SOME GEOMETRIC PROPERTIES OF CONVEX BODIES. II

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ABSTRACT. Topological means are used for the study of approximation of 2-dimensional sections of a 3-dimensional convex body by affine-regular pentagons and approximation of a centrally symmetric convex body by a prism. Also, the problem of estimating the relative surface area of the sphere in a normed 3-space, the problem on universal covers for sets of unit diameter in Euclidean space, and some related questions are considered.

Throughout, by a *convex body* $K \subset \mathbb{R}^n$ (a *figure* for $n = 2$) we mean a compact convex subset of $\mathbb{R}^n$ with nonempty interior.

We denote by $G_k(\mathbb{R}^n)$ (respectively, $G_k^+(\mathbb{R}^n)$) the Grassmann manifold of nonoriented (respectively, oriented) k -planes in $\mathbb{R}^n$ passing through $O \in \mathbb{R}^n$. We let

$$\gamma_k^n : E_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n) \quad \text{and} \quad (\gamma_k^n)^+ : E_k^+(\mathbb{R}^n) \rightarrow G_k^+(\mathbb{R}^n)$$

be the tautological fiber bundles, where the fiber over an (oriented) k -plane $\alpha \in G_k(\mathbb{R}^n)$ is α itself regarded as a k -dimensional vector space.

We say that a *field of convex bodies* (or *figures*; a *CB-field* or a *CF-field*) is given in a vector bundle γ if in each fiber α of γ we mark a convex body $K(\alpha)$ depending continuously on α . A CB-field is *pointed* if for each α we also mark a point $x(\alpha) \in K(\alpha)$ depending continuously on α . (In other words, $x(\alpha)$ is a section of γ .)

If $\lambda \in \mathbb{R}$ and P is an affine-regular polygon (e.g., a pentagon or a parallelogram) with center $O(P)$, then λP denotes the polygon homothetic to P with homothety ratio λ and homothety center $O(P)$.

We denote by $S(K)$ the area of a figure $K \subset \mathbb{R}^2$.

§1. FIELDS OF CONVEX FIGURES IN γ_2^3 AND $(\gamma_2^3)^+$, AND 2-DIMENSIONAL SECTIONS OF CONVEX BODIES IN $\mathbb{R}^3$

First, we prove two corollaries to the following known result.

Theorem [2]. *Each CF-field in γ_2^3 contains a figure circumscribed about an affine-regular octagon.*

Corollary 1. *Suppose C is the bounded component of a cubic surface in $\mathbb{R}^3$. Then each inner point O of C lies in a plane intersecting C along an ellipse.*

Proof. Indeed, C is convex automatically, because otherwise C intersects some line at 4 points. Consequently, the section of C by some 2-plane through O is circumscribed about an affine-regular octagon. Then this section is a component of a cubic, intersects an ellipse at 8 points, and, consequently, is an ellipse by the Bézout theorem. $\square$

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Remark. Certainly, this follows from standard facts of algebraic geometry: any plane that passes through a point inside C and a real line lying on the cubic surface intersects C along an ellipse. But actually we have proved more: each field of bounded components of cubics in γ_2^3 contains an ellipse.

Corollary 2. *Each CF-field in γ_2^3 contains a figure K containing a parallelogram P such that*

$$P \subset K \subset \left(1 + \frac{\sqrt{2}}{2}\right)P.$$

Proof. Indeed, the figure K circumscribed about an affine-regular octagon $\Omega = A_1 \dots A_8$ possesses the required property. In this case, K is contained in the octagonal star Σ bounded by segments of the rays that extend the sides of Ω . Letting P be the parallelogram $A_2A_4A_6A_8$, we easily see that $P \subset K \subset \Sigma \subset (1 + \frac{\sqrt{2}}{2})P$. $\square$

Remark. Considering fields of disks, we see that $1 + \frac{\sqrt{2}}{2}$ cannot be replaced by a constant smaller than $\sqrt{2}$.

Theorem 1. *Each pointed CF-field $(K(\alpha), x(\alpha))$ in $(\gamma_2^3)^+$ contains a figure $K(\alpha)$ circumscribed about an affine-regular pentagon with center at the marked point $x(\alpha)$.*

Corollary 3. *Each field of centrally symmetric convex figures in $(\gamma_2^3)^+$ or γ_2^3 contains a figure circumscribed about an affine-regular decagon.* $\square$

Corollary 4. *If a field of centrally symmetric convex figures in $(\gamma_2^3)^+$ consists of bounded components of curves of degree at most 4, then the field contains an ellipse.*

Indeed, by the Bézout theorem, the figure circumscribed about an affine-regular decagon is an ellipse. $\square$

Corollary 5. *Each convex bounded centrally symmetric quartic in $\mathbb{R}^3$ has a planar central section that is an ellipse.* $\square$

The proof of Theorem 1 uses two topological lemmas.

Lemma 1. *Suppose $n \in \mathbb{N}$, W is a compact oriented $2n$ -manifold, the cyclic group $\mathbb{Z}_{2n+1}$ freely acts on W , and the boundary ∂W of W with standard orientation is the union of two closed $\mathbb{Z}_{2n+1}$ -invariant manifolds M and M' . We let $\mathbb{Z}_{2n+1}$ act on $\mathbb{R}^{2n+1}$ by cyclic permutations of coordinates of points and denote by $l \subset \mathbb{R}^{2n+1}$ the line determined by the equations $x_1 = x_2 = \dots = x_{2n+1}$.*

Suppose that $\mathcal{F} : W \rightarrow \mathbb{R}^{2n+1}$ is a continuous $\mathbb{Z}_{2n+1}$ -equivariant mapping such that $\mathcal{F}(M)$ and $\mathcal{F}(M')$ do not intersect l . Then

$$\deg(\mathcal{F}| : M \rightarrow \mathbb{R}^{2n+1} \setminus l) + \deg(\mathcal{F}| : M' \rightarrow \mathbb{R}^{2n+1} \setminus l) \equiv 0 \pmod{2n+1}.$$

Proof. After a slight perturbation, we can assume that $\mathcal{F}$ is smooth and transversal to l . Then, $\mathcal{F}^{-1}(l)$ consists of a finite number of orbits of points of W . We surround the points by small balls $D_1, \dots, D_N$ such that they are mutually disjoint and disjoint from M and M' and, furthermore, the balls with centers in one $\mathbb{Z}_{2n+1}$ -orbit are mapped to one another under the action of $\mathbb{Z}_{2n+1}$.

The mapping $\mathcal{F}$ takes the $2n$ -manifold $\hat{W} := W \setminus \bigcup_{i=1}^N \overset{\circ}{D}_i$ with boundary to $\mathbb{R}^{2n+1} \setminus l \simeq S^{2n-1}$. Consequently, $\deg(\mathcal{F}| : \partial \hat{W} \rightarrow \mathbb{R}^{2n+1} \setminus l) = 0$. We have

$$\partial \hat{W} = M \cup M' \cup \bigcup_{i=1}^N \partial D_i.$$

By construction, the $\mathbb{Z}_{2n+1}$ -equivariance of $\mathcal{F}$ implies that $\deg(\mathcal{F}| : \bigcup_{i=1}^N \partial D_i \rightarrow \mathbb{R}^{2n+1} \setminus l)$ is divisible by $2n+1$, which completes the proof. $\square$

Remark. For prime $2n + 1$, Stiefel manifolds yield interesting examples in which the degree under consideration is nonzero (see [1] and below). The author does not know whether this is possible in the case where $2n + 1$ is composite.

We denote by $SO(3)$ the group of orientation-preserving rotations of $\mathbb{R}^3$ about some point.

Lemma 2. *Suppose W is a compact 4-manifold, the cyclic group $\mathbb{Z}_5$ acts freely on W , and the boundary ∂W of W with standard orientation is the union of two closed $\mathbb{Z}_5$ -invariant 3-manifolds M and M' , where M' is $SO(3)$ with standard action of $\mathbb{Z}_5$. We let $\mathbb{Z}_5$ act on $\mathbb{R}^5$ by cyclic permutations of the coordinates of points.*

Suppose that $\mathcal{F} : W \rightarrow \mathbb{R}^5$ is a continuous $\mathbb{Z}_5$ -equivariant mapping. Then $\mathcal{F}(M)$ contains a point of the form (x, x, x, x, y) , where $y \leq x$ (or, optionally, $y \geq x$).

Proof. Let $l \subset \mathbb{R}^5$ be the line determined by the equations $x_1 = x_2 = \cdots = x_5$. If $F(M) \cap l \neq \emptyset$, then we are done. Otherwise, it suffices to prove the following assertion (see [1]).

Assertion. *If $F(M) \cap l = \emptyset$, then the degree of the mapping*

$$F| : M \rightarrow \mathbb{R}^5 \setminus l \simeq S^3$$

is not divisible by 5.

Proof. After a slight perturbation, we can assume that the mapping $F' : M' \rightarrow \mathbb{R}^5$ is a $\mathbb{Z}_5$ -equivariant mapping with image in $\mathbb{R}^5 \setminus l$. It is well known that in this case $\deg(F' : M' \rightarrow \mathbb{R}^5 \setminus l)$ is not divisible by 5 (see [1]; actually, $\deg(F') = -1$). Now, the required result follows from Lemma 1. $\square$

Proof of Theorem 1. We consider positively oriented affine-regular pentagons inscribed in the figures $K(\alpha)$. If $K(\alpha)$ is a generic smooth field of smooth convex figures in $(\gamma_2^3)^+$, then the pentagons constitute a compact oriented smooth 3-manifold M , on which the cyclic group $\mathbb{Z}_5$ acts by cyclic permutations of the vertices of the pentagons.

If all figures $K(\alpha)$ are (C^1) -close to disks, then, obviously, $M \cong SO(3)$.

We define a continuous mapping

$$F : M \rightarrow \mathbb{R}^5$$

as follows. If $P \in \alpha$ is a pentagon $A_1 \dots A_5$ inscribed in $K(\alpha)$ and with center $O(P)$, then the i th coordinate of $F(P)$ is the orthogonal projection of $x(\alpha)$ to the oriented axis $O(P)A_i$ with origin at $O(P)$, $i = 1, \dots, 5$.

By construction, F is $\mathbb{Z}_5$ -equivariant if $\mathbb{Z}_5$ acts on $\mathbb{R}^5$ by cyclic permutations of the coordinates of points.

Using a smooth generic deformation $K_t(\alpha)$, $t \in [0, 1]$, we deform the initial field $K_0(\alpha) := K(\alpha)$ into a field of figures $K_1(\alpha)$ close to disks. Then the oriented affine-regular pentagons inscribed in the figures $K_t(\alpha)$ form a cobordism between M and $M' \cong SO(3)$. By Lemma 2, $F(M)$ contains a point $F(P) = (x, x, x, x, y)$. Simple geometric arguments show that $x = 0$, whence $O(P) = x(\alpha)$. $\square$

Theorem 2. *Each CF-field $K(\alpha)$ in $(\gamma_2^3)^+$ contains a convex figure K circumscribed about an affine-regular pentagon P and such that*

$$(*) \quad P \subset K \subset \left(1 + \frac{1}{2 \sin 54^\circ}\right) P \subset 1.6181 P.$$

This estimate is sharp for the field consisting of the sections of a tetrahedron T that pass through an inner point of T .

FIGURE 1.

Proof. If P is a pentagon $A_1 \dots A_5$ inscribed in $K(\alpha)$, then we draw the support lines of $K(\alpha)$ parallel to $A_1A_2, A_2A_3, \dots, A_5A_1$ and denote by $B_1, \dots, B_5$ the points where these support lines touch $K(\alpha)$. We define

$$F(P) := (S(\triangle A_1A_2B_1), \dots, S(\triangle A_5A_1B_5)).$$

As in the proof of Theorem 1, we see that there is a 2-plane α such that for a certain pentagon P inscribed in $K(\alpha)$ we have

$$f(P) = (x, x, x, x, y),$$

where $y \leq x$. We show that P is the required pentagon.

After an affine transformation, we may assume that $P = A_1 \dots A_5$ is a regular pentagon. Obviously, the altitude of the triangles $\triangle A_1A_2B_1, \dots, \triangle A_4A_5B_4$ is maximal possible if $S(\triangle A_5A_1B_5) = 0$, and the figure $K(\alpha)$ circumscribed about P is the isosceles trapezoid shown in Figure 1.

Simple calculations show that in this case we have (*).

Remark. From [11] it follows that each CF-field in $(\gamma_2^3)^+$ contains a figure K circumscribed about a *regular* pentagon P , in which case we have

$$P \subset K \subset (1 + \sqrt{5})P.$$

(Certainly, the same is true for any *affine-regular* pentagon inscribed in K .) In the general case, the constant $1 + \sqrt{5} = 1 + \tan(\pi/5) \cot(\pi/10)$ here cannot be made smaller. Indeed, suppose that the point O lies near the apex of a regular triangular pyramid T the lateral edge of which is many times longer than the edge of the base. We consider the CF-field in $(\gamma_2^3)^+$ consisting of sections of T by planes through O . In this case, the sections intersecting the base of T are very prolate, while for the other triangular sections the above constant, obviously, cannot be improved, because two sides of a pentagon inscribed into a triangle lie on the sides of the triangle.

If in Theorem 2 and in the situation considered above we lift the condition that the pentagon P is inscribed, then the sharp values of the constants are not known to the author.

§2. THE RELATIVE SURFACE AREA OF THE SPHERE IN A NORMED 3-SPACE

Definition. Suppose P is a polyhedron in a finite-dimensional normed space with unit ball K . For each hyperface F of P , we take the ratio of the area of F and the area of the central section of K parallel to F . The sum of these ratios over all hyperfaces of P is the *relative surface area* of P .

In [10], the author constructed a one-parameter family of affine images of a cube-octahedron which are inscribed in the unit ball of a normed 3-space and have relative surface area of at least $5/2$.

FIGURE 2.

To obtain an upper estimate, we approximate the ball of unit diameter in a normed 3-space by a circumscribed hexagonal prism.

We need the following result.

Theorem [10]. *Suppose $\Pi = A_1 \dots A_{12}$ is a regular hexagonal prism, and A_{13} and A_{14} are points that lie outside Π on the symmetry axis and are symmetric to each other with respect to the center. Each centrally symmetric convex body $K \subset \mathbb{R}^3$ centered at O is circumscribed about an affine image of the 14-tope $P_{14} = A_1 \dots A_{14}$ with the same center and such that the parallel support planes of K at the images of A_{13} and A_{14} are parallel to the images of the base planes of Π .*

Corollary 6. *Each centrally symmetric convex body $K \subset \mathbb{R}^3$ is circumscribed about an affine-regular hexagonal bipyramid $A_1 \dots A_8$ with the same center and such that the support planes of K at A_7 and A_8 are parallel to the plane of the base $A_1 \dots A_6$, while the support planes of K at $A_1, \dots, A_6$ are parallel to the axis A_7A_8 of the bipyramid.*

Proof. We apply the above theorem. As the lateral edge of Π in P_{14} tends to zero while the length of $A_{13}A_{14}$ remains constant, in the limit we obtain the required bipyramid. $\square$

Theorem 3. *The unit ball K in a normed 3-space is inscribed in a centrally symmetric hexagonal prism with relative surface area not exceeding $\frac{32}{3}$.*

Corollary 7. *The relative surface area of K is at most $32/3$.* $\square$

Remark. If K is a Euclidean ball, then the regular prism is a hexagonal prism circumscribed about K with minimal relative surface area, which is equal to $12\sqrt{3}/\pi$.

The proof involves the following lemma.

Lemma 2. *Suppose that the unit disk in a 2-dimensional normed space is an affine-regular hexagon H . Then the area of any centrally symmetric convex figure K circumscribed about H is at most $\frac{4}{3}S(H)$, and the perimeter of K (with respect to the norm) is at most 8. Both estimates are sharp.*

Proof. Drawing the support lines of K at the vertices of H , we reduce the proof to the case where K is a centrally symmetric hexagon.

1) Simple variational arguments show that the area of K is maximal if the vertices of H are the midpoints of the sides of K , or K is a parallelogram. In both cases, we have $S(K) = \frac{4}{3}S(H)$.

2) Let ED and $E'D'$ be a pair of parallel sides of K , and let CC' be the diameter of H parallel to them (see Figure 2). Continuing the opposite sides AB and $A'B'$ that contain (respectively) C and C' , to the intersection with the lines ED and $E'D'$, we obtain a

parallelogram $FGF'G'$ circumscribed about H . Obviously,

$$\frac{ED + E'D'}{OC} = 8 \frac{S(OED) + S(OE'D')}{S(FGF'G')} \leq 8 \frac{S(OED) + S(OE'D')}{S(K)}.$$

Writing this inequality for each pair of parallel sides of K and summing up, we obtain the assertion of the lemma concerning the perimeter of K . $\square$

Remark. If in the lemma we do not assume the central symmetry of K , then it is well known that $S(K) \leq \frac{3}{2}S(H)$, and arguments similar to ours show that the perimeter of K is at most 9.

Proof of Theorem 3. We assume that K is smooth. In the general case, the result is obtained by passage to the limit.

Let $A_1 \dots A_8$ be the affine-regular hexagonal bipyramid constructed in Corollary 6. Obviously, the support planes of K at $A_1, \dots, A_8$ bound a hexagonal prism Π with centrally symmetric base.

1) The relative areas of the bases of Π are at most $4/3$ by Lemma 2, because they are circumscribed about the affine-regular hexagon $A_1 \dots A_6$ with unit side, which is inscribed in the central section of K parallel to the bases.

2) Suppose P is one of the lateral faces of Π . Then the area of the parallelogram P does not exceed the length of the base of P . (Indeed, the area of the central section of K parallel to P cannot be smaller than the area of the parallelogram P' having the same directions of sides and such that the lateral side of P' is equal to that of P , while the base of P is a unit radius of the central section.)

Thus, by Lemma 2, the relative area of the lateral surface of Π is at most 8, and, consequently, the complete relative surface area of Π is at most $8 + 2 \cdot \frac{4}{3} = \frac{32}{3}$. $\square$

Theorem 4. *Each convex body $K \subset \mathbb{R}^3$ is circumscribed about an affine-regular hexagonal bipyramid $A_1 \dots A_8$ such that the support planes of K at A_7 and A_8 are parallel to the plane of the base $A_1 \dots A_6$.*

Proof. We prove this theorem for strictly convex smooth bodies K ; in the general case, the theorem is obtained by passage to the limit.

For $\alpha \in G_2(\mathbb{R}^3)$, we draw the support planes α_1 and α_2 of K parallel to α , and also the secant plane α_3 equidistant from α_1 and α_2 and parallel to them. We join the points of tangency of α_1 and α_2 with K by a segment I and denote by $A(\alpha)$ the orthogonal projection of the point $I \cap \alpha_3$ to the plane α . Let $B(\alpha)$ be the orthogonal projection to α of the set of the centers of the affine-regular hexagons inscribed in $\alpha_3 \cap K$.

It suffices to prove that for some $\alpha \in G_2(\mathbb{R}^3)$ we have $A(\alpha) \in B(\alpha)$. By construction,

$$C_1 := \{A(\alpha) \mid \alpha \in G_2(\mathbb{R}^3)\}$$

is the image of a section of γ_2^3 that realizes the generator of $H_2(E_2(\mathbb{R}^3); \mathbb{Z}_2)$.

We denote by Ω the 8-manifold of affine-regular hexagons lying in the planes α_3 . If K is generic, then the affine-regular hexagons inscribed into all possible sections $\alpha_3 \cap K$ constitute a compact smooth 2-manifold $\mathcal{H}$ in Ω . Then

$$C_2 := \bigcup \{B(\alpha) \mid \alpha \in G_2(\mathbb{R}^3)\}$$

is a continuous image of $\mathcal{H}$ that intersects a generic fiber $\alpha \in E_2(\mathbb{R}^3)$ at an odd number of points, because a generic convex figure is circumscribed about an odd number of affine-regular hexagons (see [2]). Thus, C_2 also realizes the generator of $H_2(E_2(\mathbb{R}^3); \mathbb{Z}_2)$. Consequently, the $\mathbb{Z}_2$ intersection number of C_1 and C_2 is equal to 1, whence $C_1 \cap C_2 \neq \emptyset$. $\square$

Remark. It is easily seen that the volume of any inscribed affine-regular bipyramid in Theorem 4 is at least $\text{Vol}(K)/6$.

§3. UNIVERSAL COVERS FOR SETS OF UNIT DIAMETER IN EUCLIDEAN SPACE

Definitions and examples. A subset $A \subset \mathbb{R}^n$ is a *universal cover* for sets of unit diameter if each subset of $\mathbb{R}^n$ of diameter not exceeding 1 is contained in a congruent image of A .

A universal cover A is *rigid* if the set of bodies of constant unit width and such that each of them is contained in finitely many congruent images of A is dense in the Hausdorff metric.

Rigid covers are well known in dimensions 2 and 3. Each centrally symmetric hexagon of unit width is a rigid cover in dimension 2. Borsuk's solution of the Borsuk problem in the plane involved the cover having the form of a regular hexagon of unit width (see [3]).

A regular rhombo-dodecahedron of unit width is a rigid cover in 3-space (see [4–7]). The one-parameter families of rigid covers constructed in [7, 8] consist of centrally symmetric dodecahedra circumscribed about a ball of unit diameter. Is it true that each universal cover contains a rigid universal cover?

A universal cover A is an *s-cover* if there is a C^1 open and dense set of smooth bodies having constant unit width and such that each of them is contained in an odd number of congruent images of A .

By definition, all *s-covers* are rigid. All rigid covers mentioned above are *s-covers*. For $n \geq 3$, the author knows no examples of rigid covers that are not *s-covers*.

The following theorem yields infinite series of *s-covers* in Euclidean spaces.

Theorem 5. *Suppose that $A_n \subset \mathbb{R}^n$ is a centrally and mirror-symmetric s-cover bounded by a finite number of regular hypersurfaces. Let Π be the right prism in $\mathbb{R}^{n+1}$ with base A_n and unit height, and let A_{n+1} denote the intersection of Π with two unit balls centered at the centers of the bases of Π . Then A_{n+1} is an s-cover in $\mathbb{R}^{n+1}$.*

Proof. Suppose that $K \subset \mathbb{R}^{n+1}$ is a generic smooth body of constant unit width. In the total space $E_n(\mathbb{R}^{n+1})$ of γ_n^{n+1} , we construct two n -dimensional cycles C_1 and C_2 intersecting the generic fiber at an odd number of points.

For $\alpha \in G_n(\mathbb{R}^{n+1})$, we denote by $s(\alpha)$ the point of intersection of α with the line containing the diameter of K perpendicular to α . Obviously, s is a section of γ_n^{n+1} , and its image $C_1 = s(G_n(\mathbb{R}^{n+1}))$ intersects each fiber at a unique point.

Consider the fiber bundle $\xi : E(\xi) \rightarrow G_n(\mathbb{R}^{n+1})$ such that the fiber over a hyperplane $\alpha \in G_n(\mathbb{R}^{n+1})$ is the set of the congruent images of A_n that lie in α . (By the mirror symmetry of A_n , no problems with orientation arise.) For $\alpha \in G_n(\mathbb{R}^{n+1})$, let B_α be the set of the congruent images of A_n that contain the orthogonal projection of K to α , and let $B = \bigcup_\alpha B_\alpha \subset E(\xi)$. If K is a generic smooth body, then B is a smooth compact n -manifold in $E(\xi)$ intersecting the generic fiber at an odd number of points.

We consider the fiberwise mapping $p : E(\xi) \rightarrow E_n(\mathbb{R}^{n+1})$ that takes the congruent image of A lying in α to its center in α . Then $C_2 = p(B)$ is an n -dimensional cycle in $E_n(\mathbb{R}^{n+1})$ intersecting the generic fiber at an odd number of points. Consequently, C_2 is $\mathbb{Z}_2$ -homologous to C_1 .

As before, the $\mathbb{Z}_2$ intersection number of C_1 and C_2 in $E_n(\mathbb{R}^{n+1})$ is nonzero, i.e., in the generic situation they intersect at an odd number of points, which precisely means that the initial body K is contained in an odd number of congruent images of A_{n+1} . $\square$

Corollary 8 (to the proof). *Suppose $A_n \subset \mathbb{R}^n$ is a centrally and mirror-symmetric s-cover bounded by a finite number of regular hypersurfaces. Then each pointed CB-field $(K(\alpha), x(\alpha))$ of constant unit width in γ_n^{n+1} contains a body $K(\alpha)$ that is contained*

in a congruent image of A_n centered at $x(\alpha)$. If $K(\alpha)$ is a generic field consisting of smooth convex bodies of constant unit width, then the number of such fibers (and covers) is odd. $\square$

Remarks. 1. The s -cover A_{n+1} itself satisfies the assumptions of Theorem 5, which allows us to use it for constructing an s -cover $A_{n+2} \subset \mathbb{R}^{n+2}$, etc.

2. If we take one of the 1-, 2-, or 3-dimensional covers mentioned above as a “basis” cover, then Theorem 5 yields an infinite series of s -covers that are intersections of half-spaces, cylinders, and spheres of unit diameter.

3. In the text above, we presented polygonal and polyhedral s -covers in dimensions ≤ 3 . The author does not know of any polyhedral s -covers in dimensions 4, 5, $\dots$.

In [9], it was proved that each centrally symmetric 14-hedron P circumscribed about a ball of unit diameter is a universal cover. However, these covers are certainly not rigid, because for each P and each body K of constant unit width in $\mathbb{R}^4$ there is a 3-parametric family of congruent images of P each of which is circumscribed about K .

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