Алгебра и анализ
Том. 16 (2004), вып. 2

St. Petersburg Math. J.
Vol. 16 (2005), No. 2, Pages 341–348
S 1061-0022(05)00853-8
Article electronically published on March 9, 2005

ON SPACES OF POLYNOMIAL GROWTH WITH NO CONJUGATE POINTS

N. D. LEBEDEVA

Abstract. The following generalization of the Hopf conjecture is proved: if the fundamental group of an n-dimensional compact polyhedral space M without boundary and with no conjugate points has polynomial growth, then there exists a finite covering of M by a flat torus.

§1. Introduction

By an *n-dimensional polyhedral space* we mean a metric space M (with an inner metric) covered by n-simplexes; each simplex is endowed with a smooth Riemannian metric, and these metrics coincide on the common $(n-1)$-faces of the n-simplexes. The precise definition is given at the end of this section. In the definitions below, it is assumed that we deal with a fixed triangulation.

A *polyhedral pseudomanifold* is an n-dimensional polyhedral space in which the $(n-1)$-simplexes of the triangulation are adjacent to at most two n-simplexes. The *boundary* of a polyhedral space is the union of the $(n-1)$-simplexes of the triangulation that are adjacent to only one n-simplex. We say that *M has no conjugate points* if any two points in the universal covering space of M are connected by a unique geodesic. All polyhedral spaces considered in this paper are assumed to be connected.

Let M be a compact polyhedral space without boundary and with no conjugate points. It is well known that M is isometric to the quotient space $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is the universal covering space of M, and Γ is a subgroup of the group of isometries of $\widetilde{M}$; recall that Γ is isomorphic to $\pi_1(M)$.

Our aim in this paper is to prove the following two theorems.

Theorem 1. *Let M be an n-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group $\pi_1(M)$ of M is nilpotent, then M is a flat torus.*

Theorem 2. *Let M be an n-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group $\pi_1(M)$ of M is of polynomial growth, then there exists a finite covering of M by a flat torus.*

Theorem 2 can be derived from Theorem 1. Indeed, let M satisfy the assumptions of Theorem 1. Then $\pi_1(M)$ is of polynomial growth. The well-known result by Gromov (see [G2]) says that $\pi_1(M)$ is virtually nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup G of finite index. Consequently, there exists a finite covering $\overline{M} \to M$ such that $\pi_1(\overline{M}) = G$.

2000 *Mathematics Subject Classification.* Primary 57N16.
Key words and phrases. n-dimensional polyhedral space, polyhedral pseudomanifold, fundamental group.
Partially supported by RFBR (grant no. 02-01-00090), by CRDF (grant no. RM1-2381-ST-02), and by SS (grant no. 1914.2003.1).

Since $\overline{M}$ is a compact polyhedral space without boundary and with no conjugate points, $\overline{M}$ is flat by Theorem 1. In the remaining part of the paper we prove Theorem 1. The proof is organized as follows.

In §2 we prove that M^n is a pseudomanifold and that it is homotopy equivalent to an n-dimensional torus.

In §3 we construct a map $f : M \to T^n$, where T^n is a flat torus. We show that f is a local isometry on the complement of the $(n-2)$-skeleton of M. This step of the proof is similar to a version of the proof of the Hopf conjecture (see [I]). For the first time, the Hopf conjecture was proved by D. Burago and S. Ivanov in [BI].

In §4 we prove that the map $f : M \to T^n$ is an isometry. In contrast to the case of Riemannian manifolds considered in [I], this step is not trivial for Riemannian polyhedra.

Now we explain more precisely what we mean by polyhedral spaces.

An *n-dimensional Riemannian simplex* is an n-simplex in $\mathbb{R}^n$ equipped with a smooth Riemannian metric (as usual, we assume that the metric is defined in a neighborhood of this simplex), as well as any metric space isometric to such a simplex.

An *n-dimensional polyhedral space* is a connected metric space that can be obtained by gluing together n-dimensional Riemannian simplexes along some isometries between their faces.

§2. Homotopy type of M

In the proof of Theorem 1 we use the following results obtained earlier (see [L1, L2]).

Claim 1 ([L1]). *Let M be a compact locally simply connected space without conjugate points. Then every nilpotent subgroup of the fundamental group of M is Abelian and torsion free.*

Claim 2 ([L2]). *Let M be an n-dimensional compact polyhedral space without boundary and with no conjugate points. If the triangulation of M contains three n-simplexes with a common $(n-1)$-face, then the fundamental group $\pi_1(M)$ is of exponential growth.*

Our aim in this section is to prove the following auxiliary statement.

Lemma 1. *Let M be as in Theorem* 1. *Then M is a pseudomanifold that is homotopy equivalent to an n-dimensional torus.*

Proof. Since the fundamental group of a compact metric space with intrinsic metric is finitely generated, from Claim 1 it follows that $\pi_1(M) = \mathbb{Z}^m$ for some m. Applying Claim 2, we see that at most two n-simplexes of M may have a common $(n-1)$-face, i.e., M is a pseudomanifold. Since the universal covering space of M is contractible, the fundamental group of M determines the homotopy type of M. Hence, M is homotopy equivalent to an m-torus T^m. It follows that $H_k(M, \cdot) = H_k(T^m, \cdot)$ for every k.

We prove that $m = n$, where n is the dimension of M.

Suppose that $n > m$. Since M is a pseudomanifold, we have $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$. This contradicts the relation $H_n(T^m, \mathbb{Z}_2) = 0$.

Suppose $n < m$; then $H_m(M, \mathbb{Z}) = 0$. This contradicts the relation $H_m(T^m, \mathbb{Z}) = \mathbb{Z}$. Thus, $\pi_1(M) = \mathbb{Z}^n$. □

§3. Constructing a local isometry

We denote by M' the complement of the $(n-2)$-skeleton of M; then M' is an open dense subset of M. In this section we shall prove the following statement.

Proposition 1. *Under the assumptions of Theorem* 1, *there exists a map $f : M \to T^n$, where T^n is a flat n-torus, with the following properties*:

(1) $f|_{M'}$ *is a local isometry on* M', *i.e.,* $f|_{M'}$ *is an open map preserving distances*;
(2) f *is Lipschitz*;
(3) f *induces an isomorphism between the corresponding fundamental groups.*

We start with several lemmas.

Let SM denote the space of all unit tangent vectors of M. A canonical measure μ_L on the space SM is defined in a standard way as the product of two measures: the normalized Riemannian volume on M and the normalized Riemannian volume on the unit $(n-1)$-sphere. This measure is called the *Liouville measure*.

Since for almost every unit vector $e \in SM$ there exists a unique generic geodesic γ with $\gamma'(0) = e$ (see [L1]), the geodesic flow transformation is well defined almost everywhere on SM, and it is known that the Liouville measure is invariant with respect to this transformation (see [L1]).

We recall that M is isometric to the quotient space $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is the universal covering space of M and Γ is a deck transformation group isomorphic to $\pi_1(M) = \mathbb{Z}^n$ and acting by isometries on $\widetilde{M}$.

Consider the vector space $V = \Gamma \otimes \mathbb{R}$; it is isomorphic to $\mathbb{R}^n$. There exists a canonical immersion of $\Gamma = \mathbb{Z}^n \hookrightarrow V$, and its image is an integral lattice in $V = \mathbb{R}^n$. Below we shall denote elements of Γ and the corresponding points of the lattice by the same symbol. Fix a point $x_0 \in \widetilde{M}$. The orbit of Γ is a lattice in $\widetilde{M}$; there is a one-to-one correspondence between the points of the lattice and the elements of Γ. For $k \in \Gamma$ and $x \in \widetilde{M}$, we denote by $x + k$ the image of x under the isometry k. When studying distances between remote points, it is convenient to approximate points of $\widetilde{M}$ by elements of the lattice. We define a map $\overline{k} : \widetilde{M} \to \Gamma$ commuting with Γ. For this, we fix a bounded fundamental domain F containing the point x_0. For an arbitrary $x \in \widetilde{M}$, we put $\overline{k}(x) = k$, where k is a unique element of Γ such that $x \in F + k$.

Consider the function $\|\cdot\|\colon \Gamma \to [0, \infty)$ given by the formula

$$\|k\| = \lim_{n\to\infty} \frac{\widetilde{\rho}(x_0, x_0 + nk)}{n},$$

where $\widetilde{\rho}$ is the lift of the metric ρ. The function $\|\cdot\|$ is well known to be a norm on Γ; therefore, it extends to a norm on V, called the stable norm. For a linear function $L : V \to \mathbb{R}$ we set $\|L\| = \max\{L(x) | \|x\| = 1\}$.

Lemma 2. *Let* $L : V \to \mathbb{R}$ *be a linear function with* $\|L\| = 1$. *There exists a function* $\widetilde{B}_L : \widetilde{M} \to \mathbb{R}$ *such that*

1) $\widetilde{B}_L$ *is Lipschitz with Lipschitz constant* 1;
2) $\widetilde{B}_L(x + k) = \widetilde{B}_L(x) + L(k)$ *for every* $x \in \widetilde{M}$, $k \in \Gamma$.

Proof. Indeed, let

$$\widetilde{B}_L(x) = \inf_{k\in\Gamma}(L(k) + \rho(x, x_0 + k)).$$

We prove that the function $\widetilde{B}_L$ is well defined. Since $\|L\| = 1$, from the definition of the stable norm it follows that

$$-\rho(x_0 + k, x_0) \le -\|k\| \le L(k),$$

whence

$$L(k) + \rho(x, x_0 + k) \ge -\rho(x_0 + k, x_0) + \rho(x, x_0 + k) \ge -\rho(x, x_0).$$

The required properties of $\widetilde{B}_L$ immediately follow from the definition. □

For a linear function $L : V \to \mathbb{R}$, let $\widetilde{B}_L$ denote the function constructed in Lemma 2. Since $\widetilde{B}_L$ is Lipschitz, it has a gradient almost everywhere; this gradient will be denoted by $\widetilde{v}_L$.

For $\widetilde{v} \in \widetilde{SM}$, let $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ be a geodesic with $\gamma'(0) = \widetilde{v}$. We define the direction at infinity $\widetilde{R}(\widetilde{v}) = \widetilde{R}(\widetilde{\gamma}) \in V$ by

$$\widetilde{R}(\widetilde{v}) = \lim_{T\to\infty} \frac{\overline{k}(\widetilde{\gamma}(T)) - \overline{k}(\widetilde{\gamma}(0))}{T}.$$

By definition, for $v \in SM$ we put $R(v) = \widetilde{R}(\widetilde{v})$, where $\widetilde{v}$ is a lifting of v.

Since M has no conjugate points, it is clear that $\|R(v)\| = 1$.

Lemma 3. *The functions R and $\widetilde{R}$ are defined almost everywhere on SM and $\widetilde{SM}$, respectively.*

Proof. Let $\phi : M \to V/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n$ be a homotopy equivalence; we may assume that ϕ is simplicial. Since ϕ induces an isomorphism between fundamental groups, the lifting function $\widetilde{\phi} : \widetilde{M} \to V$ commutes with Γ.

Since the functions $\widetilde{\phi}$ and $\overline{k}$ commute with Γ, we have $\|\widetilde{\phi} - \overline{k}\| \leq$ const. Thus, in the definition of $\widetilde{R}$ we can replace $\overline{k}$ by $\widetilde{\phi}$. Since the differential $d\widetilde{\phi}$ is defined almost everywhere on $T\widetilde{M}$ and is Γ-invariant, it is the lift of some measurable function $\omega : TM \to V$. For a geodesic γ in M and its lifting $\widetilde{\gamma}$, we have

$$\widetilde{\phi}(\widetilde{\gamma}(T)) - \widetilde{\phi}(\widetilde{\gamma}(0)) = \int_0^T d\widetilde{\phi}(\widetilde{\gamma}') = \int_0^T \omega(\gamma').$$

Thus, $R(v)$ is equal to the average of ω along γ. The Birkhoff ergodic theorem shows that $R(v)$ is defined for almost all $v \in SM$. □

Lemma 4. *Let $L : V \to R$ be a linear function with $\|L\| = 1$. Recall that $\widetilde{v}_L$ denotes the gradient field of B_L. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \gamma', \widetilde{v}_L\rangle = L(R(\gamma))$$

if both sides are well defined.

Proof. Since $B_L \circ \gamma$ is Lipschitz, the Newton–Leibniz formula yields

$$\int_0^T \langle \gamma', \widetilde{v}_L\rangle = \int_0^T (B_L \circ \gamma)' = B_L(\gamma(T)) - B_L(\gamma(0)).$$

Since the function $B_L(x) - L(\overline{k}(x))$ is bounded on the fundamental domain and periodic, it is bounded. This implies that $B_L(\gamma(T)) - B_L(\gamma(0))$ differs from $L(\overline{k}(\gamma(T))) - L(\overline{k}(\gamma(0)))$ by a constant. So, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \gamma', \widetilde{v}_L\rangle = \lim_{T\to\infty} L\left(\frac{\overline{k}(\gamma(T)) - \overline{k}(\gamma(0))}{T}\right) = L(R(\gamma)). \qquad \square$$

Let F denote the unit sphere of the norm $\|\cdot\|$, and let m be the measure on F that is the image of μ_L under $R : SM \to F$.

Lemma 5. *If $L : V \to R$ is a linear function with $\|L\| = 1$, then*

$$\int_F L^2 \, dm \leq \frac{1}{n}.$$

Equality occurs if and only if $\langle \widetilde{v}_L, w\rangle = L(\widetilde{R}(w))$ for almost every $w \in \widetilde{SM}$.

Proof. Consider the average of $\langle v_L, \cdot\rangle$ along geodesics. By Lemma 3, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \gamma', v_L\rangle = L\circ R.$$

By the Schwartz inequality,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \gamma', v_L\rangle^2 \geq (L\circ R)^2.$$

Since R is constant on every trajectory of the geodesic flow, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle v_L, w\rangle^2 = (L\circ R)^2 + \lim_{T\to\infty} \frac{1}{T}\int_0^T (\langle v_L, \cdot\rangle - L\circ R)^2.$$

Integrating and using the Birkhoff ergodic theorem, we obtain

$$\int_{SM} \langle v_L, \cdot\rangle^2 \, d\mu_L = \int_{SM} (L\circ R)^2 \, d\mu_L + \int_{SM} (\langle v_L, \cdot\rangle - L\circ R)^2 \, d\mu_L.$$

From the inequality $|v_L| < 1$ it follows that $\int_{SM} \langle v_L, \cdot\rangle^2 \, d\mu_L \leq 1/n$. Consequently,

$$\int_F L^2 \, dm = \int_{SM} (L\circ R)^2 \, d\mu_L \leq \frac{1}{n} - \int_{SM} (\langle v_L, \cdot\rangle - L\circ R)^2 \, d\mu_L.$$

The integral on the right is nonnegative, and it vanishes if and only if $\langle v_L, w\rangle = L(R(w))$ for almost every $w \in SM$. The lemma is proved. □

We use the following known result (for its proof, see, e.g., [BI]).

Lemma 6. *Let $(V, \|\cdot\|)$ be an n-dimensional Banach space, let F be the unit sphere of the norm $\|\cdot\|$, and let F^* be the set of linear functions L such that $\|L\| = 1$. Then there exists an ("inscribed") quadratic form $Q : V \to \mathbb{R}$ representable as a finite sum*

$$Q = \sum a_i L_i^2, \quad L_i \in F^*, \ a_i > 0, \ \sum a_i = n,$$

and such that $Q(x) \geq \|x\|^2$ for every $x \in V$. In particular, Q is positive.

Remark 1. The unit ball of Q is the ellipsoid of maximal volume inscribed in F.

Let $Q = \sum a_i L_i^2$ be the corresponding (inscribed) quadratic form for the stable norm $\|\cdot\|$ associated with $\widetilde{\rho}$. We denote by B_i the functions constructed as in Lemma 2 for the linear functions L_i, and by $\widetilde{v}_i$ their gradients.

Lemma 7. *For all i, we have*

$$\langle \widetilde{v}_i, w\rangle = L_i(R(w)) \tag{1}$$

for almost every $w \in \widetilde{SM}$.

Proof. Applying Lemma 5 to L_i, we obtain

$$\int_F Q \, dm = \sum a_i \int_F L_i^2 \, dm \leq \frac{1}{n} \sum a_i = 1.$$

But $Q|_F \geq 1$ on F. Therefore, $\int_F Q \, dm = 1$, so that $\int_F L_i^2 \, dm = \frac{1}{n}$ for every i. By Lemma 5, it follows that $\langle \widetilde{v}_i, w\rangle = L_i(\widetilde{R}(w))$ for almost every $w \in \widetilde{SM}$. □

The lemma just proved implies that (1) is true almost everywhere for almost every trajectory of the geodesic flow; this means that for almost every $w \in \widetilde{SM}$, if γ is a geodesic with $\gamma'(0) = w$, then the function $\langle \widetilde{v}_i, \gamma'\rangle = (B_i \circ \gamma)'$ is defined almost everywhere. Moreover it is equal to the constant $L_i(R(\gamma))$. Since this function is Lipschitz, it is linear. Thus,

$$(B_i \circ \gamma)' \equiv L_i(R(\gamma)), \quad t \in \mathbb{R}. \tag{2}$$

Since $Q|_F \geq 1$, the relation $\int_F Q\,dm = 1$ implies that m-almost everywhere on F we have $Q = 1$. By the definition of m, this means that

$$Q(R(w)) = 1 \tag{3}$$

for almost all $w \in S\widetilde{M}$. Since Q is nondegenerate, there is no loss of generality in assuming that $L_1, \dots, L_n$ are linearly independent.

Consider the map

$$\widetilde{f} = (\widetilde{B}_1, \dots, \widetilde{B}_n) : \widetilde{M} \to \mathbb{R}^n.$$

We endow $\mathbb{R}^n$ with the Euclidean structure corresponding to the quadratic form Q under the isomorphism

$$I = (L_1, \dots, L_n) : V \to \mathbb{R}^n.$$

For almost every geodesic $\gamma : \mathbb{R} \to \widetilde{M}$ we obtain

$$(\widetilde{f} \circ \gamma)' = (L_1(\widetilde{R}(\gamma)), \dots, L_n(\widetilde{R}(\gamma))) = I(\widetilde{R}(\gamma)).$$

Since for almost every geodesic γ the vector $I(\widetilde{R}(\gamma))$ is a unit vector with respect to the new Euclidean structure, the image $\widetilde{f}(\gamma)$ is a straight line with constant unit velocity.

Now we prove Proposition 1.

Proof. Since $\widetilde{f}$ commutes with the group Γ of integral translations on $\widetilde{M}$ and $\mathbb{R}^n$, $\widetilde{f}$ induces a map $f : M \to T^n$, where T^n is a flat torus. The homomorphism of fundamental groups induced by f is an isomorphism, which implies statement (3) of Proposition 1. The map f is Lipschitz because so is $\widetilde{f}$.

Recall that M' denotes the complement of the $(n-2)$-skeleton of M.

We show that $f|_{M'} \to T^n$ is a local isometry. Consider a convex neighborhood $U \in M'$ and fix two points $x, y \in U$. For any neighborhoods $U_x, U_y \subset U$ of x and y, let $V(U_x, U_y)$ be the set of initial velocity vectors of all shortest paths starting in U_x and ending in U_y. Since for almost every geodesic $\gamma : [a, b] \to M$ the image $f \circ \gamma$ is a straight line with a constant unit speed and $\mu_L V(U_x, U_y) > 0$, there exist two points $x' \in U_x$ and $y' \in U_y$ such that f preserves the distance between them. Since U_x and U_y are arbitrary and f is continuous, f preserves the distance between x and y. Thus, $f|_U$ preserves distances.

Since M' and T^n are n-dimensional manifolds, and $f|_{M'}$ preserves the distances, for any $x \in M'$ the image of some neighborhood of x is a neighborhood of $f(x)$, and we see that f is an open map. □

§4. f is an isometry

The following Lemma 8 is an obvious consequence of Proposition 1(1).

Lemma 8. *$f|_{M'}$ preserves the lengths of curves.*

Lemma 9. *The map $f|_{M'} : M' \to f(M')$ is bijective, and $f : M \to T^n$ is surjective. As a consequence* (*because $f|_{M'}$ is a local isometry*), *the map $(f|_{M'})^{-1}$ is well defined, is continuous, and preserves the lengths of curves.*

Proof. Recall that M is homotopy equivalent to an n-dimensional torus. Consequently, the n-homology group of M is isomorphic to $\mathbb{Z}$. We fix an isomorphism between $H_n(T^n)$ and $\mathbb{Z}$ and choose a generator of $H_n(M)$. The induced homomorphism $f_* : H_n(M) \to H_n(T^n) = \mathbb{Z}$ takes the generator of $H_n(M)$ to some integer; this integer is called the degree of f. We show that the degree of f is ± 1. Since the universal covering space of M is contractible, the induced homomorphism f_* determines the homotopy type of f. Proposition 1(3) shows that f_* is an isomorphism; then f is a homotopy equivalence. Thus, the degree of f is ± 1.

The choice of generators of the homology group fixes orientations of the manifolds $M' \subset M$ and T^n. We define the degree of f at $x \in M'$ to be equal to 1 if $d_x f$ preserves the orientations of the tangent spaces at x, and to -1 if $d_x f$ reverses the orientations. Suppose $y \in T^n$ is a regular point, i.e., the preimage $f^{-1}(y) = x_1, \dots, x_l$ is contained in M'. As in the case of Riemannian manifolds, it can be proved that the degree of f is the sum of the degrees of f at the points $x_1, \dots, x_l$. Hence, f is surjective.

Since M is a pseudomanifold that is homotopy equivalent to an n-dimensional torus, the space M' is connected. Indeed, assume the contrary; then the group $H_n(M, \mathbb{Z}_2)$ contains two nonzero elements. Since $f|_{M'}$ is a local isometry, it preserves the orientation of tangent spaces everywhere, or it reverses these orientations. Consequently, the degree of f is constant at the points $x_1, \dots, x_l$. Since the degree of f is 1, this means that each regular point has a unique preimage. By the definition of a regular point, it follows that all points having two or more preimages are contained in $f^{-1}(f(M \setminus M'))$. We put $J = f^{-1}(f(M \setminus M'))$. Observe that the dimension of J does not exceed $n-2$.

Suppose that $f|_{M'}$ is not injective. Let $y \in f(M')$ be a point with more than one preimage in M', and let x_1, x_2 be two such preimages. Let $D_{r_0}(x_1)$, $D_{r_0}(x_2)$ be balls centered at x_1 and x_2 and such that the restriction of f to these balls is an isometry. Since the dimension of J is at most $n-2$, there exists a point $x_3 \in D_{r_0}(x_1) \in M \setminus J$. The image of this point coincides with an image of some point contained in $D_{r_0}(x_2)$, which contradicts the fact that f is injective on $M \setminus J$ ($x_3 \in M \setminus J$). □

We complete the proof of Theorem 1 by the following statement.

Lemma 10. *The map $f : M \to T^n$ is an isometry.*

Proof. We show that f is noncontracting and nonexpanding. Every path in M can be approximated by a piecewise differentiable path of almost the same length. We can move each of the corresponding pieces to the interior of an appropriate n-simplex, leaving the endpoints fixed and almost length preserving.

The map f preserves the lengths of these pieces (see Lemma 8). Therefore, the map is nonexpanding.

Now we show that f is noncontracting. Let $x, y \in M$ be arbitrary points. Given $\varepsilon > 0$, we let $x', y' \in M'$ be points such that $\rho(x, x') < \varepsilon$ and $\rho(y, y') < \varepsilon$. Since f is nonexpanding, we have $|(f(x), f(x'))| < \varepsilon$ and $|(f(y), f(y'))| < \varepsilon$, where $|(\cdot, \cdot)|$ denotes the metric on the flat torus.

Since f is Lipschitz and surjective, the Hausdorff dimension of the set $T^n \setminus f(M')$ does not exceed $n-2$. Therefore, the shortest path $[f(x'), f(y')] \in T^n$ can be approximated by a path in $f(M')$ with almost the same length and the same endpoints. Let $s : [a, b] \to f(M')$ be a path that joins $f(x')$ and $f(y')$ and such that the length of s differs from $|f(x'), f(y')|$ by less than ε. Since $(f|_{M'})^{-1}$ preserves distances, the length of the path $s \circ (f|_{M'})^{-1} : [a, b] \to M'$, which joins x' and y', differs from $|f(x'), f(y')|$ by less than ε. Thus,

$$\rho(x, y) < \rho(x', y') + 2\varepsilon < |f(x'), f(y')| + 3\varepsilon < |f(x), f(y)| + 5\varepsilon.$$

Therefore, f is noncontracting. □

References

[BB] W. Ballmann and M. Brin, *Orbihedra of nonpositive curvature*, Inst. Hautes Études Sci. Publ. Math. No. 82 (1995), 169–209 (1996). MR1383216 (97i:53049)

[BI] D. Burago and S. Ivanov, *Riemannian tori without conjugate points are flat*, Geom. Funct. Anal. **4** (1994), no. 3, 259–269. MR1274115 (95h:53049)

[C] C. Croke, *Volumes of balls in manifolds without conjugate points*, Internat. J. Math. **3** (1992), 455–467. MR1168355 (93e:53048)

[G1] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. No. 53 (1981), 53–73. MR0623534 (83b:53041)

[G2] ———, *Asymptotic invariants of infinite groups*, Geometric Group Theory. Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544 (95m:20041)

[I] S. V. Ivanov, *The geometry of periodic metrics, and volumes of limit Finsler manifolds*, Abstract of Thesis Prepared by Candidate, St. Petersburg, 1995.

[TG] J. Tits, *Appendix to: "Groups of polynomial growth and expanding maps" by M. Gromov*, Inst. Hautes Études Sci. Publ. Math. No. 53 (1981), 74–78. MR0623535 (83b:53042)

[L1] N. D. Lebedeva, *Exponential growth of spaces without conjugate points*, Algebra i Analiz **15** (2003), no. 1, 184–200; English transl., St. Petersburg Math. J. **15** (2004), no. 1, 127–137. MR1979721

[L2] ———, *On the fundamental group of a compact space without conjugate points*, PDMI Preprint no. 05/2002, S.-Peterburg. Otdel. Mat. Inst. (POMI), St. Petersburg, 2002.

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191011, Russia

E-mail address: lebed@pdmi.ras.ru

Received 18/FEB/2003

Translated by THE AUTHOR