CELL STRUCTURE OF THE SPACE OF REAL POLYNOMIALS

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ABSTRACT. The space of real polynomials is endowed with cell decompositions such that all polynomials in a single cell have the same root structure on the unit interval, the half-line, or the real line. These decompositions are used to investigate relationship between the roots and extrema of a polynomial, to construct an interpolation polynomial with free knots that increases or decreases simultaneously with the data, and to classify the Abel equations arising in the problem of Chebyshev approximation with fixed coefficients.

§0. INTRODUCTION

Consider the (p, q) -plane $\mathbb{R}^2$ of real polynomials

$$f(t) = t^2 + pt + q.$$

In the plane $\mathbb{R}^2$, the curve $p^2 - 4q = 0$ separates the polynomials with different real roots and those with complex roots. Adding the point at infinity ∞ to the plane $\mathbb{R}^2$, we obtain a cell decomposition of the sphere S^2 ; this decomposition has one zero-dimensional cell ∞ , one one-dimensional cell $p^2 - 4q = 0$, and two two-dimensional cells $p^2 - 4q > 0$ and $p^2 - 4q < 0$.

In each of these cells, the polynomial $f(t)$ has a fixed structure of real roots. Similar cell decompositions can be constructed for polynomials of an arbitrary degree.

Consider a real polynomial

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n.$$

We define the *root structure* of $f(t)$ on the line $\mathbb{R} = (-\infty, \infty)$, on the half-line $\mathbb{R}_+ = [0, \infty)$, and on the unit interval $[-1, 1]$ as follows.

In the case of the line $\mathbb{R}$, we write

$$f(t) = (t - x_1)^{q_1} \cdots (t - x_k)^{q_k} g(t),$$

where $x_1 < \cdots < x_k$, $q_1 \geq 1, \dots, q_k \geq 1$, and the polynomial $g(t)$ has no roots on $\mathbb{R}$. By the *root structure* of the polynomial $f(t)$ on the line $\mathbb{R}$ we shall mean the collection

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$(q_1, \dots, q_k)$. The set of polynomials $f(t)$ with a given structure $(q_1, \dots, q_k)$ is connected. We note that the polynomial $g(t)$ can be written in the form

$$g(t) = u^2(t) + v^2(t),$$

where the polynomials $u(t)$ and $v(t)$ have interlacing roots on $\mathbb{R}$.

In the case of the half-line $\mathbb{R}_+$, we express the polynomial $f(t)$ as

$$f(t) = t^p(t - x_1)^{q_1} \cdots (t - x_k)^{q_k} g(t),$$

where $0 < x_1 < \cdots < x_k$, $p \geq 0$, $q_1 \geq 1, \dots, q_k \geq 1$, and the polynomial $g(t)$ has no roots on $\mathbb{R}_+$. The root structure of $f(t)$ on the half-line $\mathbb{R}_+$ is defined to be the collection $(p; q_1, \dots, q_k)$. The set of polynomials $f(t)$ with a given structure $(p; q_1, \dots, q_k)$ is connected. We note that the polynomial $g(t)$ can be written in the form

$$g(t) = u^2(t) + tv^2(t),$$

where the polynomials $u(t)$ and $v(t)$ have interlacing roots on the half-line $(0, \infty)$.

In the case of the interval $[-1, 1]$, we write

$$f(t) = (t + 1)^p(t - x_1)^{q_1} \cdots (t - x_k)^{q_k}(t - 1)^r g(t),$$

where $-1 < x_1 < \cdots < x_k < 1$, $p \geq 0$, $q_1 \geq 1, \dots, q_k \geq 1$, $r \geq 0$, and the polynomial $g(t)$ has no roots on the interval $[-1, 1]$. We define the root structure of the polynomial $f(t)$ on the interval $[-1, 1]$ to be the collection $(p; q_1, \dots, q_k; r)$. If

$$p + q_1 + \cdots + q_k + r = n,$$

then $g(t) \equiv 1$, so that the set of polynomials $f(t)$ with the structure $(p; q_1, \dots, q_k; r)$ is connected. If

$$p + q_1 + \cdots + q_k + r < n,$$

then the set of polynomials $g(t)$ with the structure $(p; q_1, \dots, q_k; r)$ has two components. In accordance with the sign of $g(t)$ on the interval $[-1, 1]$, we denote these components by $(p; q_1, \dots, q_k; r)_+$ and $(p; q_1, \dots, q_k; r)_-$. Observe that if $g(t)$ has even degree, then

$$g(t) = \pm[u^2(t) + (1 - t^2)v^2(t)],$$

and if $g(t)$ has odd degree, then

$$g(t) = \pm[(1 + t)u^2(t) + (1 - t)v^2(t)],$$

where, in both cases, the polynomials $u(t)$ and $v(t)$ have interlacing roots on the interval $(-1, 1)$.

The root structure of $f(t)$ does not depend on the roots of $g(t)$. However, the representation of $g(t)$ as the sum of squares of $v(t)$ and $u(t)$ plays an important role in the topological constructions below.

We identify the polynomial

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

with the point $a = (a_1, \dots, a_n)$ of the space $\mathbb{R}^n$. Adding the point ∞ at infinity to the space $\mathbb{R}^n$, we identify the extended space $\mathbb{R}^n \cup \infty$ with the unit sphere S^n via the stereographic projection. We agree that for any cell decomposition of the sphere S^n the point ∞ is a zero-dimensional cell. Consequently, the finite cells decompose the space $\mathbb{R}^n$, and this decomposition of $\mathbb{R}^n$ will also be called a cell decomposition.

Let S_1^n denote any cell decomposition of the sphere S^n such that all polynomials in a single cell have the same root structure on the interval $[-1, 1]$. Let S_2^n denote any cell decomposition of the sphere S^n such that all polynomials in a single cell have the same root structure on the half-line $\mathbb{R}_+$. Finally, let S_3^n denote any cell decomposition of the sphere S^n such that all polynomials in a single cell have the same root structure on the line $\mathbb{R}$.

If a cell decomposition of any of the types S_1^n , S_2^n , S_3^n has the minimum number of cells, then we talk about a *large* cell decomposition. All other cell decompositions will be called *small*. The large cell decomposition S_1^n has $1 + 2n$ cells of dimension n , the large cell decomposition S_2^n has $1 + n$ cells of dimension n , and the large cell decomposition S_3^n has $1 + [n/2]$ cells of dimension n .

These large cell decompositions have a rather complicated structure. On the other hand, there exist small decompositions of quite a simple geometric nature. We construct such small decompositions and obtain large decompositions by uniting cells.

All interesting small decompositions will have a simplicial structure, and in this case the meaning of the union of cells is self-evident.

We shall prove the existence of cell decompositions S_1^n , S_2^{n+1} , S_3^{n+2} such that

$$\begin{aligned} S_2^{n+1} &= \text{su } S_1^n, \\ S_3^{n+2} &= / \text{su } S_2^{n+1}. \end{aligned}$$

Here su stands for suspension and $/\text{su}$ is suspension with a marked point. The cell decomposition $S_2^{n+1} = \text{su } S_1^n$ has precisely two zero-dimensional cells. Therefore, the cell decomposition $\text{su } S_2^{n+1}$ has precisely two one-dimensional cells. The factorization $/$ shrinks a closed one-dimensional cell to a point. This gives the cell decomposition $S_3^{n+2} = / \text{su } S_2^{n+1}$ with one zero-dimensional cell. All the topological constructions that we need can be found in the textbook [1] by Rokhlin and Fuks.

The suspension su and the suspension with a marked point $/\text{su}$ reduce the problem to the construction of cell decompositions S_1^0 , S_1^1 , S_1^2 , $\dots$. The cell decomposition S_1^0 is of simplicial nature and can be obtained by attaching the n -dimensional octahedron to the n -dimensional tetrahedron via a simplicial mapping of the boundary of the octahedron onto the boundary of the tetrahedron. For example, the cell decomposition S_1^3 is obtained by attaching the octahedron to the tetrahedron:

via the simplicial mapping

$$\begin{aligned}
 -(1, 0, 0) &\longmapsto -(1, 0, 0, 0), \\
 (1, 0, 0) &\longmapsto (0, 1, 0, 0), \\
 (0, 1, 0) &\longmapsto (0, 1, 0, 0), \\
 -(0, 1, 0) &\longmapsto -(0, 0, 1, 0), \\
 -(0, 0, 1) &\longmapsto -(0, 0, 1, 0), \\
 (0, 0, 1) &\longmapsto (0, 0, 0, 1).
 \end{aligned}$$

In this way we obtain a small cell decomposition S_1^3 with $9 = 1 + 2^3$ cells of dimension 3. The large cell decomposition S_1^3 with $7 = 1 + 2 \cdot 3$ cells of dimension 3 is obtained by uniting two pairs of three-dimensional cells in the octahedron. The tetrahedron $\Delta_{\pm}^3$ consists of alternating vectors the coordinates of which change sign in such a way that the last coordinate is positive. The case of $n = 3$ is easily extended to the general case, and this gives the topological classification of all cell decompositions $S_1^0, S_1^1, S_1^2, \dots$.

It is essential that the n -dimensional octahedron Q^n and the n -dimensional alternating tetrahedron $\Delta_{\pm}^n$ are regular polyhedra. The points of Q^n and of $\Delta_{\pm}^n$ can be identified with piecewise linear functions having n and $n + 1$ links. For example, we associate the barycenter of $\Delta_{\pm}^3$ with the saw-tooth polygonal line:

Glued together, Q^n and $\Delta_{\pm}^n$ can be embedded in the sphere S^{n+1} in such a way that the corresponding piecewise linear functions have the same sequence of minimum and maximum values as the polynomials on $[-1, 1]$ associated with them. For example, the barycenter of $\Delta_{\pm}^3$ corresponds to the Chebyshev polynomial depicted in the figure.

Such a linearization of polynomials proves to be quite useful in many problems and allows one to use the techniques of piecewise linear topology. As applications, we shall consider the problem on roots and extrema, the problem on comonotone interpolation, and the problem on classification of the Abel equations.

In the problem on roots and extrema, we consider polynomials of degree $m + n$ on the real line with $m \geq 2$ fixed roots and n free roots. In a natural way, we define a mapping from the space $\mathbb{R}^n$ of free roots to the space $\mathbb{R}^{n+m-1}$ of extrema. The image of this mapping is a variety of dimension n in the space of dimension $n + m - 1$. We describe

the topology of this variety. In particular, for $m = 2$ this variety contains an embedded n -dimensional sphere.

The Lagrange interpolation with uniform step has an essential drawback: if the data consist of many points, then the interpolation polynomial oscillates strongly. We construct an interpolation polynomial with free knots that increases or decreases simultaneously with the data. If the data are alternating, then such an interpolation polynomial is unique. For example, the classical Chebyshev polynomial corresponds to the saw-tooth data.

The Chebyshev approximation problem with m fixed coefficients is reduced to finitely many Abel equations. For $m = 0, 1, 2$ fixed coefficients, the problem was solved by Chebyshev, Zolotarev, and Akhieser. In this paper we give a topological classification of the Abel equations arising in the problem of Chebyshev approximation with an arbitrary number m of fixed coefficients. For this, we construct a cell decomposition of the m -dimensional sphere in such a way that each cell is associated with a certain Abel equation.

§1. EXAMPLES IN SMALL DIMENSIONS

Let us fill the table

$$\begin{array}{cccc} S_1^0 & S_1^1 & & \\ S_2^0 & S_2^1 & S_2^2 & \\ S_3^0 & S_3^1 & S_3^2 & S_3^3. \end{array}$$

The cell decompositions $S_1^0 = S_2^0 = S_3^0$. Two zero-dimensional cells:

$$\begin{array}{cc} \bullet & \bullet \\ 0 & \infty \end{array}$$

In what follows we depict all cells except for the cell ∞ . For the finite cells we present typical representatives.

The cell decomposition S_1^1 . Three zero-dimensional cells ∞ , $(1; \emptyset; 0)$, $(0; \emptyset; 1)$, and three one-dimensional cells $(0; \emptyset; 0)_+$, $(0; 1; 0)$, $(0; \emptyset; 0)_-$.

The cell decomposition S_2^1 . Two zero-dimensional cells ∞ and $(1; \emptyset)$, and two one-dimensional cells $(0; \emptyset)$, $(0; 1)$.

The cell decomposition S_3^1 . One zero-dimensional cell ∞ and one one-dimensional cell (1).

The cell decomposition S_2^2 . Two zero-dimensional cells ∞ and $(2; \emptyset)$, three one-dimensional cell $(1; \emptyset)$, $(1; 1)$, $(0; 2)$, and three two-dimensional cell $(0; \emptyset)$, $(1; 1)$, $(0; 1, 1)$.

The cell decomposition S_3^2 . One zero-dimensional cell ∞ , one one-dimensional cell (2), and two two-dimensional cells $(\emptyset)$ and $(1, 1)$.

The cell decomposition S_3^3 . The large decomposition: one zero-dimensional cell ∞ , one one-dimensional cell (3), two two-dimensional cells $(2, 1)$ and $(1, 2)$, and two three-dimensional cells (1), $(1, 1, 1)$. A small decomposition: it is convenient to decompose the large three-dimensional cell (1) into two small three-dimensional cells $(1)'$ and $(1)''$ by

additional two-dimensional cell $(1)^0$ consisting of strictly increasing symmetric polynomials.

All these cell decompositions are shown in the figure.

We explain the construction of S^3_3 . We take the unit disk D^2 in the plane $x = 0$ of the space $\mathbb{R}^3$. This disk is shaded. The zero-dimensional cell ∞ is marked. The one-dimensional cell (3) is the boundary circle of D^2 without the marked point. The two-dimensional cell $(1)^0$ is the interior of D^2 . The two-dimensional cell $(2, 1)$ is the left open hemisphere of S^2 . The two-dimensional cell $(1, 2)$ is the right open hemisphere of S^2 . The three-dimensional cell $(1)'$ is the left open half-ball of D^3 . The three-dimensional cell $(1)''$ is the right open half-ball of D^3 . The three-dimensional cell $(1, 1, 1)$ is the exterior of D^3 together with the point at infinity of the space $\mathbb{R}^3$.

These cell decompositions are related by suspension:

and by suspension with a marked point:

We recall the definition of the suspension su . Let X be a cell space. Take the cylinder $X \times [0, 1]$ over X . The cells in $X \times [0, 1]$ have the form $e \times 0$, $e \times (0, 1)$, and $e \times 1$, where e is a cell of X . To obtain the suspension $\text{su} X$ from the cylinder $X \times [0, 1]$, we shrink the base $X \times 0$ to the point 0 and shrink the base $X \times 1$ to the point ∞ . Thus, $\text{su} X$ has precisely two zero-dimensional cells 0 and ∞ . The other cells of $\text{su} X$ are of the form $e \times (0, 1)$, where e is a cell of X .

The operation of suspension $/\text{su}$ with a marked point can be described as follows. Let Y be a cell space with a marked zero-dimensional cell ∞ . Then the suspension $\text{su} Y$ has a marked one-dimensional cell $\infty \times (0, 1)$. We shrink the closure $\infty \times [0, 1]$ of the one-dimensional cell $\infty \times (0, 1)$ to a point and denote the resulting cell space by $/\text{su} Y$. In our case $Y = \{0, \infty\}$ or $Y = \text{su} X$. Therefore, the suspension $/\text{su} Y$ with a marked point has precisely one zero-dimensional cell; as before, this cell is denoted by ∞ .

§2. POSITIVE POLYNOMIALS

In the section we give uniform proofs of theorems on the representation of positive polynomials on the line, on the half-line, and on the unit interval. The reader can find other proofs in the works of Pólya and Szegő [2], Karlin and Shapley [3], Karlin and Studden [4], Videnskii [5], and Szegő [6]. For the first time, the representation of positive polynomials on the interval was given by Markov [7] for polynomials of even degree. In

the general case, the representation of positive polynomials on the interval is obtained via Bernstein's construction of the generalized Chebyshev polynomials (see [8]).

Theorem 1. *Suppose that a real polynomial*

$$f(t) = t^{2m} + a_1 t^{2m-1} + \cdots + a_{2m}$$

has no roots on $\mathbb{R}$. Then there are unique points $x_1 < x_2 < \cdots < x_{2m-1}$ and a number $r > 0$ such that

$$f(t) = \prod_{k=1}^m (t - x_{2k-1})^2 + r \prod_{k=1}^{m-1} (t - x_{2k})^2.$$

Proof. Let $z_1, \dots, z_m$ be the roots of $f(t)$ in the upper half-plane. Consider the rational function

$$F(t) = \prod_{k=1}^m \frac{t - z_k}{t - \bar{z}_k}.$$

Then the points $-\infty < x_1 < x_2 < \cdots < x_{2m-1} < +\infty$ are determined by the relations

$$F(x_k) = (-1)^k,$$

where $k = 1, 2, \dots, 2m-1$, and $r > 0$ is determined by the formula

$$r^2 = \sum_{k=1}^m \operatorname{Im} z_k.$$

The proof is based on the representation

$$f(t) = (t - z_1) \cdots (t - z_m)(t - \bar{z}_1) \cdots (t - \bar{z}_m).$$

Put

$$\varphi(t) = (t - z_1) \cdots (t - z_m)$$

and

$$\psi(t) = (t - \bar{z}_1) \cdots (t - \bar{z}_m).$$

Then the real polynomials

$$u(t) = \frac{\varphi(t) + \psi(t)}{2}$$

and

$$v(t) = \frac{\varphi(t) - \psi(t)}{2i}$$

satisfy the identity

$$f(t) = u^2(t) + v^2(t).$$

We claim that $u(t)$ and $v(t)$ have mutually different and real roots that interlace starting with a root of $u(t)$.

The rational function

$$L_k(t) = \frac{t - z_k}{t - \bar{z}_k}$$

maps the real line onto the unit circle in such a way that, as t increases from $-\infty$ to $+\infty$, the argument $\arg L_k(t)$ strictly increases from 0 to 2π . The argument of a product is the sum of the arguments of the factors. Thus, as t increases from $-\infty$ to $+\infty$, the argument $\arg F(t)$ strictly increases from 0 to $2\pi m$. This means that for all $k = 1, 2, \dots, 2m-1$ there exist real points $-\infty < x_1 < x_2 < \cdots < x_{2m-1} < +\infty$ such that

$$F(x_k) = (-1)^k.$$

Obviously, the odd points $x_1, x_3, \dots, x_{2m-1}$ are the roots of $u(t)$, and the even points $x_2, x_4, \dots, x_{2m-2}$ are the roots of $v(t)$. Since the degree of $u(t)$ is m and the degree of $v(t)$ is $m-1$, we obtain all roots of these polynomials.

Thus, we arrive at the representation of $f(t)$ in the form

$$f(t) = \prod_{k=1}^m (t - x_{2k-1})^2 + r \prod_{k=1}^{m-1} (t - x_{2k})^2.$$

Observe that

$$r^2 = \sum_{k=1}^m \operatorname{Im} z_k.$$

We prove that this representation is unique.

Assume the contrary. Then there exists another representation

$$f(t) = \prod_{k=1}^m (t - x'_{2k-1})^2 + r' \prod_{k=1}^{m-1} (t - x'_{2k})^2$$

with $x'_1 < x'_2 < \dots < x'_{2m-1}$ and $r' > 0$. We put

$$h(t) = \prod_{k=1}^m (t - x_{2k-1})^2 - \prod_{k=1}^m (t - x'_{2k-1})^2.$$

Since

$$h(t) = r' \prod_{k=1}^{m-1} (t - x'_{2k})^2 - r \prod_{k=1}^{m-1} (t - x_{2k})^2,$$

the degree of the polynomial $h(t)$ is at most $2m - 2$. Suppose that $h(t) \not\equiv 0$.

We may assume that $x'_1 \leq x_1$. Let μ_k be the number of roots of $h(t)$ that coincide with x_k . The definition of $h(t)$ shows that if $\mu_k \neq 0$, then $\mu_k \geq 2$. Let ν_k be the number of roots of $h(t)$ on (x_k, x_{k+1}) . Then the number of roots of $h(t)$ on $[x_1, x_{2m-1}]$ is

$$\frac{\mu_1}{2} + \sum_{k=1}^{2m-2} \left(\frac{\mu_k}{2} + \nu_k + \frac{\mu_{k+1}}{2} \right) + \frac{\mu_{2m-1}}{2}.$$

Obviously, $h(x_k)h(x_{k+1}) \leq 0$ for all $1 \leq k \leq 2m - 2$. Consequently, the relation $\mu_k = \mu_{k+1} = 0$ implies that $h(x_k)h(x_{k+1}) < 0$, whence $\nu_k \geq 1$. Thus,

$$\sum_{k=1}^{2m-2} \left(\frac{\mu_k}{2} + \nu_k + \frac{\mu_{k+1}}{2} \right) \geq 2m - 2.$$

If $\mu_1 \geq 2$, then $h(t)$ has $2m - 1$ roots on $[x_1, x_{2m-1}]$, a contradiction. Let $\mu_1 = 0$. Then $x'_1 < x_1$ and $h(x'_1)h(x_1) < 0$. Therefore, $h(t)$ has a root on (x'_1, x_1) . This means that $h(t)$ has $2m - 1$ roots on $[x'_1, x_{2m-1}]$, again a contradiction. $\square$

Theorem 2. *Suppose that a real polynomial*

$$f(t) = t^n + a_1 t^{n-1} + \dots + a_n$$

has no roots on $\mathbb{R}_+$. There are unique points $0 < x_1 < x_2 < \dots < x_{n-1}$ and a number $r > 0$ such that

$$f(t) = \prod_{k=1}^m (t - x_{2k-1})^2 + rt \prod_{k=1}^{m-1} (t - x_{2k})^2$$

for $n = 2m$, and

$$f(t) = t \prod_{k=1}^m (t - x_{2k})^2 + r \prod_{k=1}^m (t - x_{2k-1})^2$$

for $n = 2m + 1$.

Proof. We define $F(t) = f(t^2)$. By Theorem 1, we can write

$$F(t) = \prod_{k=1}^n (t - \lambda_{2k-1})^2 + r \prod_{k=1}^{n-1} (t - \lambda_{2k})^2$$

with some $\lambda_1 < \lambda_2 < \dots < \lambda_{2n-1}$ and $r > 0$. Since $F(t)$ is an even function, we have

$$F(t) = \prod_{k=1}^n (t + \lambda_{2k-1})^2 + r \prod_{k=1}^{n-1} (t + \lambda_{2k})^2.$$

By uniqueness, the points $\lambda_1 < \lambda_2 < \dots < \lambda_{2n-2} < \lambda_{2n-1}$ lie symmetrically relative to the point 0. Hence, $\lambda_n = 0$.

Let $x_k = \lambda_{n+k}^2$ for $1 \leq k \leq n-1$. For $n = 2m$, we have

$$F(t) = \prod_{k=1}^m (t^2 - \lambda_{n+2k-1}^2)^2 + rt^2 \prod_{k=1}^{m-1} (t^2 - \lambda_{n+2k}^2)^2,$$

whence

$$f(t) = \prod_{k=1}^m (t - x_{2k-1})^2 + rt \prod_{k=1}^{m-1} (t - x_{2k})^2.$$

If $n = 2m + 1$, then

$$F(t) = t^2 \prod_{k=1}^m (t^2 - \lambda_{n+2k}^2)^2 + r \prod_{k=1}^m (t^2 - \lambda_{n+2k-1}^2)^2,$$

whence

$$f(t) = t \prod_{k=1}^m (t - x_{2k})^2 + r \prod_{k=1}^m (t - x_{2k-1})^2.$$

The uniqueness of the resulting representations follows from Theorem 1. $\square$

Theorem 3. Suppose that a real polynomial

$$f(t) = t^n + a_1 t^{n-1} + \dots + a_n$$

has no roots on $[-1, 1]$. There are unique points $-1 < x_1 < x_2 < \dots < x_{n-1} < 1$ and a number r with $|r| > 1$ such that

$$f(t) = \frac{r+1}{2} \prod_{k=1}^m (t - x_{2k-1})^2 + \frac{r-1}{2} (1-t^2) \prod_{k=1}^{m-1} (t - x_{2k})^2$$

for $n = 2m$, and

$$f(t) = \frac{r+1}{2} (1+t) \prod_{k=1}^m (t - x_{2k})^2 + \frac{r-1}{2} (1-t) \prod_{k=1}^m (t - x_{2k-1})^2$$

for $n = 2m + 1$. If $r > 1$, then $f(t)$ is positive on $[-1, 1]$. If $r < -1$, then $f(t)$ is negative on $[-1, 1]$.

Proof. Let $z_1, \dots, z_n$ be the roots of $f(t)$. Since $z_1, \dots, z_n$ lie off $[-1, 1]$, there are unique points $w_1, \dots, w_n$ in the interior of the unit disk such that

$$\frac{1}{2} \left(w_k + \frac{1}{w_k} \right) = z_k$$

for all $k = 1, \dots, n$. Consider the rational function

$$F(w) = \frac{1}{2} \left(\prod_{k=1}^n \frac{w - w_k}{1 - w_k w} + \prod_{k=1}^n \frac{1 - w_k w}{w - w_k} \right).$$

We shall prove later that, for $-1 \leq t \leq 1$,

$$F(t + i\sqrt{1-t^2}) = \frac{g(t)}{f(t)},$$

where $g(t)$ is a real polynomial of degree n . Here $\sqrt{1-t^2}$ is positive for $-1 < t < 1$. We claim that the points $-1 < x_1 < x_2 < \dots < x_{n-1} < 1$ are determined by the formulas

$$\frac{g(x_k)}{f(x_k)} = (-1)^{n-k},$$

where $k = 1, 2, \dots, n-1$, and that r is the leading coefficient of $g(t)$ and is determined by the formula

$$r = (-1)^n \frac{1}{2} \left(\prod_{k=1}^n w_k + \prod_{k=1}^n w_k^{-1} \right).$$

Let $\sqrt{z^2 - 1}$, where $z \in \mathbb{C} \setminus [-1, 1]$, be the value of the square root $(z^2 - 1)^{1/2}$ for which

$$|z + \sqrt{z^2 - 1}| > 1.$$

Then the function

$$w = z + \sqrt{z^2 - 1}$$

maps the exterior of the unit interval onto the exterior of the unit disk, and the function

$$w = z - \sqrt{z^2 - 1}$$

maps the exterior of the unit interval onto the interior of the unit disk. In particular,

$$w_k = z_k - \sqrt{z_k^2 - 1},$$

where $k = 1, \dots, n$.

We consider a point $z \in \mathbb{C} \setminus [-1, 1]$ and put

$$w = z + \sqrt{z^2 - 1}.$$

Let $\zeta = \sqrt{z^2 - 1}$ and $\zeta_k = \sqrt{z_k^2 - 1}$. Then

$$\begin{aligned} (w - w_k)(1 - w_k w) &= -2w_k w(z - z_k), \\ (w - w_k)(w^{-1} - w_k^{-1}) &= 2(1 - z_k z + \zeta_k \zeta), \\ (w - w_k^{-1})(w^{-1} - w_k) &= 2(1 - z_k z - \zeta_k \zeta). \end{aligned}$$

Consequently,

$$\begin{aligned} F(w) &= \frac{1}{2} \left(\prod_{k=1}^n \frac{w - w_k}{1 - w_k w} + \prod_{k=1}^n \frac{1 - w_k w}{w - w_k} \right) \\ &= \frac{1}{2} \left(\prod_{k=1}^n \frac{(w - w_k)(w - w_k)}{(w - w_k)(1 - w_k w)} + \prod_{k=1}^n \frac{(1 - w_k w)(1 - w_k w)}{(w - w_k)(1 - w_k w)} \right) \\ &= \frac{1}{2} \left(\prod_{k=1}^n (1 - z_k z + \zeta_k \zeta) + \prod_{k=1}^n (1 - z_k z - \zeta_k \zeta) \right) \frac{1}{f(z)}. \end{aligned}$$

The polynomial

$$\Phi(z, \zeta) = \frac{1}{2} \left(\prod_{k=1}^n (1 - z_k z + \zeta_k \zeta) + \prod_{k=1}^n (1 - z_k z - \zeta_k \zeta) \right)$$

has real coefficients, and its degree in the variables z and ζ is at most n . The polynomial $\Phi(z, \zeta)$ is an even function of ζ . Therefore,

$$g(z) = \Phi(z, \sqrt{z^2 - 1})$$

is a real polynomial of degree at most n . For the leading coefficient r of $g(z)$ we have

$$r = \lim_{t \rightarrow +\infty} \frac{g(t)}{t^n} = (-1)^n \frac{1}{2} \left(\prod_{k=1}^n w_k + \prod_{k=1}^n w_k^{-1} \right).$$

Hence, $|r| > 1$ and the degree of $g(t)$ is equal to n . If $f(t) > 0$ on $[-1, 1]$, then the parity of the number of roots of $f(t)$ on the negative real half-line is equal to the parity of n , whence $r > 1$. If $f(t) < 0$ on $[-1, 1]$, then the parity of the number of roots of $f(t)$ on the negative real half-line is not equal to the parity of n , whence $r < -1$.

By continuity, since

$$F(z + \sqrt{z^2 - 1}) = \frac{g(z)}{f(z)}$$

for $z \in \mathbb{C} \setminus [-1, 1]$, we have

$$F(t + i\sqrt{1 - t^2}) = F(t - i\sqrt{1 - t^2}) = \frac{g(t)}{f(t)}$$

for $-1 \leq t \leq 1$.

We consider the rational function

$$P(w) = \prod_{k=1}^n \frac{w - w_k}{1 - \bar{w}_k w}.$$

Observe that $P(1) = 1$ and $P(-1) = (-1)^n$. Since the polynomial $(w - w_1) \cdots (w - w_n)$ has real coefficients, we have

$$P(w) = \prod_{k=1}^n \frac{w - w_k}{1 - \bar{w}_k w}.$$

Let $|w| = 1$. The identity $P(\bar{w}) = \overline{P(w)}$ means that the images of the upper and the lower half-circle under $P(w)$ lie symmetrically relative to the real line.

We see that the rational function

$$L_k(w) = \frac{w - w_k}{1 - \bar{w}_k w}$$

maps the unit circle onto itself. If the point w goes around the circle in the positive direction, then the argument $\arg L_k(w)$ strictly increases, with the increment 2π . The function $P(w)$ also maps the unit circle onto itself. But if the point w goes around the circle in the positive direction, then the argument $\arg P(w)$ strictly increases with the increment $2\pi n$.

We write the points on the upper half-circle in the form

$$w = t + i\sqrt{1 - t^2},$$

where $-1 \leq t \leq 1$. Let the point t run over the unit interval in the positive direction from -1 to 1 . Then the point

$$z = P(t + i\sqrt{1 - t^2})$$

runs over the unit circle in the negative direction from $(-1)^n$ to 1 . By symmetry, the point on the circle makes precisely n half-turns. If n is even, then z starts from 1 , makes $n/2$ turns, and comes to the initial point 1 . If n is odd, then z starts from -1 , makes $(n - 1)/2$ turns plus one half-turn and comes to the opposite point 1 .

This means that the rational function

$$\frac{g(t)}{f(t)} = F(t + i\sqrt{1-t^2}),$$

where $-1 \leq t \leq 1$, satisfies the inequality

$$-1 \leq \frac{g(t)}{f(t)} \leq 1$$

and possesses the Chebyshev alternance property. There exist points

$$-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$$

such that $f(t)/g(t)$ is strictly monotone on $[x_{k-1}, x_k]$, $k = 1, \dots, n$, and

$$\frac{g(x_k)}{f(x_k)} = (-1)^{n-k}$$

for $k = 0, 1, \dots, n$.

Let

$$f^*(t) = \frac{f(t) + g(t)}{2} \quad \text{and} \quad f_*(t) = \frac{f(t) - g(t)}{2}.$$

We see that

$$f(t) = f^*(t) + f_*(t).$$

Let $n = 2m$. Then

$$f^*(t) = \alpha \prod_{k=1}^m (t - x_{2k-1})^2 \quad \text{and} \quad f_*(t) = \beta(1-t^2) \prod_{k=1}^{m-1} (t - x_{2k})^2.$$

Let $n = 2m + 1$. Then

$$f^*(t) = \alpha(1+t) \prod_{k=1}^m (t - x_{2k})^2 \quad \text{and} \quad f_*(t) = \beta(1-t) \prod_{k=1}^m (t - x_{2k-1})^2.$$

In both cases we have

$$\alpha = \frac{r+1}{2} \quad \text{and} \quad \beta = \frac{r-1}{2}.$$

We prove uniqueness.

Assume the contrary. Then there exists another representation with $-1 < x'_1 < x'_2 < \cdots < x'_{n-1} < 1$ and $r' > 1$. We introduce the polynomial

$$h(t) = \frac{r+1}{2} \prod_{k=1}^m (t - x_{2k-1})^2 - \frac{r'+1}{2} \prod_{k=1}^m (t - x'_{2k-1})^2$$

for $n = 2m$ and the polynomial

$$h(t) = \frac{r+1}{2} (1+t) \prod_{k=1}^m (t - x_{2k})^2 - \frac{r'+1}{2} (1+t) \prod_{k=1}^m (t - x'_{2k})^2$$

for $n = 2m + 1$. Suppose that $h(t) \not\equiv 0$.

We may assume that $x'_1 \leq x_1$. Let $x_0 = -1$ and $x_n = 1$. For $k = 0, 1, \dots, n$, let μ_k be the number of roots of $h(t)$ that coincide with x_k . The definition of $h(t)$ shows that if $\mu_k \neq 0$, then $\mu_k \geq 2$, where $k = 1, \dots, n-1$. Let ν_k be the number of roots of $h(t)$ on (x_k, x_{k+1}) , $k = 0, 1, \dots, n-1$. Then the number of roots of $h(t)$ on $[-1, 1]$ is

$$\mu_0 + \nu_0 + \frac{\mu_1}{2} + \sum_{k=1}^{n-2} \left(\frac{\mu_k}{2} + \nu_k + \frac{\mu_{k+1}}{2} \right) + \frac{\mu_{n-1}}{2} + \nu_{n-1} + \mu_n.$$

Obviously, $h(x_k)h(x_{k+1}) \leq 0$ for $1 \leq k \leq n-2$. Consequently, the relation $\mu_k = \mu_{k+1} = 0$ implies that $h(x_k)h(x_{k+1}) < 0$, whence $\nu_k \geq 1$. Thus,

$$\sum_{k=1}^{n-2} \left(\frac{\mu_k}{2} + \nu_k + \frac{\mu_{k+1}}{2} \right) \geq n-2.$$

We claim that

$$\mu_0 + \nu_1 + \frac{\mu_1}{2} \geq 2.$$

Clearly, $\mu_0 \geq 1$. Assume that $\mu_1 = 0$. Then $x'_1 < x_1$ and $h(x'_1)h(x_1) < 0$. Therefore, $h(t)$ has a root on (x'_1, x_1) , so that $\nu_1 \geq 1$. Since $\mu_n \geq 1$, the nonzero polynomial $h(t)$ of degree at most n has $n+1$ roots on $[-1, 1]$, a contradiction. $\square$

§3. POLYNOMIALS OVER THE INTERVAL

From this section on, to distinguish between the Euclidean space $\mathbb{R}^n$ and the space of real polynomials

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n;$$

we denote the latter by $\mathbb{R}^n(t)$. We add the polynomial at infinity $\infty(t)$ to the space $\mathbb{R}^n(t)$ and denote the extended space $\mathbb{R}^n(t) \cup \infty(t)$ by $S^n(t)$.

Let

$$\omega_n(t, x) = \prod_{i=1}^n (t - x_i),$$

where $x \in \mathbb{R}^n$. (For $n = 0$ we assume that the empty product is equal to 1.)

We introduce the following polynomials: for $n = 2m$

$$\mu_n(t, y, r) = \frac{r+1}{2} \prod_{i=1}^m (t - y_{2i-1})^2 + \frac{r-1}{2} (1-t^2) \prod_{i=1}^{m-1} (t - y_{2i})^2,$$

and for $n = 2m+1$

$$\mu_n(t, y, r) = \frac{r+1}{2} (1+t) \prod_{i=1}^m (t - y_{2i})^2 + \frac{r-1}{2} (1-t) \prod_{i=1}^m (t - y_{2i-1})^2,$$

where $y \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$. The first four polynomials look like this:

$$\begin{aligned} \mu_1(t, y, r) &= \frac{r+1}{2} (1+t) + \frac{r-1}{2} (1-t), \\ \mu_2(t, y, r) &= \frac{r+1}{2} (t - y_1)^2 + \frac{r-1}{2} (1-t^2), \\ \mu_3(t, y, r) &= \frac{r+1}{2} (1+t) (t - y_2)^2 + \frac{r-1}{2} (1-t) (t - y_1)^2, \\ \mu_4(t, y, r) &= \frac{r+1}{2} (t - y_1)^2 (t - y_3)^2 + \frac{r-1}{2} (1-t^2) (t - y_2)^2. \end{aligned}$$

Consider the simplex of increasing vectors

$$\nabla^n = \{x \in \mathbb{R}^n : -1 \leq x_1 \leq \cdots \leq x_n \leq 1\}$$

with vertices

$$\begin{aligned} A_0^n &= (1, 1, 1, \dots, 1), \\ A_1^n &= (-1, 1, 1, \dots, 1), \\ &\vdots \\ A_n^n &= (-1, -1, -1, \dots, -1). \end{aligned}$$

Let $f \in \mathbb{R}^n(t)$. If $f(t)$ has n roots on the interval $[-1, 1]$, then

$$f(t) = \omega_n(t, x),$$

where $x \in \nabla^n$. If $f(t)$ has $0 \leq k \leq n-1$ roots on the interval $[-1, 1]$, then

$$f(t) = \omega_k(t, x) \mu_{n-k}(t, y, r),$$

where $x \times y \in \nabla^k \times \text{Int } \nabla^{n-1-k}$ and $|r| > 1$. Hence, the space $\mathbb{R}^n(t)$ is obtained by gluing $1 + 2n$ polyhedra

$$\begin{aligned} & \nabla^n, \\ & \nabla^0 \times \nabla^{n-1} \times (-\infty, -1], \quad \nabla^0 \times \nabla^{n-1} \times [1, \infty), \\ & \nabla^1 \times \nabla^{n-2} \times (-\infty, -1], \quad \nabla^1 \times \nabla^{n-2} \times [1, \infty), \\ & \vdots \\ & \nabla^{n-1} \times \nabla^0 \times (-\infty, -1], \quad \nabla^{n-1} \times \nabla^0 \times [1, \infty) \end{aligned}$$

along their boundaries. We identify the boundary points corresponding to one and the same polynomial $f(t)$. If we glue only the last $2n$ polyhedra, then we obtain the n -dimensional octahedron with center deleted. Therefore, the sphere $S^n(t)$ is obtained by gluing the octahedron and the tetrahedron of dimension n along their boundaries. This construction is nontrivial because the attaching mapping is not a homeomorphism if $n \geq 2$.

Any Euclidean simplex is a polyhedron with obvious triangulation. Any polyhedron is a cell space with obvious cell decomposition. By the faces of a convex polyhedron (bounded and unbounded) we mean its faces in the usual sense. In this connection, the interior of a face will be called a cell of the corresponding dimension.

For $x \in \mathbb{R}^n$ we denote

$$|x| = |x_1| + \cdots + |x_n|.$$

We introduce the octahedron

$$Q^n = \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

Let $E_1^n, \dots, E_n^n$ be the unit basis vectors of $\mathbb{R}^n$. The points

$$-E_1^1, \dots, -E_n^n, E_1^1, \dots, E_n^n$$

are the vertices of Q^n . We recall the definition of the unit simplex:

$$\Delta^n = \text{conv}\{E_1^{n+1}, E_2^{n+1}, \dots, E_{n+1}^{n+1}\}.$$

For a vector $\delta = (\delta_1, \dots, \delta_n)$ with coordinates ± 1 , we put

$$\Delta^{n-1}(\delta) = \text{conv}\{\delta_1 E_1^n, \delta_2 E_2^n, \dots, \delta_n E_n^n\}$$

and

$$Q^n(\delta) = \text{conv}\{0, \delta_1 E_1^n, \delta_2 E_2^n, \dots, \delta_n E_n^n\}.$$

Note that the face $\Delta^{n-1}(\delta)$ of the simplex $Q^n(\delta)$ lies opposite the origin 0. We view the octahedron Q^n as a polyhedron with the triangulation formed by triangulations of all simplexes $Q^n(\delta)$.

Let $k = \text{alt}(\delta)$ be the number of the sign alternations in the sequence $\delta_1, \dots, \delta_n$. We write a point $\lambda \in \Delta^{n-1}(\delta)$ in the form

$$\lambda = \sum_{\nu=1}^n \gamma_\nu \delta_\nu E_\nu^n,$$

where $\gamma \in \Delta^{n-1}$. Suppose that in the sequence $\delta_1, \dots, \delta_n$ the sign changes at the indices $i_1, \dots, i_k$ and preserves at $j_1, \dots, j_{n-1-k}$. Thus, $\delta_{i_\nu} \delta_{i_\nu+1} = -1$ for $\nu = 1, \dots, k$, and $\delta_{j_\nu} \delta_{j_\nu+1} = 1$ for $\nu = 1, \dots, n-1-k$. We define a vector $x \in \nabla^k$ by

$$x_\nu = -1 + 2(\gamma_1 + \dots + \gamma_{i_\nu}),$$

where $\nu = 1, \dots, k$, and introduce a vector $y \in \nabla^{n-1-k}$ by

$$y_\nu = -1 + 2(\gamma_1 + \dots + \gamma_{j_\nu}),$$

where $\nu = 1, \dots, n-1-k$. We put

$$T_\delta(\lambda) = x \times y.$$

Obviously, the mapping

$$T_\delta : \Delta^{n-1}(\delta) \longrightarrow \nabla^k \times \nabla^{n-1-k}$$

transforms $\Delta^{n-1}(\delta)$ simplicially into one of the $\binom{n-1}{k}$ simplexes of dimension $n-1$ in the standard triangulation of $\nabla^k \times \nabla^{n-1-k}$.

Let $x \in \mathbb{R}^n$. By $x_\leq$ we denote the vector in $\mathbb{R}^n$ obtained by the rearrangement of the coordinates of x in increasing order. For example,

$$(4, 3, 2, 4, 2, 1)_\leq = (1, 2, 2, 3, 4, 4).$$

The mapping $x \mapsto x_\leq$ will turn out to be very useful.

We recall the operation of attaching (gluing). Let $A \subset X$ and $B \subset Y$ be subsets of topological spaces X and Y , and let

$$\varphi : A \longrightarrow B$$

be a continuous surjection. Denote by $Y \cup_\varphi X$ the factor-space of the direct sum $Y \sqcup X$ modulo the decomposition the classes of which are the singletons $x \in X \setminus A$, the singletons $y \in Y \setminus B$, and the sets $y \cup \varphi^{-1}(y)$ with $y \in B$. We say that the topological space $Y \cup_\varphi X$ is obtained by attaching the space X to the space Y via the mapping φ . If X and Y are cell spaces, A and B are cell subspaces, and φ is a cell mapping, then the space $Y \cup_\varphi X$ has a natural cell structure.

We introduce the simplicial mapping

$$\varphi : \partial Q^n \longrightarrow \partial \nabla^n$$

that transforms the vertices of Q^n into those of ∇^n for even n by the rule

$$\begin{aligned} +E_1^n &\longmapsto A_0^n, \\ -E_2^n, -E_1^n &\longmapsto A_1^n, \\ +E_3^n, +E_2^n &\longmapsto A_2^n, \\ &\vdots \\ -E_n^n, -E_{n-1}^n &\longmapsto A_{n-1}^n, \\ +E_n^n &\longmapsto A_n^n, \end{aligned}$$

and for odd n by the rule

$$\begin{aligned} -E_1^n &\longmapsto A_0^n, \\ +E_2^n, +E_1^n &\longmapsto A_1^n, \\ -E_3^n, -E_2^n &\longmapsto A_2^n, \\ &\vdots \\ -E_n^n, -E_{n-1}^n &\longmapsto A_{n-1}^n, \\ +E_n^n &\longmapsto A_n^n. \end{aligned}$$

Theorem 4. *The sphere $S^n(t)$ has a cell decomposition $S_1^n(t)$ that is topologically equivalent to the cell decomposition $\nabla^n \cup_\varphi Q^n$.*

Proof. We define a mapping

$$\Phi_\nabla : \nabla^n \longrightarrow S^n(t)$$

by

$$\Phi_\nabla : \lambda \longmapsto \omega_n(t, \lambda),$$

where $\lambda \in \nabla^n$. Then Φ_∇ is an embedding.

Let $\delta = (\delta_1, \dots, \delta_n)$ be a vector with coordinates ± 1 , and let an auxiliary mapping

$$F_\delta : Q^n(\delta) \longrightarrow S^n(t)$$

be defined as follows. Put $F_\delta(0) = \infty(t)$. For $\lambda \in Q^n(\delta) \setminus 0$ we define

$$F_\delta(\lambda) = \omega_k(t, x) \mu_{n-k}(t, y, r),$$

where $k = \text{alt}(\delta)$, $x \times y = T_\delta(|\lambda|^{-1}\lambda)$, and $r = |\lambda|^{-1}\delta_n$.

Now, let

$$\Phi_Q : Q^n \longrightarrow S^n(t)$$

denote the mapping given by

$$\Phi_Q(\lambda) = F_\delta(\lambda),$$

where $\lambda \in Q^n(\delta)$. We check that $\Phi_Q(\lambda)$ does not depend on the choice of δ for which $\lambda \in Q^n(\delta)$.

Let $\lambda \in Q^n(\delta) \cap Q^n(\delta')$. We must prove that $F_\delta(\lambda) = F_{\delta'}(\lambda)$. Let $k = \text{alt}(\delta)$ and $k' = \text{alt}(\delta')$. For definiteness, we assume that $k' \geq k$. If $\lambda = 0$, then the statement is obvious. Let $\lambda \neq 0$.

To prove that $F_\delta(\lambda) = F_{\delta'}(\lambda)$, it suffices to consider the case where δ and δ' differ in one coordinate only. Let this coordinate be the p th, $1 \leq p \leq n$. From $\delta'_p = -\delta_p$ it follows that in the vector $\lambda = (\lambda_1, \dots, \lambda_n)$ the coordinate λ_p is equal to 0. Consequently, $n \geq 2$. We put $x \times y = T_\delta(|\lambda|^{-1}\lambda)$, $x' \times y' = T_{\delta'}(|\lambda|^{-1}\lambda)$, and $r = |\lambda|^{-1}\delta_n$. The cases where $p = 1$, $2 \leq p \leq n-1$, and $p = n$ will be analyzed separately.

Let $p = 1$. Then $k' = k + 1$, whence

$$x' = (-1, x), \quad y = (-1, y').$$

Then

$$\begin{aligned} F_\delta(\lambda) &= \omega_k(t, x) \mu_{n-k}(t, y, r) \\ &= (t+1) \omega_k(t, x) \mu_{n-1-k}(t, y', r) \\ &= \omega_{k+1}(t, x') \mu_{n-1-k}(t, y', r) \\ &= F_{\delta'}(\lambda). \end{aligned}$$

Here we carry the factor $t+1$ from the polynomial $\mu_{n-k}(t, y, r)$ to the polynomial $\omega_{k+1}(t, x')$.

Let $2 \leq p \leq n-1$. Then $k' = k$ or $k' = k+2$. If $k' = k$, then

$$x' = x, \quad y' = y,$$

so that $F_\delta(\lambda) = F_{\delta'}(\lambda)$. Let $k' = k+2$. Then

$$\begin{aligned} x' &= (x_1, \dots, x_{i_{p-2}}, y_{j_{p-1}}, y_{j_p}, x_{i_{p-2}+1}, \dots, x_k), \\ y &= (y'_1, \dots, y'_{j_{p-2}}, y_{j_{p-1}}, y_{j_p}, y'_{j_{p-2}+1}, \dots, y'_{n-3-k}), \end{aligned}$$

whence

$$\begin{aligned} F_\delta(\lambda) &= \omega_k(t, x) \mu_{n-k}(t, y, r) \\ &= (t - y_{j_p})^2 \omega_k(t, x) \mu_{n-2-k}(t, y', r) \\ &= \omega_{k+2}(t, x') \mu_{n-2-k}(t, y', r) \\ &= F_{\delta'}(\lambda). \end{aligned}$$

Here we carry the factor $(t - y_{j_p})^2$ from the polynomial $\mu_{n-k}(t, y, r)$ to the polynomial $\omega_{k+2}(t, x')$.

Finally, if $p = n$, then $k' = k + 1$, whence

$$x' = (x, 1), \quad y = (y', 1).$$

Consequently,

$$\begin{aligned} F_\delta(\lambda) &= \omega_k(t, x) \mu_{n-k}(t, y, r) \\ &= (t - 1) \omega_k(t, x) \mu_{n-1-k}(t, y', -r) \\ &= \omega_{k+1}(t, x) \mu_{n-1-k}(t, y', -r) \\ &= F_{\delta'}(\lambda). \end{aligned}$$

Here we carry the factor $t - 1$ from the polynomial $\mu_{n-k}(t, y, r)$ to the polynomial $\omega_{k+1}(t, x')$.

We claim that the mapping Φ_Q is continuous. It suffices to prove that the mapping F_δ is continuous for any δ . For

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

we denote

$$\|f(t)\| = (a_1^2 + \cdots + a_n^2)^{1/2}.$$

Obviously,

$$\mu_{n-k}(t, y, r) = \mu_{n-k}(t, y, 0) + r \frac{\partial}{\partial r} \mu_{n-k}(t, y, r).$$

The polynomial

$$\frac{\partial}{\partial r} \mu_{n-k}(t, y, r)$$

is independent of r , its degree in t is at most $n - 1 - k$, and it is not identically zero for any $y \in \nabla^{n-1-k}$. By compactness, there exists $C > 0$ such that

$$\|\omega_k(t, x) \mu_{n-k}(t, y, r)\| \geq C|r|$$

for all $x \times y \in \nabla^k \times \nabla^{n-1-k}$ and all sufficiently large $|r|$. Therefore,

$$\|F_\delta(\lambda)\| \geq C|\lambda|^{-1}$$

for all sufficiently small $\lambda \in Q^n(\delta)$. This means that the mapping F_δ is continuous at the point $\lambda = 0$. Obviously, the mapping F_δ is continuous at all other points.

Observe that

$$\Phi_\nabla(\nabla^n) \cup \Phi_Q(Q^n) = S^n(t)$$

and

$$\Phi_\nabla(\text{Int } \nabla^n) \cap \Phi_Q(\text{Int } Q^n) = \emptyset.$$

Moreover,

$$\Phi_\nabla(\partial \nabla^n) = \Phi_Q(\partial Q^n).$$

We check that the mapping Φ_Q is injective on $\text{Int } Q^n$. To this end, for the mapping

$$\Phi_Q : \text{Int } Q^n \longrightarrow S^n(t) \setminus \Phi_\nabla(\nabla^n)$$

we construct the inverse mapping

$$\Phi_Q^{-1} : S^n(t) \setminus \Phi_\nabla(\nabla^n) \longrightarrow \text{Int } Q^n.$$

Let $f \in \mathbb{R}^n(t) \setminus \Phi_\nabla(\nabla^n)$. Since the polynomial $f(t)$ has $k \leq n-1$ roots on the interval $[-1, 1]$, we have

$$f(t) = \omega_k(t, x) \mu_{n-k}(t, y, r),$$

where $x \times y \in \nabla^k \times \text{Int } \nabla^{n-1-k}$ and $|r| > 1$. We introduce the vector $z = (x, y)_\leq$. Let $z_0 = -1$ and $z_n = 1$. We put

$$\lambda_{\nu+1} = \frac{z_{\nu+1} - z_\nu}{2},$$

where $\nu = 0, \dots, n-1$. Obviously, $\lambda \in \Delta^{n-1}$. Let $Y = \{y_1, \dots, y_{n-1-k}\}$, let

$$\varepsilon_\nu = \begin{cases} 1 & \text{if } \lambda_{\nu+1} > 0 \text{ and } z_\nu \in Y, \\ -1 & \text{if } \lambda_{\nu+1} = 0 \text{ or } z_\nu \notin Y, \end{cases}$$

where $\nu = 1, \dots, n-1$, and let $\varepsilon_n = \text{sgn } r$. We define $\delta_1, \dots, \delta_n$ by the rule

$$\delta_\nu = \varepsilon_\nu \varepsilon_{\nu+1} \cdots \varepsilon_n,$$

where $\nu = 1, \dots, n$. Then

$$\Phi_Q^{-1}(f) = |r|^{-1} \sum_{\nu=1}^n \lambda_\nu \delta_\nu E_\nu^n.$$

The identity

$$\Phi_\nabla(\partial \nabla^n) = \Phi_Q(\partial Q^n)$$

allows us to define a mapping

$$\tilde{\varphi} : \partial Q^n \longrightarrow \partial \nabla^n$$

by the formula

$$\tilde{\varphi}(\lambda) = \Phi_\nabla^{-1} \circ \Phi_Q(\lambda).$$

We claim that $\tilde{\varphi}$ is a simplicial mapping.

Let $\delta = (\delta_1, \dots, \delta_n)$ be a vector with coordinates ± 1 . We claim that the restriction

$$\tilde{\varphi} : \Delta^{n-1}(\delta) \longrightarrow \partial \nabla^n$$

is a simplicial mapping. Let $k = \text{alt } \delta$. We define a mapping

$$P_\delta : \nabla^k \times \nabla^{n-1-k} \longrightarrow \partial \nabla^n$$

as follows. Let $x \times y \in \nabla^k \times \nabla^{n-1-k}$. If $n-k = 2m$, we put

$$P_\delta(x \times y) = \begin{cases} (x_1, \dots, x_k, y_1, y_1, y_3, y_3, \dots, y_{2m-1}, y_{2m-1})_\leq & \text{if } \delta_n = 1, \\ (x_1, \dots, x_k, -1, y_2, y_2, y_4, y_4, \dots, y_{2m-2}, y_{2m-2}, 1)_\leq & \text{if } \delta_n = -1. \end{cases}$$

If $n-k = 2m+1$, we put

$$P_\delta(x \times y) = \begin{cases} (x_1, \dots, x_k, -1, y_2, y_2, y_4, y_4, \dots, y_{2m}, y_{2m})_\leq & \text{if } \delta_n = 1, \\ (x_1, \dots, x_k, y_1, y_1, y_3, y_3, \dots, y_{2m-1}, y_{2m-1}, 1)_\leq & \text{if } \delta_n = -1. \end{cases}$$

It is easy to check that

$$\tilde{\varphi}(\lambda) = P_\delta \circ T_\delta(\lambda)$$

for all $\lambda \in \Delta^{n-1}(\delta)$. Hence, $\tilde{\varphi}$ is a simplicial mapping.

Direct inspection shows that the simplicial mappings φ and $\tilde{\varphi}$ take equal values at all vertices of the polyhedron ∂Q^n . Consequently,

$$\varphi(\lambda) = \Phi_\nabla^{-1} \circ \Phi_Q(\lambda)$$

for all $\lambda \in \partial Q^n$.

$$\Phi : \nabla^n \sqcup Q^n \longrightarrow S^n(t)$$
$$\Phi(\lambda) = \begin{cases} \Phi_{\nabla}(\lambda) & \text{if } \lambda \in \nabla^n, \\ \Phi_Q(\lambda) & \text{if } \lambda \in Q^n. \end{cases}$$
$$S^n(t) = (\nabla^n \sqcup Q^n)/\Phi = \nabla^n \cup_{\circ} Q^n.$$

The space $\nabla^n \cup_{\varphi} Q^n$ inherits a cell structure from the simplex ∇^n and the octahedron Q^n . The cells of $\nabla^n \cup_{\varphi} Q^n$ are all the cells of the simplex ∇^n and all the cells lying in the interior $\text{Int } Q^n$ of the octahedron Q^n . Thus, $\nabla^n \cup_{\varphi} Q^n$ has $1 + 2^n$ cells of dimension n . This yields a small cell decomposition of $S_1^n(t)$ with $1 + 2^n$ cells of dimension n . The large cell decomposition $S_1^n(t)$ with $1 + 2n$ cells of dimension n is obtained by uniting small cells. The first cell is

$$(0; \underbrace{1, \dots, 1}_n; 0) \leftrightarrow \text{Int } \nabla^n.$$

$$(0; \underbrace{1, \dots, 1}_k; 0)_\varepsilon \leftrightarrow \text{Int} \left\{ \bigcup Q^n(\delta) \right\},$$

For example, the cell decomposition $S_1^2(t)$ is shown in the figure.

In a similar way, we can construct a model of the cell decomposition $S_1^n(t)$ for any n . Consider a tetrahedron in $\mathbb{R}^n$ and draw n oriented lines through its edges.

We assume that these lines are parallel to the coordinate axes. The orientations are chosen to alternate, and the last line is oriented positively.

We describe a small cell decomposition with $1 + 2^n$ cells of dimension n . The first cell is the tetrahedron. The other 2^n cells are constructed in the following way. On each line we take a ray with origin at a vertex of the tetrahedron. The choice of rays depends on the choice of signs $\delta = (\delta_1, \dots, \delta_n)$ that mark the rays. Then, on each ray, we take a point at a distance $r > 0$ from the origin. These points generate an $(n-1)$ -dimensional simplex. If r grows from 0 to ∞ , then this simplex sweeps out an open set $e(\delta)$ in the space $\mathbb{R}^n$. This set is one of 2^n cells of dimension n .

Now we describe the large cell decomposition with $1 + 2n$ cells of dimension n . The cell

$$(0; \underbrace{1, \dots, 1}_n; 0) \leftrightarrow \nabla^n$$

is the tetrahedron. The cell

$$(0; \underbrace{1, \dots, 1}_k; 0)_- \leftrightarrow \nabla^k \times \nabla^{n-1-k} \times (-\infty, -1],$$

where $0 \leq k \leq n-1$, is the union of all $\binom{n-1}{k}$ cells $e(\delta)$ such that $\text{alt } \delta = k$ and $\delta_n = -1$.

For $n = 2$, the transformation of the cell decomposition $\mathbb{R}^2(t)$ into the cell decomposition $S_1^2(t)$ is shown in the figure.

The inversion of the exterior of the square onto its interior gives the following:

Now we attach the square Q^2 to the triangle ∇^2 by the rule

$$\begin{aligned} (1, 0) &\mapsto (1, 1), \\ -(1, 0) &\mapsto (-1, 1), \\ -(0, 1) &\mapsto (-1, 1), \\ (0, 1) &\mapsto (-1, 1). \end{aligned}$$

§4. POLYNOMIALS OVER THE HALF-LINE

We introduce the cone of increasing positive vectors

$$\mathbb{R}_{\leq+}^n = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq \dots \leq x_n\}.$$

Let $\delta = (\delta_1, \dots, \delta_n)$ be a vector with coordinates ± 1 , and let $\mathbb{R}^n(\delta)$ be the cone with the generators $\delta_1 E_1^n, \dots, \delta_n E_n^n$. Then the positive half-space

$$\mathbb{R}^{n-1} \times \mathbb{R}_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$$

can be represented as

$$\mathbb{R}^{n-1} \times \mathbb{R}_+ = \bigcup_{\delta: \delta_n=1} \mathbb{R}^n(\delta).$$

By the faces of the half-space $\mathbb{R}^{n-1} \times \mathbb{R}_+$ we mean the faces of all cones $\mathbb{R}^n(\delta)$ occurring in the above union. In particular, $\mathbb{R}^{n-1} \times \mathbb{R}_+$ has 2^{n-1} cells of dimension n .

For $n = 2m$ we consider the polynomial

$$\rho_n(t, y, r) = \prod_{k=1}^m (t - y_{2k-1})^2 + rt \prod_{k=1}^{m-1} (t - y_{2k})^2,$$

and for $n = 2m + 1$ the polynomial

$$\rho_n(t, y, r) = t \prod_{k=1}^m (t - y_{2k})^2 + r \prod_{k=1}^m (t - y_{2k-1})^2,$$

where $y \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$. The first four polynomials look like this:

$$\begin{aligned} \rho_1(t, y, r) &= t & + r, \\ \rho_2(t, y, r) &= (t - y_1)^2 & + rt, \\ \rho_3(t, y, r) &= t(t - y_2)^2 & + r(t - y_1)^2, \\ \rho_4(t, y, r) &= (t - y_1)^2(t - y_3)^2 & + rt(t - y_2)^2. \end{aligned}$$

Let $f \in \mathbb{R}^n(t)$. If the polynomial $f(t)$ has n roots on $\mathbb{R}_+$, then

$$f(t) = \omega_n(t, x),$$

where $x \in \mathbb{R}_{\leq+}^n$. If the polynomial $f(t)$ has $k \leq n - 1$ roots on $\mathbb{R}_+$, then

$$f(t) = \omega_k(t, x) \rho_{n-k}(t, y, r),$$

where $x \times y \in \mathbb{R}_{\leq+}^k \times \text{Int } \mathbb{R}_{\leq+}^{n-1-k}$ and $r > 0$. Consequently, the space $\mathbb{R}^n(t)$ is obtained by gluing $1 + n$ polyhedra

$$\begin{aligned} & \mathbb{R}_{\leq+}^n, \\ & \mathbb{R}_{\leq+}^0 \times \mathbb{R}_{\leq+}^{n-1} \times \mathbb{R}_+, \\ & \mathbb{R}_{\leq+}^1 \times \mathbb{R}_{\leq+}^{n-2} \times \mathbb{R}_+, \\ & \vdots \\ & \mathbb{R}_{\leq+}^{n-1} \times \mathbb{R}_{\leq+}^0 \times \mathbb{R}_+ \end{aligned}$$

along their boundaries.

Consider the simplex of positive increasing vectors

$$\nabla_0^{n-1} = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = 1\}$$

with the vertices

$$\begin{aligned} B_0^n &= (1, 1, 1, \dots, 1, 1), \\ B_1^n &= (0, 1, 1, \dots, 1, 1), \\ &\vdots \\ B_{n-1}^n &= (0, 0, 0, \dots, 0, 1). \end{aligned}$$

Let ϕ be the mapping that transforms the vertices of Q^{n-1} into those of ∇_0^{n-1} for even n by the rule

$$\begin{aligned} +E_1^{n-1} &\mapsto B_0^n, \\ -E_2^{n-1}, -E_1^{n-1} &\mapsto B_1^n, \\ +E_3^{n-1}, +E_2^{n-1} &\mapsto B_2^n, \\ &\vdots \\ +E_{n-1}^{n-1}, +E_{n-2}^{n-1} &\mapsto B_{n-2}^n, \\ -E_{n-1}^{n-1} &\mapsto B_{n-1}^n, \end{aligned}$$

and for odd n by the rule

$$\begin{aligned} -E_1^{n-1} &\mapsto B_0^n, \\ +E_2^{n-1}, +E_1^{n-1} &\mapsto B_1^n, \\ -E_3^{n-1}, -E_2^{n-1} &\mapsto B_2^n, \\ &\vdots \\ +E_{n-1}^{n-1}, +E_{n-2}^{n-1} &\mapsto B_{n-2}^n, \\ -E_{n-1}^{n-1} &\mapsto B_{n-1}^n. \end{aligned}$$

We extend ϕ to a piecewise linear mapping

$$\phi : \mathbb{R}^{n-1} \longrightarrow \partial \mathbb{R}_{\leq+}^n$$

by the formula

$$\phi \left(\sum_{i=1}^{n-1} \lambda_i \delta_i E_i^{n-1} \right) = \sum_{i=1}^{n-1} \lambda_i \phi \left(\delta_i E_i^{n-1} \right),$$

where $\lambda \in \mathbb{R}_+^{n-1}$. Observe that $\mathbb{R}^{n-1} = \partial(\mathbb{R}^{n-1} \times \mathbb{R}_+)$.

Theorem 5. *There is a homeomorphism*

$$\mathbb{R}^n(t) = \mathbb{R}_{\leq+}^n \cup_{\phi} (\mathbb{R}^{n-1} \times \mathbb{R}_+)$$

such that $S_2^n(t) = \text{su } S_1^{n-1}(t)$.

Proof. Since the proof is similar to that of Theorem 4, we only explain the identity $S_2^n = \text{su } S_1^{n-1}$. We consider the polyhedron

$$Q_0^{n-1} = \bigcup_{\delta: \delta_n=1} \Delta^{n-1}(\delta)$$

with obvious triangulation. We see that

$$\phi : r\partial Q_0^{n-1} \longrightarrow r\partial \nabla_0^{n-1}$$

is the simplicial mapping that, for any $r > 0$, attaches rQ_0^{n-1} to $r\nabla_0^{n-1}$ in the same way as the simplicial mapping φ attaches Q^{n-1} to ∇^{n-1} . Factorization at $r = 0$ and $r = \infty$ leads to the identity $S_2^n = \text{su } S_1^{n-1}$.

The case of $n = 3$ is shown below.

In this case the attaching mapping acts as follows:

$$\begin{aligned} -(1, 0, 0) &\mapsto (1, 1, 1), \\ (1, 0, 0) &\mapsto (0, 1, 1), \\ (0, 1, 0) &\mapsto (0, 1, 1), \\ -(0, 1, 0) &\mapsto (0, 0, 1). \end{aligned}$$

The theorem is proved. $\square$

The relation $S_2^n(t) = \text{su } S_1^{n-1}(t)$ transforms the large cell decomposition $S_1^{n-1}(t)$ into a small cell decomposition $S_2^n(t)$. In particular, this small cell decomposition has $1+2(n-1)$ cells of dimension n . Another obvious small cell decomposition $S_2^n(t)$ has $1+2^{n-1}$ cells of dimension n . The large cell decomposition $S_2^n(t)$ is obtained by uniting cells of the small cell decomposition $S_2^n(t)$. We list all $1+n$ cells of dimension n of the large cell decomposition $S_2^n(t)$. The first large cell is $\text{Int } \mathbb{R}_{\leq+}^n$. The other n large cells are of the form $\text{Int}\{\bigcup \mathbb{R}^n(\delta)\}$, where $0 \leq k \leq n-1$. Here the union is taken over all δ such that $\text{alt } \delta = k$ and $\delta_n = 1$.

§5. POLYNOMIALS OVER THE LINE

Consider the wedge of increasing vectors

$$\mathbb{R}_{\leq}^n = \{x \in \mathbb{R}^n : x_1 \leq \dots \leq x_n\}.$$

We define the polynomial

$$\sigma_{2m}(t, y, r) = \prod_{k=1}^m (t - y_{2k-1})^2 + r \prod_{k=1}^{m-1} (t - y_{2k})^2,$$

where $y \in \mathbb{R}^{2m-1}$ and $r \in \mathbb{R}$. The first three polynomials look like this:

$$\begin{aligned} \sigma_2(t, y, r) &= (t - y_1)^2 + r, \\ \sigma_4(t, y, r) &= (t - y_1)^2(t - y_3)^2 + r(t - y_2)^2, \\ \sigma_6(t, y, r) &= (t - y_1)^2(t - y_3)^2(t - y_5)^2 + r(t - y_2)^2(t - y_4)^2. \end{aligned}$$

Let $f \in \mathbb{R}^n(t)$. If the polynomial $f(t)$ has n roots on $\mathbb{R}$, then

$$f(t) = \omega_n(t, x),$$

where $x \in \mathbb{R}_{\leq}^n$. If the polynomial $f(t)$ has $k \leq n-1$ roots on $\mathbb{R}$, then k and n have equal parity, and

$$f(t) = \omega_k(t, x) \sigma_{n-k}(t, y, r),$$

where $x \times y \in \mathbb{R}_{\leq}^k \times \text{Int } \mathbb{R}_{\leq}^{n-1-k}$ and $r > 0$. Therefore, the space $\mathbb{R}^n(t)$ is obtained by gluing $1 + [n/2]$ polyhedra along their boundaries. For $n = 2m$, we glue the polyhedra

$$\begin{aligned} &\mathbb{R}_{\leq}^{2m}, \\ &\mathbb{R}_{\leq}^0 \times \mathbb{R}_{\leq}^{2m-1} \times \mathbb{R}_+, \\ &\mathbb{R}_{\leq}^2 \times \mathbb{R}_{\leq}^{2m-3} \times \mathbb{R}_+, \\ &\vdots \\ &\mathbb{R}_{\leq}^{2m-2} \times \mathbb{R}_{\leq}^1 \times \mathbb{R}_+. \end{aligned}$$

For $n = 2m + 1$, we glue the polyhedra

$$\begin{aligned} &\mathbb{R}_{\leq}^{2m+1}, \\ &\mathbb{R}_{\leq}^1 \times \mathbb{R}_{\leq}^{2m-1} \times \mathbb{R}_+, \\ &\mathbb{R}_{\leq}^3 \times \mathbb{R}_{\leq}^{2m-3} \times \mathbb{R}_+, \\ &\vdots \\ &\mathbb{R}_{\leq}^{2m-1} \times \mathbb{R}_{\leq}^1 \times \mathbb{R}_+. \end{aligned}$$

For polynomials on the line, the cell structure of the space $\mathbb{R}^n(t)$ is shift invariant. To employ this property, we consider the polynomial

$$\psi_{2m-1}(t, y, r) = \sigma_{2m}[t, (0, y), r],$$

where $y \in \mathbb{R}^{2m-2}$, and the polynomial

$$\psi_{2m}(t, y, r) = t \sigma_{2m}(t, y, r),$$

where $y \in \mathbb{R}^{2m-1}$. The first four polynomials are as follows:

$$\begin{aligned}\psi_1(t, y, r) &= t^2 + r, \\ \psi_2(t, y, r) &= t(t - y_1)^2 + rt, \\ \psi_3(t, y, r) &= t^2(t - y_2)^2 + r(t - y_1)^2, \\ \psi_4(t, y, r) &= t(t - y_1)^2(t - y_3)^2 + rt(t - y_2)^2.\end{aligned}$$

Let $f \in \mathbb{R}^n(t)$. If the polynomial $f(t)$ has n roots on $\mathbb{R}$, then

$$f(t) = (t - \tau)\omega_{n-1}(t - \tau, x),$$

where $\tau \in \mathbb{R}$ and $x \in \mathbb{R}_{\leq+}^{n-1}$. If the polynomial $f(t)$ has at most $n - 2$ roots on $\mathbb{R}$, then

$$f(t) = \omega_k(t - \tau, x)\psi_{n-1-k}(t - \tau, y, r),$$

where $\tau \in \mathbb{R}$, $x \times y \in \mathbb{R}_{\leq+}^k \times \text{Int } \mathbb{R}_{\leq+}^{n-2-k}$, and $r > 0$. Here $0 \leq k \leq n - 2$. If $n - 1 - k$ is odd, then $f(t)$ has k roots on $\mathbb{R}$. If $n - 1 - k$ is even, then $f(t)$ has $k + 1$ roots on $\mathbb{R}$.

The shift τ is uniquely determined by the polynomial $f(t)$. Let $\mathbb{R}_\tau^{n-1}(t)$ denote the set of polynomials $f \in \mathbb{R}^n(t)$ with the shift τ . All $\mathbb{R}_\tau^{n-1}(t)$ are identical. Moreover, $\mathbb{R}_\tau^{n-1}(t)$ is obtained by gluing n polyhedra

$$\begin{aligned}& \mathbb{R}_{\leq+}^{n-1}, \\ & \mathbb{R}_{\leq+}^0 \times \mathbb{R}_{\leq+}^{n-2} \times \mathbb{R}_+, \\ & \mathbb{R}_{\leq+}^1 \times \mathbb{R}_{\leq+}^{n-3} \times \mathbb{R}_+, \\ & \vdots \\ & \mathbb{R}_{\leq+}^{n-2} \times \mathbb{R}_{\leq+}^0 \times \mathbb{R}_+\end{aligned}$$

along their boundaries. The identification rule is the same as in §4. In particular, the mapping occurring in the next theorem is the same as in §4.

Theorem 6. *There is a homeomorphism*

$$\mathbb{R}^n(t) = \{\mathbb{R}_{\leq+}^{n-1} \cup_\phi (\mathbb{R}^{n-2} \times \mathbb{R}_+)\} \times \mathbb{R}$$

such that $S_3^n(t) = /su S_2^{n-1}(t)$.

Proof. Since the space of polynomials is shift invariant, we have a homeomorphism

$$\mathbb{R}^n(t) = \{\mathbb{R}_{\leq+}^{n-1} \cup_\phi (\mathbb{R}^{n-2} \times \mathbb{R}_+)\} \times \mathbb{R}.$$

For any τ and any $n \geq 2$ the sphere $\mathbb{R}_\tau^{n-1}(t) \cup \infty(t)$ has a cell decomposition $S_2^{n-1}(t)$ topologically equivalent to $su S_1^{n-2}(t)$. Using invariance relative to the shift again, we obtain the desired identity $S_3^n(t) = /su S_2^{n-1}(t)$. The factorization $/$ is necessary, because Euclidean space has only one point at infinity. The theorem is proved. $\square$

The relation $S_3^n(t) = /su S_2^{n-1}(t)$ transforms the large cell decomposition $S_2^{n-1}(t)$ into a small cell decomposition $S_3^n(t)$. In particular, this small cell decomposition has n cells of dimension n . Other two obvious small cell decompositions have $1 + 2(n - 2)$ and $1 + 2^{n-2}$ cells of dimension n , respectively. The large cell decomposition $S_3^n(t)$ is obtained by uniting cells of the small cell decomposition $S_3^n(t)$. We describe all $1 + [n/2]$ cells of dimension n . The first large cell is $\text{Int } \mathbb{R}_{\leq+}^{n-1} \times \mathbb{R}$. The other $[n/2]$ large cells are of the form $\text{Int}\{\bigcup \mathbb{R}^{n-1}(\delta)\} \times \mathbb{R}$, where $k = n - 2, n - 4, \dots, n - 2[n/2]$. Here the union is taken over all δ such that $k - 1 \leq \text{alt } \delta \leq k$ and $\delta_{n-1} = 1$.

§6. CELL HOMEOMORPHISMS

Let $x \in \nabla^n$. For $i = 0, 1, \dots, n$, we put

$$\xi_i = \int_{x_i}^{x_{i+1}} \omega_n(t, x) dt,$$

where $x_0 = -1$ and $x_{n+1} = 1$. The definition of the mapping

$$x \longmapsto \xi$$

is illustrated in the figure.

Consider the simplex of alternating vectors

$$\Delta_{\pm}^n = \text{conv}\{(-1)^{n+1}E_1^{n+1}, (-1)^n E_2^{n+1}, \dots, E_{n+1}^{n+1}\}.$$

We put

$$\mathcal{L}_{\nabla}(x) = \frac{\xi}{|\xi|}.$$

Then $\mathcal{L}_{\nabla}(x) \in \Delta_{\pm}^n$. It is easily seen that

$$\mathcal{L}_{\nabla} : \nabla^n \longrightarrow \Delta_{\pm}^n$$

is a cell mapping.

Let $\lambda \in Q^n \setminus 0$, and let $\delta = (\delta_1, \dots, \delta_n)$ be such that $\lambda \in Q^n(\delta) \setminus 0$. We denote $k = \text{alt}(\delta)$ and $r = |\lambda|^{-1}\delta_n$.

Using the mapping

$$T_{\delta} : \Delta^{n-1}(\delta) \longrightarrow \nabla^k \times \nabla^{n-1-k}$$

defined in §3, we put $x \times y = T_{\delta}(|\lambda|^{-1}\lambda)$.

Let $z = (x, y)_{\leq}$. We recall that the operator $(\cdot)_{\leq}$ rearranges the coordinates of a vector in increasing order. We define

$$\eta_i = \int_{z_{i-1}}^{z_i} \omega_k(t, x) \mu_{n-k}(t, y, r) dt,$$

where $i = 1, \dots, n$. Here $z_0 = -1$ and $z_n = 1$. The definition of the mapping

$$\lambda \longmapsto \eta$$

is illustrated in the figure.

We put

$$\mathcal{L}_Q(\lambda) = \frac{1}{|r|} \frac{\eta}{|\eta|}.$$

Obviously, $\mathcal{L}_Q(\lambda) \in Q^n(\delta) \setminus 0$. Letting $\mathcal{L}_Q(0) = 0$, we easily see that

$$\mathcal{L}_Q : Q^n \longrightarrow Q^n$$

is a cell mapping.

Theorem 7. *The mappings*

$$\begin{aligned} \mathcal{L}_\nabla &: \nabla^n \longrightarrow \Delta_\pm^n, \\ \mathcal{L}_Q &: Q^n \longrightarrow Q^n \end{aligned}$$

are cell homeomorphisms.

Proof. Since $\mathcal{L}_\nabla$ and $\mathcal{L}_Q$ are cell mappings, it suffices to check that they are bijections. We prove this by induction on the dimension of cells. For zero-dimensional cells the statement is obvious. Assume that these mappings are bijections on the $(k-1)$ -dimensional skeleton. Each k -dimensional cell is an open set in $\mathbb{R}^k$. The mappings in question are differentiable in all k -dimensional cells. Later it will be shown that these mappings are nonsingular at any point of any k -dimensional cell. Obviously, each closed k -dimensional cell is homeomorphic to the k -dimensional ball D^k . Now, by induction, the fact that our mappings are bijections on the k -dimensional skeleton is a consequence of the following statement. Let

$$F : D^k \longrightarrow D^k$$

be a continuous mapping such that $F(S^{k-1}) \subset S^{k-1}$ and $F(\text{Int } D^k) \subset \text{Int } D^k$. If the restriction $F|_{S^{k-1}}$ is a homeomorphism and the restriction $F|_{\text{Int } D^k}$ is a local homeomorphism, then F is a homeomorphism. Indeed, under these conditions the restriction

$$F|_{\text{Int } D^k} : \text{Int } D^k \longrightarrow \text{Int } D^k$$

is a universal covering, hence, it is a homeomorphism. $\square$

Remark. The proof that the mappings $\mathcal{L}_\nabla$ and $\mathcal{L}_Q$ are nonsingular at all points of all k -dimensional cells constitutes the computational part of the theorem. The proof is rather bulky and is placed in the Appendix.

Let Ψ_∇ be the composition mapping

$$\Psi_\nabla : \Delta_\pm^n \xrightarrow{\mathcal{L}_\nabla^{-1}} \nabla^n \xrightarrow{\Phi_\nabla} S^n(t),$$

and let Ψ_Q be the composition mapping

$$\Psi_Q : Q^n \xrightarrow{\mathcal{L}_Q^{-1}} Q^n \xrightarrow{\Phi_Q} S^n(t),$$

where Φ_∇ and Φ_Q are as in the proof of Theorem 4. Since

$$\Psi_\nabla(\partial\Delta_\pm^n) = \Psi_Q(\partial Q^n),$$

we can define a composition mapping

$$\psi : \partial Q^n \longrightarrow \partial\Delta_\pm^n$$

by the formula

$$\psi(\lambda) = \Psi_\nabla^{-1} \circ \Psi_Q(\lambda).$$

Let $\lambda = \psi(\mu)$, where $\mu \in \partial Q^n$ and $\lambda \in \partial\Delta_\pm^n$. In this case the points μ and λ will be called *equivalent*.

The mapping ψ is simplicial and takes the vertices of the octahedron Q^n to the vertices of the simplex $\Delta_\pm^n$ for even n by the rule

$$\begin{aligned} +E_1^n &\longmapsto +E_1^{n+1}, \\ -E_2^n, -E_1^n &\longmapsto -E_2^{n+1}, \\ +E_3^n, +E_2^n &\longmapsto +E_3^{n+1}, \\ &\vdots \\ -E_n^n, -E_{n-1}^n &\longmapsto -E_n^{n+1}, \\ +E_n^n &\longmapsto +E_{n+1}^{n+1}, \end{aligned}$$

and for odd n by the rule

$$\begin{aligned} -E_1^n &\longmapsto -E_1^{n+1}, \\ +E_2^n, +E_1^n &\longmapsto +E_2^{n+1}, \\ -E_3^n, -E_2^n &\longmapsto -E_3^{n+1}, \\ &\vdots \\ -E_n^n, -E_{n-1}^n &\longmapsto -E_n^{n+1}, \\ +E_n^n &\longmapsto +E_{n+1}^{n+1}. \end{aligned}$$

As a result, we arrive at another representation $S_1^n(t) = \Delta_\pm^n \cup_\psi Q^n$ of the cell decomposition $S_1^n(t)$.

The cell decomposition $S_1^2(t)$ is obtained by attaching the square to the triangle:

via the simplicial mapping

$$\begin{aligned} (1, 0) &\mapsto (1, 0, 0), \\ -(1, 0) &\mapsto -(0, 1, 0), \\ -(0, 1) &\mapsto -(0, 1, 0), \\ (0, 1) &\mapsto (0, 0, 1). \end{aligned}$$

The cell decomposition $S_1^3(t)$ is obtained by attaching the octahedron to the tetrahedron:

via the simplicial mapping

$$\begin{aligned} -(1, 0, 0) &\mapsto -(1, 0, 0, 0), \\ (1, 0, 0) &\mapsto (0, 1, 0, 0), \\ (0, 1, 0) &\mapsto (0, 1, 0, 0), \\ -(0, 1, 0) &\mapsto -(0, 0, 1, 0), \\ -(0, 0, 1) &\mapsto -(0, 0, 1, 0), \\ (0, 0, 1) &\mapsto (0, 0, 0, 1). \end{aligned}$$

§7. LINEARIZATION OF POLYNOMIALS

By linearization of a polynomial we mean a transformation of the polynomial to a piecewise linear function. Naturally, this piecewise linear function must depend on the polynomial continuously, and different polynomials must give rise to different piecewise linear functions. A simplest linearization arises when a polynomial of degree n is taken to the piecewise linear function with n links and with $n+1$ knots at the points $0, 1, \dots, n$.

The relationship between this linearization and the Lagrange interpolation is obvious.

For many problems, only the extremal values of a polynomial are of interest. Moreover, often only the relative position of these values is important. The extremal values of a polynomial depend on its coefficients in a fairly complicated way. Therefore, it is desirable

to replace a polynomial by a simpler function with the same maximum and minimum values or with the same relative position of these values. For this, a piecewise linear function with a unit step is most convenient. Difficulties arise when the polynomial in question has few extremal points. We describe such a linearization for polynomials of degree 3.

Example. Consider the space of polynomials

$$F(t) = \frac{1}{3}t^3 + a_1t^2 + a_2t + a_3$$

that satisfy the condition

$$F(-1) = 0.$$

Suppose that the derivative $f(t) = F'(t)$ has k roots on the interval $[-1, 1]$.

If $k = 2$, then

$$f(t) = (t - x_1)(t - x_2).$$

The linearization of $F(t)$ is as depicted.

For $k = 1$, the derivative has the form

$$f(t) = (t - x_1)(t + r),$$

and the linearization of $F(t)$ is as shown in the next figure.

If $k = 0$, for the derivative we have

$$f(t) = \frac{r+1}{2}(t - y_1)^2 + \frac{r-1}{2}(1 - t^2).$$

The linearization of $F(t)$ is as depicted.

Thus, for $k = 2$ the polynomial $F(t)$ is linearized by a three-link polygonal line, and for $0 \leq k \leq 1$ the polynomial $F(t)$ is linearized by a two-link polygonal line. We explain the position of these polygonal lines in $\mathbb{R}^3$ and $\mathbb{R}^2$. For $k = 2$, with $F(t)$ we associate the vector

$$\xi = \begin{bmatrix} L(1) - L(0) \\ L(2) - L(1) \\ L(3) - L(2) \end{bmatrix}.$$

In the space $\mathbb{R}^3$, the vectors $\xi = (\xi_0, \xi_1, \xi_2)$ sweep a curvilinear triangle.

If $0 \leq k \leq 1$, then with $F(t)$ we associate the vector

$$\eta = \begin{bmatrix} L(1) - L(0) \\ L(2) - L(1) \end{bmatrix}.$$

In the plane $\mathbb{R}^2$, the vectors $\eta = (\eta_1, \eta_2)$ sweep the open exterior of a curvilinear quadrangle.

If we add the boundary to that exterior, then some polynomials will correspond simultaneously to polygonal lines with three and two links. Naturally, we need to identify these polygonal lines. In geometric terms, this means that we attach the exterior of a curvilinear quadrangle to a curvilinear triangle. The curves $[B_0, B_1']$, $[B_1'', B_2]$, $[B_2, B_0]$ are glued to the identical curves $[A_0, A_1]$, $[A_1, A_2]$, $[A_2, A_0]$, and the interval $[B_1', B_1'']$ becomes the point A_1 .

Now we describe linearization for polynomials of an arbitrary degree n . We consider the tetrahedron $\Delta_{\pm}^n$ and the octahedron Q^n separately.

Let $\lambda \in \Delta_{\pm}^n$. We put

$$z = \mathcal{L}_{\nabla}^{-1}(\lambda).$$

With the point λ we associate the polynomial

$$F_{\lambda}(t) = \int_{-1}^t \omega_n(\tau, z) d\tau.$$

Simultaneously, with the point λ we associate the $(n+1)$ -link piecewise linear function $L_{\lambda}(t)$ such that

$$\begin{aligned} L_{\lambda}(0) &= 0, \\ L_{\lambda}(1) &= \lambda_0, \\ L_{\lambda}(2) &= \lambda_0 + \lambda_1, \\ &\vdots \\ L_{\lambda}(n+1) &= \lambda_0 + \lambda_1 + \cdots + \lambda_{n+1}. \end{aligned}$$

Obviously,

$$F_{\lambda}(z_i) = \rho L_{\lambda}(i)$$

for all $i = 0, 1, \dots, n+1$. Here $z_0 = -1$, $z_{n+1} = 1$, and

$$\rho = \int_{-1}^1 |f_{\lambda}(t)| dt.$$

The points $z_0, z_1, \dots, z_{n+1}$ will be called the *knots* of $F_{\lambda}(t)$ corresponding to the point $\lambda \in \Delta_{\pm}^n$.

Let $\lambda \in Q^n \setminus 0$. We put

$$\gamma = \mathcal{L}_Q^{-1}(\lambda).$$

With the point λ we associate the polynomial

$$F_\lambda(t) = \int_{-1}^t \omega_k(\tau, x) \mu_{n-k}(\tau, y, r) d\tau,$$

where $\gamma \in Q^n(\delta)$, $k = \text{alt}(\delta)$, $x \times y = T_\delta(|\gamma|^{-1}\gamma)$, and $r = |\gamma|^{-1}\delta_n$. Simultaneously, with the point λ we associate the n -link piecewise linear function $L_\lambda(t)$ such that

$$\begin{aligned} L_\lambda(0) &= 0, \\ L_\lambda(1) &= \lambda_1, \\ L_\lambda(2) &= \lambda_1 + \lambda_2, \\ &\vdots \\ L_\lambda(n) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n. \end{aligned}$$

Let $z = (x, y)_\leq$. Obviously,

$$F_\lambda(z_i) = \rho L_\lambda(i)$$

for all $i = 0, 1, \dots, n$. Here $z_0 = -1$ and $z_n = 1$, and

$$\rho = |r| \int_{-1}^1 |f_\lambda(t)| dt.$$

The points $z_0, z_1, \dots, z_n$ will be called the *knots* of $F_\lambda(t)$ corresponding to the point $\lambda \in Q^n \setminus 0$. With the point $\lambda = 0$ we associate the polynomial at infinity $F_0(t) = \infty(t)$ and the zero piecewise linear function $L_0(t) = 0$.

Thus, each nonzero $\lambda \in \Delta_\pm^n \sqcup Q^n$ is associated with a polynomial

$$F_\lambda(t) = \frac{1}{n+1} t^{n+1} + \cdots$$

such that $F_\lambda(-1) = 0$, and also with a piecewise linear function $L_\lambda(t)$ such that $L_\lambda(0) = 0$. If $\mu \in \partial Q^n$ and $\lambda \in \partial \Delta_\pm^n$ are equivalent, then

$$F_\mu(t) = F_\lambda(t).$$

Therefore, we identify the piecewise linear functions $L_\mu(t)$ and $L_\lambda(t)$ if μ and λ are equivalent.

By construction, for any nonzero $\lambda \in \Delta_\pm^n \sqcup Q^n$ there exists $\rho > 0$ such that the sequence of maximum and minimum values of the piecewise linear function $\rho L_\lambda(t)$ is the same as the sequence of maximum and minimum values of the polynomial $F_\lambda(t)$ on the interval $[-1, 1]$. Here we do not take into account the multiplicities of the extremal values of $L_\lambda(t)$ and $F_\lambda(t)$. Observe that equivalent piecewise linear functions have equal sequences of maximum and minimum values up to multiplicities.

Later we shall need a relationship between the multiplicities of the knots of $F_\lambda(t)$ and the multiplicities of the knots of the piecewise linear function $L_\lambda(t)$.

Given a polynomial $F(t)$, by the multiplicity $\text{mul}(F, z)$ of a point z we mean the multiplicity of z as a root of $F'(t)$. If z is not a root of $F'(t)$, then the multiplicity of z is equal to zero.

Now we define the multiplicity of a knot k of a piecewise linear function $L(t)$. Consider the minimum $i \leq k$ and the maximum $j \geq k$ such that

$$L(i) = \dots = L(k) = \dots = L(j).$$

The number $j - i + 1$ will be called the multiplicity $\text{mul}(L, k)$ of the knot k of $L(t)$. For example, in the case of the polygonal line shown in the picture,

the multiplicity of 0, 1, and 2 is one, the multiplicity of 3, 4, 5, 6, and 7 is five, and the multiplicity of 8 and 9 is two.

Let $\lambda \in \Delta_{\pm}^n \sqcup \text{Int } Q^n$, and let $\lambda \neq 0$. It is easy to check the following statements. If a knot $z_k \in (-1, 1)$ is not an extremal point of the polynomial $F_{\lambda}(t)$, then

$$\text{mul}(F_{\lambda}, z_k) = 2 \left\lceil \frac{\text{mul}(L_{\lambda}, k)}{2} \right\rceil.$$

If a knot $z_k \in (-1, 1)$ is an extremal point of the polynomial $F_{\lambda}(t)$, then

$$\text{mul}(F_{\lambda}, z_k) = 2 \left\lceil \frac{\text{mul}(L_{\lambda}, k) - 1}{2} \right\rceil + 1.$$

If $z_k = \pm 1$, then $\text{mul}(F_{\lambda}, z_k) = \text{mul}(L_{\lambda}, k) - 1$.

§8. INTERPOLATION WITHOUT OSCILLATION

The Lagrange interpolation with uniform step has an essential drawback. The interpolation polynomial oscillates strongly for large arrays of data. The oscillation of the Lagrange polynomial depicted at the beginning of the preceding section is admissible. But the Lagrange polynomial constructed for the data

has a strong oscillation at the ends of the interval.

For such saw-tooth data, oscillation is easily eliminated by passage to the Chebyshev polynomial;

of course, the latter interpolation polynomial has a nonuniform step.

The data $L_0, L_1, \dots, L_{n+1}$ are said to be *alternating* if

$$(L_{i+1} - L_i)(L_i - L_{i-1}) < 0$$

for all $i = 1, \dots, n$.

Theorem 8. *For any alternating data $L_0, L_1, \dots, L_{n+1}$, there exists a unique polynomial*

$$F(t) = a_0 t^{n+1} + a_1 t^n + \dots + a_{n+1}$$

with extremal points $-1 < x_1 < \dots < x_n < 1$ such that

$$F(x_i) = L_i$$

for all $i = 0, 1, \dots, n+1$. Here $x_0 = -1$ and $x_{n+1} = 1$.

Proof. Given alternating data $L_0, L_1, \dots, L_{n+1}$, we introduce the point

$$\lambda = \frac{1}{\rho} \begin{bmatrix} L_1 - L_0 \\ L_2 - L_1 \\ \vdots \\ L_{n+1} - L_n \end{bmatrix},$$

where

$$\rho = L_{n+1} - L_n + L_{n-1} - \dots - (-1)^n L_0.$$

Obviously, $\lambda \in \text{Int } \Delta_{\pm}^n$. Let $F_{\lambda}(t)$ be the polynomial corresponding to this point. Then the desired polynomial $F(t)$ can be written as

$$F(t) = L_0 + \rho F_{\lambda}(t).$$

The uniqueness of $F(t)$ is obvious. □

Remark. The resulting polynomial is a natural generalization of the Chebyshev polynomial. In a unique way, each point in $\text{Int } \Delta_{\pm}^n$ gives rise to such a generalized polynomial. Observe that the barycenter of $\Delta_{\pm}^n$ corresponds to the classical Chebyshev polynomial.

Example. For the alternating data

the generalized Chebyshev polynomial has the form

§9. ROOTS AND EXTREMA

Consider the polynomial

$$f(t) = (t - a_1) \cdots (t - a_m)(t - x_1) \cdots (t - x_n).$$

Here the roots $a_1 < a_2 < \cdots < a_m$ are fixed and the roots $x_1 \leq x_2 \leq \cdots \leq x_n$ are free. In this section we study how the extrema of $f(t)$ depend on the free roots $x_1, \dots, x_n$. Since the trivial cases where $m = 0$ or $m = 1$ are not interesting, we assume that $m \geq 2$.

We arrange $n + m - 1$ roots of the derivative $f'(t)$ in increasing order

$$t_1 \leq t_2 \leq \cdots \leq t_{n+m-1}$$

and put

$$y_k = (-1)^{n+m-k} f(t_k),$$

where $1 \leq k \leq n + m - 1$. Then $y_k \geq 0$ for all $1 \leq k \leq n + m - 1$. in the figure we see the case where $m = n = 2$.

We define a mapping

$$F : \mathbb{R}_{\leq}^n \longrightarrow \mathbb{R}_{+}^{n+m-1}$$

by the rule

$$F : (x_1, \dots, x_n) \longmapsto (y_1, \dots, y_{n+m-1}).$$

Observe that $F(\mathbb{R}_{\leq}^n)$ is a variety. Indeed, the functions

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+m-1} (-1)^{i_1+i_2+\dots+i_k} y_{i_1} y_{i_2} \dots y_{i_k},$$

where $1 \leq k \leq n + m - 1$, are symmetric relative to the roots $t_1, t_2, \dots, t_{n+m-1}$ of $f'(t)$, and the coefficients of $f'(t)$ depend on the roots $x_1, \dots, x_n$ in a rational way. This leads to the system of algebraic equations

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+m-1} (-1)^{i_1+i_2+\dots+i_k} y_{i_1} y_{i_2} \dots y_{i_k} = H_k(x_1, \dots, x_n),$$

where $1 \leq k \leq n + m - 1$. Eliminating the variables $x_1, \dots, x_n$, we obtain a system of algebraic equations

$$\begin{aligned} G_1(y_1, \dots, y_{n+m-1}) &= 0, \\ &\vdots \\ G_{m-1}(y_1, \dots, y_{n+m-1}) &= 0. \end{aligned}$$

In the space $\mathbb{R}^{n+m-1}$ these equations determine a variety. We study the part $F(\mathbb{R}_{\leq}^n)$ of this variety in the positive cone $\mathbb{R}_{+}^{n+m-1}$.

Example. Consider the polynomial

$$f(t) = (t^2 - 1)(t - x).$$

Then the curve $F(\mathbb{R})$ is given by the equation

$$G(y_1, y_2) = 0,$$

where

$$\begin{aligned} G(y_1, y_2) &= 78732y_1^3y_2^3 - 531441(y_1^4 + y_2^4) - 236196y_1y_2(y_1^2 + y_2^2) \\ &\quad + 30618y_1^2y_2^2 + 1492992(y_1^2 + y_2^2) + 4313088y_1y_2 - 1048576. \end{aligned}$$

In the positive quadrant, the curve $F(\mathbb{R})$ looks like this:

In particular, the curve $F(\mathbb{R})$ contains an embedded circle. It is curious that the change of variables

$$\begin{aligned} X &= y_1 y_2, \\ Y &= y_1^2 + y_2^2 \end{aligned}$$

results in the singular cubic curve

$$78732X^3 + 1093500X^2 + 4313088X - 236196XY - 531441Y^2 + 1492992Y - 1048576 = 0$$

with a double point

$$X = \frac{2^8}{3^3}, \quad Y = \frac{2^9}{3^3}.$$

In higher dimensions, the system of algebraic equations for the extrema $y_1, \dots, y_{n+m-1}$ is quite bulky. The topology of the variety $F(\mathbb{R}_{\leq}^n)$ can be studied with the help of piecewise linear objects.

Let

$$\Lambda : \mathbb{R}_{\leq}^n \longrightarrow \mathbb{R}_+^{n+m-1}$$

be the mapping defined by

$$\Lambda : x \longmapsto \Delta\{(a, x)_{\leq}\},$$

where Δ is the operator

$$\Delta(x_1, x_2, \dots, x_n) = (x_2 - x_1, \dots, x_n - x_{n-1}).$$

Theorem 9. *There is a homeomorphism $F(\mathbb{R}_{\leq}^n) = \Lambda(\mathbb{R}_{\leq}^n)$.*

Proof. Let $x \in \mathbb{R}_{\leq}^n$, and let

$$\begin{aligned} \alpha &= \min\{a_1, \dots, a_m, x_1, \dots, x_n\}, \\ \beta &= \max\{a_1, \dots, a_m, x_1, \dots, x_n\}. \end{aligned}$$

The change of variables

$$t \longmapsto \frac{\beta + \alpha}{2} + \frac{\beta - \alpha}{2}t$$

transforms the polynomial

$$f(t) = (t - a_1) \cdots (t - a_m)(t - x_1) \cdots (t - x_n)$$

into a polynomial

$$g(t) = \left(\frac{\beta - \alpha}{2}\right)^{n+m} (t - \lambda_1) \cdots (t - \lambda_{n+m})$$

with

$$-1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+m-1} \leq \lambda_{n+m} = 1.$$

Next, we define numbers

$$-1 \leq \tau_1 \leq \cdots \leq \tau_{n+m-1} \leq 1$$

by the condition

$$(t - \tau_1) \cdots (t - \tau_{n+m-1}) = \frac{1}{n+m} \frac{d}{dt} [(t - \lambda_1) \cdots (t - \lambda_{n+m})].$$

As a result, we obtain mappings $x \mapsto \lambda$ and $x \mapsto \tau$.

Let $x^0, x^1 \in \mathbb{R}_{\leq}^n$. We claim that the identities

$$F(x^0) = F(x^1), \quad \Lambda(x^0) = \Lambda(x^1)$$

are equivalent.

Let $F(x^0) = F(x^1)$. Then $\mathcal{L}_{\nabla}(\tau^0) = \mathcal{L}_{\nabla}(\tau^1)$, where τ^0 corresponds to x^0 and τ^1 corresponds to x^1 . By Theorem 7, we have $\tau^0 = \tau^1$. Then $\lambda^0 = \lambda^1$, where λ^0 corresponds to x^0 and λ^1 corresponds to x^1 . Consequently, $\beta^0 - \alpha^0 = \beta^1 - \alpha^1$, which leads to the identity $\Lambda(x^1) = \Lambda(x^0)$.

Let $\Lambda(x^0) = \Lambda(x^1)$. We put $z^0 = (a, x^0)_{\leq}$ and $z^1 = (a, x^1)_{\leq}$. There exists a constant r such that $z_k^1 = z_k^0 + r$ for all $k = 1, \dots, n+m$. Therefore, $F(x^1) = F(x^0)$.

Since the mappings

$$\begin{aligned} F : \mathbb{R}_{\leq}^n &\longrightarrow F(\mathbb{R}_{\leq}^n), \\ \Lambda : \mathbb{R}_{\leq}^n &\longrightarrow \Lambda(\mathbb{R}_{\leq}^n) \end{aligned}$$

are closed, we have homeomorphisms

$$\begin{aligned} F(\mathbb{R}_{\leq}^n) &= \mathbb{R}_{\leq}^n / F, \\ \Lambda(\mathbb{R}_{\leq}^n) &= \mathbb{R}_{\leq}^n / L. \end{aligned}$$

But $F^{-1}\{F(x)\} = \Lambda^{-1}\{\Lambda(x)\}$ for all $x \in \mathbb{R}_{\leq}^n$, whence we see that $\mathbb{R}_{\leq}^n / F = \mathbb{R}_{\leq}^n / \Lambda$. This means that $F(\mathbb{R}_{\leq}^n)$ and $\Lambda(\mathbb{R}_{\leq}^n)$ are homeomorphic. $\square$

In order to describe the set $\Lambda(\mathbb{R}_{\leq}^n)$, we introduce the $(n+m)$ -plane

$$\begin{aligned} \lambda_{i_1} + \cdots + \lambda_{i_2-1} &= a_2 - a_1, \\ &\vdots \\ \lambda_{i_{m-1}} + \cdots + \lambda_{i_m-1} &= a_m - a_{m-1}, \end{aligned}$$

where $1 \leq i_1 < i_2 < \cdots < i_m \leq n+m$. The set $\Lambda(\mathbb{R}_{\leq}^n)$ is equal to the intersection of the cone $\mathbb{R}_{+}^{n+m-1}$ with the union of all such planes. In particular, the set $\Lambda(\mathbb{R}_{\leq}^n)$ is representable as a union of finitely many convex polyhedra. This permits us to describe the topology of the variety $F(\mathbb{R}_{\leq}^n)$. As an application, we prove the following statement.

Theorem 10. *For $m = 2$, the variety $F(\mathbb{R}_{\leq}^n)$ contains a sphere of dimension n .*

Proof. Indeed, a linear change of the variable t reduces the problem to the polynomials of the form

$$f(t) = t(t-1)(t-x_1) \cdots (t-x_n).$$

In this case the mapping

$$\Lambda : \mathbb{R}_{\leq}^n \longrightarrow \mathbb{R}_{+}^{n+1}$$

has the form

$$\Lambda(x_1, \dots, x_n) = \Delta\{(0, 1, x_1, \dots, x_n)_{\leq}\}.$$

Then the boundary of the convex polyhedron with the faces

$$\begin{aligned}\lambda_1 &= 1, \\ \lambda_1 + \lambda_2 &= 1, \\ \lambda_2 &= 1, \\ \lambda_2 + \lambda_3 &= 1, \\ \lambda_3 &= 1, \\ &\vdots \\ \lambda_n + \lambda_{n+1} &= 1, \\ \lambda_{n+1} &= 1\end{aligned}$$

is a sphere in $\Lambda(\mathbb{R}_{\leq}^n)$. The theorem is proved. $\square$

§10. THE SPHERE OF ALTERNATING POLYNOMIALS

A continuous function $f(t)$ is said to be *alternating* on an interval $[a, b]$ if

$$\min_{a \leq t \leq b} f(t) = - \max_{a \leq t \leq b} f(t).$$

Let $f(t)$ be any continuous function on $[a, b]$. If

$$f^{\text{alt}}(t) = f(t) - \frac{1}{2} \left(\min_{a \leq t \leq b} f(t) + \max_{a \leq t \leq b} f(t) \right),$$

then $f^{\text{alt}}(t)$ is alternating. Now, let $f(t)$ be any alternating function. By $\text{alt } f$ we denote the maximum number of alternations between the points of maximum and minimum. Obviously, $\text{alt } f \geq 1$.

In this section we construct a cell decomposition of the sphere

$$S_{\text{alt}}^{n-1} = \{F \in S^n(t) : \min_{-1 \leq t \leq 1} F(t) = - \max_{-1 \leq t \leq 1} F(t)\}$$

of alternating polynomials on the interval $[-1, 1]$. We agree that the polynomial at infinity is alternating.

In §7, every point $\lambda \in \Delta_{\pm}^{n-1} \sqcup Q^{n-1}$ was associated with a polynomial

$$F_{\lambda}(t) = \frac{1}{n}t^n + a_1t^{n-1} + \dots + a_n$$

such that $F_{\lambda}(-1) = 0$ and, simultaneously, with a piecewise linear function $L_{\lambda}(t)$ such that $L_{\lambda}(0) = 0$. Recall that the piecewise linear function $L_{\lambda}(t)$ has the same mutual location of the maximum and minimum values as the polynomial $F_{\lambda}(t)$ on the interval $[-1, 1]$.

We see that the mapping

$$\lambda \mapsto nF_{\lambda}^{\text{alt}}(t)$$

is a homeomorphism of the sphere $\Delta_{\pm}^{n-1} \cup_{\psi} Q^{n-1}$ onto the sphere S_{alt}^{n-1} . Therefore, for constructing a cell decomposition of S_{alt}^{n-1} , we may use the alternating piecewise linear functions $L_{\lambda}^{\text{alt}}(t)$ instead of the alternating polynomials $F_{\lambda}^{\text{alt}}(t)$.

Consider a k -link nonzero alternating piecewise linear function $L(t)$ with the knots $0, 1, \dots, k$. Putting

$$\mu = \max_{0 \leq i \leq k} L(i),$$

we write the set

$$I = \{0 \leq i \leq k : |L(i)| = \mu\}$$

in the form $I = I_0 \cup \dots \cup I_p$, where $I_0, \dots, I_p$ are the maximal integral intervals such that for each $I_\nu = \{i_\nu, i_\nu + 1, \dots, j_\nu\}$ we have

$$L(i_\nu) = L(i_\nu + 1) = \dots = L(j_\nu).$$

The system of integral intervals $I_0, \dots, I_p$ will be called the *segmentation* of the piecewise linear function $L(t)$. Observe that $\text{mul}(L, i) = j_\nu - i_\nu + 1$ for all $i \in I_\nu$.

For example, the segmentation of the piecewise linear function

is as follows: $I_0 = \{0, 1\}$, $I_1 = \{2, 3\}$, $I_2 = \{5, 6, 7\}$, $I_3 = \{9, 10, 11\}$.

We shall construct cell decompositions of $\Delta_\pm^k$ and Q^k such that for any cell e all piecewise linear functions L_λ^{alt} have a constant segmentation for all $\lambda \in e$. In this case the quantity $\text{alt } L_\lambda^{\text{alt}}$ is also constant for all $\lambda \in e$; we denote this quantity by $\text{alt } e$.

Cell decomposition of $\Delta_\pm^k$. The cell decomposition of $\Delta_\pm^{-1} = \emptyset$ is empty. The cell decomposition of $\Delta_\pm^0 = 1$ consists of one zero-dimensional cell $e = 1$. The cell e has the segmentation $\{0\} \cup \{1\}$. In particular, $\text{alt } e = 1$.

For $k \geq 1$, assume that the required cell decompositions of $\Delta_\pm^{k-2}$ and $\Delta_\pm^{k-1}$ have been constructed. We describe a cell decomposition of $\Delta_\pm^k$.

Let $\Delta_\pm^k(i)$ denote the face $\lambda_i = 0$, and let $\Delta_\pm^k(i, i+1)$ be the face $\lambda_i = \lambda_{i+1} = 0$. The mapping

$$\phi_0 : (\lambda_0, \lambda_1, \dots, \lambda_{k-1}) \mapsto (0, \lambda_0, \lambda_1, \dots, \lambda_{k-1})$$

identifies $\Delta_\pm^{k-1}$ and $\Delta_\pm^k(0)$. The mapping

$$\phi_k : (\lambda_0, \lambda_1, \dots, \lambda_{k-1}) \mapsto -(\lambda_0, \lambda_1, \dots, \lambda_{k-1}, 0)$$

identifies $\Delta_\pm^{k-1}$ and $\Delta_\pm^k(k)$. For $0 \leq i \leq k-1$, the mapping

$$\varphi_i : (\lambda_0, \dots, \lambda_{i-1}, \lambda_i, \dots, \lambda_{k-2}) \mapsto (\lambda_0, \dots, \lambda_{i-1}, 0, 0, \lambda_i, \dots, \lambda_{k-2})$$

identifies $\Delta_\pm^{k-2}$ and $\Delta_\pm^k(i, i+1)$. The mappings ϕ_0 and ϕ_k transfer the cell decomposition of $\Delta_\pm^{k-1}$ to the faces $\Delta_\pm^k(0)$ and $\Delta_\pm^k(k)$, and the mapping φ_i transfers the cell decomposition $\Delta_\pm^{k-2}$ to the face $\Delta_\pm^k(i, i+1)$, where $0 \leq i \leq k-1$.

Consider the face $\Delta_\pm^k(0)$. Suppose that the segmentation of a cell e of $\Delta_\pm^{k-1}$ is $I_0, \dots, I_p$. The segmentation of the cell $E = \phi_0(e)$ is obtained as follows. We add 1 to all numbers in $I_0, \dots, I_p$. If the new set I_0 contains 1, we supplement it with the new element 0.

Consider the face $\Delta_\pm^k(k)$. Let $I_0, \dots, I_p$ be the segmentation of a cell e of $\Delta_\pm^{k-1}$. The segmentation of the cell $E = \phi_k(e)$ is obtained as follows. We do not change the sets $I_0, \dots, I_{p-1}$. If the set I_p contains k , then we supplement I_p with the new element $k+1$.

Consider the face $\Delta_\pm^k(i, i+1)$. Let e be a cell of $\Delta_\pm^{k-2}$, and let $I_0, \dots, I_p$ be the segmentation of e . The segmentation of the cell $E = \varphi_i(e)$ is obtained as follows. We add 2 to all numbers $j \in I_0 \cup \dots \cup I_p$ such that $j \geq i+1$. If $0 \leq \nu \leq p$ for some $i \in I_\nu$, then we supplement I_ν with the new elements $i+1$ and $i+2$.

Consider a face $\Delta_{\pm}^k(i)$ with $1 \leq i \leq k-1$. Let e be a cell of $\Delta_{\pm}^{k-2}$. With the cell e we associate a cell E of the face $\Delta_{\pm}^k(i)$. The cell E will consist of all points such that

$$\tau\varphi_{i-1}(\lambda) + (1-\tau)\varphi_i(\lambda),$$

where $0 < \tau < 1$ and $\lambda \in e$. Let $I_0, \dots, I_p$ be the segmentation of e . We describe the segmentation of the cell E . We add 2 to all numbers $j \in I_0 \cup \dots \cup I_p$ such that $j \geq i$. If $\{i-1, i\} \subset I_{\nu}$ for some $0 \leq \nu \leq p$, then we supplement I_{ν} with the new element $i+1$.

Now we construct a cell decomposition of the entire simplex $\Delta_{\pm}^k$. Let $\text{bar } \Delta_{\pm}^k$ denote the barycenter of $\Delta_{\pm}^k$. With any cell e of the cell decomposition of the boundary $\partial\Delta_{\pm}^k$ we associate a cell E of the simplex $\Delta_{\pm}^k$. The cell E will consist of all points such that

$$\tau\lambda + (1-\tau)\text{bar } \Delta_{\pm}^k,$$

where $0 < \tau < 1$ and $\lambda \in e$. Let $I_0, \dots, I_p$ be the segmentation of e , where

$$I_{\nu} = \{i_{\nu}, i_{\nu}+1, i_{\nu}+2, i_{\nu}+3, \dots, j_{\nu}\}.$$

We describe the segmentation of E . If $i_0 = 0$, then we replace the segment I_0 by the system of segments

$$\{j_0 - 2[j_0/2]\}, \dots, \{j_0 - 4\}, \{j_0 - 2\}, \{j_0\}.$$

If $0 < i_{\nu}$ and $j_{\nu} < k+1$, then i_{ν} and j_{ν} are of equal parity, and we replace I_{ν} by the system of segments

$$\{i_{\nu}\}, \{i_{\nu}+2\}, \{i_{\nu}+4\}, \dots, \{j_{\nu}\}.$$

If $j_p = k+1$, then we replace I_p by the system of segments

$$\{i_p\}, \{i_p+2\}, \{i_p+4\}, \dots, \{i_p+2[(k+1-i_p)/2]\}.$$

Observe that $\text{alt } E = \text{alt } e$ in all cases.

To finish decomposition, to the resulting cells we add yet another zero-dimensional cell, namely, $\text{bar } \Delta_{\pm}^k$. The cell $\text{bar } \Delta_{\pm}^k$ has the obvious segmentation

$$I = \{0\} \cup \{1\} \cup \dots \cup \{k+1\}.$$

In particular, $\text{alt bar } \Delta_{\pm}^k = k+1$. This completes the construction of the required cell decomposition of the simplex $\Delta_{\pm}^k$.

Cell decomposition of Q^k . First, we construct a cell decomposition of the boundary ∂Q^k . Let $\delta = (\delta_1, \dots, \delta_k)$ have coordinates ± 1 . We only construct a decomposition of the interior of the simplex $\Delta^{k-1}(\delta)$. The other simplexes of the triangulation of ∂Q^k are decomposed in a similar way. Let

$$\delta = (\underbrace{\varepsilon_0, \dots, \varepsilon_0}_{q_0}, \underbrace{\varepsilon_1, \dots, \varepsilon_1}_{q_1}, \dots, \underbrace{\varepsilon_l, \dots, \varepsilon_l}_{q_l}),$$

where $\varepsilon_{i-1}\varepsilon_i = -1$ for all $i = 1, \dots, l$. We define a mapping

$$\varphi : \Delta^{k-1}(\delta) \longrightarrow \Delta_{\pm}^l$$

by the following rule. The mapping φ takes the vector

$$(\lambda_{01}, \dots, \lambda_{0q_0}, \lambda_{11}, \dots, \lambda_{1q_1}, \dots, \lambda_{l1}, \dots, \lambda_{lq_l})$$

to the vector

$$\varepsilon_l(\lambda_{01} + \dots + \lambda_{0q_0}, \lambda_{11} + \dots + \lambda_{1q_1}, \dots, \lambda_{l1} + \dots + \lambda_{lq_l}).$$

Let e be a cell of $\Delta_{\pm}^l$ lying in $\text{Int } \Delta_{\pm}^l$, and let $I_0, \dots, I_p$ be the segmentation of e . With e we associate the cell

$$E = \text{Int } \Delta^k(\delta) \cap \varphi^{-1}(e).$$

To obtain the segmentation of E , we replace $i \geq 1$ in $I_0, \dots, I_p$ by the sum $q_0 + \dots + q_{i-1}$.

As a result, we obtain a cell decomposition of ∂Q^k such that the attaching mapping

$$\psi : \partial Q^k \longrightarrow \partial \Delta_{\pm}^k$$

is a cell mapping.

Now we construct the cell decomposition of the entire octahedron Q^k . Let e be a cell of the resulting cell decomposition of the boundary ∂Q^k .

With e we associate a cell E of the octahedron Q^k . The cell E will consist of all points of the form $\tau\lambda$ with $0 < \tau < 1$ and $\lambda \in e$. We see that e and E have equal segmentations. To finish decomposition, to the resulting collection of cells we add the zero-dimensional cell 0. We assume that $\text{alt } 0 = \infty$. The required cell decomposition of the octahedron Q^k is constructed.

We agree to write points of the simplex $\Delta_{\pm}^k$ and points of the boundary of the octahedron Q^k in homogeneous coordinates. In particular, the barycenters of the faces in the simplex $\Delta_{\pm}^k$ and on the boundary of the octahedron Q^k are written in terms of the integers $-1, 0, 1$. For instance, $\text{bar } \Delta_{\pm}^2 = (1, -1, 1)$.

The case where $n = 1$. The cell decompositions $\Delta_{\pm}^1$ and Q^1 are as shown in the figure.

At the barycenter $(-1, 1)$ the function alt is equal to 2. At all other cells except for the cell 0, the function alt is equal to 1.

The case where $n = 2$. The cell decompositions $\Delta_{\pm}^2$ and Q^2 are as shown in the figure.

At the barycenter $(1, -1, 1)$ the function alt is equal to 3, at the zero-dimensional cells $(1, -1, 0)$ and $(0, -1, 1)$, as well as at the one-dimensional cells connecting the zero-dimensional cells $(1, -1, 0)$ and $(0, -1, 1)$ with the barycenter $(1, -1, 1)$, the function alt is equal to 2. At all other cells of the simplex $\Delta_{\pm}^2$ the function alt is equal to 1. At the zero-dimensional cells $(1, -1)$ and $(-1, 1)$, as well as at the one-dimensional cells connecting the zero-dimensional cells $(1, -1)$ and $(-1, 1)$ with the origin $(0, 0)$, the function alt is equal to 2. At all other cells of the octahedron Q^2 except for the cell $(0, 0)$, the function alt is equal to 1.

§11. TOPOLOGICAL CLASSIFICATION OF THE ABEL EQUATIONS

In this section we treat the Chebyshev approximation problem with m fixed coefficients. Consider a polynomial

$$F(t) = t^n + x_1 t^{n-1} + \dots + x_m t^{n-m} + y_1 t^{n-m-1} + \dots + y_{n-m}.$$

For fixed coefficients $x_1, \dots, x_m$, we need to find the free coefficients $y_1, \dots, y_{n-m}$ such that the value

$$R = \max_{-1 \leq t \leq 1} |F(t)|$$

is the least possible. For $m = 0, 1, 2$, this problem was solved by Chebyshev [9], Zolotarev [10], and Akhieser [11].

By the Chebyshev theorem, for any $x_1, \dots, x_m$ there exists a unique solution $y_1, \dots, y_{n-m}$ of the above minimization problem. The continuous mapping

$$x \longmapsto y$$

of the space $\mathbb{R}^m$ to the space $\mathbb{R}^{n-m}$ arises, and this mapping is piecewise analytic. The space $\mathbb{R}^m$ is decomposed into cells of dimensions $0, 1, \dots, m$ such that on each cell, y depends on x analytically.

Let

$$F(t) = t^n + a_1 t^{n-1} + \dots + a_n$$

be an alternating polynomial on the interval $[-1, 1]$, and let $k = \text{alt } F$. Obviously, $1 \leq k \leq n$. Put $m = n - k$. We see that $F(t)$ is the solution of the Chebyshev problem with m fixed coefficients $x_1 = a_1, \dots, x_m = a_m$ and with free coefficients $y_1 = a_{m+1}, \dots, y_{n-m} = a_n$. Let

$$R = \max_{-1 \leq t \leq 1} |F(t)|.$$

Then $F(t)$ and R satisfy the Abel equation

$$F^2(t) - R^2 = (t+1)^\alpha (t-1)^\beta G(t) H^2(t),$$

where $H(t)$ has all roots in the interval $(-1, 1)$, and $G(t)$ is a polynomial of degree γ such that no roots of $G(t)$ lie in the interval $[-1, 1]$.

The integers α , β , and γ are determined by the alternating polynomial $F \in S_{\text{alt}}^{n-1}$ uniquely. This gives a mapping

$$F \longmapsto (\alpha, \beta, \gamma).$$

In §10 we constructed a cell decomposition of the sphere S_{alt}^{n-1} . Here we prove that the above mapping is constant on all cells of that decomposition.

Let $\lambda \in \Delta_{\pm}^{n-1} \cup \text{Int } Q^{n-1}$. Consider the alternating polynomial $F(t) = nF_{\lambda}^{\text{alt}}(t)$ and the alternating piecewise linear function $L(t) = L_{\lambda}^{\text{alt}}(t)$. The polynomial $F(t)$ attains its maximum and minimum values on the interval $[-1, 1]$ at the knots of the polynomial $F_{\lambda}(t)$. Let $x_0 < x_1 < \dots < x_p$ be these extreme points. We see that

$$\alpha = \begin{cases} 0 & \text{if } -1 < x_0, \\ \text{mul}(F, -1) + 1 & \text{if } -1 = x_0, \end{cases}$$

and

$$\beta = \begin{cases} 0 & \text{if } +1 > x_p, \\ \text{mul}(F, +1) + 1 & \text{if } +1 = x_p. \end{cases}$$

Similarly,

$$\deg H = \frac{1}{2} \sum_{|x_{\nu}| < 1} \{\text{mul}(F, x_{\nu}) + 1\}.$$

Consequently,

$$\gamma = 2n - \alpha - \beta - \sum_{|x_{\nu}| < 1} \{\text{mul}(F, x_{\nu}) + 1\}.$$

Let $I_0, \dots, I_p$ be the segmentation of the piecewise linear function $L(t)$ in question. Observe that $L(t)$ has $N = n$ links for $\lambda \in \Delta_{\pm}^{n-1}$ and has $N = n-1$ links for $\lambda \in \text{Int } Q^{n-1}$. Using the formulas written at the end of §7, we see that

$$\begin{aligned}\alpha &= \begin{cases} 0 & \text{if } 0 \notin I_0, \\ j_0 - 0 + 1 & \text{if } 0 \in I_0, \end{cases} \\ \beta &= \begin{cases} 0 & \text{if } N \notin I_p, \\ N - i_p + 1 & \text{if } N \in I_p, \end{cases} \\ \gamma &= 2n - \alpha - \beta - 2 \sum_{\nu} \left(\left\lfloor \frac{j_{\nu} - i_{\nu}}{2} \right\rfloor + 1 \right),\end{aligned}$$

where the sum is taken over all $\nu = 0, 1, \dots, p$ such that $0 \notin I_{\nu}$ and $N \notin I_{\nu}$.

This implies that the mapping

$$F \longmapsto (\alpha, \beta, \gamma)$$

is constant on all cells of the decomposition S_{alt}^{n-1} .

For any n , the segmentations of $L_{\lambda}^{\text{alt}}(t)$, where $\lambda \in \Delta_{\pm}^{n-1} \cup \text{Int } Q^{n-1}$, can be constructed in finitely many steps. The integers α, β, γ can be calculated in terms of segmentations, and the Abel equations can be written in terms of α, β, γ . Therefore, the construction of all segmentations gives a classification of the Abel equations. Additional information about Abel–Pell equations can be found in the surveys [12] and [13].

To distinguish the case of m fixed coefficients, consider the sphere

$$S_{\text{alt}}^{n-1}(m) = \{F \in S_{\text{alt}}^{n-1} : \text{alt } F \geq n - m\}.$$

This sphere consists of all solutions of the Chebyshev problem with m fixed coefficients $(a_1, \dots, a_m) \in \mathbb{R}^m$. The sphere $S_{\text{alt}}^{n-1}(m)$ is homeomorphic to S^m and has a natural cell decomposition induced by the cell decomposition of the sphere S_{alt}^{n-1} . In particular, each cell of the sphere $S_{\text{alt}}^{n-1}(m)$ corresponds to some Abel equation arising in the Chebyshev problem with m fixed coefficients.

The Chebyshev case: $m = 0$. It is easily seen that

$$S_{\text{alt}}^{n-1}(0) = \{0, \text{bar } \Delta_{\pm}^{n-1}\}.$$

The cell 0 is not interesting. The cell $\text{bar } \Delta_{\pm}^{n-1}$ has the segmentation

$$\{0\} \cup \{1\} \cup \dots \cup \{n-1\} \cup \{n\}.$$

For this segmentation we have $\alpha = 1$, $\beta = 1$, and $\gamma = 0$. Consequently, for $m = 0$ we have only one Abel equation

$$F^2(t) - R^2 = (t^2 - 1)H^2(t).$$

The solution $F(t)$ of this equation is the classical Chebyshev polynomial $T_n(t)$.

The Zolotarev case: $m = 1$. The sphere $S_{\text{alt}}^{n-1}(1)$ has four zero-dimensional cells

$$0, \quad \text{bar } \Delta_{\pm}^{n-1}(0), \quad \text{bar } \Delta_{\pm}^{n-1}, \quad \text{bar } \Delta_{\pm}^{n-1}(n-1)$$

and four one-dimensional cells between the zero-dimensional cells.

The cell 0 is not interesting. The segmentations of the other three zero-dimensional cells look like this:

$$\begin{aligned} & \{0\} \cup \{1\} \cup \cdots \cup \{n-1\} \cup \{n\}, \\ & \{0, 1\} \cup \{2\} \cup \cdots \cup \{n-1\} \cup \{n\}, \\ & \{0\} \cup \{1\} \cup \cdots \cup \{n-2\} \cup \{n-1, n\}. \end{aligned}$$

The segmentations of two one-dimensional cells belonging to $\Delta_{\pm}^{n-1}$ are as follows:

$$\begin{aligned} & \{1\} \cup \{2\} \cup \cdots \cup \{n-1\} \cup \{n\}, \\ & \{0\} \cup \{1\} \cup \cdots \cup \{n-2\} \cup \{n-1\}. \end{aligned}$$

The segmentation of two one-dimensional cells belonging to $\text{Int } Q^{n-1}$ is of the form

$$\{0\} \cup \{1\} \cup \cdots \cup \{n-2\} \cup \{n-1\}.$$

Thus, for $m = 1$ we have six Abel equations,

$$\begin{aligned} F^2(t) - R^2 &= (t^2 - 1)H^2(t), \\ F^2(t) - R^2 &= (t+1)^2(t-1)(t+p)H^2(t), \\ F^2(t) - R^2 &= (t+1)(t-1)^2(t+p)H^2(t), \\ F^2(t) - R^2 &= (t-1)(t+p)H^2(t), \\ F^2(t) - R^2 &= (t+1)(t+p)H^2(t), \\ F^2(t) - R^2 &= (t^2 - 1)(t^2 + pt + q)H^2(t). \end{aligned}$$

§12. APPENDIX

Let $e(A_0, A_1, \dots, A_k)$ denote the open simplex with vertices $A_0, A_1, \dots, A_k$. For $k = 0$, we assume that $e(A_0) = A_0$.

Theorem 11. *The cell mapping*

$$\mathcal{L}_{\nabla} : \nabla^n \longrightarrow \Delta_{\pm}^n$$

is nonsingular in each cell of positive dimension.

Proof. The vertices of the simplexes ∇^n and $\Delta_{\pm}^n$ will be denoted by

$$A_m = (\underbrace{-1, \dots, -1}_m, \underbrace{1, \dots, 1}_{n-m})$$

and

$$B_m = (-1)^{n-m}(\underbrace{0, \dots, 0}_{m+1}, \underbrace{1, 0, \dots, 0}_{n-m}),$$

where $m = 0, 1, \dots, n$.

Let $1 \leq l \leq n$, and let $0 \leq m_0 < m_1 < \cdots < m_l \leq n$. The definition of the mapping $\mathcal{L}_{\nabla}$ shows that

$$\mathcal{L}_{\nabla}\{e(A_{m_0}, \dots, A_{m_l})\} \subset e(B_{m_0}, \dots, B_{m_l}).$$

We must check that $\mathcal{L}_\nabla$ maps $e(A_{m_0}, \dots, A_{m_l})$ onto $e(B_{m_0}, \dots, B_{m_l})$ and has no singular points.

Let $k_i = m_i - m_{i-1}$, where $i = 0, 1, \dots, l+1$. Here $m_{-1} = 0$ and $m_{l+1} = n$. The cell $e(A_{m_0}, \dots, A_{m_l})$ consists of all points of the form

$$x = (\underbrace{-1, \dots, -1}_{k_0}, \underbrace{u_1, \dots, u_1}_{k_1}, \dots, \underbrace{u_l, \dots, u_l}_{k_l}, \underbrace{1, \dots, 1}_{k_{l+1}}),$$

where $u \in \text{Int } \nabla^l$.

We put

$$\omega(t, u) = (t+1)^{k_0} \prod_{i=1}^l (t-u_i)^{k_i} (t-1)^{k_{l+1}}.$$

For $u \in \text{Int } \nabla^l$, the polynomial $\omega(t, u)$ has n roots in the interval $[-1, 1]$. Observe that the interior roots $u_1, \dots, u_l$ have multiplicities $k_1 \geq 1, \dots, k_l \geq 1$.

For $i = 0, 1, \dots, l$, we introduce the polynomials

$$F_i(u) = (-1)^{n-m_i} \int_{u_i}^{u_{i+1}} \omega(t, u) dt$$

in the variables $u_1, \dots, u_l$. Here $u_0 = -1$ and $u_{l+1} = 1$. Obviously, $F_i(u) > 0$ for $u \in \text{Int } \nabla^l$. Let

$$S(u) = \sum_{i=0}^l F_i(u).$$

Then

$$\mathcal{L}_\nabla(x) = \frac{1}{S(u)} \sum_{i=0}^l F_i(u) B_{m_i},$$

where the vectors $x \in e(A_{m_0}, \dots, A_{m_l})$ and $u \in \text{Int } \nabla^l$ are related to each other as above. We put

$$H(u) = \frac{1}{S(u)} \begin{bmatrix} F_1(u) \\ \vdots \\ F_l(u) \end{bmatrix}.$$

It suffices to prove that the mapping $H(u)$ has no singular points.

We consider the mapping

$$F(u) = \begin{bmatrix} F_0(u) \\ F_1(u) \\ \vdots \\ F_l(u) \end{bmatrix}$$

and denote by

$$\partial_u F(u) = \left\{ \frac{\partial F_i(u)}{\partial u_j} \right\}_{i=0, j=1}^l$$

the matrix of partial derivatives.

We claim that the determinant

$$D(u) = \det[F(u), \partial_u F(u)]$$

is nonsingular for all $u \in \text{Int } \nabla^l$.

Indeed,

$$\frac{\partial F_i(u)}{\partial u_j} = \pm k_j \int_{u_i}^{u_{i+1}} \frac{\omega(t, u)}{t - u_j} dt,$$

where $0 \leq i \leq l$ and $1 \leq j \leq l$. Consider the $(l+1) \times (l+1)$ -matrix

$$A(\tau, u) = \begin{bmatrix} 1 & (\tau_0 - u_1)^{-1} & \cdots & (\tau_0 - u_l)^{-1} \\ 1 & (\tau_1 - u_1)^{-1} & \cdots & (\tau_1 - u_l)^{-1} \\ & & \ddots & \\ 1 & (\tau_l - u_1)^{-1} & \cdots & (\tau_l - u_l)^{-1} \end{bmatrix}.$$

It is easy to check that

$$\det A(\tau, u) = (-1)^l \prod_{0 \leq i < j \leq l} (\tau_j - \tau_i) \prod_{1 \leq i < j \leq l} (u_i - u_j) \prod_{i=0}^l \prod_{j=1}^l (\tau_i - u_j)^{-1}.$$

We see that

$$D(u) = \pm \prod_{i=1}^l k_i \int_{u_0}^{u_1} \int_{u_1}^{u_2} \cdots \int_{u_l}^{u_{l+1}} \det A(\tau, u) \prod_{i=0}^l \omega(\tau_i, u) d\tau_0 d\tau_1 \cdots d\tau_l.$$

If

$$u_0 < \tau_0 < u_1 < \tau_1 < u_2 < \cdots < u_l < \tau_l < u_{l+1},$$

then the expression

$$\det A(\tau, u) \prod_{i=0}^l \omega(\tau_i, u)$$

is of constant sign. Consequently, $D(u) \neq 0$.

It is easily seen that

$$\frac{\partial H_i(u)}{\partial u_j} = \frac{1}{S(u)} \left(\frac{\partial F_i(u)}{\partial u_j} - \frac{F_i(u)}{S(u)} \frac{\partial S(u)}{\partial u_j} \right),$$

whence

$$\det \partial_u H(u) = \frac{1}{S^{l+1}(u)} \det[F(u), \partial_u F(u)] \neq 0$$

for all $u \in \text{Int } \nabla^l$. Thus, the mapping

$$\mathcal{L}_\nabla : e(A_{m_0}, \dots, A_{m_l}) \longrightarrow e(B_{m_0}, \dots, B_{m_l})$$

has no singular points. $\square$

Theorem 12. *The cell mapping*

$$\mathcal{L}_Q : Q^n \longrightarrow Q^n$$

is nonsingular on each cell of positive dimension.

Proof. Let $1 \leq p \leq n$ and $1 \leq m_1 < \cdots < m_p \leq n$. Let $\delta_{m_1}, \dots, \delta_{m_p}$ be numbers equal to ± 1 . The open p -dimensional simplexes $e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n)$ and the open $(p-1)$ -dimensional simplexes $e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n)$ are the cells of the octahedron Q^n that lie in its interior and on its boundary, respectively. By the definition of the mapping $\mathcal{L}_Q$, we have

$$\begin{aligned} \mathcal{L}_Q \{e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n)\} &\subset e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n), \\ \mathcal{L}_Q \{e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n)\} &\subset e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_p} E_{m_p}^n). \end{aligned}$$

Since $\mathcal{L}_Q$ is identical on the cells $e(0, \delta_m E_m^n)$ and $e(\delta_m E_m^n)$, we only consider the mapping $\mathcal{L}_Q$ on the cells

$$e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$$

and

$$e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$$

for $1 \leq p \leq n-1$.

Let δ_i be defined as ± 1 for the indices $1 \leq i \leq n$ different from $m_1, \dots, m_{p+1}$ in such a way that the value $k = \text{alt } \delta$ be maximal. It is clear that

$$e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n) \subset Q^n(\delta)$$

and

$$e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n) \subset \Delta^{n-1}(\delta).$$

Consider the mapping $\mathcal{L}_Q$ on the cell $e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$. It is convenient to use a different coordinate system. We write $\lambda \in e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$ in the form

$$\lambda = \sum_{i=1}^{p+1} \lambda_{m_i} \delta_{m_i} E_{m_i}^n$$

and write the coordinates $\lambda_{m_1}, \dots, \lambda_{m_{p+1}}$ as

$$\begin{aligned} \lambda_{m_1} &= \rho \gamma_1, \\ &\vdots \\ \lambda_{m_p} &= \rho \gamma_p, \\ \lambda_{m_{p+1}} &= \rho(1 - \gamma_1 - \dots - \gamma_p), \end{aligned}$$

where $0 < \rho < 1$ and $(\gamma_1, \dots, \gamma_{p+1}) \in \text{Int } \Delta^p$. It should be noted that $\rho = |\lambda|$. We put

$$\begin{aligned} u_1 &= \gamma_1, \\ u_2 &= \gamma_1 + \gamma_2, \\ &\vdots \\ u_p &= \gamma_1 + \gamma_2 + \dots + \gamma_p, \\ r &= \rho^{-1} \delta_n, \end{aligned}$$

where $|r| > 1$ and $(u_1, \dots, u_p) \in \text{Int } \nabla^p$. Clearly, the change of coordinates

$$(\lambda_{m_1}, \dots, \lambda_{m_{p+1}}) \longmapsto (r, u_1, \dots, u_p)$$

is nonsingular. The subsequent calculations will be done in the coordinates $r, u_1, \dots, u_p$.

Consider a point $x \times y = T_\delta(|\lambda|^{-1} \lambda)$. Let z be the vector formed by the coordinates of the vectors x and y placed in increasing order. There are precisely p distinct points in the sequence $z_1, \dots, z_{n-1}$, and these points are $u_1, \dots, u_p$.

The sequence of equalities and inequalities for the points $z_1, \dots, z_{n-1}$ is independent of $\lambda \in e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$. This means that for some $0 \leq l \leq k$ we have two strictly monotone increasing mappings

$$\begin{aligned} \nu &: \{1, 2, \dots, l\} \longrightarrow \{1, 2, \dots, p\}, \\ \sigma &: \{1, 2, \dots, n - k - 1\} \longrightarrow \{1, 2, \dots, p\} \end{aligned}$$

with

$$\text{Im } \nu \cup \text{Im } \sigma = \{1, \dots, p\},$$

and also integers

$$k_0 \geq 0, \quad k_{\nu(1)} \geq 1, \quad \dots, \quad k_{\nu(l)} \geq 1, \quad k_{p+1} \geq 0$$

with

$$k_0 + k_{\nu(1)} + \dots + k_{\nu(l)} + k_{p+1} = k$$

and such that

$$x = \underbrace{(-1, \dots, -1)}_{k_0} \underbrace{u_{\nu(1)}, \dots, u_{\nu(1)}}_{k_{\nu(1)}} \underbrace{u_{\nu(l)}, \dots, u_{\nu(l)}}_{k_{\nu(l)}} \underbrace{1, \dots, 1}_{k_{p+1}}$$

and

$$y = (u_{\sigma(1)}, \dots, u_{\sigma(n-k-1)}).$$

For $l = 0$ the mapping ν does not exist; we assume that $\text{Im } \nu = \emptyset$.

Let

$$\omega(t, u) = (t+1)^{k_0} \prod_{i=1}^l (t - u_{\nu(i)})^{k_{\nu(i)}} (t-1)^{k_{p+1}}.$$

For $n - k = 2m$ we put

$$f(t, u) = \prod_{i=1}^m (t - u_{\sigma(2i-1)})^2,$$

$$g(t, u) = (1 - t^2) \prod_{i=1}^{m-1} (t - u_{\sigma(2i)})^2,$$

and for $n - k = 2m + 1$ we put

$$f(t, u) = (1 + t) \prod_{i=1}^m (t - u_{\sigma(2i)})^2,$$

$$g(t, u) = (1 - t) \prod_{i=1}^m (t - u_{\sigma(2i-1)})^2.$$

We denote

$$\Omega(t, u, r) = \frac{r+1}{2} \omega(t, u) f(t, u) + \frac{r-1}{2} \omega(t, u) g(t, u)$$

and, for $i = 0, \dots, p$, introduce the polynomials

$$F_i(u, r) = \int_{u_i}^{u_{i+1}} \Omega(t, u, r) dt$$

in the variables $r, u_1, \dots, u_p$. Here $u_0 = -1$ and $u_{p+1} = 1$.

Consider the mapping

$$F(u, r) = \begin{bmatrix} F_0(u, r) \\ F_1(u, r) \\ \vdots \\ F_p(u, r) \end{bmatrix}.$$

We claim that the determinant

$$D(u, r) = \det[F(u, r), \partial_u F(u, r)]$$

is nonsingular for all $(u, r) \in R^p \times R$ such that $-1 < u_1 < \dots < u_p < 1$ and $|r| > 1$.

For definiteness, suppose that $r > 1$. The case where $r < -1$ is treated in a similar way. Put

$$\alpha = \frac{r+1}{2} \quad \text{and} \quad \beta = \frac{r-1}{2};$$

then $\alpha > 1$ and $\beta > 0$.

Let $J_0 = \text{Im } \sigma \setminus \text{Im } \nu$. For $n - k = 2m$ we define

$$J_1 = \sigma\{1, 3, \dots, 2m-1\} \setminus \text{Im } \nu,$$

$$J_2 = \sigma\{2, 4, \dots, 2m-2\} \setminus \text{Im } \nu,$$

and for $n - k = 2m + 1$ we define

$$J_1 = \sigma\{2, 4, \dots, 2m\} \setminus \text{Im } \nu,$$

$$J_2 = \sigma\{1, 3, \dots, 2m-1\} \setminus \text{Im } \nu.$$

It is convenient to index the rows and columns of a $(p+1) \times (p+1)$ -matrix by $i, j = 0, 1, \dots, p$. In other cases we use other ways of indexing.

We write the matrix under study as a sum:

$$[F(u, r), \partial_u F(u, r)] = A(u, r) + B(u, r).$$

The matrix $A(u, r)$ arises as a result of differentiation under the integral sign. The elements of $A(u, r)$ are

$$A_{i,0} = \int_{u_i}^{u_{i+1}} \Omega(t, u, r) dt$$

for $0 \leq i \leq p$, and

$$A_{i,j} = \int_{u_i}^{u_{i+1}} \frac{\partial \Omega(t, u, r)}{\partial u_j} dt$$

for $0 \leq i \leq p$ and $1 \leq j \leq p$. The matrix $B(u, r)$ arises as a result of differentiation with respect to the limits of integration. All elements of $B(u, r)$ are equal to zero except for the superdiagonal and diagonal elements

$$\begin{aligned} B_{i-1,i} &= \Omega(u_i, u, r), \\ B_{i,i} &= -\Omega(u_i, u, r) \end{aligned}$$

with $i \in J_0$. The presence of $B(u, r)$ makes the calculation of the determinant $D(u, r)$ fairly bulky.

For $J \subset J_0$, let $A_J(u, r)$ be the matrix obtained from $A(u, r)$ by replacing its columns indexed by $j \in J$ with the corresponding columns of $B(u, r)$. Calculating the determinant, we get

$$D(u, r) = \sum_{J \subset J_0} \det A_J(u, r).$$

We claim that for all $-1 < u_1 < \dots < u_p < 1$ and all $r > 1$ the determinants $\det A_J(u, r)$ have one and the same sign.

We put

$$W(t, u, r) = \frac{1}{\omega(t, u)} [\Omega(t, u, r), -\partial_u \Omega(t, u, r)].$$

Then the row $W(t, u, r)$ is of length $p+1$ and consists of the elements

$$\begin{aligned} W_0 &= \alpha f(t, u) + \beta g(t, u), \\ W_j &= \begin{cases} \frac{k'_j \alpha f(t, u) + k''_j \beta g(t, u)}{t - u_j} & \text{if } j \in \text{Im } \nu, \\ \frac{2\alpha f(t, u)}{t - u_j} & \text{if } j \in J_1, \\ \frac{2\beta g(t, u)}{t - u_j} & \text{if } j \in J_2. \end{cases} \end{aligned}$$

Here $k'_j, k''_j \geq k_j$.

Suppose $J \subset J_0$ has $p-q$ elements. Let $W_J(t, u, r)$ denote the row of length $q+1$ obtained from $W(t, u, r)$ by deleting the elements with indices $j \in J$.

We introduce the $(q+1) \times (q+1)$ -matrix

$$a_J(\tau, u, r) = \begin{bmatrix} W_J(\tau_0, u, r) \\ W_J(\tau_1, u, r) \\ \vdots \\ W_J(\tau_q, u, r) \end{bmatrix},$$

where $\tau = (\tau_0, \tau_1, \dots, \tau_q)$. In the matrix $a_J(\tau, u, r)$, the rows are indexed by $i = 0, 1, \dots, q$ and the columns by $j = \{0, 1, \dots, p\} \setminus J$.

Let $v = (v_1, \dots, v_q)$ be the vector obtained from the vector u by deleting the elements with indices $j \in J$. We put $v_0 = -1$ and $v_{q+1} = 1$.

We represent J as the union of κ integral intervals

$$J = \bigcup_{s=1}^{\kappa} \{j_s^1, j_s^1 + 1, \dots, j_s^2 - 1, j_s^2\},$$

assuming that $j_s^2 + 1 < j_{s+1}^1$ for all $s = 1, \dots, \kappa - 1$. Then the coordinates of v look like this:

$$(v_1, \dots, v_q) = (\dots, u_{j_1^1-1}, u_{j_1^1+1}, \dots, u_{j_2^1-1}, u_{j_2^1+1}, \dots, u_{j_\kappa^1-1}, u_{j_\kappa^1+1}, \dots).$$

We show that

$$\det A_J(u, r) = (-1)^p \prod_{i \in J} \Omega(u_i, u, r) \int_{v_0}^{v_1} \int_{v_1}^{v_2} \dots \int_{v_q}^{v_{q+1}} \prod_{i=1}^q \omega(\tau_i, u) \det a_J(\tau, u, r) d\tau.$$

To prove this identity, we eliminate the superdiagonal elements of the matrix $A_J(u, r)$ in the columns with indices $j \in J$. For this, for each $s = 1, \dots, \kappa$ we add the j_s^2 th row to the $(j_s^2 - 1)$ st row, the $(j_s^2 - 1)$ st row to the $(j_s^2 - 2)$ nd row, and so on. This process finishes when we add the j_s^1 th row to the $(j_s^1 - 1)$ st row. After this, we can factor out the integrals from the determinants of the diagonal minors.

We prove that $\det a_J(\tau, u, r) \neq 0$ for all τ such that

$$v_0 < \tau_0 < v_1 < \tau_1 < v_2 < \dots < v_q < \tau_q < v_{q+1}$$

and

$$\tau_i \notin \bigcup_{j \in J} u_j,$$

where $0 \leq i \leq q$.

Let τ satisfy these conditions. Assume that $\det a_J(\tau, u, r) = 0$. Since the columns of the matrix $a_J(\tau, u, r)$ are linearly independent, there exists a nontrivial collection C_j , where $j \in \{0, 1, \dots, p\} \setminus J$, such that the function

$$\Phi(t) = R(t)\alpha f(t, u) + Q(t)\beta g(t, u)$$

with

$$R(t) = C_0 + \sum_{j \in \text{Im } \nu} C_j \frac{k_j'}{t - u_j} + \sum_{j \in J_1 \setminus J} C_j \frac{2}{t - u_j}$$

and

$$Q(t) = C_0 + \sum_{j \in \text{Im } \nu} C_j \frac{k_j''}{t - u_j} + \sum_{j \in J_2 \setminus J} C_j \frac{2}{t - u_j}$$

vanishes at $q + 1$ points $\tau_0, \tau_1, \dots, \tau_q$. We see that $\alpha f(\tau_i, u) > 0$ and $\beta g(\tau_i, u) > 0$ for all $i = 0, 1, \dots, q$. Hence, from the identity $\Phi(\tau_i) = 0$ it follows that $R(\tau_i)Q(\tau_i) < 0$ for all $i = 0, 1, \dots, q$. Writing the rational function $R(t)Q(t)$ as a sum of partial fractions, we get

$$R(t)Q(t) = C_0^2 + \sum_{j=1}^q \frac{\zeta_j}{t - v_j} + \sum_{j \in \text{Im } \nu} C_j^2 \frac{k_j' k_j''}{(t - u_j)^2}$$

with some $\zeta_1, \dots, \zeta_q$. Thus, for the rational function

$$H(t) = \sum_{j=1}^q \frac{\zeta_j}{t - v_j}$$

we have $H(\tau_i) < 0$ for all $i = 0, 1, \dots, q$. We represent $H(t)$ in the form

$$H(t) = \frac{h(t)}{(t - v_1) \dots (t - v_q)}$$

with a polynomial $h(t)$ of degree at most $q-1$. Then $h(\tau_{i-1})h(\tau_i) < 0$ for all $i = 1, \dots, q$. Consequently, the polynomial $h(t)$ of degree at most $q-1$ has q roots. But $h(t)$ is not identically zero. This contradiction proves that $\det a_J(\tau, u, r) \neq 0$.

Now we check that $\det a_J(\tau, u, r) > 0$ for all τ in question. This is proved by reduction to a special case. First, we consider this special case.

Let $p = n-1$ and $k = 0$. In this case, the mapping ν does not exist, and the mapping

$$\sigma : \{1, 2, \dots, n-1\} \longrightarrow \{1, 2, \dots, n-1\}$$

is the identity. We claim that if

$$-1 = u_0 < \tau_0 < u_1 < \tau_1 < u_2 < \tau_2 < \dots < u_{n-1} < \tau_{n-1} < u_n = 1,$$

then $\det a_\emptyset(\tau, u, r) \neq 0$.

To prove this, for each $i = 1, 2, \dots, n-1$ we perform the following transformations, which preserve the sign of the determinant $\det a_\emptyset(\tau, u, r) \neq 0$. We let τ_i tend to u_{i+1} from the left, divide the i th row by $u_{i+1} - u_i$, and let u_i tend to u_{i+1} from the left. After that, all elements of the i th row will become zero except for a strictly positive element in the i th column. By continuity, we conclude that the determinant $\det a_\emptyset(\tau, u, r)$ is strictly positive.

Now we treat the general case. Let $1 \leq s \leq \kappa$, and let i be such that $u_{j_s^1-1} < \tau_i < u_{j_s^2+1}$. Then

$$u_{j_s^1-1} < u_{j_s^1} < \dots < u_{i_s} < \tau_i < u_{i_s+1} < \dots < u_{j_s^2} < u_{j_s^2+1}$$

for some i_s . We let the points $u_{j_s^1}, \dots, u_{i_s}$ tend to $u_{j_s^1-1}+0$ and the points $u_{i_s+1}, \dots, u_{j_s^2}$ to $u_{j_s^2+1}-0$. As above, we easily check that this limit preserves the relation $\det a_J(\tau, u, r) \neq 0$. Hence, it suffices to verify that $\det a_J(\tau, u, r) > 0$ for the vectors u with

$$u_{j_s^1} = \dots = u_{i_s} = u_{j_s^1-1}$$

and

$$u_{i_s+1} = \dots = u_{j_s^2} = u_{j_s^2+1}$$

for all $s = 1, \dots, \kappa$. Let u be such a vector, and let $\varphi(t, u)$ denote the greatest common divisor of the polynomials $f(t, u)$ and $g(t, u)$ in the variable t . We divide the i th row of the matrix $a_J(\tau, u, r)$ by $\varphi(\tau_i, u)$ and denote the resulting matrix by $b(\tau, u, r)$. Obviously,

$$\operatorname{sgn} \det a_J(\tau, u, r) = \operatorname{sgn} \det b(\tau, u, r).$$

This reduces the problem to the case where $J = \emptyset$. Here the role of $f(t, u)$ and $g(t, u)$ is played by

$$f_0(t, u) = \frac{f(t, u)}{\varphi(t, u)}$$

and

$$g_0(t, u) = \frac{g(t, u)}{\varphi(t, u)}.$$

Now we return to the initial notation and prove that $\det a_\emptyset(\tau, u, r) > 0$. Here the rows and columns of the matrix $a_\emptyset(\tau, u, r)$ are indexed by $i, j = 0, 1, \dots, p$.

Let $c(\tau, u, r)$ be the matrix obtained from $a_\emptyset(\tau, u, r)$ by deleting the rows and columns with indices $i, j \in \operatorname{Im} \nu$. Clearly, $\det c(\tau, u, r) \neq 0$.

Let $i \in \operatorname{Im} \nu$. Then the i th diagonal element of $a_\emptyset(\tau, u, r)$ is of the form

$$\frac{k'_i \alpha f(\tau_i, u) + k''_i \beta g(\tau_i, u)}{\tau_i - u_i},$$

and the numerator does not vanish at $\tau_i = u_i$. We multiply the i th row of the matrix $a_{\emptyset}(\tau, u, r)$ by

$$\frac{\tau_i - u_i}{k'_i \alpha f(\tau_i, u) + k''_i \beta g(\tau_i, u)}$$

and then let τ_i tend to $u_i + 0$. The result is a matrix $b(\tau, u, r)$ with the following rows. All elements of the row with index $i \in \text{Im } \nu$ are equal to zero except for the unit diagonal element. It is clear that

$$\det b(\tau, u, r) = \det c(\tau, u, r).$$

By continuity,

$$\text{sgn } \det a_{\emptyset}(\tau, u, r) = \text{sgn } \det c(\tau, u, r).$$

Up to notation, the matrix $c(\tau, u, r)$ is the same as that in the special case considered before. Consequently, $\det c(\tau, u, r) > 0$.

Thus, if $v_0 < \tau_0 < v_1 < \tau_1 < v_2 < \dots < v_q < \tau_q < v_{q+1}$ and

$$\tau_i \notin \bigcup_{j \in J} u_j,$$

where $0 \leq i \leq q$, then $\det a_J(\tau, u, r) > 0$. This means that the sign of $\det A_J(u, r)$ does not depend on the choice of $J \subset J_0$. Indeed, the identity

$$\text{sgn} \left\{ \prod_{i \in J} \Omega(u_i, u, r) \prod_{i=1}^q \omega(\tau_i, u) \right\} = \text{sgn} \prod_{i=0}^p \omega \left(\frac{u_i + u_{i+1}}{2}, u \right)$$

shows that all determinants $\det A_J(u, r)$ are nonsingular and have one and the same sign. Thus, $D(u, r) \neq 0$.

In order to check that the mapping

$$\mathcal{L}_Q : e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n) \longrightarrow e(0, \delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$$

has no singular points, we introduce

$$H(u, r) = \frac{1}{r} \frac{F(u, r)}{S(u, r)},$$

where

$$S(u, r) = \sum_{i=1}^{p+1} \delta_{m_i} F_{i-1}(u, r),$$

and prove that

$$\det \partial_{(r,u)} H(u, r) \neq 0$$

if $-1 < u_1 < \dots < u_p < 1$ and $r > 1$.

Differentiating, we get

$$\partial_{(r,u)} H = \left[-\frac{1}{r^2} \frac{F}{S} + \frac{1}{rS} \left(\partial_r F - \frac{F}{S} \partial_r S \right), \frac{1}{rS} \left(\partial_u F - \frac{F}{S} \partial_u S \right) \right].$$

In this matrix, the comma separates the first column. Since

$$\det \left[\partial_r F - \frac{F}{S} \partial_r S, \partial_u F - \frac{F}{S} \partial_u S \right] = \det \begin{bmatrix} \frac{1}{S} & \partial_r S & \partial_u S \\ \frac{F}{S} & \partial_r F & \partial_u F \end{bmatrix} = 0,$$

we have

$$\det \partial_{(r,u)} H(u, r) = -\frac{1}{r^{p+2} S^{p+1}} \det \left[F, \partial_u F - \frac{F}{S} \partial_u S \right].$$

Now it suffices to observe that

$$\det \left[F, \partial_u F - \frac{F}{S} \partial_u S \right] = \det [F, \partial_u F] \neq 0.$$

Consider the mapping $\mathcal{L}_Q$ on the cell $e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$. We write a point $\lambda \in e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$ in the form

$$\lambda = \sum_{i=1}^{p+1} \lambda_{m_i} \delta_{m_i} E_{m_i}^n$$

and introduce the more convenient coordinates

$$\begin{aligned} u_1 &= \lambda_{m_1}, \\ u_2 &= \lambda_{m_1} + \lambda_{m_2}, \\ &\vdots \\ u_p &= \lambda_{m_1} + \lambda_{m_2} + \dots + \lambda_{m_p}. \end{aligned}$$

Clearly, the change of variables

$$(\lambda_{m_1}, \dots, \lambda_{m_{p+1}}) \longmapsto (u_1, \dots, u_p)$$

is nonsingular.

Recalling the earlier notation, we introduce the polynomials

$$F_i(u) = \int_{u_i}^{u_{i+1}} \Omega(t, u, \delta_n) dt$$

in the variables $u_1, \dots, u_p$. Observe that

$$\Omega(t, u, \delta_n) = \begin{cases} \omega(t, u) f(t, u) & \text{if } \delta_n = 1, \\ -\omega(t, u) g(t, u) & \text{if } \delta_n = -1. \end{cases}$$

Consider the mapping

$$F(u) = \begin{bmatrix} F_0(u) \\ F_1(u) \\ \vdots \\ F_p(u) \end{bmatrix}.$$

From the proof of the preceding theorem it follows that the determinant

$$D(u) = \det[F(u), \partial_u F(u)]$$

is nonsingular for all $u \in \text{Int } \nabla^p$.

Now, let

$$H(u) = \frac{1}{S(u)} \begin{bmatrix} F_1(u) \\ \vdots \\ F_p(u) \end{bmatrix},$$

where

$$S(u) = \sum_{i=1}^{p+1} \delta_{m_i} F_{i-1}(u).$$

It is easy to check that

$$\det \partial_u H(u) \neq 0$$

for all $u \in \text{Int } \nabla^p$. Consequently, the mapping

$$\mathcal{L}_Q : e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n) \longrightarrow e(\delta_{m_1} E_{m_1}^n, \dots, \delta_{m_{p+1}} E_{m_{p+1}}^n)$$

has no singular points. □

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