\vec{x}_*d\tau$$
$$=\frac{O(1)}{1+t^{1+\frac{l}{2}}(t-|\vec{x}|+\sqrt{t})^{1/2}}.\quad \text{(C.3)}$$

Here, (C.1) was obtained in [22]. The computation is basic for obtaining most of the coupling waves. For the readability of this paper, we include the proof in Appendix D.

LEMMA C.2. For $\vec{x}\in\mathbb{R}^2$, it is satisfied with $|\vec{x}|<t$ and $l=1,2$,

$$\int_0^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^4}d\vec{x}_*d\tau$$
$$=O(1)\left(\frac{1}{1+(|\vec{x}|+\sqrt{t})^2}+\frac{1}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}}\right),\quad \text{(C.4)}$$

$$\int_0^{t/2}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)^{(3+l)/2}(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^4}d\vec{x}_*d\tau$$
$$=O(1)\frac{1}{1+t^{1+l/2}(t-|\vec{x}|+\sqrt{t})^{1/2}},\quad \text{(C.5)}$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{(l+1)/2}(|\vec{x}_*|+\sqrt{\tau})^4}d\vec{x}_*d\tau$$
$$=O(1)\left(\frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})^2}+\frac{1}{1+t^{1+l/2}(t-|\vec{x}|+\sqrt{t})^{1/2}}\right),\quad \text{(C.6)}$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{(l+4)/2}(|\vec{x}_*|+\sqrt{\tau})^2}d\vec{x}_*d\tau$$
$$=O(1)\left(\frac{1}{1+t^{(l+1)/2}(|\vec{x}|+\sqrt{t})^2}+\frac{1}{1+t^{1+l/2}(t-|\vec{x}|+\sqrt{t})^{1/2}}\right).\quad \text{(C.7)}$$

LEMMA C.3. For $\vec{\boldsymbol{x}} \in \mathbb{R}^2$, it is satisfied with $|\vec{\boldsymbol{x}}| < t$ and $l = 1, 2$

$$\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\left(\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}\right), \quad \text{(C.8)}$$

$$\int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\left(\frac{1}{1+t^{l/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^2} + \frac{1}{1+t^{1+l/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}\right), \quad \text{(C.9)}$$

$$\int_{t/2}^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^{(6+l)/2}} d\vec{\boldsymbol{x}}_* d\tau = O(1)\frac{1}{1+t^{1+l/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}. \quad \text{(C.10)}$$

LEMMA C.4. For $\vec{\boldsymbol{x}} \in \mathbb{R}^n (n = 2, 3)$, it is satisfied with $|\vec{\boldsymbol{x}}| < t$ and $l = 1, 2$,

$$\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^{2n}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}, \quad \text{(C.11)}$$

$$\int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^{2n}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\frac{1}{1+t^{l/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}, \quad \text{(C.12)}$$

$$\int_{t/2}^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+\tau^{(l+1)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^{2n}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\frac{1}{1+t^{l/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}, \quad \text{(C.13)}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n}\frac{1}{1+\tau^{(l+n+2)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^n}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\frac{1}{1+t^{l/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}. \tag{C.14}$$

LEMMA C.5. For $\vec{\boldsymbol{x}}\in\mathbb{R}^2$, and $l=1,2$,

$$\int_{0}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{5/2}}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)+e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{3/2}}, \tag{C.15}$$

$$\int_{0}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{(3+l)/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{5/2}}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)+e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3+l)/2}}, \tag{C.16}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{(6+l)/2}}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)+e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3+l)/2}}. \tag{C.17}$$

LEMMA C.6. For $\vec{\boldsymbol{x}}\in\mathbb{R}^n (n=2,3)$, and $l=1,2$,

$$\int_{0}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n}\frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\left(\frac{\mathbf{H}(2-n)\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}+\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}+\frac{\mathbf{H}(|\vec{\boldsymbol{x}}|-t)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3n-1)/4}}\right), \tag{C.18}$$

$$\int_0^{t/2} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|>\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{x}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{x}_* d\tau$$
$$= O(1)\frac{1}{1+t^{l/2}}\left(\frac{\mathbf{H}(2-n)\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^n} + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{(3n-1)/4}}\right), \tag{C.19}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|>\tau}} \frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{x}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n+l)/2}} d\vec{x}_* d\tau$$
$$= O(1)\frac{1}{1+t^{l/2}}\left(\frac{\mathbf{H}(2-n)\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^n} + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{(3n-1)/4}}\right). \tag{C.20}$$

LEMMA C.7. For $\vec{x}\in\mathbb{R}^2$ and $l=1,2$,

$$\int_0^{t} \int_{\substack{|\vec{x}-\vec{x}_*|>t-\tau \\ |\vec{x}_*|<\tau}} \frac{e^{-\frac{(|\vec{x}-\vec{x}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}} \left(\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^4} + \frac{1}{1+\tau^2(\tau-|\vec{x}_*|+\sqrt{\tau})}\right) d\vec{x}_* d\tau$$
$$= O(1)\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2} + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}}\right), \tag{C.21}$$

$$\int_0^{t/2} \int_{\substack{|\vec{x}-\vec{x}_*|>t-\tau \\ |\vec{x}_*|<\tau}} \frac{e^{-\frac{(|\vec{x}-\vec{x}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+(1+l)/2}} \left(\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^4} + \frac{1}{1+\tau^2(\tau-|\vec{x}_*|+\sqrt{\tau})}\right) d\vec{x}_* d\tau$$
$$= O(1)\frac{1}{1+t^{l/2}}\left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2} + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}}\right), \tag{C.22}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{x}-\vec{x}_*|>t-\tau \\ |\vec{x}_*|<\tau}} \frac{e^{-\frac{(|\vec{x}-\vec{x}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4}} \frac{1}{1+\tau^{\frac{l+1}{2}}} \left(\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^4} + \frac{1}{1+\tau^{\frac{5}{2}}} \right) d\vec{x}_* d\tau$$

$$= O(1) \frac{1}{1+t^{l/2}} \left(\frac{\mathbf{H}(t-|\vec{x}|)}{1+t(t-|\vec{x}|+\sqrt{t})^{1/2}} + \frac{\mathbf{H}(t-|\vec{x}|)}{1+(|\vec{x}|+\sqrt{t})^2} + \frac{\mathbf{H}(|\vec{x}|-t)e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}} \right). \tag{C.23}$$

The above lemmas are mainly for the closure of the nonlinearity when $n = 2$. The following lemmas are for case $n = 3$.

Lemma C.8. For $\vec{x} \in \mathbb{R}^3$, it is satisfied with $|\vec{x}| < t$ and $l = 1, 2$,

$$\int_{0}^{t} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2}$$

$$\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^2(\tau-|\vec{x}_*|+\sqrt{\tau})^4} d\vec{x}_* d\tau = O(1) \frac{1}{1+(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2}, \tag{C.24}$$

$$\int_{0}^{t/2} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|<\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2}$$

$$\frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^2(\tau-|\vec{x}_*|+\sqrt{\tau})^4} d\vec{x}_* d\tau = O(1) \frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2}, \tag{C.25}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|<\tau}} \frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^{4+(l+1)/2}} d\vec{x}_* d\tau$$

$$= O(1) \frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2}. \tag{C.26}$$

Lemma C.9. For $\vec{x} \in \mathbb{R}^3$, it is satisfied with $|\vec{x}| < t$ and $l = 1, 2$,

$$\int_{0}^{t} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2} \frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^6} d\vec{x}_* d\tau$$

$$= O(1) \left(\frac{1}{1+(|\vec{x}|+\sqrt{t})^3} + \frac{1}{1+(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2} \right), \tag{C.27}$$

$$\int_{0}^{t/2} \int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau \\ |\vec{x}_*|<\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2} \frac{1}{1+(|\vec{x}_*|+\sqrt{\tau})^6} d\vec{x}_* d\tau$$

$$= O(1) \frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})(t-|\vec{x}|+\sqrt{t})^2}, \tag{C.28}$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2}\frac{1}{1+\tau^{(l+1)/2}(|\vec{x}_*|+\sqrt{\tau})^6}d\vec{x}_*d\tau$$
$$=O(1)\frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})^3},\quad (C.29)$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2}\frac{1}{1+\tau^{(l+5)/2}(|\vec{x}_*|+\sqrt{\tau})^3}d\vec{x}_*d\tau$$
$$=O(1)\frac{\log(1+t)}{1+t^{(l+1)/2}(|\vec{x}|+\sqrt{t})^3}.\quad (C.30)$$

LEMMA C.10. For $\vec{x}\in\mathbb{R}^3$, it is satisfied with $|\vec{x}|<t$ and $l=1,2$

$$\int_{0}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^3}\frac{1}{1+\tau^4}d\vec{x}_*d\tau =O(1)\frac{1}{1+(|\vec{x}|+\sqrt{t})^3},\quad (C.31)$$

$$\int_{0}^{t/2}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^3}\frac{1}{1+\tau^4}d\vec{x}_*d\tau =O(1)\frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})^3},\quad (C.32)$$

$$\int_{t/2}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|<\tau}}\frac{1}{1+(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^3}\frac{1}{1+\tau^{4+(1+l)/2}}d\vec{x}_*d\tau =O(1)\frac{1}{1+t^{l/2}(|\vec{x}|+\sqrt{t})^3}.\quad (C.33)$$

LEMMA C.11. For $\vec{x}\in\mathbb{R}^3$, and $l=1,2$,

$$\int_{0}^{t}\int_{\substack{|\vec{x}-\vec{x}_*|<t-\tau\\|\vec{x}_*|>\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})(t-\tau-|\vec{x}-\vec{x}_*|+\sqrt{t-\tau})^2}\frac{e^{-\frac{(|\vec{x}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^4}d\vec{x}_*d\tau$$
$$=O(1)\frac{\mathbf{H}(t-|\vec{x}|)+e^{-\frac{(|\vec{x}|-t)^2}{C_0(1+t)}}}{1+t^2},\quad (C.34)$$

$$\int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{(1+l)/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1)\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)+e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3+l)/2}}, \quad \text{(C.35)}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-t)^2}{C_1(1+\tau)}}}{(1+\tau)^{(6+l)/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1)\frac{\mathbf{H}(t-|\vec{\boldsymbol{x}}|)+e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3+l)/2}}. \quad \text{(C.36)}$$

LEMMA C.12. For $\vec{\boldsymbol{x}}\in\mathbb{R}^3$ and $l=1,2$,

$$\int_0^t \int_{|\vec{\boldsymbol{x}}_*|<\tau} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{2+1/2}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}+\frac{1}{1+\tau^4}\right) d\vec{\boldsymbol{x}}_* d\tau = O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{(1+t)^2}, \quad \text{(C.37)}$$

$$\int_0^{t/2} \int_{|\vec{\boldsymbol{x}}_*|<\tau} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{2+(1+l)/2}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}+\frac{1}{1+\tau^4}\right) d\vec{\boldsymbol{x}}_* d\tau = O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{(1+t)^{2+l/2}}, \quad \text{(C.38)}$$

$$\int_{t/2}^{t} \int_{|\vec{\boldsymbol{x}}_*|<\tau} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^2} \frac{1}{1+\tau^{\frac{l+1}{2}}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}+\frac{1}{1+\tau^4}\right) d\vec{\boldsymbol{x}}_* d\tau = O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{(1+t)^{2+l/2}}. \quad \text{(C.39)}$$

Appendix D. Proofs for coupling waves.

Proof of Lemma C.1. When $\tau \in (0,(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4)$, for $|\vec{\boldsymbol{x}}|<t$, $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau$, and $|\vec{\boldsymbol{x}}_*|<\tau$, it follows from the triangle inequality

$$t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}>t-\tau-|\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}_*|+\sqrt{3t}/2>t-2\tau-|\vec{\boldsymbol{x}}|+\sqrt{3t}/2>(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4.$$

Thus,

$$\begin{aligned}
&\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{3/2}\sqrt{t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}}\frac{1}{1+\tau^{5/2}}d\vec{\boldsymbol{x}}_*d\tau\\
&\qquad=O(1)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{3/2}\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}}\frac{1}{1+\tau^{5/2}}d\vec{\boldsymbol{x}}_*d\tau\\
&=O(1)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\frac{1}{1+(t-\tau)^{3/2}\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}}\frac{\tau^2}{\tau^{5/2}}d\tau=O(1)\frac{1}{1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}.
\end{aligned}$$

When $\tau\in((t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4,t/2)$ and $|\vec{\boldsymbol{x}}|<t/2$, one uses the polar coordinate (r,θ) for $\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*$:

$$\begin{aligned}
&\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{3/2}\sqrt{t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}}\frac{1}{1+\tau^{5/2}}d\vec{\boldsymbol{x}}_*d\tau\\
&\qquad=O(1)\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_0^{t-\tau}\frac{r}{1+(t-\tau)^{3/2}\sqrt{t-\tau-r+\sqrt{t-\tau}}}\frac{1}{1+\tau^{5/2}}drd\tau\\
&\quad=O(1)\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\frac{1}{1+\tau^{5/2}}d\tau=O(1)\left(1+(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{3/2}\right)^{-1}=O(1)(1+t^{3/2})^{-1}.
\end{aligned}$$

When $\tau\in((t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4,t/2)$ and $|\vec{\boldsymbol{x}}|\in(t/2,t)$, one uses the polar coordinate (r,θ) for $\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*$ and takes the angle into consideration. Then, by the diagram in Figure 1 the integration interval of the angle $\Delta\theta$ is of the order $\tau/|\vec{\boldsymbol{x}}|$.

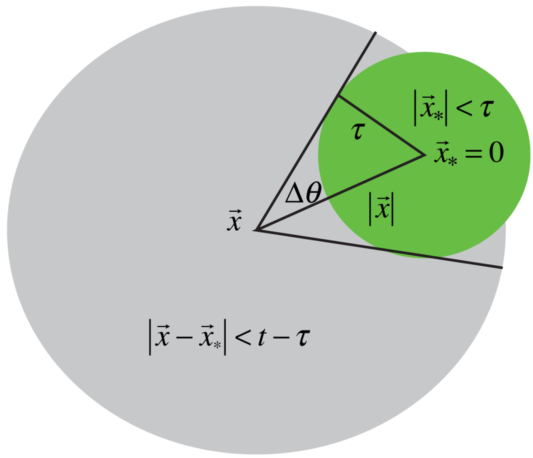

FIG. 1. The diagram for $\vec{\boldsymbol{x}}_*$ with $|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_*| < t - \tau$ and $|\vec{\boldsymbol{x}}_*| < \tau$

Thus,

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{3/2}\sqrt{t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}}} \frac{1}{1+\tau^{5/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \frac{\tau}{|\vec{\boldsymbol{x}}|} \left(\int_0^{t-\tau} \frac{r}{1+(t-\tau)^{3/2}\sqrt{t-\tau-\rho}} \frac{1}{1+\tau^{5/2}} dr \right) d\tau$$

$$= O(1) \int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \frac{1}{|\vec{\boldsymbol{x}}|} \frac{1}{1+\tau^{3/2}} d\tau$$

$$= O(1) \frac{1}{1+|\vec{\boldsymbol{x}}|\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}} = O(1) \frac{1}{1+t\sqrt{t-|\vec{\boldsymbol{x}}|+\sqrt{t}}}.$$

When $\tau \in (t/2, t)$,

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{3/2}\sqrt{t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-\sqrt{t-\tau}}} \frac{1}{1+\tau^{5/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \int_0^{t-\tau} \frac{r}{1+(t-\tau)^{3/2}\sqrt{t-\tau-r+\sqrt{t-\tau}}} \frac{1}{1+\tau^{5/2}} dr d\tau$$

$$= O(1) \int_{t/2}^{t} \frac{1}{1+\tau^{5/2}} d\tau = O(1)(1+t^{3/2})^{-1}.$$

The above cases conclude (C.1) for $k = 2$. The estimate (C.1) for $k = 1$ will follow by the same calculation.

The estimates (C.2) and (C.3) for derivatives follow directly from (C.1), since the extra factors $(t-\tau)^{-l/2}$ and $\tau^{-l/2}$ are $O(1)t^{-l/2}$ in the regions for the integrations. The lemma follows. □

Proof of Lemma C.2. When $\tau \in (0, t/2)$, it is trivial:

$$\begin{aligned}
&\int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1) \int_0^{t/2} \int_0^{\tau} \frac{1}{1+t^{3/2+1/4}} \frac{r}{1+(r+\sqrt{\tau})^4} dr d\tau \\
&\qquad = O(1) \int_0^{t/2} \frac{1}{1+t^{3/2+1/4}} \frac{1}{1+\tau} d\tau = O(1)(1+t^{7/4})^{-1} \log(1+t).
\end{aligned} \tag{D.1}$$

When $\tau \in (t/2, t)$, if $|\vec{\boldsymbol{x}}| < \sqrt{t}$, it is also trivial:

$$\begin{aligned}
&\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1) \int_{t/2}^{t} \int_0^{t-\tau} \frac{r}{1+(t-\tau)^{3/2}(t-\tau-r+\sqrt{t-\tau})^{1/2}} \frac{1}{1+(\sqrt{t})^4} dr d\tau \\
&\qquad = O(1) \int_{t/2}^{t} \frac{1}{1+t^2} d\tau = O(1)(1+t)^{-1} = O(1)(1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2)^{-1}.
\end{aligned}$$

If $|\vec{\boldsymbol{x}}| > \sqrt{t}$, one takes the integration interval of the angle into consideration after using the polar coordinate:

$$\begin{aligned}
&\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1) \int_{t/2}^{t} \int_0^{\tau} \frac{1}{1+(t-\tau)^{3/2+1/4}} \frac{r}{1+(r+\sqrt{\tau})^4} \frac{t-\tau}{|\vec{\boldsymbol{x}}|} dr d\tau \\
&\qquad = O(1) \int_{t/2}^{t} \frac{1}{1+(t-\tau)^{3/4}} \frac{1}{1+t} \frac{1}{|\vec{\boldsymbol{x}}|} d\tau = O(1)(1+t^{3/4}|\vec{\boldsymbol{x}}|)^{-1}.
\end{aligned}$$

If $t^{1/2} < |\vec{\boldsymbol{x}}| < t^{3/4}$, then

$$(1+t^{3/4}|\vec{\boldsymbol{x}}|)^{-1} < (1+|\vec{\boldsymbol{x}}|^2)^{-1} < C(1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2)^{-1}.$$

If $t^{3/4} < |\vec{\boldsymbol{x}}| < t$, one has

$$(1+t^{3/4}|\vec{\boldsymbol{x}}|)^{-1} < (1+t^{3/2})^{-1} < C(1+t(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2})^{-1}.$$

The above cases conclude (C.4).

The estimates (C.5) and (C.6) result directly from (D.1) and (C.4) respectively. The final estimate (C.7) can be obtained by a similar computation: if $|\vec{\boldsymbol{x}}|<\sqrt{t}$,

$$\begin{aligned}&\int_{t/2}^{t}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{(l+4)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2}d\vec{\boldsymbol{x}}_*d\tau\\&=O(1)\int_{t/2}^{t}\int_0^{t-\tau}\frac{r}{1+(t-\tau)(t-\tau-r+\sqrt{t-\tau})^{1/2}}\frac{1}{1+t^{(l+6)/2}}drd\tau=O(1)\int_{t/2}^{t}\frac{1}{1+t^{(l+5)/2}}d\tau\\&=O(1)(1+t)^{-(l+3)/2}=O(1)\left(1+t^{(l+1)/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^2\right)^{-1}.\end{aligned}$$

If $|\vec{\boldsymbol{x}}|>\sqrt{t}$, one takes the integration interval of the angle into consideration after using the polar coordinate:

$$\begin{aligned}&\int_{t/2}^{t}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}}\frac{1}{1+\tau^{(l+4)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2}d\vec{\boldsymbol{x}}_*d\tau\\&=O(1)\int_{t/2}^{t}\int_0^{\tau}\frac{1}{1+(t-\tau)^{1+1/4}}\frac{r}{1+\tau^{(l+4)/2}(r+\sqrt{\tau})^2}\frac{t-\tau}{|\vec{\boldsymbol{x}}|}drd\tau=O(1)\int_{t/2}^{t}\frac{1}{1+(t-\tau)^{1/4}}\frac{\log(1+t)}{1+t^{(l+4)/2}}\frac{1}{|\vec{\boldsymbol{x}}|}d\tau\\&=O(1)\frac{\log(1+t)}{1+t^{1/4}}\frac{1}{1+t^{(l+2)/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})}.\end{aligned}$$

The above cases conclude (C.7) and thus we finish the proof of the lemma. □

Proof of Lemma C.3. When $\tau\in(t/2,t)$, the decay is fast:

$$\begin{aligned}&\int_{t/2}^{t}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})}d\vec{\boldsymbol{x}}_*d\tau\\&=O(1)\int_{t/2}^{t}\int_0^{t-\tau}\frac{r}{1+\sqrt{t-\tau}(r+\sqrt{t-\tau})^2}\frac{1}{1+\tau^{5/2}}drd\tau=O(1)(1+t^{3/2})^{-1}.\quad\text{(D.2)}\end{aligned}$$

When $\tau\in(0,(|\vec{\boldsymbol{x}}|+\sqrt{t})/4)$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}\geq|\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}_*|+\sqrt{3t}/2\geq|\vec{\boldsymbol{x}}|-\tau+\sqrt{3t}/2\geq(|\vec{\boldsymbol{x}}|+\sqrt{t})/2,$$

and

$$\begin{aligned}&\int_0^{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})}d\vec{\boldsymbol{x}}_*d\tau\\&\qquad=O(1)\int_0^{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_0^{\tau}\frac{1}{1+\sqrt{t}(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}\frac{r}{1+\tau^2(\tau-r+\sqrt{\tau})}drd\tau\\&=O(1)\int_0^{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\frac{1}{1+\sqrt{t}(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}\frac{\log(1+\tau)}{1+\tau}d\tau=O(1)\frac{(\log(1+t))^2}{1+t^{1/2}}\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}.\quad\text{(D.3)}\end{aligned}$$

When $\tau \in ((|\vec{\boldsymbol{x}}|+\sqrt{t})/4, t/2)$, one has

$$\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_0^{t-\tau} \frac{r}{1+\sqrt{t}(r+\sqrt{t})^2}\frac{1}{1+\tau^{5/2}} dr d\tau$$
$$= O(1)\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \frac{\log(1+t)}{1+\sqrt{t}}\frac{1}{1+\tau^{5/2}} d\tau = O(1)\frac{\log(1+t)}{1+t^{1/2}}\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^{3/2}}. \quad (D.4)$$

If $t^{17/24} < |\vec{\boldsymbol{x}}| < t$, one has

$$\frac{\log(1+t)}{1+t^{1/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^{3/2}} = O(1)\frac{\log(1+t)}{1+t^{1/16}}\frac{1}{1+t^{3/2}}.$$

If $|\vec{\boldsymbol{x}}| < t^{17/24}$, one uses

$$\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_0^{\tau} \frac{1}{1+t^{3/2}}\frac{r}{1+\tau^2(r+\sqrt{\tau})} dr d\tau$$
$$= O(1)\frac{\log(1+t)}{1+t^{3/2}} < O(1)\frac{1}{1+t^{17/12}} = O(1)\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}.$$

This together with (D.2), (D.3) and (D.4) results in (C.8).

The estimates (C.9) and (C.10) are trivial conclusions of (C.8) and (D.2) respectively. Thus we finish the proof of the lemma. □

Proof of Lemma C.4. This part is similar to a heat kernel and (C.11) can be obtained by direct computations:

$$\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^{2n}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\int_0^t \frac{1}{1+(t-\tau)^{1/2}}\left(\int_0^{t-\tau} \frac{r^{n-1}}{1+(r+\sqrt{t-\tau})^n}\frac{1}{1+|\vec{\boldsymbol{x}}|^{2n}} dr + \int_0^{\tau} \frac{1}{1+(|\vec{\boldsymbol{x}}|)^n}\frac{r^{n-1}}{1+(r+\sqrt{\tau})^{2n}} dr\right) d\tau$$
$$= O(1)\frac{1}{1+|\vec{\boldsymbol{x}}|^n}\left(\frac{\sqrt{t}\log(1+t)}{1+|\vec{\boldsymbol{x}}|^n} + \frac{1}{1+t^{(n-1)/2}} + \frac{(\log(1+t))^{3-n}}{1+t^{1/2}}\right). \quad (D.5)$$

When $|\vec{\boldsymbol{x}}| > \sqrt{t}$, (D.5) is fast enough to yield (C.11). The case for $|\vec{\boldsymbol{x}}| < \sqrt{t}$ is trivial:

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^{2n}} d\vec{\boldsymbol{x}}_* d\tau \\
&= O(1)\int_0^{t/2}\int_0^{\tau} \frac{1}{1+t^{(n+1)/2}} \frac{r^{n-1}}{1+(r+\sqrt{\tau})^{2n}} dr d\tau \\
&+ O(1)\int_{t/2}^{t}\int_0^{t-\tau} \frac{r^{n-1}}{1+\sqrt{t-\tau}(r+\sqrt{t-\tau})^n} \frac{1}{1+t^n} dr d\tau \\
&= O(1)\frac{(\log(1+t))^{3-n}}{1+t^{(n+1)/2}} + O(1)\frac{\log(1+t)}{1+t^{n-1/2}}. \qquad \text{(D.6)}
\end{aligned}$$

The cases in (D.5) and (D.6) yield (C.11) and (C.12) and (C.13) are direct results of (C.11).

The proof of (C.14) is similar to (D.5):

$$\begin{aligned}
&\int_{t/2}^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{1}{1+\tau^{(l+n+2)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^n} d\vec{\boldsymbol{x}}_* d\tau \\
&\qquad = O(1)\int_{t/2}^t \frac{1}{1+t^{(l+n+2)/2}} \frac{1}{1+|\vec{\boldsymbol{x}}|^n} \left(\int_0^{t-\tau} \frac{r^{n-1}}{1+(r+\sqrt{t-\tau})^n} dr \right. \\
&\qquad\qquad \left. + \int_0^{\tau} \frac{r^{n-1}}{1+(r+\sqrt{\tau})^n} dr \right) d\tau \\
&\qquad\qquad\qquad = O(1)\frac{\log(1+t)}{1+t^{n/2}} \frac{1}{1+t^{l/2}|\vec{\boldsymbol{x}}|^n}.
\end{aligned}$$

Thus, we finish the proof. □

Proof of Lemma C.5. When $|\vec{\boldsymbol{x}}| < t$,

$$\begin{aligned}
&\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{5/2}} d\vec{\boldsymbol{x}}_* d\tau \\
&= O(1)\int_0^{t/2} \frac{1}{1+t^{7/4}} \frac{1}{1+\tau} d\tau + O(1)\int_{t/2}^t \int_0^{t-\tau} \frac{r}{1+(t-\tau)^{3/2}(t-\tau-r+\sqrt{t-\tau})^{1/2}} \frac{1}{(1+t)^{5/2}} dr d\tau \\
&\qquad = O(1)\frac{1}{1+t^{3/2}}. \qquad \text{(D.7)}
\end{aligned}$$

When $|\vec{\boldsymbol{x}}| > t$, for $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| < t-\tau$, one has

$$|\vec{\boldsymbol{x}}_*| - \tau \geq |\vec{\boldsymbol{x}}| - |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| + (t-\tau) - t > |\vec{\boldsymbol{x}}| - t > 0,$$

and similar to (D.7),

$$\int_0^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^{1/2}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{5/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}} \left(\int_0^{t/2} \frac{1}{1+t^{7/4}} \frac{1}{1+\tau} d\tau \right.$$

$$\left. + \int_{t/2}^t \int_0^{t-\tau} \frac{r}{1+(t-\tau)^{3/2}(t-\tau-r+\sqrt{t-\tau})^{1/2}} \frac{1}{(1+t)^{5/2}} dr d\tau \right)$$

$$= O(1) \frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{3/2}}.$$

The above cases yield (C.15) and (C.16) and (C.17) can be resulted directly from (C.15). □

Proof of Lemma C.6. We first consider the case $|\vec{\boldsymbol{x}}|<t$: when $\tau \in (t/2,t)$,

$$\int_{t/2}^t \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^t \int_0^{t-\tau} \frac{r^{n-1}}{1+(t-\tau)^{1/2}(r+\sqrt{t-\tau})^n} \frac{1}{(1+t)^{(3n-1)/2}} dr d\tau$$

$$= O(1) \frac{\log(1+t)}{1+t^{1/2}} \frac{1}{1+t^{(3n-3)/2}}.$$

When $\tau \in (0, (|\vec{\boldsymbol{x}}|+\sqrt{t})/8)$, one has

$$\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8} \frac{1}{1+t^{1/2}(|\vec{\boldsymbol{x}}|+\sqrt{t})^n} \frac{1}{1+\tau^{n/2}} d\tau = O(1) \frac{\big(\log(1+t)\big)^{3-n}}{1+t^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^n}.$$

Here, we use the following facts: for $\tau \in \big(0, (|\vec{\boldsymbol{x}}|+\sqrt{t})/8\big)$ and $\tau < |\vec{\boldsymbol{x}}_*| < 2\tau$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau} > |\vec{\boldsymbol{x}}|-3\tau+\sqrt{7t/8} > (|\vec{\boldsymbol{x}}|+\sqrt{t})/2.$$

For $\tau \in (0, (|\vec{\boldsymbol{x}}|+\sqrt{t})/8)$ and $|\vec{\boldsymbol{x}}_*| > 2\tau$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}+|\vec{\boldsymbol{x}}_*|-\tau > |\vec{\boldsymbol{x}}|-\tau+\sqrt{7t/8} > (|\vec{\boldsymbol{x}}|+\sqrt{t})/2.$$

When $\tau \in \big((|\vec{\boldsymbol{x}}|+\sqrt{t})/8, t/2\big)$, one has

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8}^{t/2} \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8}^{t/2} \frac{\log(1+t)}{1+t^{1/2}} \frac{1}{1+\tau^{(3n-1)/2}} d\tau = O(1)\frac{\log(1+t)}{1+t^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^{(3n-3)/2}}, \tag{D.8}$$

and

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8}^{t/2} \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8}^{t/2} \frac{1}{1+t^{(n+1)/2}} \frac{1}{1+\tau^{n/2}} d\tau = O(1)\frac{\log(1+t)^{3-n}}{1+t^{(n+1)/2}}. \tag{D.9}$$

When $n=3$, (D.8) is fast enough. When $n=2$, for $|\vec{\boldsymbol{x}}| \in (t^{17/24}, t)$ and $|\vec{\boldsymbol{x}}| \in (0, t^{17/24})$, one uses (D.8) and (D.9) respectively to yield

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/8}^{t/2} \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{5/2}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)\frac{1}{1+t^{3/2}} + O(1)\frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^2}.$$

When $|\vec{\boldsymbol{x}}|>t$, since $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau$, one has

$$|\vec{\boldsymbol{x}}_*|-\tau \geq |\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+(t-\tau)-t > |\vec{\boldsymbol{x}}|-t>0,$$

and similar to (D.7),

$$\int_0^t \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|>\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^n} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}_*|-\tau)^2}{C_1(1+\tau)}}}{(1+\tau)^{(3n-1)/2}} d\vec{\boldsymbol{x}}_* d\tau$$
$$= O(1)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}} \bigg(\int_0^{t/2} \frac{1}{1+t^{(n+1)/2}} \frac{1}{1+\tau^{n/2}} d\tau$$
$$+ \int_{t/2}^t \int_0^{t-\tau} \frac{r^{n-1}}{1+(t-\tau)^{1/2}(r+\sqrt{t-\tau})^n} \frac{1}{(1+t)^{(3n-1)/2}} dr d\tau\bigg)$$
$$= O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{(3n-1)/4}}.$$

The above cases yield (C.18) and (C.19) and (C.20) can result directly from (C.18). $\square$

Proof of Lemma C.7. When $|\vec{\boldsymbol{x}}| < t$ and $\tau \in (0, t/2)$,

$$\int_0^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}} \left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} + \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} \right) d\vec{\boldsymbol{x}}_* d\tau$$

$$= \int_0^{t/2} \int_0^{\tau} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}} \left(\frac{r}{1+(r+\sqrt{\tau})^4} + \frac{r}{1+\tau^2(\tau-r+\sqrt{\tau})} \right) dr d\tau$$

$$= O(1) \int_0^{t/2} \frac{1}{1+t^{7/4}} \left(\frac{1}{1+\tau} + \frac{\log\tau}{1+\tau} \right) d\tau = O(1) \frac{(\log(1+t))^2}{1+t^{7/4}}.$$

When $\tau \in (t/2, t)$, one has

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}} \frac{1}{1+\tau^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \frac{1}{1+(t-\tau)^{1/4}} \frac{1}{1+t^{5/2}} d\tau = O(1) \frac{1}{1+t^{7/4}},$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t-|\vec{\boldsymbol{x}}|/8} \frac{1}{1+(t-\tau)^{7/4}} \frac{1}{1+t} d\tau + O(1) \int_{t-|\vec{\boldsymbol{x}}|/8}^{t} \frac{1}{1+(t-\tau)^{1/4}} \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^4} d\tau$$

$$= O(1) \frac{1}{1+t|\vec{\boldsymbol{x}}|^{3/4}} + O(1) \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}. \quad \text{(D.10)}$$

In (D.10), one uses the following triangle inequalities: for $\tau \in (t-|\vec{\boldsymbol{x}}|/8, t)$ and $t-\tau < |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| < 2(t-\tau)$,

$$|\vec{\boldsymbol{x}}_*| + \sqrt{\tau} \ge |\vec{\boldsymbol{x}}| - |\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| + \sqrt{t}/2 \ge |\vec{\boldsymbol{x}}| - 3(t-\tau) + \sqrt{t}/2 > (|\vec{\boldsymbol{x}}|+\sqrt{t})/2,$$

while for $\tau \in (t-|\vec{\boldsymbol{x}}|/8, t)$ and $|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| > 2(t-\tau)$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| - (t-\tau) + |\vec{\boldsymbol{x}}_*| + \sqrt{\tau} \ge |\vec{\boldsymbol{x}}| - (t-\tau) + \sqrt{t}/2 \ge (|\vec{\boldsymbol{x}}|+\sqrt{t})/2.$$

When $|\vec{\boldsymbol{x}}| > t$, since $|\vec{\boldsymbol{x}}_*| < \tau$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| - (t-\tau) > |\vec{\boldsymbol{x}}| - t + \tau - |\vec{\boldsymbol{x}}_*| > |\vec{\boldsymbol{x}}| - t > 0.$$

Thus,

$$\begin{aligned}
&\int_0^t \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau\\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}}\left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4}+\frac{1}{1+\tau^{5/2}}\right)d\vec{\boldsymbol{x}}_*d\tau\\
&=O(1)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}\left(\int_0^{t/2}\frac{1}{1+t^{7/4}}\left(\frac{1}{1+\tau}+\frac{1}{1+\tau^{1/2}}\right)d\tau\right.\\
&\qquad\left.+\int_{t/2}^t\frac{1}{1+(t-\tau)^{1/4}}\left(\frac{1}{1+t^2}+\frac{1}{1+t^{5/2}}\right)d\tau\right)\\
&=O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}}.
\end{aligned}$$

The above cases conclude (C.21) and the estimates (C.22) and (C.23) can be obtained directly from (C.21) and we finish the proof. □

Proof of Lemma C.8. One has

$$\begin{aligned}
&\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\\
&\qquad=O(1)\left(\frac{1}{1+(t-\tau)^3}+\frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\right). \qquad \text{(D.11)}
\end{aligned}$$

When $\tau\in(0,(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4)$, one has

$$t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau}\ge t-\tau-|\vec{\boldsymbol{x}}|-|\vec{\boldsymbol{x}}_*|+\sqrt{3t}/2\ge t-|\vec{\boldsymbol{x}}|+\sqrt{3t}/2-2\tau\ge(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4. \qquad \text{(D.12)}$$

Thus,

$$\begin{aligned}
&\int_0^t \int\limits_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|>t-\tau\\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{e^{-\frac{(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|-(t-\tau))^2}{C_1(1+t-\tau)}}}{1+(t-\tau)^{5/4+1/2}}\left(\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4}+\frac{1}{1+\tau^{5/2}}\right)d\vec{\boldsymbol{x}}_*d\tau\\
&=O(1)e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}\left(\int_0^{t/2}\frac{1}{1+t^{7/4}}\left(\frac{1}{1+\tau}+\frac{1}{1+\tau^{1/2}}\right)d\tau+\int_{t/2}^t\frac{1}{1+(t-\tau)^{1/4}}\left(\frac{1}{1+t^2}+\frac{1}{1+t^{5/2}}\right)d\tau\right)\\
&=O(1)\frac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{C_0(1+t)}}}{1+t^{5/4}},
\end{aligned}$$

and

$$\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^3} \frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4} \int_0^{\tau} \frac{1}{1+t^3} \frac{r^2}{1+(r+\sqrt{\tau})^2(\tau-r+\sqrt{\tau})^4} dr d\tau$$

$$= O(1) \frac{1}{1+t^3} \int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4} \frac{1}{1+\tau^{3/2}} d\tau = O(1) \frac{1}{1+t^3}.$$

When $\tau \in ((t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4, t/2)$, one has

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}$$

$$\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_0^{\tau} \frac{1}{1+t^{5/2}} \frac{r^2}{1+(r+\sqrt{\tau})^2(\tau-r+\sqrt{\tau})^4} dr d\tau$$

$$= O(1) \frac{1}{1+t^{5/2}} \int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \frac{1}{1+\tau^{3/2}} d\tau = O(1) \frac{1}{1+t^{5/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}.$$

When $\tau \in (t/2, t)$, it is trivial: since $\dfrac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} = O(1)\dfrac{1}{1+\tau^{2+2}}$, one has

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}$$

$$\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^2(\tau-|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \int_0^{t-\tau} \frac{r^2}{1+(t-\tau)^{1/2}(r+\sqrt{t-\tau})(t-\tau-r+\sqrt{t-\tau})^2} \frac{1}{1+t^4} dr d\tau$$

$$= O(1) \int_{t/2}^{t} \frac{1}{1+t^4} d\tau = O(1)(1+t^3)^{-1}.$$

The above cases conclude (C.24) and thus (C.25) and (C.26) follows. □

Proof of Lemma C.9. From (D.11) and (D.12), when $\tau \in \big(0,(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4\big)$, one has

$$\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{3/2}(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$+O(1)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^3}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\int_0^{\tau}\left(\frac{1}{1+t^{3/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\frac{1}{1+t^3}\right)\frac{r^2}{1+(r+\sqrt{\tau})^6}drd\tau$$
$$=O(1)\left(\frac{1}{1+t^{3/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\frac{1}{1+t^3}\right)\int_0^{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}\frac{1}{1+\tau^{3/2}}d\tau=O(1)\left(\frac{1}{1+t^{3/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^2}+\frac{1}{1+t^3}\right).$$

When $\tau\in\big((t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4,t/2\big)$, one has

$$\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\int_0^{\tau}\frac{1}{1+t^{5/2}}\frac{r^2}{1+(r+\sqrt{\tau})^6}drd\tau$$
$$=O(1)\frac{1}{1+t^{5/2}}\int_{(t-|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2}\frac{1}{1+\tau^{3/2}}d\tau=O(1)\frac{1}{1+t^{5/2}(t-|\vec{\boldsymbol{x}}|+\sqrt{t})^{1/2}}.$$

When $\tau\in(t/2,t)$, similar to the 2-D case, we take the angle into consideration when we use the polar coordinates:

$$\int_{t/2}^{t}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\int_{t/2}^{t}\int_0^{\tau}\frac{1}{1+(t-\tau)^{5/2}}\frac{r^2}{1+(r+\sqrt{\tau})^6}\frac{(t-\tau)^2}{|\vec{\boldsymbol{x}}|^2}drd\tau$$
$$=O(1)\frac{1}{1+t^{3/2}|\vec{\boldsymbol{x}}|^2}\int_{t/2}^{t}\frac{1}{1+\tau^{1/2}}d\tau=O(1)\frac{1}{1+t|\vec{\boldsymbol{x}}|^2}.\tag{D.13}$$

Meanwhile, one has

$$\int_{t/2}^{t}\int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau\\|\vec{\boldsymbol{x}}_*|<\tau}}\frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2}\frac{1}{1+(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^6}d\vec{\boldsymbol{x}}_*d\tau$$
$$=O(1)\int_{t/2}^{t}\int_0^{t-\tau}\frac{r^2}{1+(t-\tau)^{1/2}(r+\sqrt{t-\tau})(t-\tau-r+\sqrt{t-\tau})^2}\frac{1}{1+t^3}drd\tau=O(1)\frac{1}{1+t^3}\int_{t/2}^{t}1d\tau=O(1)\frac{1}{1+t^2}.\tag{D.14}$$

If $|\vec{\boldsymbol{x}}|<\sqrt{t}$, (D.14) yields the pointwise structure (C.27), since

$$(1+t^2)^{-1}<(1+t^{3/2})^{-1}=O(1)\left(1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3\right)^{-1}.$$

For $|\vec{\boldsymbol{x}}|>\sqrt{t}$, one uses (D.13):

$$(1+t|\vec{\boldsymbol{x}}|^2)^{-1}<(1+|\vec{\boldsymbol{x}}|^3)^{-1}=O(1)\left(1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3\right)^{-1}.$$

Thus we finish the proof of (C.27). The estimates (C.28) and (C.29) are direct results of (D.13) and (D.14) respectively.

For the final estimate (C.30), similar to (D.13) and (D.14):

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^{(l+5)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^3} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \int_0^{\tau} \frac{1}{1+(t-\tau)^2} \frac{r^2}{1+t^{(l+5)/2}(r+\sqrt{\tau})^3} \frac{(t-\tau)^2}{|\vec{\boldsymbol{x}}|^2} dr d\tau$$

$$= O(1) \frac{\log(1+t)}{1+t^{(l+5)/2}|\vec{\boldsymbol{x}}|^2} \int_{t/2}^{t} 1 d\tau = O(1) \frac{\log(1+t)}{1+t^{(l+3)/2}|\vec{\boldsymbol{x}}|^2}, \quad \text{(D.15)}$$

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})(t-\tau-|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^2} \frac{1}{1+\tau^{(l+5)/2}(|\vec{\boldsymbol{x}}_*|+\sqrt{\tau})^3} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \int_0^{t-\tau} \frac{r^2}{1+(r+\sqrt{t-\tau})(t-\tau-r+\sqrt{t-\tau})^2} \frac{1}{1+t^{(l+8)/2}} dr d\tau$$

$$= O(1) \frac{1}{1+t^{(l+8)/2}} \int_{t/2}^{t} \sqrt{t-\tau} d\tau = O(1) \frac{1}{1+t^{(l+5)/2}}. \quad \text{(D.16)}$$

The above cases (D.15) and (D.16) yield (C.30) for $|\vec{\boldsymbol{x}}| > \sqrt{t}$ and $|\vec{\boldsymbol{x}}| < \sqrt{t}$ respectively. Thus we finish the proof of the lemma. ☐

Proof of Lemma C.10. When $\tau \in (t/2, t)$, the decay is fast:

$$\int_{t/2}^{t} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^3} \frac{1}{1+\tau^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{t/2}^{t} \int_0^{t-\tau} \frac{r^2}{1+\sqrt{t-\tau}(r+\sqrt{t-\tau})^3} \frac{1}{1+\tau^4} dr d\tau = O(1)(1+t^3)^{-1}. \quad \text{(D.17)}$$

When $\tau \in \big(0, (|\vec{\boldsymbol{x}}|+\sqrt{t})/4\big)$, one has

$$|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*| + \sqrt{t-\tau} \geq |\vec{\boldsymbol{x}}| - |\vec{\boldsymbol{x}}_*| + \sqrt{3t}/2 \geq |\vec{\boldsymbol{x}}| - \tau + \sqrt{3t}/2 \geq (|\vec{\boldsymbol{x}}|+\sqrt{t})/2,$$

and

$$\int_0^{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^3} \frac{1}{1+\tau^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_0^{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4} \frac{1}{1+\sqrt{t}(|\vec{\boldsymbol{x}}|+\sqrt{t})^3} \frac{1}{1+\tau} d\tau = O(1) \frac{\log(1+t)}{1+t^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}. \quad \text{(D.18)}$$

When $\tau \in \big((|\vec{\boldsymbol{x}}|+\sqrt{t})/4, t/2\big)$, one has

$$\int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_{\substack{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|<t-\tau \\ |\vec{\boldsymbol{x}}_*|<\tau}} \frac{1}{1+(t-\tau)^{1/2}(|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_*|+\sqrt{t-\tau})^3} \frac{1}{1+\tau^4} d\vec{\boldsymbol{x}}_* d\tau$$

$$= O(1) \int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \int_0^{t-\tau} \frac{r^2}{1+\sqrt{t}(r+\sqrt{t})^3} \frac{1}{1+\tau^4} dr d\tau$$

$$= O(1) \int_{(|\vec{\boldsymbol{x}}|+\sqrt{t})/4}^{t/2} \frac{\log(1+t)}{1+\sqrt{t}} \frac{1}{1+\tau^4} d\tau = O(1) \frac{\log(1+t)}{1+t^{1/2}} \frac{1}{1+(|\vec{\boldsymbol{x}}|+\sqrt{t})^3}. \quad \text{(D.19)}$$

The estimate (D.19) together with (D.17) and (D.18) is sharp enough to yield (C.31).

The estimates (C.32) and (C.33) are direct results of (C.31) and (D.17) respectively. Thus, we finish the proof. □

Remark D.1. The proofs of Lemma C.11 and Lemma C.12 are similar to that for Lemma C.5 and Lemma C.7 respectively and we omit the details. ■

Acknowledgments. The first author was supported by the National Nature Science Foundation of China 11201296 and 11571230 and the Shanghai Chen Guang project 13CG09. The second author was supported by the MOE tier I grant R-146-000-221-112.

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