SYMMETRIC GEODESICS ON CUBES: ALGORITHMS FOR FINDING THEM

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ABSTRACT. An algorithm is presented for finding simple closed symmetric geodesics on the boundary of an n -cube.

§1. INTRODUCTION

1.1. A geodesic γ on the boundary ∂K^n of an n -cube K^n is said to be *symmetric* if γ is closed and symmetric with respect to the center O_n of K^n . The following assertions were proved in [1].

1) The closed geodesics on ∂K^n intersect no $(n-3)$ -faces of K^n . Each closed geodesic is a broken line the links of which lie on $(n-1)$ -faces (*facets*), and the vertices lie at inner points of $(n-2)$ -faces.

2) If a cube K^k , where $2 \leq k < n$, is the section of K^n by a k -plane passing through O_n (a *central* section), then each symmetric geodesic on ∂K^k is also a symmetric geodesic on ∂K^n . Conversely, if a symmetric geodesic γ on ∂K^n has no links on at least one of the facets of K^n , then γ is a symmetric geodesic on the boundary of a certain cube K^{n-1} which is a central section of K^n .

Therefore, if we pass from the dimension $n-1$ to the dimension n , the only “new” geodesics are the symmetric geodesics that pass *along each* of the $2n$ facets of K^n . The problem that we solve consists in finding symmetric geodesics on ∂K^n with precisely this property for various n . (For $n=3$, such a geodesic is the hexagon obtained as the central section of ∂K^3 by the plane orthogonal to one of the four great diagonals of K^3 .)

As in [1], we are interested only in the symmetric geodesics that are *simple*, i.e., have neither self-intersections nor self-overappings.

1.2. The above problem was considered in [1] only for small dimensions $n=3, 4, 5$. This allowed us to use intuitive geometric methods.

Already for $n=4$, it turned out that besides a “1-turn” symmetric geodesic γ passing only once along each of the eight facets of K^4 , there also exist “3-turn” symmetric geodesics. They “surround” γ , filling (together with γ) a certain torus with a cross-section in the form of an open triangle, and thus transforming this torus into a *filled torus with a Seifert fibration*. (See [2, the beginning of §3].)

For $n=5$, in [1] we found a 1-turn symmetric geodesic by an artificial trick.

1.3. In the present paper, we present algorithms for finding 1- and many-turn symmetric geodesics on the surfaces of cubes of higher dimensions in finitely many steps.

In §§6 and 7, we consider the cases of $n=4, 5, 6, 7$ as examples of application of these algorithms.

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1.4. In [1], we presented a method for finding symmetric geodesics on ∂K^4 . This method involves rolling the unit cube K^4 along the integer lattice in $\mathbb{R}^3$. As a result, a symmetric geodesic on ∂K^4 is unfolding in $\mathbb{R}^3$ along some ray. The ray was found by geometric arguments.

If for $n > 4$ we similarly roll the unit cube K^n along the integer lattice in $\mathbb{R}^{n-1}$, then the required geodesic is also unfolding along a certain ray l .

The crucial role for the present paper was played by the following fact: a thorough examination of the results obtained in [1] for $n = 3, 4$ suggested not only the direction of such a ray, but also the choice of its initial point. This allowed us in §3 to immediately indicate a chain of $2n$ unit $(n-1)$ -cubes of the integer lattice in $\mathbb{R}^{n-1}$ along which the cube K^n rolls in the course of rectification of a 1-turn geodesic.

Rolling along the chosen ray confirms that the 1-turn geodesic obtained is symmetric, suggests which many-turn closed geodesics can be expected for a given n , and helps to construct them and to verify whether they (or at least some of them) are symmetric with respect to O_n .

As soon as we have found a symmetric geodesic γ on ∂K^n of a certain shape, the other geodesics of the same shape are obtained from γ via self-congruences of K^n .

1.5. Describing the process of rolling for $n > 4$ involves some specific combinatorial means. For this reason, our notation here differs much from that used in [1].

§2. THE INITIAL POSITION OF THE CUBE K^n AND NOTATION FOR ITS VERTICES AND FACES

2.1. We assume that K^n is a unit cube lying in $\mathbb{R}^n(x_1, \dots, x_n)$ so that one of its vertices is at the origin and the edges of K^n are parallel to the coordinate axes. We call this the *initial position* of K^n .

2.2. The cube K^n has 2^n vertices. In the initial position, the coordinates $(x_1, \dots, x_n)$ of any vertex A of K^n form one of the 2^n possible n -tuples of zeros and units. We denote this n -tuple by σ and denote A by A_σ . The n -tuple σ is the *index* of A_σ . In what follows, we rotate and shift K^n in $\mathbb{R}^n$ as a rigid body. However, we preserve the index σ of a vertex adopted in the initial position. So, we will say: “The vertex A_σ comes to a certain point”.

2.3. The cube K^n has $2n$ facets, which are unit $(n-1)$ -cubes. We denote them by G_{i0} and G_{i1} , where $1 \leq i \leq n$. The vertices of G_{i0} are the vertices of K^n with $x_i = 0$ in the initial position. For the vertices of G_{i1} we have $x_i = 1$. The facets G_{i0} and G_{i1} are opposite to each other. In the initial position, they both are orthogonal to the x_i -axis.

§3. PREPARATION FOR ROLLING

3.1. Throughout, we let $\mathbb{R}^{n-1}$ be the hyperplane $x_n = 0$ in $\mathbb{R}^n(x_1, \dots, x_n)$, which we call the *plane*. We regard this plane as a space $\mathbb{R}^{n-1} = \mathbb{R}^{n-1}(x_1, \dots, x_{n-1}, 0)$, retaining the zero n th coordinate of all points of this $(n-1)$ -dimensional space, which is convenient for the algorithms in the sequel. In the initial position in $\mathbb{R}^n$, the facet G_{n0} of K^n lies on an $(n-1)$ -cube $Q_1 \subset \mathbb{R}^{n-1}$. In this position, the index σ of any vertex of K^n in G_{n0} coincides with the index σ of the corresponding vertex of $Q_1 \subset \mathbb{R}^{n-1}$.

3.2. In the space $\mathbb{R}^{n-1}(x_1, \dots, x_{n-1}, 0)$, we consider the $(n-2)$ -simplex Δ_0^{n-2} with vertices

$$(1) \quad \begin{aligned} &D_1^0(0, 0, 0, \dots, 0, 0, 0), \\ &D_2^0(1, 0, 0, \dots, 0, 0, 0), \\ &D_3^0(1, 1, 0, \dots, 0, 0, 0), \\ &\quad \vdots \\ &D_{n-1}^0(1, 1, 1, \dots, 1, 0, 0). \end{aligned}$$

We denote by p_0 the barycenter of Δ_0^{n-2} :

$$p_0 = \left(\frac{n-2}{n-1}, \frac{n-3}{n-1}, \dots, \frac{1}{n-1}, 0, 0 \right).$$

3.3. Let l be the ray in $\mathbb{R}^n$ (actually, in $\mathbb{R}^{n-1}$) emanating from p_0 and having the parametric equation

$$(2) \quad \begin{aligned} x_1 &= \frac{n-2}{n-1} + t, \\ x_2 &= \frac{n-3}{n-1} + t, \\ &\quad \vdots \\ x_{n-2} &= \frac{1}{n-1} + t, \\ x_{n-1} &= t, \\ x_n &= 0. \end{aligned} \quad 0 \leq t < \infty.$$

The ray l passes successively through a sequence of unit $(n-1)$ -cubes of the integer lattice in $\mathbb{R}^{n-1}$. We are interested only in the first $2n$ of them, which we denote successively by $Q_1, \dots, Q_{2n}$.

Under passage from the $(n-1)$ -cube Q_i to the $(n-1)$ -cube Q_{i+1} , the ray l intersects their common $(n-2)$ -face at the point p_i corresponding to $t = \frac{i}{n-1}$.

At the points $p_1, \dots, p_{n-1}$, the coordinates $x_1, \dots, x_{n-1}$ successively turn out to be integers. After that, the same order repeats for the points $p_n, \dots, p_{2n-2}$, and once again the coordinate x_1 for p_{2n-1} becomes an integer. Accordingly, the $(n-1)$ -cubes $Q_1, \dots, Q_{2n}$ in the integer lattice in $\mathbb{R}^{n-1}$ differ from each other by successive shifts by 1 in the direction of the axes $x_1, \dots, x_{n-1}, x_1, \dots, x_{n-1}, x_1$.

3.4. We roll the cube K^n as a rigid body along the plane $\mathbb{R}^{n-1}$ so that the $(n-1)$ -faces of K^n lie successively onto the $(n-1)$ -cubes $Q_1, \dots, Q_{2n}$.

When we pass Q_i , the segment of l contained in Q_i is “printed” on the $(n-1)$ -face of K^n that lies on Q_i .

In the favorable case, if the prints of p_0 and p_{2n} on ∂K^n coincide, we obtain a closed and symmetric 1-turn geodesic on ∂K^n with vertices $p_0, p_1, \dots, p_{2n} = p_0$.

3.5. If a ray l' parallel to l emanates not from the point p_0 , but from an interior point $p'_0 \in \text{Int } \Delta_0^{n-2}$, then l' intersects the same $(n-1)$ -cubes $Q_1, \dots, Q_{2n}$ in the same order, and enters and leaves them through the same $(n-2)$ -faces as the ray l .

The “flow” of the rays parallel to l and emanating from all points in Δ_0^{n-2} successively projects the simplex Δ_0^{n-2} to the $(n-2)$ -faces of $Q_1, \dots, Q_{2n}$ through which the ray l leaves these $(n-1)$ -cubes. We denote the projections by $\Delta_1^{n-2}, \dots, \Delta_{2n}^{n-2}$.

For $1 \leq i \leq 2n$, the projection $\varphi_i : \Delta_0^{n-2} \rightarrow \Delta_i^{n-2}$ is an affine mapping. We denote

$$D_j^i := \varphi_i(D_j^0) \in \Delta_i^{n-2}, \quad j = 1, \dots, n-1.$$

Since the mapping φ_i is affine, the barycenter of Δ_i^{n-2} is precisely the point p_i . Therefore, if for each $j = 1, \dots, n-1$ we know which of the vertices A_σ of K^n comes to D_j^i at the moment when, in the course of rolling, the cube K^n stays on the $(n-1)$ -cube Q_i , then we shall determine not only the (already known) position of the point p_i on the plane $\mathbb{R}^{n-1}$, but also the “print” p'_i of p_i on ∂K^n in the initial position of K^n . We need precisely these “prints” of the points p_i : they will be the vertices of the required 1-turn geodesic.

3.6. Remark. For each i , rolling (see §5) tells us which of the vertices A_σ come to the vertices of Q_i in the course of rolling at the moments when K^n stays on Q_i . There are $2n \cdot 2^{n-1}$ such vertices, while the simplexes $\Delta_1^{n-2}, \dots, \Delta_{2n}^{n-2}$ have only $2n(n-1)$ vertices. It suffices to know what vertices A_σ come to these $2n(n-1)$ points, which makes the algorithms polynomial in n .

§4. SHIFTING THE SIMPLEXES Δ_i^{n-2}

4.1. The objective of “Algorithm 1” presented below is as follows: for each i , to determine the coordinates in $\mathbb{R}^n$ acquired by the vertices of the simplex Δ_i^{n-2} if we shift the $(n-1)$ -cube Q_i containing Δ_i^{n-2} to the position in $\mathbb{R}^{n-1}$ “initial” for the cubes K^{n-1} .

The algorithm takes into account the fact that the vertices

$$D_j^0 \mapsto D_j^1 \mapsto D_j^2 \mapsto \dots$$

are related by projection along the vector $(1, 1, \dots, 1, 0)$. The algorithm also takes into account the shift of Q_i relative to Q_{i-1} for $i = 1, \dots, n-1$.

4.2. Algorithm 1. 1) We start with the $((n-1) \times n)$ -matrix M_0 the rows of which coincide with the coordinates of the vertices $D_1^0, \dots, D_{n-1}^0$ of Δ_0^{n-2} ; see (1).

2) Replacing a unique zero row in M_0 by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_1 ; it is formed by the coordinates of the vertices of the simplex Δ_1^{n-2} after shifting it.

3) We replace a unique column in M_1 consisting only of units by a zero column, after which a unique zero row arises. Replacing it by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_2 , the rows of which are the coordinates of the vertices of the simplex Δ_2^{n-2} after shifting it.

We proceed further as in step 3). Replacing a unique column of units in the matrix M_{i-1} by a zero column and, after that, replacing a unique zero row by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_i .

As a result, Algorithm 1 provides matrices $M_0, M_1, \dots, M_{2n}$ formed by the coordinates of the vertices of the simplexes $\Delta_0^{n-2}, \Delta_1^{n-2}, \dots, \Delta_{2n}^{n-2}$ after shifting them.

§5. ROLLING

5.1. First, we eliminate the shift component of K^n arising in the course of rolling.

Suppose that in the initial position the cube K^n stays on an $(n-1)$ -cube $Q_1 \subset \mathbb{R}^{n-1}$. We roll K^n along the plane $\mathbb{R}^{n-1}$ onto the cell of the integer lattice in $\mathbb{R}^{n-1}$ adjacent to Q_1 in the direction of the x_i -axis, and after that, shift K^n back (by 1 in the direction opposite to that of the x_i -axis) to Q_1 . Then K^n undergoes a certain rotation α_i , which gives us a self-congruence of K^n interchanging the vertices of K^n .

Now we describe how, knowing the index σ of a vertex A_σ , we find the index $\alpha_i(\sigma)$ of the vertex $\alpha_i(A_\sigma) = A_{\alpha_i(\sigma)}$. The operator α_i preserves all coordinates of A_σ except x_i and x_n and transforms the pair (x_i, x_n) by the following rule:

$$\alpha_i(0, 0) \mapsto (0, 1) \mapsto (1, 1) \mapsto (1, 0) \mapsto (0, 0).$$

This is illustrated in the figure.

“Algorithm 2” lists the vertices A_σ of the initial cube K^n that come in the course of rolling to the vertices of Δ_i^{n-2} when K^n stays on Q_i . It involves the operators β_i inverse to the operators α_i . The operator β_i transforms a pair of coordinates (x_i, x_n) as follows:

$$(3) \quad \beta_i(0,0) \mapsto (1,0) \mapsto (1,1) \mapsto (0,1) \mapsto (0,0).$$

5.2. Algorithm 2. By definition, the operator β_i acts on an $((n-1) \times n)$ matrix M consisting only of zeros and ones as follows. In each row of M , the operator β_i preserves all entries except the i th and the n th, which it transforms by the rule (3).

The objective of Algorithm 2 is as follows: for each $i = 0, 1, \dots, 2n$, to construct an $((n-1) \times n)$ -matrix $A(M_i)$ the rows of which are the indexes σ of the vertices A_σ of K^n that come to the vertices of Δ_i^{n-2} in the course of rolling when K^n stays on Q_i .

After each step of rolling, we shift K^n back to Q_1 . Then the entire process of rolling K^n from Q_1 to $Q_2, \dots, Q_{2n}$ reduces to a sequence of rotations $\alpha_1, \dots, \alpha_{n-1}, \alpha_1, \dots, \alpha_{n-1}, \alpha_1$ of K^n about O_n . The simplexes Δ_i^{n-2} together with the $(n-1)$ -cubes Q_i have already been shifted to Q_1 : Algorithm 1 gave us the matrices M_i of coordinates of their vertices $D_1^i, \dots, D_{n-1}^i$ thus shifted.

Therefore, Algorithm 2 constructs the matrices $A(M_i)$ by applying the corresponding sequences of operators β_j to the matrices M_i :

$$\begin{aligned}
 A(M_0) &= M_0, \\
 A(M_1) &= M_1, \\
 A(M_2) &= \beta_1(M_2), \\
 A(M_3) &= \beta_1\beta_2(M_3), \\
 &\vdots \\
 A(M_n) &= \beta_1\beta_2 \cdots \beta_{n-1}(M_n), \\
 A(M_{n+1}) &= \beta_1\beta_2 \cdots \beta_{n-1}\beta_1(M_{n+1}), \\
 A(M_{n+2}) &= \beta_1 \cdots \beta_{n-1}\beta_1\beta_2(M_{n+2}), \\
 &\vdots \\
 A(M_{2n-1}) &= \beta_1 \cdots \beta_{n-1}\beta_1 \cdots \beta_{n-1}(M_{2n-1}), \\
 A(M_{2n}) &= \beta_1 \cdots \beta_{n-1}\beta_1 \cdots \beta_{n-1}\beta_1(M_{2n}).
 \end{aligned}
 \tag{4}$$

5.3. The knowledge of the matrices $A(M_i)$ makes it possible to easily construct a 1-turn symmetric geodesic on ∂K^n with vertices $p_0, p_1, \dots, p_{2n-1}, p_{2n} = p_0$, as well as obtain many-turn closed geodesics and look for symmetric ones among them. We proceed to examples, where we denote the “prints” of Δ_i^j on ∂K^n by $\bar{\Delta}_i^j$.

§6. 1-TURN SYMMETRIC GEODESICS

6.1. For $n = 4$, Algorithm 1 gives us the matrices

$$\begin{array}{ccc}
 M_0 & M_1 & M_2 \\
 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
 M_3 & M_4 & M_5 \\
 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
 M_6 & M_7 & M_8 \\
 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
 \end{array}$$

Algorithm 2 transforms them to the matrices

$$\begin{array}{ccc}
 A(M_0) & A(M_1) & A(M_2) \\
 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \\
 A(M_3) & A(M_4) & A(M_5) \\
 \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \\
 A(M_6) & A(M_7) & A(M_8) \\
 \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{array}
 \tag{5}$$

The rows of these matrices give us the coordinates of the vertices of the 2-simplexes $\bar{\Delta}_0^2, \bar{\Delta}_1^2, \dots, \bar{\Delta}_8^2 \subset \partial K^4$. Their barycenters

$$\begin{array}{ll}
 p_8 = p_0\left(\frac{2}{3}, \frac{1}{3}, 0, 0\right), & p_4\left(\frac{1}{3}, \frac{2}{3}, 1, 1\right), \\
 p_1\left(1, \frac{2}{3}, \frac{1}{3}, 0\right), & p_5\left(0, \frac{1}{3}, \frac{2}{3}, 1\right), \\
 p_2\left(1, 1, \frac{2}{3}, \frac{1}{3}\right), & p_6\left(0, 0, \frac{1}{3}, \frac{2}{3}\right), \\
 p_3\left(\frac{2}{3}, 1, 1, \frac{2}{3}\right), & p_7\left(\frac{1}{3}, 0, 0, \frac{1}{3}\right)
 \end{array}
 \tag{6}$$

are the vertices of a closed geodesic on ∂K^4 . This geodesic is symmetric, because the midpoints of the segments p_0p_4 , p_1p_5 , p_2p_6 , and p_3p_7 lie at O_4 . (All the coordinates of O_4 are equal to $1/2$.)

Their rows give us the coordinates of the vertices of the 3-simplexes $\bar{\Delta}_0^3, \bar{\Delta}_1^3, \dots, \bar{\Delta}_{10}^3 \subset \partial K^5$. Their barycenters

$$(8) \quad \begin{aligned} p_{10} = p_0 \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0 \right), & \quad p_5 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1 \right), \\ p_1 \left(1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0 \right), & \quad p_6 \left(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \right), \\ p_2 \left(1, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right), & \quad p_7 \left(0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right), \\ p_3 \left(\frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2} \right), & \quad p_8 \left(\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{2} \right), \\ p_4 \left(\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4} \right), & \quad p_9 \left(\frac{1}{2}, \frac{1}{4}, 0, 0, \frac{1}{4} \right) \end{aligned}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^5 .

6.3. For $n = 6$, applying Algorithms 1 and 2, we obtain the matrices

$$(9) \quad \begin{aligned} & \begin{matrix} A(M_0) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_1) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_2) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}, \\ & \begin{matrix} A(M_3) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_4) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_5) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}, \\ & \begin{matrix} A(M_6) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_7) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_8) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}, \\ & \begin{matrix} A(M_9) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_{10}) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \\ & \begin{matrix} A(M_{11}) \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} A(M_{12}) \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}. \end{aligned}$$

Their rows give us the coordinates of the vertices of the 4-simplexes $\bar{\Delta}_0^4, \bar{\Delta}_1^4, \dots, \bar{\Delta}_{12}^4 \subset \partial K^6$. Their barycenters

$$(10) \quad \begin{aligned} p_{12} = p_0 & \left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, 0 \right), & p_6 & \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1 \right), \\ p_1 & \left(1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0 \right), & p_7 & \left(0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1 \right), \\ p_2 & \left(1, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5} \right), & p_8 & \left(0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right), \\ p_3 & \left(\frac{4}{5}, 1, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5} \right), & p_9 & \left(\frac{1}{5}, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5} \right), \\ p_4 & \left(\frac{3}{5}, \frac{4}{5}, 1, 1, \frac{4}{5}, \frac{3}{5} \right), & p_{10} & \left(\frac{2}{5}, \frac{1}{5}, 0, 0, \frac{1}{5}, \frac{2}{5} \right), \\ p_5 & \left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, \frac{4}{5} \right), & p_{11} & \left(\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, 0, \frac{1}{5} \right) \end{aligned}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^6 .

6.4. Let $n = 7$. Applying Algorithm 1 and then Algorithm 2, we obtain the following matrices:

$$(11) \quad \begin{aligned} & \begin{pmatrix} A(M_0) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, & \begin{pmatrix} A(M_1) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, \\ & \begin{pmatrix} A(M_2) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, & \begin{pmatrix} A(M_3) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, \\ & \begin{pmatrix} A(M_4) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, & \begin{pmatrix} A(M_5) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}, \\ & \begin{pmatrix} A(M_6) \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{pmatrix}, & \begin{pmatrix} A(M_7) \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{pmatrix} \end{aligned}$$

$$\begin{array}{cc}
A(M_8) & A(M_9) \\
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \\
\\
A(M_{10}) & A(M_{11}) \\
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \\
\\
A(M_{12}) & A(M_{13}) \\
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\\
A(M_{14}) \\
\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{array}$$

Their rows give us the coordinates of the vertices of the 5-simplexes $\bar{\Delta}_0^5, \bar{\Delta}_1^5, \dots, \bar{\Delta}_{14}^5 \subset \partial K^7$. Their barycenters

$$\begin{array}{ll}
p_{14} = p_0 \left(\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0 \right), & p_7 \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, 1 \right), \\
p_1 \left(1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0 \right), & p_8 \left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \right), \\
p_2 \left(1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right), & p_9 \left(0, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right), \\
p_3 \left(\frac{5}{6}, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \right), & p_{10} \left(\frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \right), \\
p_4 \left(\frac{2}{3}, \frac{5}{6}, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2} \right), & p_{11} \left(\frac{1}{3}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right), \\
p_5 \left(\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, 1, \frac{5}{6}, \frac{2}{3} \right), & p_{12} \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{3} \right), \\
p_6 \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, 1, \frac{5}{6} \right), & p_{13} \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0, \frac{1}{6} \right)
\end{array}
\tag{12}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^7 .

§7. MANY-TURN SYMMETRIC GEODESICS

7.1. Suppose the cube K^n , rolled along the ray l on the plane $\mathbb{R}^{n-1}$, runs over $2nk$ (rather than $2n$) $(n-1)$ -cubes $Q_1, \dots, Q_{2nk}$. Let k be the minimum number such that K^n stays on Q_{2nk} in a position obtained from the initial position by a parallel shift. From (2) it follows that this happens when $t = \frac{2nk}{n-1}$ is an integer for the first time. Therefore,

$$(13) \quad k = \begin{cases} n-1 & \text{for } n \text{ even,} \\ \frac{n-1}{2} & \text{for } n \text{ odd.} \end{cases}$$

7.2. If l' is the ray parallel to l and emanating from any point $p'_0 \in \text{Int } \Delta_0^{n-2}$, then the length of the segment of l' from p'_0 to the exit from Q_{2nk} is one and the same, independently of the choice of p'_0 . Transferring these segments by Algorithms 1 and 2 to ∂K^n , we obtain k -turn *closed* geodesics. (And 1-turn geodesics of type (6), (8), (10), (12) are passed k times.)

The closed geodesics obtained may be nonsymmetric. For example, for $n = 3$ we have $k = 1$. All these 1-turn geodesics are closed. However, only one of them is symmetric, namely, that emanating from the barycenter p_0 of $\bar{\Delta}_0^2$.

7.3. Necessary conditions for symmetry of a closed k -turn geodesic. Lemma 3.6 in [1] leads to the following two conclusions.

1) The number of turns of a symmetric simple closed geodesic in question is necessarily odd.

2) In order that a k -turn simple closed geodesic in question be symmetric, the point $p'_{nk} \in \partial K^n$ must be symmetric to p'_0 with respect to O_n .

These conditions reduce the class of closed geodesics that must be checked for symmetry.

We proceed to examples where we characterize the initial point p'_0 by its barycentric coordinates $\lambda_1, \dots, \lambda_{n-1}$ in the simplex $\bar{\Delta}_0^{n-2}$ and denote p'_0 by $p_0(\lambda_1, \dots, \lambda_{n-1})$.

7.4. Let $n = 4$. We consider the geodesic γ on ∂K^4 emanating in the direction of the ray l from the point

$$p_0(\lambda_1, \lambda_2, \lambda_3) = \lambda_1(0, 0, 0, 0) + \lambda_2(1, 0, 0, 0) + \lambda_3(1, 1, 0, 0).$$

The matrix $A(M_8)$ in (5) shows that after passing along all eight facets of K^4 , γ comes to the point

$$p'_8 = \lambda_1(1, 0, 0, 0) + \lambda_2(1, 1, 0, 0) + \lambda_3(0, 0, 0, 0) = p_8(\lambda_3, \lambda_1, \lambda_2).$$

The points p'_0 and p'_8 coincide only for $\lambda_1 = \lambda_2 = \lambda_3$, which brings us back to the 1-turn geodesic (6).

In the other cases, γ goes to the second turn, which starts at the point $p_8(\lambda_3, \lambda_1, \lambda_2)$ and ends at $p_{16}(\lambda_2, \lambda_3, \lambda_1)$, after which the third turn starts, ending at the point $p_{24}(\lambda_1, \lambda_2, \lambda_3) = p'_0$, and γ closes.

The knowledge of the matrices (5) makes it possible to list all the 24 vertices of such a 3-turn geodesic γ . The list confirms the fact already known from [1]: γ is symmetric for any positive barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ of the initial point p'_0 .

7.5. Let $n = 5$. Then $k = 2$, and each geodesic γ emanating from a point $p'_0 = p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \text{Int } \bar{\Delta}_0^3$ closes after 2 turns. Only the geodesics that close already after the first turn can be symmetric.

The matrix $A(M_{10})$ in (6) shows that γ comes to the point $p_{10}(\lambda_3, \lambda_4, \lambda_1, \lambda_2)$ after passing along all ten facets of K^5 . It closes if and only if $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$. One of such geodesics is the symmetric geodesic (8), for which we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$.

Consider a 1-turn geodesic γ such that $\lambda_1 = \lambda_3 = \frac{1}{4} + \varepsilon$ and $\lambda_2 = \lambda_4 = \frac{1}{4} - \varepsilon$, where $0 < |\varepsilon| < \frac{1}{4}$. The matrices $A(M_0)$ and $A(M_5)$ in (6) give us the coordinates of p'_0 and p'_5 , so that we can verify the condition of Subsection 7.3:

$$\begin{aligned} p'_0 &= \left(\frac{1}{4} + \varepsilon\right)(0, 0, 0, 0, 0) + \left(\frac{1}{4} - \varepsilon\right)(1, 0, 0, 0, 0) \\ &\quad + \left(\frac{1}{4} + \varepsilon\right)(1, 1, 0, 0, 0) + \left(\frac{1}{4} - \varepsilon\right)(1, 1, 1, 0, 0) \\ &= \left(\frac{3}{4} - \varepsilon, \frac{1}{2}, \frac{1}{4} - \varepsilon, 0, 0\right), \\ p'_5 &= \left(\frac{1}{4} + \varepsilon\right)(0, 0, 0, 1, 1) + \left(\frac{1}{4} - \varepsilon\right)(1, 1, 1, 1, 1) \\ &\quad + \left(\frac{1}{4} + \varepsilon\right)(0, 1, 1, 1, 1) + \left(\frac{1}{4} - \varepsilon\right)(0, 0, 1, 1, 1) \\ &= \left(\frac{1}{4} - \varepsilon, \frac{1}{2}, \frac{3}{4} - \varepsilon, 1, 1\right), \end{aligned}$$

and the midpoint of the segment $p'_0 p'_5$ is the point $(\frac{1}{2} - \varepsilon, \frac{1}{2}, \frac{1}{2} - \varepsilon, \frac{1}{2}, \frac{1}{2})$, and not $O_5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Therefore, (8) is a unique symmetric geodesic in the class considered.

7.6. Let $n = 6$. Then $k = 5$. In accordance with the matrix $A(M_{12})$ in (9), the geodesic γ emanating from a point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \text{Int } \bar{\Delta}_0^4$ and parallel to the ray l comes to the point $p_{12}(\lambda_3, \lambda_4, \lambda_5, \lambda_1, \lambda_2)$ after the first turn. It closes after one turn only for

$$\lambda_1 = \lambda_3 = \lambda_5 = \lambda_2 = \lambda_4 = \frac{1}{5},$$

which gives us the geodesic (10).

For all the other initial points p_0 , γ closes only after 5 turns, which consist of the vertices $p_0, \dots, p_{11}, p_{12}, \dots, p_{23}, p_{24}, \dots, p_{35}, p_{36}, \dots, p_{47}$, and $p_{48}, \dots, p_{59}$, respectively. Below we present the list of the vertices $p_0, \dots, p_{11}$ of the first turn, which starts at the point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. The vertices $p_{12}, \dots, p_{23}, p_{24}, \dots, p_{35}, p_{36}, \dots, p_{47}$, and $p_{48}, \dots, p_{59}$ differ from the points $p_0, \dots, p_{11}$ only by the replacement of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ with $(\lambda_3, \lambda_4, \lambda_5, \lambda_1, \lambda_2)$, $(\lambda_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $(\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_1)$, and $(\lambda_4, \lambda_5, \lambda_1, \lambda_2, \lambda_3)$, respectively; $p_{60} = p_0$. The list confirms that the points $p_{30}, \dots, p_{59}$ are symmetric to the points $p_0, \dots, p_{29}$ with respect to O_6 .

$$\begin{aligned} p_0 &= \lambda_1(000000) + \lambda_2(100000) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_1 &= \lambda_1(111110) + \lambda_2(100000) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_2 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_3 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(111000) + \lambda_5(111100), \\ p_4 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(111100), \\ p_5 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_6 &= \lambda_1(000011) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_7 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_8 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(001111) + \lambda_5(000111), \\ p_9 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(000111), \end{aligned}$$

$$p_{10} = \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(110000),$$

$$p_{11} = \lambda_1(111000) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(110000).$$

The 1-turn geodesic (10) and the family of 5-turn geodesics fill a 5-dimensional topological solid torus $T \subset \partial K^6$ and form a Seifert fibration in T [2]. The simplexes $\bar{\Delta}_0^2, \dots, \bar{\Delta}_{12}^4 = \bar{\Delta}_0^4$ are sections of T by 4-planes *not orthogonal* to rectilinear parts of this family of geodesics.

7.7. Let $n = 7$. Then $k = 3$. The matrix $A(M_{14})$ in (11) shows that the geodesic γ emanating in the direction of l from a point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \bar{\Delta}_0^5$ comes to the point $p_{14}(\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_1, \lambda_2)$ after the first turn. Such a geodesic γ closes after the first turn only if $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$. One such geodesic is the symmetric geodesic (12), for which we have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1}{6}.$$

Suppose that

$$\lambda_1 = \lambda_3 = \lambda_5 = \frac{1}{6} + \varepsilon \quad \text{and} \quad \lambda_2 = \lambda_4 = \lambda_6 = \frac{1}{6} - \varepsilon,$$

where $0 < |\varepsilon| < \frac{1}{6}$. The matrices $A(M_0)$ and $A(M_7)$ in (11) give us the coordinates of p'_0 and p'_7 :

$$p'_0 = \left(\frac{5}{6} - \varepsilon, \frac{2}{3}, \frac{1}{2} - \varepsilon, \frac{1}{3}, \frac{1}{6}, 0, 0 \right),$$

$$p'_7 = \left(\frac{1}{6} - \varepsilon, \frac{1}{3}, \frac{1}{2} - \varepsilon, \frac{2}{3}, \frac{5}{6} - \varepsilon, 0, 0 \right),$$

so that we can verify the conditions of Subsection 7.3, and the midpoint of the segment $p_0 p_7$ does not lie at $O_7 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence, (12) is a unique 1-turn geodesic in the class considered.

All other geodesics, for which the relations $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$ are violated, close only after the third turn. Are some of them symmetric? The answer turns out to be rather unexpected.

The necessary condition (see Subsection 7.3) requiring that p_0 and p_{21} be symmetric with respect to O_7 is fulfilled only if

$$(14) \quad \lambda_1 = \lambda_4, \quad \lambda_2 = \lambda_5, \quad \lambda_3 = \lambda_6,$$

i.e., for the initial points of the form

$$(15) \quad p_0 \left(\frac{\mu_1}{2}, \frac{\mu_2}{2}, \frac{\mu_3}{2}, \frac{\mu_1}{2}, \frac{\mu_2}{2}, \frac{\mu_3}{2} \right),$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$, and $\mu_1 + \mu_2 + \mu_3 = 1$. Such points form the interior of the triangle with vertices at the midpoints of the edges $D_1^0 D_4^0$, $D_2^0 D_5^0$, and $D_3^0 D_6^0$ of $\bar{\Delta}_0^4$.

The matrices (11) give us all the 42 vertices of the geodesic γ emanating from the point (15) in the direction of the ray l . The list of these vertices confirms the symmetry of γ independently of the choice of μ_1 , μ_2 , and μ_3 . Here is only the list of vertices $p_0, \dots, p_6$ of the first half of the first turn and of vertices $p_{21}, \dots, p_{27}$ in the middle of the second turn, which are symmetric to $p_0, \dots, p_6$:

$$p_0 = \frac{\mu_1}{2}(1110000) + \frac{\mu_2}{2}(2111000) + \frac{\mu_3}{2}(2211100),$$

$$p_1 = \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2111000) + \frac{\mu_3}{2}(2211100),$$

$$p_2 = \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(2211100),$$

$$\begin{aligned}
p_3 &= \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(1222211), \\
p_4 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(1222211), \\
p_5 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(1112222) + \frac{\mu_3}{2}(1222211), \\
p_6 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(1112222) + \frac{\mu_3}{2}(0111222), \\
p_{21} &= \frac{\mu_1}{2}(1112222) + \frac{\mu_2}{2}(0111222) + \frac{\mu_3}{2}(0011122), \\
p_{22} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0111222) + \frac{\mu_3}{2}(0011122), \\
p_{23} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(0011122), \\
p_{24} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(1000011), \\
p_{25} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(1000011), \\
p_{26} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(1110000) + \frac{\mu_3}{2}(1000011), \\
p_{27} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(1110000) + \frac{\mu_3}{2}(2111000).
\end{aligned}$$

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