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DYNAMICAL SYSTEM APPROACH TO SYNCHRONIZATION OF THE COUPLED SCHRÖDINGER–LOHE SYSTEM

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Abstract. We study wave function synchronization of the Schrödinger–Lohe model, which describes the dynamics of the ensemble of coupled quantum Lohe oscillators with infinite states. To do this, we first derive a coupled system of ordinary differential equations for the L^2_x inner products between distinct wave functions. For the same one-body potentials, we show that the inner products of two wave functions converge to unity for some restricted class of initial data, so complete wave function synchronization emerges asymptotically when the dynamical system approach is used. Moreover, for the family of one-body potentials consisting of real-value translations of the same base potential, we show that the inner products for a two-oscillator system follow the motion of harmonic oscillators in a small coupling regime, and then as the coupling strength increases, the inner products converge to constant values; this behavior yields convergence toward constant values for the L^2_x differences between distinct wave functions.

1. Introduction. The collective synchronous behaviors of classical complex systems are ubiquitous in nature, e.g., the flashing of fireflies, clapping of hands in a concert hall, and heartbeat regulation by pacemaker cells [1–3, 24, 25]. However, rigorous mathematical studies of these collective phenomena were performed only several decades ago by Winfree

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[28] and Kuramoto [16, 17]. They provided continuous dynamical systems based on heuristic and intuitive arguments. In this paper, we are interested in the collective behaviors of quantum Lohe oscillators with all-to-all couplings under one-body external force fields. To establish the idea, we consider a complete network consisting of N nodes, where each pair of nodes is connected with equal capacity. We also assume that quantum Lohe oscillators with the same unit mass are positioned on the nodes of the complete network. Let $\psi_i = \psi_i(x,t)$ be the wave function of the i-th Lohe oscillator on a periodic spatial domain $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$. Then, the dynamics of Lohe oscillators with unit mass is governed by the Schrödinger–Lohe (S–L) model: for $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$,

$$\mathrm{i}\partial_t \psi_i + \Delta \psi_i = V_i(x)\psi_i + \frac{\mathrm{i}K}{N}\sum_{k=1}^{N}\Big(\frac{\|\psi_i\|\psi_k}{\|\psi_k\|} - \frac{\langle \psi_i, \psi_k\rangle \psi_i}{\|\psi_i\|\|\psi_k\|}\Big), \quad 1 \le i \le N, \tag{1.1}$$

where $\|\cdot\| = ||\cdot||_{L^2}$ and $\langle\cdot,\cdot\rangle$ are the standard L^2 norm and inner product on $\mathbb{T}^d$, and $V_i = V_i(x)$ and K correspond to the real-valued one-body potential and coupling strength, respectively. The S–L model (1.1) was first introduced by Australian physicist Max Lohe [19] several years ago as an infinite state generalization of the Lohe matrix model [20]. As discussed in [19, 20], quantum synchronization has received much attention from the physics community [12, 18, 21, 27, 29] because of its possible applications in quantum computing and quantum information [7, 13–15, 22, 23, 26, 30, 31]. From the modeling perspective, the first question that one can ask is

> "Under what conditions, if any, will the S–L model (1.1) exhibit desired synchronous behaviors ?"

This question was partially treated in [4, 5] for some restricted class of initial data and a large coupling strength regime. However, it seems that the answer to this question is still far from complete. We first recall the definition of wave function synchronization as follows.

Definition 1.1 ([4, 5]). Let $\Psi = (\psi_1, \dots, \psi_N)$ be a solution to the S–L model (1.1).

(1) The model (1.1) exhibits complete wave function synchronization if and only if the following estimate holds:

$$\lim_{t\to\infty} ||\psi_i(t) - \psi_j(t)|| = d_{ij}, \quad 1 \le i, j \le N,$$

where d_{ij} is a nonnegative constant.

(2) The model (1.1) exhibits practical wave function synchronization if and only if the following estimate holds:

$$\lim_{K\to\infty} \limsup_{t\to\infty} ||\psi_i(t) - \psi_j(t)|| = 0, \qquad 1 \le i, j \le N.$$

In the previous study performed in [4, 5], global analysis based on the diameter of the wave functions was conducted, so we need full infinite-dimensional results on the wave functions themselves. Note that for some classical oscillator systems, say for Landau–Stuart oscillators, the asymptotic dynamics can be effectively described by the Kuramoto model governing the phase evolutions of limit cycle oscillators [1]. Thus, we can naturally ask whether a similar asymptotic picture can hold, i.e., whether some partial information on the wave functions can be employed in the relaxation process. Note that for two wave functions ψ_i and ψ_j with unit L^2 norms,

$$\lim_{t\to\infty} ||\psi_i(t) - \psi_j(t)|| = d_{ij} \quad \Longleftrightarrow \quad \lim_{t\to\infty} \mathrm{Re}\langle\psi_i(t), \psi_j(t)\rangle = 1 - \frac{d_{ij}^2}{2}, \tag{1.2}$$

where $\langle\psi_i, \psi_j\rangle := \int_{\mathbb{T}^d} \psi_i(x)\overline{\psi_j(x)}dx$. Thus, it suffices to study the evolution of $\mathrm{Re}\langle\psi_i, \psi_j\rangle$.

The main result of this paper is as follows. We provide a finite-dimensional dynamical system approach based on the explicit dynamics of the inner products of two wave functions. More precisely, we set $h_{ij}(t) := \langle\psi_i(t), \psi_j(t)\rangle$. Then, for the special case where

$$V_i(x) = V(x) + \omega_i, \quad \omega_i \in \mathbb{R},$$

h_{ij} satisfies a coupled system of ordinary differential equations (ODEs) (see Lemma 3.3):

$$\frac{dh_{ij}}{dt} = -\mathrm{i}\omega_{ij}h_{ij} + \frac{K}{N}\left[2 + \sum_{k\neq i}^{N} h_{ik} + \sum_{k\neq j}^{N} h_{kj}\right](1 - h_{ij}), \quad 1 \le i < j \le N, \tag{1.3}$$

where $\omega_{ij} = \omega_i - \omega_j$. Several advantages of using this finite-dimensional system instead of the S–L system (1.1) are obvious. First, we do not need to solve the coupled partial differential equation system for synchronization; instead, we solve only the ODE system (1.3), which reduces the computation cost for practical purposes. Second, we might be able to improve our weak synchronization results in [4] for distinct potentials. For the same one-body potential with $\omega_i = \omega_j,\ 1 \le i < j \le N$ in (1.3), we provide an admissible class of initial data leading to the exponential convergence of h_{ij} to 1 for any positive coupling strength. In contrast, for the distributed case where $\omega_i \neq \omega_j$ for some i and j, the linear term $-\mathrm{i}\omega_{ij}h_{ij}$ on the R.H.S. of (1.3) generates rotational motion, whereas the coupling component induces convergence toward the equilibrium points. Hence, naturally the system (1.3) has two competing mechanisms, *rotation* versus *synchronization*. For a small coupling strength $K \ll 1$, the solution to (1.3) can be periodic (see Section 4.2), whereas for a large coupling strength $K \gg 1$, system (1.3) exhibits synchronous behaviors. Hence, our new results improve the earlier results available in [4, 5].

The rest of this paper is organized as follows. In Section 2, we briefly review the basic properties of the S–L model and previous results. We also discuss the relationship between the S–L model (1.1) and the Kuramoto model of synchronization. In Section 3, we derive our governing finite-dimensional model (1.3) and study its dynamics for the identical potential case. In Section 4, we study the dynamic bifurcation-like phenomena of (1.3) from a small coupling strength to a large coupling strength. Finally, Section 5 summarizes our main results. In Appendix A, we briefly discuss the global existence of smooth solutions using a standard energy method.

2. Preliminaries. In this section, we study a priori L^2 conservation of the S–L model and its relationship with the Kuramoto model. We also briefly review earlier results on the synchronization problem.

2.1. *The S–L model.* In [19], Lohe proposed a coupled Schrödinger-type model with an infinite state space, namely, the Schrödinger–Lohe model (1.1) in [19]: for $(x,t) \in \mathbb{T}^d \times \mathbb{R}_+$ and $1 \le i \le N$,

$$\partial_t \psi_i = \mathrm{i}\Delta\psi_i - \mathrm{i}V_i\psi_i + \frac{K}{N}\sum_{k=1}^{N}\Big(\frac{\|\psi_i\|\psi_k}{\|\psi_k\|} - \frac{\langle \psi_i, \psi_k\rangle \psi_i}{\|\psi_i\|\|\psi_k\|}\Big). \tag{2.1}$$

LEMMA 2.1. For a given $T \in (0,\infty]$, let $\Psi = (\psi_1, \cdots, \psi_N)$ be a smooth solution to (2.1) with initial data $\Psi^0 = (\psi_1^0, \cdots, \psi_N^0)$ in the time interval $[0,T)$. Then, the L^2 norm of ψ_i is constant along the flow (2.1):

$$\|\psi_i(t)\| = \|\psi_i^0\| \ \text{ for } \ t \in [0,T), \ 1 \le i \le N.$$

Proof. From system (2.1), we obtain

$$\langle \psi_i, \partial_t\psi_i\rangle = -\mathrm{i}\langle \psi_i, \Delta\psi_i\rangle + \mathrm{i}\langle \psi_i, V_i\psi_i\rangle + \frac{K}{N}\sum_{k=1}^{N}\Big(\frac{\|\psi_i\|\langle \psi_i, \psi_k\rangle}{\|\psi_k\|} - \frac{\langle \psi_i, \psi_k\rangle\|\psi_i\|}{\|\psi_k\|}\Big). \tag{2.2}$$

We take the complex conjugate of (2.2) and use $\overline{\langle f, g\rangle} = \langle g, f\rangle$ to get

$$\langle \partial_t\psi_i, \psi_i\rangle = \mathrm{i}\langle \Delta\psi_i, \psi_i\rangle - \mathrm{i}\langle V_i\psi_i, \psi_i\rangle + \frac{K}{N}\sum_{k=1}^{N}\Big(\frac{\|\psi_i\|\langle \psi_k, \psi_i\rangle}{\|\psi_k\|} - \frac{\langle \psi_k, \psi_i\rangle\|\psi_i\|}{\|\psi_k\|}\Big). \tag{2.3}$$

We add (2.2) and (2.3), use integration by parts twice using the periodic boundary conditions to obtain

$$\langle \psi_i, \Delta\psi_i\rangle = \langle \Delta\psi_i, \psi_i\rangle, \quad \langle \psi_i, V_i\psi_i\rangle = \langle V_i\psi_i, \psi_i\rangle,$$

to find

$$\frac{d}{dt}\|\psi_i\|^2 = 0.$$

This leads to the desired result. □

REMARK 2.2. We refer to Appendix A for the global existence of smooth solutions to (2.1).

We next discuss how the S–L model can be reduced to the Kuramoto model for classical synchronization following [5]. Suppose that the quantum system at each i-th node is a one-body system and assume spatial homogeneity of the system:

$$V_i(x) = \omega_i : \text{constant}, \qquad \psi_i(t,x) = \psi_i(t), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+.$$

In this setting, the S–L system becomes

$$\mathrm{i}\frac{d\psi_i}{dt} = \omega_i\psi_i + \frac{\mathrm{i}K}{N}\sum_{k=1}^{N}\Big(\frac{|\psi_i|\psi_k}{|\psi_k|} - \frac{\langle \psi_i, \psi_k\rangle\psi_i}{|\psi_i||\psi_k|}\Big). \tag{2.4}$$

To derive the Kuramoto model from (2.4), we simply take the following ansatz for ψ_i:

$$\psi_i := e^{-\mathrm{i}\theta_i}, \quad 1 \le i \le N, \tag{2.5}$$

and substitute this ansatz into (2.4) to obtain

$$\dot{\theta}_i\psi_i = \omega_i\psi_i + \frac{\mathrm{i}K}{N}\sum_{k=1}^{N}\Big(\psi_k - e^{-\mathrm{i}(\theta_i - \theta_k)}\psi_i\Big).$$

Then, we take the inner product of the above relation with ψ_i and compare the real part of the resulting relation to get the Kuramoto model for classical synchronization [1, 2, 6, 9–11]:

$$\dot{\theta}_i = \omega_i + \frac{2K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i).$$

2.2. *Previous results.* In this subsection, we briefly review the previous results [4, 5] on wave function synchronization for the S–L model. As mentioned in the Introduction, the S–L model was first considered in Lohe's work [19] for the non-Abelian generalization of the Kuramoto model. However, the first rigorous mathematical studies of the S–L model were treated by the second author and his collaborators in [4, 5]. Below, we briefly summarize the main results from these works. We first set

$$D(\Psi) := \max_{i,j} ||\psi_i - \psi_j||, \quad D(\mathcal{V}) := \max_{i,j} ||V_i - V_j||_{L^\infty}.$$

In [5], they derived a differential inequality for the diameter $D(\Psi)$:

$$\frac{d}{dt}D(\Psi)^2 \le K\Big[(D(\Psi))^2(2D(\Psi) - 1) + \frac{2D(\mathcal{V})}{K}\Big], \quad t > 0. \tag{2.6}$$

2.2.1. *Same potentials.* For the same potentials

$$V_i = V, \quad 1 \le i \le N, \quad \text{i.e.,} \quad D(\mathcal{V}) = 0,$$

equation (2.6) becomes

$$\frac{d}{dt}D(\Psi) \le K(D(\Psi))(D(\Psi) - \frac{1}{2}), \quad t > 0.$$

Then, the ODE comparison principle and the explicit solution to the Riccati equation yield exponential synchronization of the ensemble of wave functions for system (1.1) with $D(\mathcal{V}) = 0$.

THEOREM 2.3 ([5]). Suppose that the coupling strength and initial data satisfy

$$K > 0, \quad V_i = V, \quad \|\psi_i^0\|_2 = 1, \quad 1 \le i \le N, \quad D(\Psi^0) < \frac{1}{2}.$$

Then, for any solution $\Psi = (\psi_1, \ldots, \psi_N)$ to (1.1), the diameter $D(\Psi)$ satisfies

$$D(\Psi(t)) \le \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{Kt}}, \quad t \ge 0.$$

2.2.2. *Distinct potentials.* For distinct potentials with $D(\mathcal{V}) > 0$, we have practical synchronization. Consider the cubic equation

$$f(x) := 2x^3 - x^2 + \frac{2D(\mathcal{V})}{K} = 0, \quad x \in [0, \infty), \qquad K > 54D(\mathcal{V}). \tag{2.7}$$

Then, equation (2.7) has a positive local maximum $\frac{2D(\mathcal{V})}{K}$ and a negative local minimum $\frac{2D(\mathcal{V})}{K} - \frac{1}{27}$ at $x = 0$ and $\frac{1}{3}$, respectively. Moreover, (2.7) has two positive real roots, $\alpha_1 < \alpha_2$:

$$0 < \alpha_1 < \frac{1}{3} < \alpha_2 < \frac{1}{2}.$$

Clearly, the roots depend continuously on K and $D(V)$, and

$$\lim_{K\to\infty} \alpha_1 = 0, \quad \lim_{K\to\infty} \alpha_2 = \frac{1}{2}.$$

Then, we have practical synchronization for $D(\Psi)$.

THEOREM 2.4 ([4]). Suppose that the following assumptions hold.

(1) The coupling strength is sufficiently large in the sense that

$$K > 54D(\mathcal{V}).$$

(2) The initial data Ψ_0 satisfy the smallness assumption:

$$\|\psi_{i0}\| = 1, \quad j = 1, \cdots, N, \qquad D(\Psi_0) < \alpha_2.$$

Then, we can achieve practical synchronization:

$$\lim_{K\to\infty} \limsup_{t\to\infty} D(\Psi(t)) = 0.$$

3. A dynamical system for synchronization matrices. In this section, we derive a finite-dimensional dynamical system associated with the synchronization problem for (1.1) for one-body potentials of the form

$$V_i(x) = V(x) + \omega_i, \quad \omega_i \in \mathbb{R}, \quad x \in \mathbb{T}^d, \quad 1 \le i \le N. \tag{3.1}$$

3.1. *Derivation of a dynamical system.* Note that under the condition (3.1), ψ_i satisfies

$$\mathrm{i}\partial_t \psi_i = -\Delta\psi_i + (V(x) + \omega_i)\psi_i + \frac{\mathrm{i}K}{N}\sum_{k=1}^N \Big(\frac{\|\psi_i\|\psi_k}{\|\psi_k\|} - \frac{\langle\psi_i, \psi_k\rangle\psi_i}{\|\psi_i\|\|\psi_k\|}\Big). \tag{3.2}$$

Without loss of generality, we may assume that

$$\|\psi_i(t)\| = 1, \quad t \ge 0, \quad 1 \le i \le N.$$

Then, system (3.2) becomes

$$\mathrm{i}\partial_t \psi_i = -\triangle\psi_i + (V + \omega_i)\psi_i + \frac{\mathrm{i}K}{N}\sum_{k=1}^N (\psi_k - \langle\psi_i, \psi_k\rangle\psi_i). \tag{3.3}$$

Note that it follows from the relation

$$\|\psi_i - \psi_j\|^2 = 2 - \int_{\mathbb{T}^d} (\psi_i\bar{\psi_j} + \psi_j\bar{\psi_i})dx$$

that we have

$$\lim_{t\to\infty} ||\psi_i(t) - \psi_j(t)|| = d_{ij}^\infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \mathrm{Re}(\langle\psi_i(t), \psi_j(t)\rangle) = 1 - \frac{(d_{ij}^\infty)^2}{2}.$$

For notational simplicity, we set

$$h_{ij}(t) := \int_{\mathbb{T}^d} \psi_i\bar{\psi_j}\, dx \in \mathbb{C}, \quad 1 \le i, j \le N.$$

Henceforth, we consider a potential V that guarantees the global existence of a solution to (3.3). For $V = 0$, the global well-posedness of (3.3) can be studied using the standard argument. Then we have, for $1 \le i,\, j \le N$,

$$h_{ij} = \bar{h}_{ji}, \qquad h_{ii} = \|\psi_i\|^2 = 1, \quad |h_{ij}| = \left| \int_{\mathbb{T}^d} \psi_i \bar{\psi}_j dx \right| \le \|\psi_i\|_{L^2} \|\psi_j\|_{L^2} = 1. \tag{3.4}$$

PROPOSITION 3.1. Let ψ_i be a solution to (3.3). Then h_{ij} satisfies the coupled system of ODEs:

$$\begin{aligned} \frac{dh_{ij}}{dt} &= -\mathrm{i}\omega_{ij} h_{ij} + \frac{K}{N} \left[\sum_{k=1}^{N} h_{ik} + \sum_{k=1}^{N} h_{kj} \right] (1 - h_{ij}) \\ &= -\mathrm{i}\omega_{ij} h_{ij} + \frac{K}{N} \left[2 + \sum_{k \ne i}^{N} h_{ik} + \sum_{k \ne j}^{N} h_{kj} \right] (1 - h_{ij}), \quad 1 \le i, j \le N, \quad t > 0, \end{aligned} \tag{3.5}$$

where $\omega_{ij} := \omega_i - \omega_j$, and we used the fact that $h_{ii} = 1$.

Proof. It follows from (3.3) that we have

$$(\mathrm{i}\partial_t \psi_i + \Delta \psi_i)\bar{\psi}_j = (V + \omega_i)\psi_i \bar{\psi}_j + \frac{\mathrm{i}K}{N} \sum_{k=1}^{N} \left(\psi_k \bar{\psi}_j - \langle \psi_i, \psi_k \rangle \psi_i \bar{\psi}_j \right), \tag{3.6}$$

$$(\mathrm{i}\partial_t \psi_j + \Delta \psi_j)\bar{\psi}_i = (V + \omega_j)\psi_j \bar{\psi}_i + \frac{\mathrm{i}K}{N} \sum_{k=1}^{N} \left(\psi_k \bar{\psi}_i - \langle \psi_j, \psi_k \rangle \psi_j \bar{\psi}_i \right). \tag{3.7}$$

The integral $\displaystyle\int_{\mathbb{T}^d} \Big((3.6) - \overline{(3.7)} \Big) dx$ leads to

$$\begin{aligned} &\mathrm{i} \int_{\mathbb{T}^d} \partial_t (\psi_i \bar{\psi}_j) dx \\ &\quad = \int_{\mathbb{T}^d} \Big(\triangle \bar{\psi}_j \psi_i - \triangle \psi_i \bar{\psi}_j \Big) dx + \int_{\mathbb{T}^d} (\omega_i - \omega_j) \psi_i \bar{\psi}_j dx \\ &\quad + \frac{\mathrm{i}K}{N} \int_{\mathbb{T}^d} \sum_{k=1}^{N} \left(\psi_k \bar{\psi}_j - \langle \psi_i, \psi_k \rangle \psi_i \bar{\psi}_j \right) dx + \frac{\mathrm{i}K}{N} \int_{\mathbb{T}^d} \sum_{k=1}^{N} \Big(\psi_i \bar{\psi}_k - \overline{\langle \psi_j, \psi_k \rangle} \psi_i \bar{\psi}_j \Big) dx \\ &\quad =: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13} + \mathcal{I}_{14}, \end{aligned}$$

where we used the fact that $V_i = V_i(x)$ is a real-valued function. Integrating by parts, we have

$$\begin{aligned} \mathcal{I}_{11} &= \int_{\mathbb{T}^d} \Big((\Delta \bar{\psi}_j)\psi_i - (\Delta \psi_i)\bar{\psi}_j \Big) dx = \int_{\mathbb{T}^d} \Big(\nabla \psi_i \nabla \bar{\psi}_j - \nabla \bar{\psi}_j \nabla \psi_i \Big) dx = 0, \\ \mathcal{I}_{12} &= \omega_{ij} h_{ij}. \end{aligned}$$

By direct calculation, we have

$$\mathcal{I}_{13} = \frac{\mathrm{i}K}{N} \sum_{k=1}^{N} (h_{kj} - h_{ik} h_{ij}), \quad \mathcal{I}_{14} = \frac{\mathrm{i}K}{N} \sum_{k=1}^{N} (h_{ik} - h_{kj} h_{ij}).$$

Then, considering $h_{ii} = 1$, we obtain the desired system, (3.5). ☐

REMARK 3.2. In the context of the Kuramoto model, the relation (2.5) yields

$$h_{ij} = e^{-\mathrm{i}(\theta_i - \theta_j)};$$

hence, the temporal evolution of h_{ij} is equivalent to the temporal evolution of the phase difference $\theta_i - \theta_j$, which was studied in [6] for the Kuramoto model.

We now set the real and imaginary parts of h_{ij}:

$$R_{ij} := \mathrm{Re}\ h_{ij}, \qquad I_{ij} := \mathrm{Im}\ h_{ij}, \qquad h_{ij} = R_{ij} + \mathrm{i}I_{ij}, \qquad 1 \le i,j \le N,$$

and we set the synchronization matrices to be

$$\mathcal{R} := (R_{ij}) \in M_N(\mathbb{R}), \quad \mathcal{I} := (I_{ij}) \in M_N(\mathbb{R}).$$

Thus, it follows from Definition 1.1 and (1.2) that it is enough to show the convergence of the matrices $\mathcal{R}$ and $\mathcal{I}$ as $t \to \infty$. System (3.5) can be rewritten in terms of (R_{ij}, I_{ij}):

$$\begin{aligned} \frac{dR_{ij}}{dt} &= \omega_{ij} I_{ij} + \frac{K}{N}\Big[\Big(2 + 2R_{ij} + \sum_{k\neq i,j}^{N} (R_{ik} + R_{kj})\Big)(1 - R_{ij}) \\ &\qquad + \Big(2I_{ij} + \sum_{k\neq i,j}^{N} (I_{ik} + I_{kj})\Big) I_{ij}\Big], \\ \frac{dI_{ij}}{dt} &= -\omega_{ij} R_{ij} + \frac{K}{N}\Big[\Big(2I_{ij} + \sum_{k\neq i,j}^{N} (I_{ik} + I_{kj})\Big)(1 - R_{ij}) \\ &\qquad - I_{ij}\Big(2 + 2R_{ij} + \sum_{k\neq i,j}^{N} (R_{ik} + R_{kj})\Big)\Big]. \end{aligned} \tag{3.8}$$

Note that the right hand sides of (3.8) are polynomials of I_{ij}'s and R_{ij}'s, thus as long as the solution exists, it should be real analytic.

3.2. *Basic properties of the dynamical system.* In this subsection, we study basic a priori estimates of (3.8). We first show that the synchronization matrices $\mathcal{R}$ and $\mathcal{I}$ are symmetric and skew-symmetric matrices, respectively.

LEMMA 3.3. Let $(R_{ij}), (I_{ij})$ be a global solution to (3.8) with initial data $(R^0_{ij}), (I^0_{ij})$. If the initial data satisfy the conditions

$$R^0_{ij} = R^0_{ji}, \quad I^0_{ij} = -I^0_{ji}, \quad \forall\ i,j \in \{1, \cdots, N\}, \tag{3.9}$$

then we have

$$R_{ij}(t) = R_{ji}(t), \quad I_{ij}(t) = -I_{ji}(t), \quad \forall\ i,j \in \{1, \cdots, N\}, \quad t > 0.$$

Proof. Note that (R_{ji}, I_{ji}) satisfies

$$\begin{aligned}\frac{dR_{ji}}{dt} &= -\omega_{ij} I_{ji} + \frac{K}{N}\Big[\Big(2 + 2R_{ji} + \sum_{k\neq i,j}^{N}(R_{jk} + R_{ki})\Big)(1 - R_{ji}) \\ &\qquad + \Big(2I_{ji} + \sum_{k\neq i,j}^{N}(I_{jk} + I_{ki})\Big)I_{ji}\Big], \\ \frac{dI_{ji}}{dt} &= \omega_{ij} R_{ji} + \frac{K}{N}\Big[\Big(2I_{ji} + \sum_{k\neq i,j}^{N}(I_{jk} + I_{ki})\Big)(1 - R_{ji}) \\ &\qquad - I_{ji}\Big(2 + 2R_{ji} + \sum_{k\neq i,j}^{N}(R_{jk} + R_{ki})\Big)\Big],\end{aligned} \tag{3.10}$$

where we used $\omega_{ij} = -\omega_{ji}$. Then, it follows from (3.8) and (3.10) that we have

$$\begin{aligned}&\frac{d}{dt}(R_{ij} - R_{ji}) \\ &\quad = \omega_{ij}(I_{ij} + I_{ji}) + \frac{2K}{N}(R_{ji} + R_{ij})(R_{ji} - R_{ij}) \\ &\quad + \frac{K}{N}\Big[(1 - R_{ij})\sum_{k\neq i,j}^{N}\Big((R_{ik} - R_{ki}) + (R_{kj} - R_{jk})\Big) \\ &\quad + (R_{ji} - R_{ij})\sum_{k\neq i,j}(R_{jk} + R_{ki}) \\ &\quad + 2(I_{ij} + I_{ji})(I_{ij} - I_{ji}) + I_{ij}\sum_{k\neq i,j}\Big((I_{ik} + I_{ki}) + (I_{kj} + I_{jk})\Big) \\ &\quad - (I_{ij} + I_{ji})\sum_{k\neq i,j}(I_{jk} + I_{ki})\Big]\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\frac{d}{dt}(I_{ij} + I_{ji}) \\ &= \omega_{ij}(R_{ji} - R_{ij}) + \frac{2K}{N}\Big[(I_{ij} + I_{ji})(1 - R_{ij}) - I_{ji}(R_{ji} - R_{ij})\Big] \\ &+ \frac{K}{N}\Big[(1 - R_{ij})\sum_{k\neq i,j}\Big((I_{ik} + I_{ki}) + (I_{kj} + I_{jk})\Big) \\ &- \sum_{k\neq i,j}(I_{jk} + I_{ki})(R_{ij} - R_{ji}) \\ &- 2(I_{ij} + I_{ji})(1 + R_{ij}) + 2I_{ji}(R_{ij} - R_{ji}) \\ &- I_{ij}\sum_{k\neq i,j}\Big((R_{ik} - R_{ki}) + (R_{kj} - R_{jk})\Big) - (I_{ij} + I_{ji})\sum_{k\neq i,j}(R_{jk} + R_{ki})\Big].\end{aligned} \tag{3.12}$$

Note that all the terms on the R.H.S. of (3.11) and (3.12) contain the factors $R_{kl} - R_{lk}$ or $I_{kl} + I_{lk}$. Considering that all terms appearing in the R.H.S. of (3.11), (3.12) are

Lipschitz continuous with respect to $R_{ij} - R_{ji}$, $I_{ij} + I_{ji}$ and the assumption (3.9), we conclude that

$$(R_{ij} - R_{ji})(t) = 0 \quad \text{and} \quad (I_{ij} + I_{ji})(t) = 0, \quad 1 \le i, j \le N,$$

are the unique solutions to (3.11) and (3.12). As a corollary, we have $I_{ii} = 0$. $\square$

REMARK 3.4. For $\omega_{ij} = 0$, it follows from the uniqueness of the ODE system that

$$I_{ij}^0 = 0,\ 1 \le i, j \le N \quad \Longrightarrow \quad I_{ij}(t) = 0,\ 1 \le i, j \le N.$$

Thus, the set $\{I_{ij} = 0,\ 1 \le i, j \le N\}$ is a positively invariant set for the dynamics (3.8).

In the following two sections, we consider the two cases

$$D(\Omega) = 0, \qquad D(\Omega) > 0,$$

where $D(\Omega) := \max_{1 \le i,j \le N} |\omega_i - \omega_j|$.

4. Emergent dynamics: Identical potentials. In this section, we present a sufficient condition for the flow (3.3) to approach the stable equilibrium point for $D(\Omega) = 0$:

$$\omega_i = \omega_j, \quad \text{i.e.,} \quad V_i = V_j, \quad 1 \le i,\ j \le N.$$

Then, h_{ij} satisfies

$$\frac{dh_{ij}}{dt} = \frac{2K}{N}(1 - h_{ij}^2) + \frac{K}{N}\left(\sum_{k \ne i,j}^{N} h_{ik} + \sum_{k \ne i,j}^{N} h_{kj}\right)(1 - h_{ij}), \quad t > 0, \tag{4.1}$$

with initial data $h_{ij}(0) = h_{ij}^0$. Taking (3.4) into account, the initial data should be restricted. Let us define the admissible set

$$\mathcal{A}_N = \{(h_{ij})_{1 \le i < j \le N} \mid h_{ij} = \int_{\mathbb{T}^d} \psi_i \bar{\psi}_j dx \ \text{ with } \ \|\psi_i\|_{L^2} = 1 = \|\psi_j\|_{L^2}\}.$$

We can check that $(1, 1, 1) \in \mathcal{A}_3$ and $(-1, -1, -1) \notin \mathcal{A}_3$. We rewrite (4.1) in terms of its real and imaginary parts (R_{ij}, I_{ij}):

$$\begin{aligned} \frac{dR_{ij}}{dt} &= \frac{K}{N}\Big[\Big(2 + 2R_{ij} + \sum_{k \ne i,j}^{N}(R_{ik} + R_{kj})\Big)(1 - R_{ij}) + \Big(2I_{ij} + \sum_{k \ne i,j}^{N}(I_{ik} + I_{kj})\Big)I_{ij}\Big], \\ \frac{dI_{ij}}{dt} &= \frac{K}{N}\Big[(1 - R_{ij})\sum_{k \ne i,j}^{N}(I_{ik} + I_{kj}) - I_{ij}\Big(4R_{ij} + \sum_{k \ne i,j}^{N}(R_{ik} + R_{kj})\Big)\Big]. \end{aligned} \tag{4.2}$$

Note that $(R_{ij}, I_{ij}) = (1, 0)$ is a hyperbolic equilibrium for (4.2), and the linearized system near $(R_{ij}, I_{ij}) = (1, 0)$ is given by the following decoupled system:

$$\frac{dR_{ij}}{dt} = 2K(1 - R_{ij}), \qquad \frac{dI_{ij}}{dt} = -2KI_{ij}. \tag{4.3}$$

The explicit solution to (4.3) is given by

$$R_{ij}(t) = 1 + (R_{ij}^0 - 1)e^{-2Kt}, \qquad I_{ij}(t) = I_{ij}^0 e^{-2Kt}.$$

4.1. *Basin of attraction.* In this subsection, we study a basin of attraction $\mathcal{B}$ for the equilibrium $(\mathcal{R}_{eq}, \mathcal{I}_{eq})$:

$$\begin{aligned}\mathcal{R}_{eq} &:= \{(R_{ij}) \in \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ R_{ij} = 1, \quad \forall\, 1 \le i < j \le N\},\\ \mathcal{I}_{eq} &:= \{(I_{ij}) \in \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ I_{ij} = 0, \quad \forall\, 1 \le i < j \le N\},\\ \mathcal{B} &:= \{(\mathcal{R}^0, \mathcal{I}^0) \in \mathbb{R}^{\frac{N(N-1)}{2}} \times \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ \lim_{t\to\infty}(\mathcal{R}(t), \mathcal{I}(t)) = (\mathcal{R}_{eq}, \mathcal{I}_{eq})\}.\end{aligned}$$

The exact characterization of this basin $\mathcal{B}$ will be very important from the dynamical system viewpoint. However, it is very difficult to obtain it for a nonlinear system in general. Below, we present a proper subset for $\mathcal{B}$.

Using the invariance property of I_{ij} (Remark 3.2) and assuming $I_{ij} = 0$, we arrive at the following system, which is a reduced system of (4.2):

$$\frac{dR_{ij}}{dt} = \frac{K}{N}\Big[2 + \sum_{k\neq i}^{N} R_{ik} + \sum_{k\neq j}^{N} R_{kj}\Big](1 - R_{ij}), \quad 1 \le i < j \le N, \tag{4.4}$$

with initial data R_{ij}^0. We consider a set

$$\mathcal{S}(N) := \Big\{(R_{ij}) \in \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ 2 + \sum_{k\neq i}^{N} R_{ik} + \sum_{k\neq j}^{N} R_{kj} > 0, \ -1 \le R_{ij} \le 1, \ \forall\, 1 \le i < j \le N\Big\}. \tag{4.5}$$

We can check that

$$\Big\{(R_{ij}) \in \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ -\frac{1}{N-1} < R_{kj} \le 1, \quad \forall\, 1 \le i < j \le N\Big\} \subset \mathcal{S}(N).$$

LEMMA 4.1 (Existence of positively invariant sets). For $N \ge 2$, the set $\mathcal{S}(N)$ is positively invariant along the flow (4.4).

Proof. If $R_{ij} \in \mathcal{S}(N)$, then we have

$$\Big(2 + \sum_{k\neq i}^{N} R_{ik} + \sum_{k\neq j}^{N} R_{kj}\Big)(1 - R_{ij}) \ge 0.$$

Hence, we have

$$\frac{dR_{ij}}{dt} \ge 0, \quad \text{on } \mathcal{S}(N) \qquad \text{and} \qquad \frac{dR_{ij}}{dt} = 0, \quad \text{at } R_{ij} = 1.$$

These relations yield the result that the set $\mathcal{S}(N)$ is positively invariant. □

THEOREM 4.2. Let (R_{ij}) be a solution to (4.4) satisfying the initial data:

$$(R_{ij}^0) \in \mathcal{S}(N).$$

Then, there exists a positive time $t_* \ge 0$ such that, for $1 \le i < j \le N$ and $t \ge t_*$,

(*i*) $R_{ij}(t_*) > 0$.
(*ii*) $(1 - R_{ij}(t_*))e^{-2K(t-t_*)} \le 1 - R_{ij}(t) \le (1 - R_{ij}(t_*))e^{-2K(\min_{k,l} R_{kl}(t_*))(t-t_*)}$.

Proof. The proof has two steps.
(i) (Finite-time entrance into the positive real axis): First, if $R^0_{ij} = 1$ for some $1 \le i < j \le N$, then we have $R_{ij} = 1$ trivially. Therefore, we consider the case $R^0_{ij} < 1$. On this set, we have

$$\frac{dR_{ij}}{dt} = \frac{K}{N}\Big(2 + \sum_{k \neq i}^{N} R_{ik} + \sum_{k \neq j}^{N} R_{kj}\Big)(1 - R_{ij}) > 0, \quad t > 0.$$

The only equilibrium point in $\mathcal{S}(N)$ is $(1, 1, \cdots, 1)$. This clearly implies that the flow enters a positive real axis after some positive time $t_* \ge 0$.
(ii) (Decay estimates): Because the flow R_{ij} keeps increasing after $t_* \ge 0$, we have

$$0 < R_{ij}(t_*) \le \min_{i,j} R_{ij}(t) \le 1, \quad t \ge t_*.$$

• (Upper bound estimate): It follows from (4.4) that we have

$$\frac{dR_{ij}}{dt} \le 2K(1 - R_{ij}), \quad t \ge t_*.$$

This yields

$$R_{ij}(t) \le 1 - (1 - R_{ij}(t_*))e^{-2K(t - t_*)}, \quad t \ge t_*. \tag{4.6}$$

• (Lower bound estimate): Again it follows from (4.4) that we have

$$\frac{dR_{ij}}{dt} \ge 2K(\min_{k,l} R_{kl}(t_*))(1 - R_{ij}), \quad t > t_*.$$

This implies that

$$R_{ij}(t) \ge 1 - (1 - R_{ij}(t_*))e^{-2K(\min_{k,l} R_{kl}(t_*))(t - t_*)}, \quad t \ge t_*. \tag{4.7}$$

We combine (4.6) and (4.7) to obtain the desired estimates. $\Box$

REMARK 4.3. (1) Theorem 4.2 also yields the result that the equilibrium point $(\frac{-1}{N-1}, \cdots, \frac{-1}{N-1})$ is unstable.
(2) For the region

$$\mathcal{U}(N) := \Big\{(R_{ij}) \in \mathbb{R}^{\frac{N(N-1)}{2}} \ : \ 2 + \sum_{k \neq i}^{N} R_{ik} + \sum_{k \neq j}^{N} R_{kj} < 0, \ -1 \le R_{ij} \le 1, \ \forall\, 1 \le i < j \le N\Big\},$$

we have $\frac{dR_{ij}}{dt} < 0$. Thus, the trajectory will exit the region $-1 \le R_{ij} \le 1$. Then we conclude that $\mathcal{U}(N)$ does not belong to the admissible set $\mathcal{A}_N$.

4.2. *Special cases.* In this subsection, we consider the dynamics of (4.1) for $N = 2, 3, 4$. In these special cases, we can provide more detailed dynamics.

4.2.1. *A two-oscillator system.* Here we consider a two-oscillator system. In this case, system (4.1) can be reduced to a single equation for $h := h_{12}$:

$$\frac{dh}{dt} = K(1 - h^2), \quad t > 0,$$

with the initial data $h(0) = h^0$. Then, by direct calculation, we have

$$h(t) = \frac{(1 + h^0)e^{2Kt} - (1 - h^0)}{(1 - h^0) + (1 + h^0)e^{2Kt}}.$$

From this explicit formula for h, we can see that

$$\lim_{t\to\infty} h(t) = \begin{cases} -1, & h^0 = -1, \\ 1, & h^0 \neq -1. \end{cases}$$

Hence, all solutions with initial data $h^0 \neq -1$ will converge to 1 exponentially fast.

4.2.2. *A three-oscillator system.* Here we consider a three-oscillator system. Because $h_{ij} = \overline{h_{ji}}$, system (4.1) becomes the following 3×3 system:

$$\begin{aligned} \frac{dh_{12}}{dt} &= \frac{K}{3}\left(2 + 2h_{12} + \overline{h_{23}} + \overline{h_{31}}\right)(1 - h_{12}), \\ \frac{dh_{23}}{dt} &= \frac{K}{3}\left(2 + 2h_{23} + \overline{h_{31}} + \overline{h_{12}}\right)(1 - h_{23}), \\ \frac{dh_{31}}{dt} &= \frac{K}{3}\left(2 + 2h_{31} + \overline{h_{12}} + \overline{h_{23}}\right)(1 - h_{31}). \end{aligned} \tag{4.8}$$

By direct calculation, it is easy to see that the set of all equilibria for (4.8) inside a unit circle in the complex plane is given by

$$\begin{aligned} \mathcal{E}_{N=3} := \{(h_{12},\, h_{23},\, h_{31}) \in \mathbb{C}^3 \ : \ & (1,\,1,\,1), (1,\,-1,\,-1), (-1,\,1,\,-1), \\ & (-1,\,-1,\,1), (-1/2,\,-1/2,\,-1/2)\}, \end{aligned}$$

where some equilibria are excluded by the constraint $|h_{ij}| \leq 1$. Because the phase space for system (4.8) is six-dimensional, we consider the following two-dimensional reductions.

• (Two-dimensional reductions): Here we consider two-dimensional reductions of (4.8):

$$h_{31} = 1 \qquad \text{or} \qquad h_{12} = h_{23} = h_{31} \qquad \text{or} \qquad h_{23} \in \mathbb{R},\ h_{31} = h_{12} \in \mathbb{R}.$$

◇ Case A ($h_{31} = 1$): In this case, we have

$$\psi_1 = \psi_3, \quad \text{i.e.,} \quad h_{12} = h_{32} = h.$$

Thus, system (4.8), with the notation $\{(h_{12}, h_{23}, h_{31}) = (h, \bar{h}, 1)\}$, reduces to a single equation for h:

$$\frac{dh}{dt} = K(1 - h^2),$$

which is exactly the same as in the two-oscillator system in Section 4.2.1.

◇ Case B ($h_{12} = h_{23} = h_{31}$): System (4.8) reduces to a single equation:

$$\frac{dh_{12}}{dt} = \frac{K}{3}\left(2 + 2h_{12} + 2\overline{h_{12}}\right)(1 - h_{12}). \tag{4.9}$$

With the notation $h_{12} = R + \mathrm{i}I$, (4.9) is equivalent to the following two-dimensional system:

$$\frac{dR}{dt} = \frac{2K}{3}(1 + 2R)(1 - R), \qquad \frac{dI}{dt} = -\frac{2K}{3}(1 + 2R)I. \tag{4.10}$$

The first equation in (4.10) has two equilibria, $R = -\frac{1}{2}$ and $R = 1$ (see Figure 1). By direct calculation, we have the following three cases:

◇ Case B.1 ($R^0 > -\frac{1}{2}$): In this case, it is easy to see that $R(t) > -\frac{1}{2}$ and

$$R(t) = \frac{(1 + 2R^0)e^{2Kt} + R^0 - 1}{(1 + 2R^0)e^{2Kt} - 2(R^0 - 1)}, \qquad I^0 e^{-2Kt} \leq I(t) \leq I^0 e^{-\frac{2K}{3}(1 + 2R^0)t}, \quad t \geq 0. \tag{4.11}$$

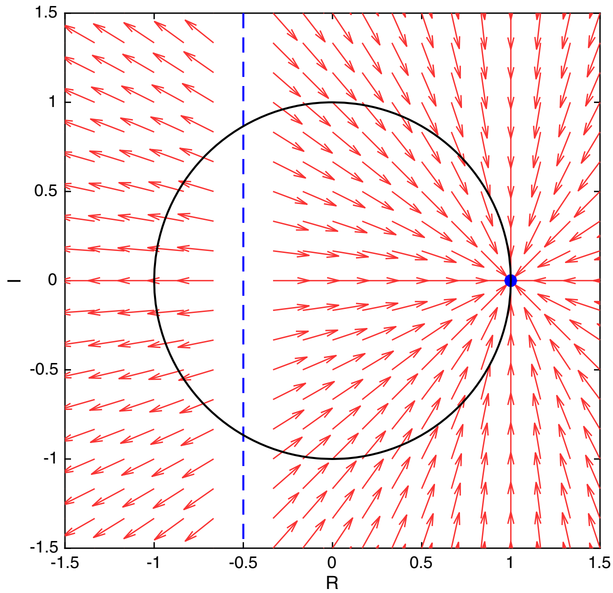

FIG. 1. Phase portrait of (4.10).

Thus, we have

$$\lim_{t\to\infty}(R(t), I(t)) = (1,0).$$

$\diamond$ Case B.2 ($R^0 = -\frac{1}{2}$): In this case, we have

$$R(t) = -\frac{1}{2}, \quad I(t) = I^0, \quad t \geq 0.$$

$\diamond$ Case B.3 ($R^0 < -\frac{1}{2}$): In this case, we can use the same formula, (4.11), to see that there is a finite-time blow-up in R:

$$\lim_{t\to t^*-} R(t) = -\infty, \quad t^* := \frac{1}{2K}\ln\left(\frac{2(R^0-1)}{2R^0+1}\right) > 0.$$

Note that the initial data in case B.3 do not belong to $\mathcal{A}_3$.
$\diamond$ Case C: For $h_{31} = h_{12}$, we have

$$\begin{aligned}\frac{dh_{12}}{dt} &= \frac{K}{3}\left(2 + 2h_{12} + \overline{h_{12}} + \overline{h_{23}}\right)(1-h_{12}),\\ \frac{dh_{23}}{dt} &= \frac{K}{3}\left(2 + 2h_{23} + 2\overline{h_{12}}\right)(1-h_{23}).\end{aligned}$$

Moreover, if we assume $h_{23} := x \in \mathbb{R}$ and $h_{12} := y \in \mathbb{R}$, then we obtain

$$\frac{dx}{dt} = \frac{2K}{3}(1+x+y)(1-x), \qquad \frac{dy}{dt} = \frac{K}{3}(2+3y+x)(1-y). \tag{4.12}$$

Thus, the equilibria with the restriction $|x|$, $|y| \leq 1$ are

$$(x,y) = (1,\,1),\ (1,\,-1),\ (-1/2,\,-1/2),$$

and the x-nullclines and y-nullclines are as follows:

$$x + y + 1 = 0, \quad x = 1, \quad x + 3y + 2 = 0, \quad y = 1.$$

Note that these nullclines divide the square $\{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, \ |y| \le 1\}$ into four subregions Ω_i,

$$\begin{aligned}
\Omega_1 &:= \{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, \ |y| \le 1, \ 1 + x + y > 0, \ 2 + x + 3y > 0\},\\
\Omega_2 &:= \{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, |y| \le 1, 1 + x + y < 0, \ 2 + 3y + x > 0\},\\
\Omega_3 &:= \{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, |y| \le 1, 1 + x + y > 0, \ 2 + 3y + x < 0\},\\
\Omega_4 &:= \{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, |y| \le 1, 1 + x + y < 0, \ 2 + 3y + x < 0\}.
\end{aligned}$$

To see the dynamics of the flow, we check the sign of $\dot{x}$ and $\dot{y}$ on the nullclines and use the Poincaré–Bendixson theorem (see Figure 2).

• On Ω_1: We can apply Theorem 4.2 to conclude $\lim_{t\to\infty}(x(t), y(t)) = (1, 1)$.

• On Ω_4: Considering Remark 4.3, the flow starting from an initial point in Ω_4 will leave the region Ω_4 in a finite-time.

• On Ω_2: Note that $\dot{x} < 0$ and $\dot{y} > 0$. If the flow $(x(t), y(t))$ crosses $x = -1$ or $2 + 3y + x = 0$ (enters Ω_4), then it violates the restriction $|x|, |y| \le 1$ for the admissible set. The only other possibility is that the flow $(x(t), y(t))$ crosses $1 + x + y = 0$, enters Ω_1, and converges to $(1, 1)$. In fact, we can construct admissible initial data in Ω_2. Using the isomorphism $L^2 = l^2$, we choose the initial data for ψ_j as follows:

$$\psi_1(x, 0) = (1, 0, 0, \cdots),$$
$$\psi_2(x, 0) = (-\frac{1}{2} + a, \ \frac{\sqrt{3}}{2} + b, \ 0, \cdots), \quad \psi_3(x, 0) = (-\frac{1}{2} + a, \ -\frac{\sqrt{3}}{2} - b, \ 0, \cdots),$$

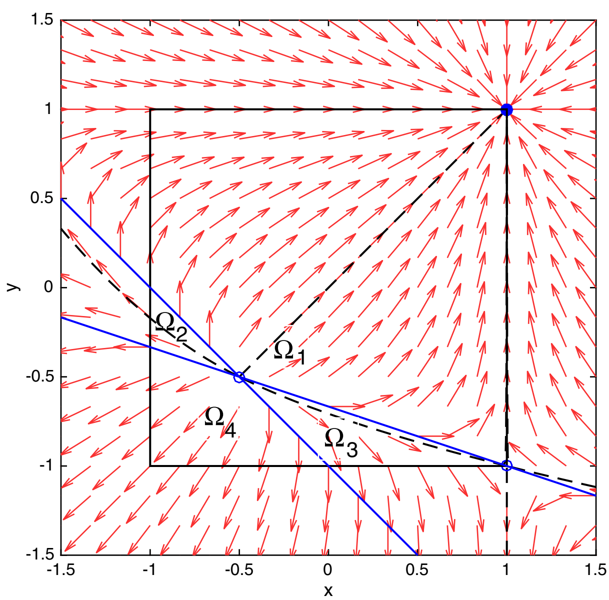

FIG. 2. Phase portrait of (4.12).

where $(a-\frac{1}{2})^2+(\frac{\sqrt{3}}{2}+b)^2=1$. Then we can check that

$$h_{23}(0)=x_0=2(a-\frac{1}{2})^2-1 \quad\text{and}\quad h_{31}(0)=h_{12}(0)=y_0=a-\frac{1}{2}.$$

For $0<a<\frac{1}{2}$, we can check that $(x^0,\,y^0)$ belongs to the region Ω_2. For initial data belonging to the boundary of Ω_2, we can apply the above argument case by case. For example, if $1+x^0+y^0=0$ and $-1\le x^0<-\frac{1}{2}$, then we have $\dot{x}(0)=0$ and $\dot{y}(0)>0$. Therefore, the flow $(x(t),\,y(t))$ converges to $(1,\,1)$.

• On Ω_3: Note that $\dot{x}>0$ and $\dot{y}<0$. For admissible initial data, we have two possibilities.
(A) $(x(t),\,y(t))$ crosses $2+3y+x=0$, enters the region Ω_1, and converges to $(1,\,1)$.
(B) $(x(t),\,y(t))$ stays in Ω_3 and converges to $(1,\,-1)$.

We can construct possibility (A) by choosing proper (x^0,y^0). Actually, for $(x^0,y^0)=(0,-\frac{2}{3}-\varepsilon)$, we have

$$x'(0)=\frac{2}{3}K(\frac{1}{3}-\varepsilon) \quad\text{and}\quad y'(0)=-\frac{K}{3}\varepsilon(5+3\varepsilon).$$

Then, we have

$$x(t)\approx\frac{2}{3}K(\frac{1}{3}-\varepsilon)t \quad\text{and}\quad y(t)\approx-\frac{K}{3}\varepsilon(5+3\varepsilon)t-\frac{2}{3}-\varepsilon \quad\text{for small } t.$$

Thus, we have

$$y+\varepsilon\frac{3}{2}\frac{5+3\varepsilon}{1-3\varepsilon}x+\frac{2}{3}+\varepsilon\approx 0.$$

Therefore $(x(t),\,y(t))$ should cross $2+3y+x=0$ for sufficiently small $\varepsilon>0$.

We do not know whether possibility (B) occurs. However, for a four-oscillator system, we find a flow $(x(t),\,y(t))$ converging to $(1,\,-1)$.

4.2.3. *A four-oscillator system.* System (4.1) reads as

$$\begin{aligned}
\frac{d}{dt}h_{12}&=\frac{K}{4}\left(2+2h_{12}+h_{13}+h_{14}+h_{32}+h_{42}\right)(1-h_{12}),\\
\frac{d}{dt}h_{23}&=\frac{K}{4}\left(2+2h_{23}+h_{21}+h_{24}+h_{13}+h_{43}\right)(1-h_{23}),\\
\frac{d}{dt}h_{31}&=\frac{K}{4}\left(2+2h_{31}+h_{32}+h_{34}+h_{21}+h_{41}\right)(1-h_{31}),\\
\frac{d}{dt}h_{14}&=\frac{K}{4}\left(2+2h_{14}+h_{12}+h_{13}+h_{24}+h_{34}\right)(1-h_{14}),\\
\frac{d}{dt}h_{24}&=\frac{K}{4}\left(2+2h_{24}+h_{21}+h_{23}+h_{14}+h_{34}\right)(1-h_{24}),\\
\frac{d}{dt}h_{34}&=\frac{K}{4}\left(2+2h_{34}+h_{31}+h_{32}+h_{14}+h_{24}\right)(1-h_{34}).
\end{aligned}\tag{4.13}$$

Let $h_{12}=h_{23}=h_{31}$ and $h_{14}=h_{24}=h_{34}$. Then, with the notation $h_{12}=z$ and $h_{14}=\nu$, (4.13) reduces to

$$\begin{aligned}
\frac{dz}{dt}&=\frac{K}{2}\big(1+2\mathrm{Re}(z)+\mathrm{Re}(\nu)\big)(1-z),\\
\frac{d\nu}{dt}&=\frac{K}{2}\big(1+2\nu+\mathrm{Re}(z)\big)(1-\nu),
\end{aligned}$$

which is, with the notation $z = x + \mathrm{i}u$ and $\nu = y + \mathrm{i}v$, equivalent to

$$\begin{aligned} x' &= \frac{K}{2}(1+2x+y)(1-x), \\ u' &= -\frac{K}{2}(1+2x+y)u, \\ y' &= \frac{K}{2}(1+2y+x)(1-y) + Kv^2, \\ v' &= -\frac{K}{2}(1+2y+x)v + K(1-y)v. \end{aligned} \tag{4.14}$$

Let $v = 0$. Then we have a reduced system of (4.14):

$$\begin{aligned} x' &= \frac{K}{2}(1+2x+y)(1-x), \\ y' &= \frac{K}{2}(1+2y+x)(1-y), \end{aligned} \tag{4.15}$$

where u satisfies $u' = -\frac{K}{2}(1+2x+y)u$. Note that system (4.15) consists of x and y only. The equilibria of (4.15) are

$$(x, y) = (1,\, 1),\ (1,\, -1),\ (-1,\, 1),\ (-1/3,\, -1/3).$$

Consider four subregions $\mathcal{R}_i$,

$$\begin{aligned} \mathcal{R}_1 &:= \{(x,y) \in \mathbb{R}^2 \ :\ |x| \le 1,\ |y| \le 1,\ 1+2x+y > 0,\ 1+2y+x > 0\}, \\ \mathcal{R}_2 &:= \{(x,y) \in \mathbb{R}^2 \ :\ |x| \le 1,\ |y| \le 1,\ 1+2x+y < 0,\ 1+2y+x > 0\}, \\ \mathcal{R}_3 &:= \{(x,y) \in \mathbb{R}^2 \ :\ |x| \le 1,\ |y| \le 1,\ 1+2x+y > 0,\ 1+2y+x < 0\}, \\ \mathcal{R}_4 &:= \{(x,y) \in \mathbb{R}^2 \ :\ |x| \le 1,\ |y| \le 1,\ 1+2x+y < 0,\ 1+2y+x < 0\}. \end{aligned}$$

For subregions $\mathcal{R}_1$ and $\mathcal{R}_4$, we can follow a similar argument to (4.12). System (4.15) is symmetric with respect to x and y. We consider only the subregion $\mathcal{R}_3$ (see Figure 3).

• On $\mathcal{R}_3$: Note that $\dot{x} > 0$ and $\dot{y} < 0$. For admissible initial data, we have two possibilities.

(A) $(x(t),\, y(t))$ crosses $1 + 2y + x = 0$, enters the region $\mathcal{R}_1$, and converges to $(1,\, 1)$.

(B) $(x(t),\, y(t))$ stays in $\mathcal{R}_3$ and converges to $(1,\, -1)$. We can construct possibility (A) by choosing a proper (x^0, y^0). Actually, for $(x^0, y^0) = (0, -\frac{1}{2} - \varepsilon)$, we have

$$x'(0) = \frac{K}{2}\Big(\frac{1}{2} - \varepsilon\Big) \quad \text{and} \quad y'(0) = -\varepsilon K\Big(\frac{3}{2} + \varepsilon\Big).$$

Then, we obtain, for small t,

$$\begin{aligned} & x(t) \approx \frac{K}{2}\frac{1-2\varepsilon}{2}t \quad \text{and} \quad y(t) \approx -\varepsilon K\Big(\frac{3}{2}+\varepsilon\Big)t - \frac{1}{2} - \varepsilon, \\ & y(t) + 2\varepsilon\frac{3+2\varepsilon}{1-2\varepsilon}x(t) + \frac{1}{2} + \varepsilon \approx 0. \end{aligned}$$

Therefore, $(x(t),\, y(t))$ should cross $1 + 2y + x = 0$ for sufficiently small $\varepsilon > 0$. Below, we can also construct possibility (B). Actually, we choose the initial data as follows:

$$\psi_1(x,0) = (\varepsilon,\, 0,\, -\sqrt{1-\varepsilon^2},\, 0, \cdots), \qquad \psi_2(x,0) = \Big(-\frac{1}{2}\varepsilon,\, \frac{\sqrt{3}}{2}\varepsilon,\, -\sqrt{1-\varepsilon^2},\, 0, \cdots\Big),$$

$$\psi_3(x,0) = \Big(-\frac{1}{2}\varepsilon,\, -\frac{\sqrt{3}}{2}\varepsilon,\, -\sqrt{1-\varepsilon^2},\, 0, \cdots\Big), \qquad \psi_4(x,0) = (0,\, 0,\, 1,\, 0, \cdots).$$

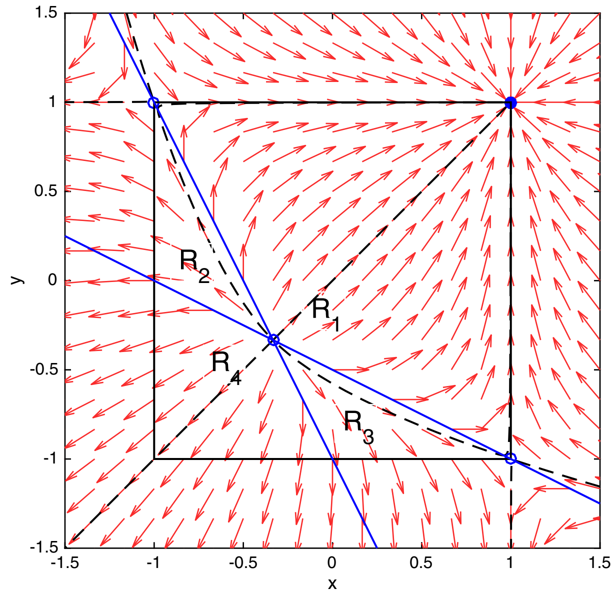

FIG. 3. Phase portrait of (4.15).

Then we can check that

$$h_{12}(0) = h_{23}(0) = h_{31}(0) = 1 - \frac{3}{2}\varepsilon^2 \quad \text{and} \quad h_{14}(0) = h_{24}(0) = h_{34}(0) = -\sqrt{1-\varepsilon^2}.$$

For $\varepsilon < 4/5$, we can check that (x^0, y^0) belongs to region $\mathcal{R}_3$ and $y^0 < -3/5$. If the flow $(x(t), y(t))$ starting by (x^0, y^0) crosses the line $1 + 2y + x = 0$ at $t = t_0$, then we should have $y''(t_0) \geq 0$. However, we can calculate

$$\begin{aligned} y''(t_0) &= (1 + 2y + x)(-y') + (2y' + a')(1 - y) \\ &= (1 - y)(2 - x + y - 7xy - 6x^2 - 5y^2) \\ &= -(1 - y)(1 + 3y)(3 + 5y), \end{aligned}$$

where $1 + 2y(t_0) + x(t_0) = 0$ is used in the second line. Therefore, we have $y''(t_0) < 0$ for $y(t_0) < -\frac{3}{5}$, which is a contradiction. Therefore, the flow $(x(t), y(t))$ starting by (x^0, y^0) should stay in $\mathcal{R}_3$ and converge to $(1, -1)$.

5. Emergent dynamics: Different external potentials. In this section, we consider the dynamics of (3.5) with $D(\Omega) > 0$:

$$\exists\, i, j \quad \text{such that} \;\; \omega_i \neq \omega_j.$$

Then, h_{ij} satisfies

$$\frac{dh_{ij}}{dt} = -\mathrm{i}\omega_{ij}h_{ij} + \frac{2K}{N}(1 - h_{ij}^2) + \frac{K}{N}\left(\sum_{k\neq i,j}^{N} h_{ik} + \sum_{k\neq i,j}^{N} h_{kj}\right)(1 - h_{ij}), \tag{5.1}$$

and its real and imaginary parts (R_{ij}, I_{ij}) satisfy

$$
\begin{aligned}
\frac{dR_{ij}}{dt} &= \omega_{ij} I_{ij} + \frac{K}{N}\Big[\Big(2 + 2R_{ij} + \sum_{k\neq i,j}^{N} (R_{ik} + R_{kj})\Big)(1 - R_{ij}) \\
&\qquad\qquad + \Big(2I_{ij} + \sum_{k\neq i,j}^{N} (I_{ik} + I_{kj})\Big) I_{ij}\Big], \qquad (5.2)\\
\frac{dI_{ij}}{dt} &= -\omega_{ij} R_{ij} + \frac{K}{N}\Big[(1 - R_{ij}) \sum_{k\neq i,j}^{N} (I_{ik} + I_{kj}) - I_{ij}\Big(4R_{ij} + \sum_{k\neq i,j}^{N} (R_{ik} + R_{kj})\Big)\Big].
\end{aligned}
$$

Note that for the zero-coupling case $K = 0$, system (5.1) becomes

$$\frac{dh_{ij}}{dt} = -\mathrm{i}\omega_{ij} h_{ij}.$$

This equation yields a closed orbit solution:

$$h_{ij}(t) = h_{ij}^0 e^{-\mathrm{i}\omega_{ij} t}.$$

We next consider a situation where the coupling strength K is sufficiently large compared with ω_{ij} that our situation is close to the situation in Section 4:

$$R_{ij} \approx 1, \qquad I_{ij} \approx 0.$$

In this regime, system (5.2) can be approximated by a linearized system:

$$
\begin{aligned}
\frac{d\tilde{R}_{ij}}{dt} &= \omega_{ij} \tilde{I}_{ij} + 2K(1 - \tilde{R}_{ij}), \quad t > 0, \\
\frac{d\tilde{I}_{ij}}{dt} &= -\omega_{ij} \tilde{R}_{ij} - 2K \tilde{I}_{ij}.
\end{aligned} \qquad (5.3)
$$

System (5.3) has a unique equilibrium

$$(\tilde{R}_{ij}^e, \tilde{I}_{ij}^e) = \Big(\frac{4K^2}{\omega_{ij}^2 + 4K^2}, \ -\frac{2K\omega_{ij}}{\omega_{ij}^2 + 4K^2}\Big),$$

and we have

$$
\begin{aligned}
\tilde{R}_{ij}(t) &= e^{-2Kt}\Big(C_1 \sin \omega_{ij} t + C_2 \cos \omega_{ij} t\Big) + \frac{4K^2}{\omega_{ij}^2 + 4K^2}, \\
\tilde{I}_{ij}(t) &= -\frac{\omega_{ij} e^{-2Kt}}{2K}\Big(C_1 \sin \omega_{ij} t + C_2 \cos \omega_{ij} t\Big) - \frac{2\omega_{ij} K}{\omega_{ij}^2 + 4K^2},
\end{aligned}
$$

where C_1 and C_2 are constants. Although we do not have a rigorous analysis of the dynamic qualitative behavior of (5.2), we can see that, as K increases, (5.2) might exhibit a bifurcation phenomenon at some critical coupling strength. This bifurcation phenomenon can be seen from the explicit example of a two-oscillator system.

For a two-oscillator system, $h := h_{12}$ and $\omega := \omega_{12}$ satisfy

$$
\begin{aligned}
&\frac{dh}{dt} = -\mathrm{i}\omega h + K(1 - h^2) = -K\left[\Big(h + \mathrm{i}\frac{\omega}{2K}\Big)^2 + \frac{\omega^2}{4K^2} - 1\right], \ t > 0, \\
&h(0) = h^0.
\end{aligned} \qquad (5.4)
$$

Depending on the relative sizes of K and ω, we consider the following three cases:

$$K > \frac{\omega}{2}, \qquad K = \frac{\omega}{2}, \qquad K < \frac{\omega}{2}.$$

• Case A ($K > \frac{\omega}{2}$): In this case, equation (5.4) has two equilibria, $h_{\infty,-}$ and $h_{\infty,+}$:

$$h_{\infty,-} := -\frac{1}{2}\sqrt{4 - \Big(\frac{\omega}{K}\Big)^2} - \mathrm{i}\frac{\omega}{2K} \quad \text{and} \quad h_{\infty,+} := \frac{1}{2}\sqrt{4 - \Big(\frac{\omega}{K}\Big)^2} - \mathrm{i}\frac{\omega}{2K}.$$

By direct calculation, the solution of (5.4) is given by the following explicit formula:

$$h(t) = \frac{h_{\infty,+}(h^0 - h_{\infty,-}) + h_{\infty,-}(h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\omega^2}t}}{h^0 - h_{\infty,-} - (h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\omega^2}t}}.$$

Thus, it is easy to see that for any initial data $h^0 \neq h_{\infty,-}$, we have

$$h(t) \to h_{\infty,+} \quad \text{as} \quad t \to \infty.$$

• Case B ($K = \frac{\omega}{2}$): In this case, the unique equilibrium is

$$h_\infty := -\mathrm{i},$$

and the solution to (5.4) is given by the following formula:

$$h(t) = \frac{h^0 - \mathrm{i}(h^0 + \mathrm{i})Kt}{1 + (h^0 + \mathrm{i})Kt}.$$

Thus, we have

$$h(t) \to h_\infty \quad \text{as} \quad t \to \pm\infty.$$

• Case C ($K < \frac{\omega}{2}$): In this case, the equilibria of system (5.4) are as follows:

$$h_{\infty,-} = \mathrm{i}\Big(-\frac{\omega}{2K} - \frac{1}{2}\sqrt{\Big(\frac{\omega}{K}\Big)^2 - 4}\Big), \quad h_{\infty,+} = \mathrm{i}\Big(-\frac{\omega}{2K} + \frac{1}{2}\sqrt{\Big(\frac{\omega}{K}\Big)^2 - 4}\Big).$$

Because $|h_{\infty,-}| > 1$, only $h_{\infty,+}$ is admissible. By direct calculation, the solution of (5.4) is given as follows:

$$h(t) = \frac{h^0 \cos\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right) - \frac{K}{\sqrt{\omega^2-4K^2}}\left(\frac{\mathrm{i}\omega h^0}{K} - 2\right)\sin\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right)}{\cos\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right) + \frac{K}{\sqrt{\omega^2-4K^2}}\left(2h^0 + \frac{\mathrm{i}\omega}{K}\right)\sin\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right)}. \tag{5.5}$$

Note that (5.5) implies that h is a periodic orbit with period $\frac{4\pi}{\sqrt{\omega^2-4K^2}}$. Thus, we can see that a two-oscillator system has a bifurcation at $K = \frac{\omega}{2}$.

6. Conclusion. In this paper, we studied complete wave synchronization of coupled quantum Lohe oscillators under potential forces with translated mother potentials. To do this, we derived an explicit coupled system of ODEs for $h_{ij} = \langle\psi_i, \psi_j\rangle$. Using phase portrait analysis of this derived ODE model, we provided an admissible class of initial configurations and conditions on the coupling strength leading to complete wave function synchronization. In previous studies, the second author and his collaborators introduced an L^2 diameter for the set of wave functions, which is a global quantity, and then derived a nonlinear Gronwall-type differential inequality. These lead to the restriction on the admissible class of initial data. On the other hand, for distinct potential functions, it seems to be very difficult to verify the emergence of complete synchronization using this global approach. In [4], a weak concept of synchronization was introduced, namely, *practical synchronization*, and they could show that for some class of initial data and coupling strength, practical synchronization can be established. However, it might be very difficult to establish complete wave function synchronization in this way. In contrast, our new approach based on the explicit dynamics of h_{ij} can tell us rather explicitly the correlation between any pair of wave functions.

Appendix A. A global existence of smooth solutions to the S-L model. In this appendix, we present a global existence of smooth solutions to the simplified S-L model (1.1) with a constraint $||\psi_i||_2 = 1$:

$$\begin{cases} \mathrm{i}\partial_t \psi_i + \Delta\psi_i = V_i(x)\psi_i + \dfrac{\mathrm{i}K}{N}\displaystyle\sum_{k=1}^{N} (\psi_k - \langle \psi_i, \psi_k\rangle \psi_i), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+, \\ \psi_i(x,0) = \psi_i^0, \quad x \in \mathbb{T}^d, \qquad ||\psi_i^0||_2 = 1. \end{cases} \tag{A.1}$$

Here $V_i = V_i(x)$ is a given smooth real-valued potential function satisfying

$$\sum_{k=0}^{m} \|\nabla^k V_i\|_{L^\infty} \le C_m < \infty, \quad \text{for a positive integer } m. \tag{A.2}$$

For $T \in (0,\infty)$, let ψ_i and $\bar{\psi}_i$ be H^m solutions to (A.1) in $\mathbb{T}^d \times [0,T]$. Then, for $m \in \mathbb{Z}_+$, we set

$$\begin{aligned} \mathcal{J}(T) &:= \max_{1\le j\le N} \sup_{0\le t\le T} \|\psi_j(t)\|_{H^m}, \\ \Delta(T) &:= \max_{1\le j\le N} \sup_{0\le t\le T} \|\psi_j(t) - \bar{\psi}_j(t)\|_{H^m}. \end{aligned} \tag{A.3}$$

The functionals $\mathcal{J}(T)$ and $\Delta(T)$ will be used in the existence and uniqueness of the local solutions, and will be estimated in the following two lemmas.

Lemma A.1. There exists a small positive constant T_1^* depending only on $\mathcal{J}(0)$ such that for a solution $(\psi_i) \in \mathcal{C}([0,T_1^*); H^m(\Omega))$ to (A.1) and $T < T_1^*$,

$$\mathcal{J}(T) \le 2\mathcal{J}(0). \tag{A.4}$$

Proof. To derive an estimate for $\mathcal{J}$ in (A.3), we study the H^1-estimate and higher-order estimates separately. Let $(\psi_i) \in \mathcal{C}([0,T_1^*); H^m(\Omega))$ be a solution to the Cauchy problem (A.1). In the sequel, we will choose T^* sufficiently small to satisfy the estimate (A.4). Let $T < T_1^*$.

• Case A ($m = 1$): We apply the energy estimates to (A.1) to obtain the following estimate: for $t \in [0,T]$,

$$\begin{aligned} \|\psi_i(t)\|_{H^1} &\le \|\psi_i^0\|_{H^1} \\ &\quad + C\int_0^T \Big(\|(V_i\psi_i)(s)\|_{H^1} + \sum_{k=1}^{N} (\|\psi_k(s)\|_{H^1} + \|\langle\psi_i,\psi_k\rangle(s)\psi_i(s)\|_{H^1}) \Big) ds. \end{aligned} \tag{A.5}$$

On the other hand, note that

$$\begin{aligned}\|V_i\psi_i\|_{H^1} &\le C(\|V_i\|_{L^\infty}\|\psi_i\|_{L^2} + \|\nabla V_i\|_{L^\infty}\|\psi_i\|_{L^2} + \|V_i\|_{L^\infty}\|\nabla\psi_i\|_{L^2}) \\ &\le C(\|V_i\|_{L^\infty} + \|\nabla V_i\|_{L^\infty})\|\psi_i\|_{H^1},\end{aligned} \tag{A.6}$$

and we use the fact that $\langle\psi_i,\psi_k\rangle$ is a function of t to see

$$\begin{aligned}\|\langle\psi_i,\psi_k\rangle\psi_i\|_{H^1} &\le C\Big(\|\langle\psi_i,\psi_k\rangle\|_{L^\infty}\|\psi_i\|_{L^2} + \|\langle\psi_i,\psi_k\rangle\|_{L^\infty}\|\nabla\psi_i\|_{L^2}\Big) \\ &\le C\|\psi_i\|_{L^2}\|\psi_k\|_{L^2}\|\psi_i\|_{H^1}.\end{aligned} \tag{A.7}$$

We combine (A.5)–(A.7) to obtain

$$\|\psi_i(t)\|_{H^1} \le \|\psi_i^0\|_{H^1} + CT\Big[\mathcal{J}(T)(\|V_i\|_{L^\infty} + \|\nabla V_i\|_{L^\infty}) + \mathcal{J}(T)^3\Big]. \tag{A.8}$$

• Case B ($2 \le |\alpha| \le m$): It follows from (A.1) that we have

$$\mathrm{i}\partial_t\partial_x^\alpha\psi_i + \Delta\partial_x^\alpha\psi_i = \partial_x^\alpha(V_i(x)\psi_i) + \frac{\mathrm{i}K}{N}\sum_{k=1}^N(\partial_x^\alpha\psi_k - \langle\psi_i,\psi_k\rangle\partial_x^\alpha\psi_i). \tag{A.9}$$

Then, we use the same energy estimate for (A.9) as in Case A to obtain

$$\|\partial_x^\alpha\psi_i(t)\|_{L^2} \le \|\partial_x^\alpha\psi_i^0\|_{L^2} + CT\Big[\mathcal{J}(T)\sum_{k=0}^m\|\nabla^k V_i\|_{L^\infty} + \mathcal{J}(T)^3\Big]. \tag{A.10}$$

Finally, we use (A.8) and (A.10) to obtain

$$\|\psi_i(t)\|_{H^m} \le \|\psi_i^0\|_{H^m} + CT\Big[\mathcal{J}(T)\sum_{k=0}^m\|\nabla^k V_i\|_{L^\infty} + \mathcal{J}(T)^3\Big]. \tag{A.11}$$

This yields

$$\mathcal{J}(T) \le \mathcal{J}(0) + CT\mathcal{J}(T)\Big[\sum_{k=0}^m\|\nabla^k V_i\|_{L^\infty} + \mathcal{J}(T)^2\Big]. \tag{A.12}$$

In (A.12), we can choose $T \ll 1$ to obtain the desired estimate:

$$\mathcal{J}(T) \le 2\mathcal{J}(0).$$

□

LEMMA A.2. There exists a small positive constant T_2^* depending only on $\mathcal{J}(0)$ and $\bar{\mathcal{J}}(0)$ such that for solutions (ψ_i) and $(\bar\psi_i)$ in $\mathcal{C}([0,T_2^*);H^m(\Omega))$ to (A.1) with initial data (ψ_i^0) and $(\bar\psi_i^0)$, respectively and $T < T_2^*$,

$$\Delta(T) \le 2\Delta(0).$$

Proof. Note that $\psi_i - \bar\psi_i$ satisfies

$$\begin{aligned}&\mathrm{i}\partial_t(\psi_i - \bar\psi_i) + \Delta(\psi_i - \bar\psi_i) = V_i(x)(\psi_i - \bar\psi_i) \\ &\quad+\frac{\mathrm{i}K}{N}\sum_{k=1}^N\Big[(\psi_k - \bar\psi_k) - \Big(\langle\psi_i - \bar\psi_i,\psi_k\rangle + \langle\bar\psi_i,\psi_k - \bar\psi_k\rangle\Big)\psi_i - \langle\bar\psi_i,\bar\psi_k\rangle(\psi_i - \bar\psi_i)\Big].\end{aligned}$$

We use the same arguments in Lemma A.1 to derive the desired estimate. We omit the details. □

Based on Lemma A.1 and Lemma A.2, we are ready to present a local existence of the H^m-solution in the following theorem.

THEOREM A.3 (Local existence). For a positive integer $m \in \mathbb{Z}_+$, suppose that initial data $\psi_i^0(x) \in H^m(\mathbb{T}^d)$. Then, there exists a positive constant $T \in (0, \infty)$ such that the initial value problem (A.1) has a unique local solution $\psi_i \in \mathcal{C}([0, T); H^m(\mathbb{T}^d))$.

Proof. The proof can be done using the standard successive approximation, Lemma A.1 and Lemma A.2 to derive a unique solvability of the local H^m-solutions to (A.1).

Related with the constraint $\|\psi_i\|_{L^2} = 1$, we can check the L^2-conservation of ψ_i. Consider initial data $\psi_i^0 \in H^m$ satisfying $\|\psi_i^0\|_{L^2} = 1$. Then the L^2 norm of ψ_i is constant along the evolution:

$$\|\psi_i(t)\|_{L^2} = \|\psi_i^0\|_{L^2} = 1.$$

This can be seen as follows. We multiply (A.1) by $\bar{\psi}_i$ and take the imaginary part and integrating by parts to obtain

$$\frac{d}{dt}(\|\psi_i\|_{L^2}^2 - 1) + \frac{K}{N}\left(\sum_{k=1}^N 2\mathrm{Re}\langle \psi_i, \psi_k\rangle\right)\left(\|\psi_i\|_{L^2}^2 - 1\right) = 0,$$

which is an ordinary differential equation for $\|\psi_i\|_{L^2}^2 - 1$. Since we have $\|\psi_i^0\|_{L^2}^2 - 1 = 0$, we can conclude that $\|\psi_i(t)\|_{L^2}^2 = 1$ for local existence time $0 \le t \le T$. $\square$

REMARK A.4. By Sobolev embedding theorem [8], it is easy to see that for $m > 2 + \frac{d}{2}$, the H^m solution is a classical solution.

Finally, we are ready to prove a global existence of a classical solution.

THEOREM A.5 (Global existence). Suppose that the initial data $\psi_i^0 \in H^m(\mathbb{T}^d)$. Then, for any $T \in (0, \infty)$, the Cauchy problem for (A.1) has a unique global solution (ψ_i) such that

$$\psi_i \in C([0, \infty),\, H^m(\mathbb{T}^d)) \cap C^1([0, \infty),\, H^{m-2}(\mathbb{T}^d)).$$

Proof. To extend our local solution to the global solution, it suffices to derive the uniform H^m-bound in any finite interval. We take the spatial derivation ∇ to (A.1), multiply $\nabla\bar{\psi}_i$ and choose the imaginary part to obtain

$$\begin{aligned}&\partial_t|\nabla\psi_i|^2 + \mathrm{i}(\nabla\psi_i\Delta\nabla\bar{\psi}_i - \nabla\bar{\psi}_i\Delta\nabla\psi_i)\\ &\qquad = 2\mathrm{Im}(\nabla V_i\psi_i\nabla\psi_i) + \frac{2K}{N}\mathrm{Re}(\nabla\psi_k\nabla\bar{\psi}_i) - \frac{2K}{N}|\nabla\psi_i|^2\mathrm{Re}(\langle\psi_i,\, \psi_k\rangle).\end{aligned}$$

We use integration by parts and use

$$\|\nabla V_i\|_{L^\infty} < C_1, \quad |\langle\psi_i, \psi_k\rangle| \le \|\psi_i\|_{L^2}\|\psi_k\|_{L^2} \le 1,$$

to obtain a Gronwall inequality:

$$\frac{d}{dt}\|\nabla\psi_i\|_{L^2}^2 \le C(\|\nabla\psi_i\|_{L^2} + \|\nabla\psi_i\|_{L^2}\|\nabla\psi_k\|_{L^2} + \|\nabla\psi_i\|_{L^2}^2), \quad i = 1, \cdots, N.$$

Then, Gronwall's lemma implies

$$\|\nabla\psi_i\|_{L^2}^2 \le Ce^{Ct}.$$

By the same argument, we obtain

$$\|\partial^\alpha\psi_i\|_{L^2}^2 \le Ce^{Ct}.$$

Thus, the H^m-norm of ψ_i does not blow up in any finite time interval. This yields the global existence of the H^m-solution. $\square$

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